On semi-linear parabolic partial differential equations

By

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We give an operator theoretical treatment of initial value problems for semi-linear parabolic partial differential equations: existence of solutions, uniqueness and regularity. We make use of the theory of fractional powers of operators, the theory of semi-groups of operators and L_p-estimates for elliptic boundary value problems.

§0. Introduction

The purpose of this paper is to derive some results on the initial value problems for semi-linear parabolic partial differential equations. The equation is as follows:

$$(0.1)\quad \begin{cases} \dfrac{\partial u}{\partial t} = -A(x\,;D)u + F(t, x, u) & (t, x)\in(0, T]\times G, \\ B_j(x\,;D)u = 0 & (t, x)\in(0, T]\times\partial G \quad (j=1, \cdots, m), \\ u|_{t=0} = u_0. \end{cases}$$

Here G is a bounded domain in Euclidean n-space E_n, A is an elliptic partial differential operator on G of order $2m$ and $\{B_j\}_{j=1}^m$ is a system of m differential operators on the boundary of G. A and $\{B_j\}$ satisfies some algebraic conditions ((R) and (C) in §3). F may contain the derivatives $D_x^\mu u$ of u of order less than $2m$ in its variables.

If we take some function space ($L_p(G)$) and realize $A(x\,;D)$ (with the boundary conditions $\{B_j\}$) as an unbounded closed operator A in it, we can rewrite (0.1) in the following abstract evolution equation:

Received December 25, 1964

(0.2)
$$\begin{cases} \dfrac{du}{dt} = -Au+F(t, u) & t\in(0, T], \\ u|_{t=0} = u_0 . \end{cases}$$

Moreover, if $-A$ generates an analytic semi-group of bounded operators, we can rewrite (0.2) in the following abstract integral equation:

(0.3)
$$u(t) = e^{-tA}u_0+\int_0^t e^{-(t-s)A}F(s, u)(s))ds .$$

Under some conditions on F and u_0 (see §2), we show that (0.3) has an unique local (in time) solution (Theorem. 2.1) and that the solution of (0.3) is a solution of (0.2) and the converse holds (Theorem. 2.2 and 2.3). Finally we study the regularity properties of such a generalized solution of (0.1).

Our treatment of (0.1) was suggested by the recent study of Kato and Fujita ([7], [8]) on the Navier Stokes initial value problem. We construct the solution of (0.3) by the successive approximation. In this procedure fractional powers A^α of A play an important role.

In Section 1 we summarize well-known results on fractional powers of closed operators (see [3], [9], [10] and [11]). In Section 2 we consider an abstract evolution equation in a Banach space. The results obtained in this section are variants of [7] which are stated in sharper forms in [8]. However we describe full proofs for the sake of self-containedness. In Section 3 we prove some a priori estimates for fractional powers of elliptic partial differential operators (Theorem 3.1, 3.2 and 3.3). In the proofs we make use of the general L_p estimates for elliptic boundary value problems established in [1], [2], [4] and [6]. These a priori estimates are main tools in this paper and enables us to sudy (0.1) as an abstract evolution equation in $L_p(G)$. Our main results are Theorem 4.1 (existence of the local solution), Theorem 5.1 (regularity of the solution in the interior) and Theorem 6.1 (regularity of the solution up to the boundary).

The author wishes to thank Prof. Yosida for his encouragement. He also wishes to express his appreciation to Dr. Komatsu, who suggested this problem, for his helpful suggestions.

§ 1. Frational powers of closed operators

Let X be a complex Banach space and A a linear operator from $D(A)\subset X$ into X. Let us assume:

(A. 1) (i) *$D(A)$ is dense in X. A is a closed operator.*

(ii) *The resolvent set $\rho(-A)$ of $-A$ contains $\{\lambda\geq 0\}$. The resolvent $(\lambda I+A)^{-1}$ of $-A$ satisfies $\|(\lambda I+A)^{-1}\|\leq M/\lambda$ for every $\lambda>0$.*

Note that the assumption (A. 1) implies

$$\|(\lambda I+A)^{-1}\|\leq M' \qquad \textit{for} \quad 0\leq\lambda\leq 1 .$$

Under the assumption (A. 1), we can define the fractional power $A^{-\alpha}$ of A for $0\leq\alpha<1$ by means of the formula

$$A^{-\alpha} = \frac{\sin\pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda I+A)^{-1}d\lambda \qquad (0<\alpha<1),$$
$$A^0 = I .$$

$A^{-\alpha}$ is a bounded linear operator on X and has an inverse. We define A^{α} by

$$A^{\alpha} = (A^{-\alpha})^{-1} .$$

For $\alpha>0$ we define $A^{-\alpha}=A^{-[\alpha]}A^{-(\alpha-[\alpha])}$ and then $A^{\alpha}=(A^{-\alpha})^{-1}$. In this way A^{α} may be defined for any real α. For $\alpha>0$ $A^{-\alpha}$ is bounded and A^{α} is closed. For any real α and β, we have

$$A^{\alpha}A^{\beta}u = A^{\alpha+\beta}u \qquad \text{for} \quad u\in D(A^{\gamma})$$

where $\gamma=\max(\beta, \alpha+\beta)$. In particular, if $\alpha<\beta$, we have $D(A^{\alpha})\supset D(A^{\beta})$ and $\|A^{\alpha}u\|\leq\|A^{\alpha-\beta}\|\,\|A^{\beta}u\|$ for every $u\in D(A^{\beta})$.

Let us assume for A instead of (A. 1):

(A. 2) (i) *The condition* (A. 1) (i) *holds.*

(ii) *$\rho(-A)$ contains a closed sector $\sum_{\pi/2+\theta}=\left\{|\arg\lambda|\leq\frac{\pi}{2}+\theta\right\}$ $\left(0<\theta<\frac{\pi}{2}\right)$. There exists a constant M such that*

$$\|(\lambda I+A)^{-1}\|\leq M/|\lambda| \qquad \textit{for} \quad \lambda\in\textstyle\sum_{\pi/2+\theta} .$$

Under the assumption (A. 2), $-A$ generates a semi-group e^{-tA} by means of the formula

$$e^{-tA} = \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I+A)^{-1}d\lambda ,$$

where Γ is a smooth contour running in $\sum_{\pi/2+\theta}$ from $\infty e^{-i(\pi/2+\theta)}$ to $\infty e^{i(\pi/2+\theta)}$. e^{-tA} is analytic in t in $\sum_\theta - \{0\}$.

For any $\alpha \geq 0$ there exists a constant M_α such that for any β with $0 \leq \beta \leq \alpha$ we have

$$||A^\beta e^{-tA}|| \leq M_\alpha |t|^{-\beta} \qquad \text{for} \quad t \in \textstyle\sum_\theta .$$

§ 2. Abstract evolution equation in a Banach space

We consider in this section the abstract evolution equation

$$(2.1) \qquad \begin{cases} \dfrac{du}{dt} = -Au + F(t, u) & 0 < t \leq T, \\ u|_{t=0} = u_0 \end{cases}$$

in a complex Banach space X.

We first state the assumptions to be made in the theorems.

(A. 2) *$-A$ is independent of t and generates an analytic semi-group e^{-tA} of bounded linear operators on X.*

(A. 3) (*Assumptions on F*) *There exists a constant α with $0 \leq \alpha < 1$ such that*:

(i) *$F(t, u)$ is a function from $(0, T] \times D(A^\alpha)$ into X;*

(ii) *$||F(t, u)|| \leq f(||A^\alpha u||)$ for $t \in (0, T]$ and $u \times D(A^\alpha)$;*

(iii) *$||F(t, u) - F(t, v)|| \leq g(||A^\alpha u|| + ||A^\alpha v||) \, ||A^\alpha(u - v)||$ for $t \in (0, T]$ and $u, v \in D(A^\alpha)$;*

(iv) *there exists γ with $0 < \gamma < 1$ such that*

$$||F(t, u) - F(t', u)|| \leq h(||A^\alpha u||) |t - t'|^\gamma \textit{ for } t, t' \in (0, T]$$

and $u \in D(A^\alpha)$,

where f, g and h are functions defined on $[0, \infty)$ which are non-negative and non-decreasing (and continuous).

In what follows, we always assume (A. 2) and (A. 3).

Now we consider the following abstract integral equation associated with (2. 1):

$$(2.2) \qquad u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} F(s, u(s)) ds \qquad 0 < t \leq T .$$

Definition 2. 1. *We call $u(t)$ a strict solution of* (2.1) *in $[0, T]$ if*

(i) *$u(t)$ is strongly continuous in $[0, T]$ and strongly continuously differentiable in $(0, T]$;*

(ii) *for each* $t\in(0, T]$ $u(t)$ *belongs to* $D(A)$ *and* $Au(t)$ *is strongly continuous in* $(0, T]$;

(iii) $u(t)$ *satisfies* (2.1).

Definition 2.2. *We call* $u(t)$ *a mild solution of* (2.2) *in* $[0, T]$ *if*

(i) $u(t)$ *is strongly continuous in* $[0, T]$;

(ii) *for each* $t\in(0, T]$ $u(t)$ *belongs to* $D(A^{\alpha})$ *and* $A^{\alpha}u(t)$ *is strongly bounded and continuous in* $(0, T]$;

(iii) $u(t)$ *satisfies* (2.2).

On the existence and uniqueness of the strict solution of (2.1) and the mild solution of (2.2), we can prove the following:

Theorem 2.1. *For every* $u_0\in D(A^{\alpha})$ *there exists a mild solution* $u(t)$ *of* (2.2) *in* $[0, T_0]$ (*for some* T_0 *with* $0<T_0\leq T$), T_0 *depending only on* $\|A^{\alpha}u_0\|$. *The mild solution of* (2.2) *is unique where it exists, if* $u_0\in D(A^{\alpha})$.

Theorem 2.2. *Any mild solution* $u(t)$ *of* (2.2) *in* $[0, T]$ *with* $u_0\in D(A^{\alpha})$ *is a strict solution of* (2.1) *in* $[0, T]$.

Theorem 2.3. *Any strict solution* $u(t)$ *of* (2.1) *in* $[0, T]$ *such that* $A^{\alpha}u(t)$ *is strongly bounded and continuous in* $(0, T]$ *is a mild solution of* (2.2) *in* $[0, T]$. *In particular, the strict solution of* (2.1) *is unique under the condition that* $A^{\alpha}u(t)$ *is strongly bounded and continuous.*

In this section we only use the strong topology of X. So, in what follows, we often write "bounded" for "strongly bouuded" and "continuous" for "strongly continuous" etc.

We also use the following notation: *For a bounded and continuous function* $w(t)$ *from* $(0, T]$ *into* X, *we put*

$$\|w\|_t=\sup_{0<s<t}\|w(s)\|.$$

Now we described two lemmas which we need in the proof of Theorem 2.1. The proofs are very easy and omitted.

Lemma 2.1. (i) *If* $A^{\alpha}u(t)$ *is bounded and continuous, so is* $F(t, u(t))$ *where it is defined.*

(ii) *If* $A^{\alpha}u(t)$ *is bounded and Hölder continuous, so is* $F(t, u(t))$.

Lemma 2. 2. *Let $w(t)$ be a bounded and continuous function from $(0, T]$ into X. Put*

$$v(t) = \int_0^t e^{-(t-s)A} w(s)\,ds \qquad 0 \leq t \leq T. \tag{2.3}$$

Then, for any α with $0 \leq \alpha < 1$,

(i) $v(t) \in D(A^\alpha) \qquad (0 \leq t \leq T)$.

(ii) $A^\alpha v(t) = \int_0^t A^\alpha e^{-(t-s)A} w(s)\,ds$ *and*

$\| A^\alpha v(t) \| \leq M_\alpha \int_0^t (t-s)^{-\alpha} \| w(s) \|\,ds$, *hence*

$\| A^\alpha v \|_t \leq M_\alpha \dfrac{t^{1-\alpha}}{1-\alpha} \| w \|_t$.

(iii) $A^\alpha(t)$ *is continuous in* $[0, T]$,

where M_α is a constant in §1.

Now we prove Theorem 2. 1. We first put

$$\begin{cases} u_0(t) = e^{-tA} u_0, \\ u_k(t) = e^{-tA} u_0 + \int_0^t e^{-(t-s)A} F(s, u_{k-1}(s))\,ds, \quad k = 1, 2, \cdots. \end{cases} \tag{2.4}$$

Because of the assumption (A. 3) and Lemma 2.1 and 2.2, each $u_k(t)$ is defined for $t \in (0, T]$ and $A^\alpha u_k(t)$ is continuous in $[0, T]$. Next we put

$$a_k(t) = \| A^\alpha u_k \|_t \qquad \text{for} \quad k = 0, 1, 2, \cdots.$$

Then, $a_k(t)$ is a continuous and non-decreasing function of t. Applying (A. 2) (ii) and Lemma 2. 2 to (2. 3), we obtain

$$a_k(t) \leq a_0 + M_\alpha \frac{t^{1-\alpha}}{1-\alpha} f(a_{k-1}(t)) \qquad (k = 1, 2, \cdots),$$

where $a_0 = M_0 \| A^\alpha u_0 \| \quad (\geq a_0(t))$.

Hence there exists $T_0 > 0$ dependent only on a_0 (when M_α, α and f are given) such that

$$\sup_k a_k(T_0) = a < \infty.$$

Now we put

$$b_k(t) = \| A^\alpha (u_{k+1} - u_k) \|_t \qquad \text{for} \quad k = 0, 1, \cdots.$$

Using (A. 3) (iii), we get

$$b_k(t) \leq M_\alpha g(2a) \int_0^t (t-s)^{-\alpha} b_{k-1}(s)\, ds\,.$$

Hence we have

$$b_k(t) \leq K^k t^{k(1-\alpha)} \prod_{j=0}^{k-1} B(1-\alpha, j(1-\alpha)+1) \times 2a$$
$$= 2a\Gamma(1)(Kt^{1-\alpha}\Gamma(1-\alpha))^k / \Gamma(k(1-\alpha)+1)\,,$$

where $K = M_\alpha g(2a)$.
Hence we have

$$\sum_{a=0}^{\infty} \| A(u_{k+1}-u_k) \|_{T_0} = \sum_{k=0}^{\infty} b_k(T_0) < \infty\,.$$

In other words, $\{A^\alpha u_k(t)\}$ converges uniformly on $[0, T_0]$, and since $A^{-\alpha}$ is a bounded operator, $\{u_k(t)\}$ converges uniformly on $[0, T_0]$. Hence there exists a function $u(t)$ from $[0, T_0]$ into $D(A^\alpha)$ such that

$$s-\lim u_k(t) = u(t) \qquad \text{and}$$
$$s-\lim A^\alpha u_k(t) = A^\alpha u(t)$$

uniformly on $[0, T_0]$. Obviously $u(t)$ and $A^\alpha u(t)$ are continuous on $[0, T_0]$ and $\|A^\alpha u\|_{T_0} \leq a$. Applying (A. 3) (iii), we can see that $F(s, u_k(s))$ converges to $F(s, u(s))$ uniformly in $s \in [0, T_0]$. Hence, passing to the limit in (2. 4), we obtain

$$u(t) = e^{-tA}u_0 + \int_0^t e^{-(t-s)A} F(s, u(s))\, ds\,, \qquad 0 < t \leq T_0\,.$$

Hence $u(t)$ is a mild solution of (2. 2) in $[0, T_0]$.
Uniqueness: Let $u(t)$ and $v(t)$ be mild solutions of (2. 2) in $[0, T_1]$ $(0 < T_1 \leq T)$ such that $u(0) = v(0) = u_0 \in D(A^\alpha)$. This implies that $A^\alpha u(t)$ and $A^\alpha v(t)$ is continuous in $[0, T_1]$. We put

$$\begin{cases} b(t) = \|A^\alpha(u-v)\|_t & \text{and} \\ K_1 = M_\alpha g(\|A^\alpha u\|_{T_1} + \|A^\alpha v\|_{T_1})\,. \end{cases}$$

Then, by (A. 3) (iii)

$$b(t) \leq K_1 \int_0^t (t-s)^{-\alpha} b(s)\, ds$$
$$\leq b(t)\Gamma(1)(K_1 t^{1-\alpha}\Gamma(1-\alpha))^k / \Gamma(k(1-\alpha)+1)$$
$$\to 0 \qquad \text{as} \quad k \to \infty\,.$$

Hence we have

$$b(t)=0 \quad \text{for} \quad 0\leq t\leq T_1.$$

Thus the proof of Theorem 2.1 is completed.

To prove Theorem 2.2, we also need some lemmas.

Lemma 2.3. *Let $v(t)$ and $w(t)$ be as in Lemma 2.2. Then for any β with $0<\beta<1$ $A^\beta v(t)$ is (uniformly) Hölder continuous with Hölder exponent $(1-\beta)$.*

Proof: Let $h>0$. Then by Lemma 2.2.

$$\begin{aligned} A^\beta v(t+h)-A^\beta v(t) &= \int_t^{t+h} A^\beta e^{-(t+h-s)A}w(s)\,ds \\ &\quad + \int_0^t A^\beta\{e^{-(t+h-s)A}-e^{-(t-s)A}\}w(s)\,ds \\ &= I_1+I_2. \end{aligned}$$

Obviously we have

$$\begin{aligned} \|I_1\| &\leq M_\beta \int_t^{t+h}(t+h-s)^{-\beta}\|w\|_T\,ds \\ &\leq M_\beta \frac{h^{1-\beta}}{1-\beta}\|w\|_T. \end{aligned}$$

On the other hand, since

$$I_2=\int_0^t\left\{A^\beta\int_0^h \frac{\partial}{\partial r}(e^{-(t-s+b)A}w(s))\,dr\right\}ds,$$

we have

$$\begin{aligned} \|I_2\| &\leq \int_0^t ds\int_0^h \|A^{1+\beta}e^{-(t-s+r)A}w(s)\|\,dr \\ &\leq M_{1+\beta}\int_0^t ds\int_0^h (t-s+r)^{-1-\beta}\,dr\,\|w\|_T. \end{aligned}$$

Changing the order of integration, we obtain

$$\begin{aligned} \|I_2\| &\leq M_{1+\beta}\int_0^h \frac{1}{\beta}r^{-\beta}\,dr\,\|w\|_T \\ &\leq M_{1+\beta}\frac{1}{\beta(1-\beta)}h^{1-\beta}\|w\|_T. \end{aligned}$$

Thus the proof is completed.

Lemma 2.4. *Let $v(t)$ and $w(t)$ be as in Lemma 2.2. Let $w(t)$ be Hölder continuous (with exponent γ_1). Then:*

(i) $v(t) \in D(A)$ *for* $t \in (0, T]$,

$$Av(t) = \int_0^t Ae^{-(t-s)A} w(s)\, ds \qquad for \quad t \in (0, T],$$

and $Av(t)$ *is continuous in* $(0, T]$.

(ii) $v(t)$ *is (strongly) continuously differentiable in* $(0, T]$, *and*

$$\frac{dv}{dt} = -Av + w \qquad for \quad 0 < t \leq T.$$

Proof: Let $\varepsilon_0 > 0$ be fixed. Let $0 < \varepsilon < \varepsilon_0 \leq t \leq T$. First we put

$$\begin{aligned} v_\varepsilon(t) &= \int_0^{t-\varepsilon} e^{-(t-s)A} w(s)\, ds \\ &= e^{-\varepsilon A} \int_0^{t-\varepsilon} e^{-(t-\varepsilon-s)A} w(s)\, ds \\ &= e^{-\varepsilon A} v(t-\varepsilon). \end{aligned}$$

Then we have

$$v_\varepsilon(t) \in D(A) \quad \text{for } t \in [\varepsilon_0, T] \quad \text{and}$$
$$Av_\varepsilon(t) = \int_0^{t-\varepsilon} Ae^{-(t-s)A} w(s)\, ds.$$

Moreover. $Av_\varepsilon(t) = Ae^{-\varepsilon A} v(t-\varepsilon)$ is continuous in $[\varepsilon_0, T]$. Next, calculating formally, we have

$$\begin{aligned} &\int_{t-\varepsilon}^t Ae^{-(t-s)A} w(s)\, ds \\ &= \int_{t-\varepsilon}^t Ae^{-(t-s)A} w(t)\, ds - \int_{t-\varepsilon}^t Ae^{-(t-s)A} \{w(t) - w(s)\}\, ds \\ &= I_1(t) + I_2(t). \end{aligned}$$

Now

$$\begin{aligned} I_1(t) &= \int_{t-\varepsilon}^t \left\{\frac{d}{ds} e^{-(t-s)A} w(t)\right\} ds \\ &= (I - e^{-\varepsilon A}) w(t). \end{aligned}$$

Hence for each $t \in [\varepsilon_0, T]$ $I_1(t)$ exists and

$$\|I_1(t)\| \to 0 \qquad \text{as} \quad \varepsilon \to 0.$$

Since $W = \{w(t);\ t \in [\varepsilon_0, T\}]$ is compact in X, and $\{I - e^{-\varepsilon A}\}$ is a sequence of equi-continuous (uniformly bounded) operators on X, $I_1(t)$ converges to 0 (as $\varepsilon \to 0$) uniformly in $t \in [\varepsilon_0, T]$. On the other hand, the Hölder continuity of $w(t)$:

$$\|w(t)-w(s)\|\leq K|t-s|^{\gamma_1} \qquad \text{for} \quad t,\, s\in[\varepsilon_0,\, T],$$

implies that $I_2(t)$ really exists for $t\in[\varepsilon_0, T]$ and

$$\begin{aligned}\|I_2(t)\| &\leq \int_{t-\varepsilon}^{t}\|Ae^{-(t-s)A}\{w(t)-w(s)\}\|ds \\ &\leq M_1\int_{t-\varepsilon}^{t}(t-s)^{-1}K(t-s)^{\gamma_1}ds \\ &= M_1K\frac{\varepsilon^{\gamma_1}}{\gamma_1}, \qquad \text{for} \quad t\in[\varepsilon_0,\, T].\end{aligned}$$

Hence $I_2(t)$ also converges to 0 as $\varepsilon\to 0$ uniformly in $t\in[\varepsilon_0, T]$. Thus there exists

$$\int_0^t Ae^{-(t-s)A}w(s)\,ds = s-\lim_{\varepsilon\to 0}\int_0^{t-\varepsilon}Ae^{-(t-s)A}w(s)\,ds\,.$$

Since the convergence is uniform in $t\in[\varepsilon_0, T]$ and $\int_0^{t-\varepsilon}Ae^{-(t-s)A}w(s)ds$ $=Av_\varepsilon(t)$ is continuous in $[\varepsilon_0, T]$, $\int_0^t Ae^{-(t-s)A}w(s)ds$ is also continuous in $[\varepsilon_0, T]$. While $s-\lim v_\varepsilon(t)=v(t)$ and A is a closed operator in X, we have

$$\begin{aligned}&v(t)\in D(A) \qquad \text{for} \quad t\in[\varepsilon_0,\, T] \quad \text{and} \\ &Av(t) = \int_0^t Ae^{-(t-s)A}w(s)\,ds\,.\end{aligned}$$

Since we can take $\varepsilon_0>0$ arbitrarily small, the proof of (i) is completed.

Now we will show (ii). Let $t\in(0, T]$ be fixed and let $h>0$. Then we have

$$\begin{aligned}v(t+h)-v(t) &= \int_t^{t+h}e^{-(t+h-s)A}w(s)\,ds \\ &\quad +\int_0^t\{e^{-(t+h-s)A}-e^{-(t-s)A}\}w(s)\,ds \\ &= \int_0^h e^{-(h-s)A}w(t)\,ds+\int_0^h e^{-(h-s)A}\{w(t+s)-w(t)\}\,ds \\ &\quad +(e^{-hA}-I)\,v(t) \\ &= I_1+I_2+I_3\,.\end{aligned}$$

It follows easily that

$$\frac{1}{h}I_1 = \frac{1}{h}(I-e^{-hA})A^{-1}w(t) \to w(t)\,,$$
$$\frac{1}{h}I_2 \to 0\,,$$
$$\frac{1}{h}I_3 \to -Av(t)$$

as $h \to 0$. Hence we have

$$s-\lim_{h \to +0} \frac{v(t+h)-v(t)}{h} = \frac{d^+}{dt}v(t) = -Av(t)+w(t) \qquad \text{for} \quad t \in (0, T]\,.$$

Since the right hand side is continuous in $t \in (0, T]$, dv/dt exists and is equal to d^+v/dt. Thus we have completed the proof.

The proof of Theorem 2.2: Let $u(t)$ be a mild solution of (2.2) in $[0, T]$. Then it follows that $A^\alpha u(t)$ is bounded and continuous in $(0, T]$, which implies that $F(t, u(t))$ is bounded and continuous in $(0, T]$ (Lemma 2.1). Therefore $A^\alpha u(t)$ is Hölder continuous in $(0, T]$ (Lemma 2.3), and so is $w(t) = F(t, u(t))$ (Lemma 2.1). Hence by Lemma 2.4

$$u(t) \in D(A) \qquad \text{for} \quad t \in (0, T]\,, \quad \text{and}$$
$$\frac{du}{dt} = -Ae^{-tA}u_0 - A\int_0^t e^{-(t-s)A}F(s, u(s))\,ds + F(t, u(t))$$
$$= -Au + F(t, u(t)) \qquad \text{for} \quad t \in (0, T]\,.$$

The continuity of du/dt in $(0, T]$ follows from the continuity of $Au(t)$ and $F(t, u(t))$, and the proof is completed.

The proof of Theorem 2.3: Let $t > s > 0$. Then we have

$$\frac{d}{ds}\{e^{-(t-s)A}u(s)\} = e^{-(t-s)A}\left\{\frac{du}{ds} + Au\right\}.$$

Since the right hand side is continuous in $(0, T]$, we can integrate both sides on $[\varepsilon, t]$ $(0 < \varepsilon < t \leq T)$. Thus we have

$$u(t) - e^{-(t-\varepsilon)A}u(\varepsilon) = \int_\varepsilon^t e^{-(t-s)A}\left\{\frac{du}{ds} + Au\right\}ds$$
$$= \int_\varepsilon^t e^{-(t-s)A}F(s, u(s))\,ds\,.$$

Letting $\varepsilon \to 0$, we have arrived at the desired relation. The convergence of the right hand side follows from the boundedness and continuity of $F(t, u(t))$ in $(0, T]$.

Now we will describe some lemmas which are used later and concern certain property of $v(t)$ and $w(t)$ in (2.3):

$$v(t) = \int_0^t e^{-(t-s)A} w(s)\,ds \qquad (0 \leq t \leq T). \tag{2.3}$$

Lemma 2.5. *Let $w(t) \in D(A^\beta)$ and $A^\beta w(t)$ be bounded and continuous in $(0, T]$. Then, for any α with $0<\alpha<1$, $v(t) \in D(A^{\alpha+\beta})$ for $t \in [0, T]$ and*

$$A^{\alpha+\beta} v(t) = \int_0^t A^\alpha e^{-(t-s)A} A^\beta w(s)\,ds \qquad (0 \leq t \leq T).$$

Moreover $A^{\alpha+\beta} v(t)$ is Hölder continnous in $[0, T]$ with exponent $(1-\alpha)$.

The proof is quite similar to the proof of Lemma 2.3. Let $w(t)$ be bounded and (not necessarily uniformly) Hölder continuous (exponent γ_1) in $(0, T]$. Then by Lemma 2.3 it follows that $v(t) \in D(A)$ for $t \in (0, T]$ and

$$Av(t) = \int_0^t Ae^{-(t-s)A} w(s)\,ds \qquad (0 < t \leq T).$$

Let $\varepsilon > 0$ be fixed. Then we have (as in the proof of Lemma 2.4)

$$\begin{aligned} Av(t) &= Ae^{-\varepsilon A} v(t-\varepsilon) + w(t) - e^{-\varepsilon A} w(t) \\ &\quad - \int_{t-\varepsilon}^t Ae^{-(t-s)A}\{w(t)-w(s)\}\,ds \\ &= I_0(t) + I_1(t) - I_2(t) - I_3(t) \qquad \text{for} \quad t \in [\varepsilon, T]. \end{aligned} \tag{2.4}$$

We have the following

Lemma 2.6. *Let $w(t)$ be bounded and (not necessarily uuiformly) γ_1-Hölder continuous in $(0, T]$. Let $\varepsilon > 0$ be fixed. Then*

(i) *$A^k I_0(t)$ and $A^k I_2(t)$ are Hölder continuous in $[\varepsilon, T]$ with exponent γ_1 for any $k \geq 0$.*

(ii) *$A^{\gamma_2} I_3(t)$ is Hölder continuous in $(\varepsilon, T]$ with exponent $\gamma_1 - \gamma_2$ for $0 < \gamma_2 < \gamma_1$.*

Proof: The proof of (i) is obvious and we have only to prove (ii). It is clear that $I_3(t) \in D(A^{\gamma_2})$ for $t \in [\varepsilon, T]$ and

$$A^{\gamma_2}I_3(t) = \int_{t-\varepsilon}^{t} A^{1+\gamma_2}e^{-(t-s)A}\{w(t)-w(s)\}\,ds \qquad (\varepsilon \leq t \leq T).$$

Let $\varepsilon/2 \geq h \geq 0$. Then

$$\begin{aligned} A^{\gamma_2}I_3(t+h)-A^{\gamma_2}I_3(t) &= \int_{t}^{t+h} A^{1+\gamma_2}e^{-(t+h-s)A}\{w(t+h)-w(s)\}\,ds \\ &\quad -\int_{t-h}^{t} A^{1+\gamma_2}e^{-(t-s)A}\{w(t)-w(s)\}\,ds \\ &\quad +\Big[\int_{t+h-\varepsilon}^{t} A^{1+\gamma_2}e^{-(t+h-s)A}\{w(t+h)-w(s)\}\,ds \\ &\quad -\int_{t-\varepsilon}^{t-h} A^{1+\gamma_2}e^{-(t-s)A}\{w(t)-w(s)\}\,ds\Big] \\ &= J_1+J_2+J_3. \end{aligned}$$

The Hölder continuity of $w(t)$:

(2.5) $\quad ||w(t)-w(s)|| \leq K|t-s|^{\gamma_1} \quad$ for $\quad t, s \in [\varepsilon', T] \ (0<\varepsilon' \leq \varepsilon/2)$,

implies

$$||J_1|| \leq M_{1+\gamma_2}K\int_{t}^{t+h}(t+h-s)^{-1-\gamma_2+\gamma_1}ds = M_{1+\gamma_2}K\frac{1}{\gamma_1-\gamma_2}h^{\gamma_1-\gamma_2}$$

and

$$||J_2|| \leq M_{1+\gamma_2}K\frac{1}{\gamma_1-\gamma_2}h^{\gamma_1-\gamma_2} \qquad (\text{for } t \in [\varepsilon, T]).$$

Now we have

$$\begin{aligned} J_3 &= A^{\gamma_2}e^{-hA}\int_{0}^{\varepsilon-h} Ae^{-sA}\{w(t+h)-w(t)\}\,ds \\ &\quad +\int_{h}^{\varepsilon} A^{1+\gamma_2}e^{-sA}\{w(t-s)-w(t+h-s)\}\,ds \\ &= A^{\gamma_2}e^{-hA}(I-e^{-(\varepsilon-h)A})\{w(t+h)-w(t)\} \\ &\quad +\int_{h}^{\varepsilon} A^{1+\gamma_2}e^{-sA}\{w(t-s)-w(t+h-s)\}\,ds \\ &= J_3'+J_3''. \end{aligned}$$

Obviously (2.5) implies

$$||J_3'|| \leq M_{\gamma_2}h^{-\gamma_2}(1+M_0)Kh^{\gamma_1} = M'h^{\gamma_1-\gamma_2} \qquad \text{for} \quad t \in [\varepsilon, T].$$

Now let $t \in [\varepsilon+\varepsilon', T]$. Then (2.5) implies

$$||J_3''|| \leq M_{1+\gamma_2}\int_h^{\varepsilon} s^{-1-\gamma_2} K h^{\gamma_1} ds$$

$$\begin{cases} \leq M_{1+\gamma_2} K \dfrac{1}{\gamma_2} h^{\gamma_1-\gamma_2} & \text{if} \quad \gamma_2 > 0\,, \\ = M_1 K \log \dfrac{\varepsilon}{h} h & \text{if} \quad \gamma_2 = 0\,. \end{cases}$$

Thus the proof is completed.

Remark: We can also show that if $w(t)$ is γ_1-Hölder continuous, so are $Av(t)$ and dv/dt.

Lemma 2.7. *Let* $w(t)$ *be continuovs in* $[0, T]$ *and (strongly) continuously differentiable in* $(0, T]$ *and* $w'(t)$ *be bounded. Then*

$$\text{(2.6)} \qquad Av(t) = w(t) - e^{-tA}w(0) - \int_0^t e^{-(t-s)A} w'(s)\,ds \qquad \textit{for} \quad t \in [0, T]\,.$$

§ 3. A priori estimates[1]

Let G be a bounded domain in Euclidean n-space E_n. We denote by ∂G the boundary of G, by $\bar{G}$ the closure of G. We denote by $x = (x_1, \cdots, x_n)$ the generic point in E_n. We use the notation

$$D_i = \frac{\partial}{\partial x_i}\,, \qquad D = (D_1, \cdots, D_n)\,,$$

denoting by

$$D^{\mu} = D_1^{\mu_1} \cdots D_n^{\mu_n}$$

a general derivative. Here μ is the n-tuple of non-negative integers $\mu = (\mu_1, \cdots, \mu_n)$ whose length $\mu_1 + \cdots + \mu_n$ is denoted by $|\mu|$.

We consider complex valued functions $u(x)$ defined in G (or $\bar{G}$). Let $C^j(\bar{G})$ be the class of functions which are j-times continuously differentiable in $\bar{G}$. For $u \in C^j(\bar{G})$ we introduce the norm:

$$||u||_{j,L_p} = \Big(\sum_{|\mu| \leq j} \int_G |D^{\mu} u|^p dx\Big)^{1/p} \qquad (1 \leq p < \infty)\,.$$

We denote by $H_{j,L_p}(G)$ the function space obtained by completion of $C^j(\bar{G})$ with the norm $||\ ||_{j,L_p}$. We denote by $C^{j+\beta}(\bar{G})$ the subclass

(1) In the statement of assumptions and a few results we follow [1].

of functions in $C^j(\bar{G})$ whose j-th derivatives are uniformly Hölder continuous with exponent β. For $u \in C^{j+\beta}(\bar{G})$ we introduce the norm:

$$|u|_{j+\beta} = \sum_{|\mu| \leq j} \max |D^\mu u| + \sum_{|\mu|=j} |D^\mu u|_\beta$$

where

$$|v|_\beta = \sup_{x,y \in \bar{G}} \frac{|v(x)-v(y)|}{|x-y|}.$$

Let $A(x; D)$ be an elliptic linear partial differential operator in $\bar{G}$ (with variable complex coefficients) of even order $2m$. Thus the characteristic polynomial associated with the principal part A_0 of A satisfies

$$A_0(x; \xi) \neq 0$$

for each $x \in \bar{G}$ and each real vector $\xi = (\xi_1, \cdots, \xi_n) \neq 0$. For $n=2$[1] we put always on A_0 the following assumption:

(R): *For every pair of linearly independent real vectors ξ, η and $x \in \bar{G}$ the polynomial in t, $A_0(x; \xi + t\eta)$, has exactly m roots with positive imaginary parts.*

Let $\{B_j(x; D); j=1, \cdots, m\}$ be a system of m linear differential operators with coefficients defined on ∂G whose orders m_j are less than $2m$. Denoting the principal part of B_j by B'_j, we assume the following:

(C) *At any $x \in \partial G$ we denote by ν the normal to ∂G and by ξ a (non-zero) tangential vector to ∂G. Let $t_k^+(\xi)$ $(k=1, \cdots, m)$ be the roots of $A_0(x; \xi + t\nu)$ with positive imaginary parts. Then the polynomials in t, $\{B'_j(x; \xi + t\nu); j=1, \cdots, m\}$, are linearly independent modulo $\prod_{k=1}^{m} (t - t_k^+(\xi))$.*

Finally we assume the following two conditions:

(S) *G is of class C^{2m}. The coefficients of A are in $C^0(\bar{G})$ and the coefficients of B_j $(j=1, \cdots, m)$ in $C^{2m-m_j}(\partial G)$.*

(N) *The boundary ∂G is non-characteristic to each B_j at any point $x \in \partial G$. For $j \neq k$ we have $m_j \neq m_k$.*

Definition 3.1. *We call $(A, \{B_j\}, G)$ a regular system (or a*

(1) For $n \geq 3$ the condition (R) always holds.

regular boundary problem), *if it satisfies* (R), (C), (S) *and* (N).

For $k \geq \max m_j$ we introduce a function space $C^k(\bar{G}; \{B_j\}) = \{u \in C^k(\bar{G}); B_j u = 0 \text{ on } \partial G \ (j=1, \cdots, m)\}$. We denote by $H_{k+1, L_p}(G; \{B_j\})$ the closure of $C^{k+1}(G; \{B_j\})$ in $H_{k+1, L_p}(G)$. The following important lemma is due to Agmon-Douglis-Nirenberg ([2]).

Lemma 3.1. (Agmon-Douglis-Nirenberg)
Let $1 < p < \infty$. *Then for each* $u \in H_{2m, L_p}(G; \{B_j\})$

$$\|u\|_{2m, L_p} \leq C(\|Au\|_{L_p} + \|u\|_{L_p}). \tag{3.1}$$

where C is a constant depending on $(A, \{B_j\}, G)$ *and* p, *but not on* u.

Lemma 3.1 is valid without the assumption (N). But in the following we always assume that $(A, \{B_j\}, G)$ is regular.

Now we define a linear (unbounded) operator A_p in $L_p(G)$ as follows:

(i) $D(A_p) = H_{2m, L_p}(G; \{B_j\})$;

(ii) For $u \in D(A_p)$, $A_p u = A(x; D)u$.

The operator A_p is clearly closed and $D(A_p)^a = L_p(G)$. We call A_p the realization of $(A, \{B_j\}, G)$ in $L_p(G)$. In what follows we always assume that $1 < p < \infty$.

For the realization A_p which satisfies (A.1) (ii), we can define fractional powers A_p^α of A_p and the following a priori estimates hold:

Theorem 3.1. *Let* A_p *satisfy* (A.1) (ii), *that is,* $\rho(-A_p) \supset \{\lambda \geq 0\}$ *and*

$$\|(\lambda I + A_p)^{-1}\| \leq M(\lambda+1)^{-1} \quad \textit{for} \quad \lambda \geq 0. \tag{3.2}$$

Then:

(i) *For any* j *and* α *with* $0 \leq j/2m < \alpha \leq 1$, *we have* $D(A_p^\alpha) \subset H_{j, L_p}(G)$ *and there exists* $G(j, \alpha) > 0$ *such that*

$$\|u\|_{j, L_p} \leq C(j, \alpha)\|A_p^\alpha u\|_{L_p} \quad \textit{for every} \quad u \in D(A_p^\alpha). \tag{3.3}$$

(ii) *For every* j, β *and* α *with* $\left(j + \beta + \frac{n}{p}\right)\Big/ 2m < \alpha \leq 1$, *we have* $D(A_p^\alpha) \subset C^{j+\beta}(\bar{G}; \{B_k\}_{m_k < j})$ *and there exists* $C'(j, \beta, \alpha) > 0$ *such that*

$$|u|_{j+\beta} \leq C'(j, \beta, \alpha)\|A_p^\alpha u\|_{L_p} \quad \textit{for every} \quad u \in D(A_p^\alpha). \tag{3.4}$$

To prove Theorem 3.1, we first prove the following:

Lemma 3. 2. *Let A_p be as in Theorem 3. 1. Then:*

(i) *For an integer j with $0 \leq j \leq 2m$, there exists $C>0$ such that*

(3.5)[1] $$||u||_{j,L_p} \leq C(\lambda+1)^{j/2m-1} ||(\lambda I+A_p)u||_{L_p} \quad \text{for } u \in D(A_p) \text{ and } \lambda \geq 0.$$

(ii) *For j and β with $0<j+\beta+\frac{n}{p}<2m$, there exists $C'>0$ such that*

(3. 6) $$|u|_{j+\beta} \leq C'(\lambda+1)^{(j+\beta+n/p)/(2\varepsilon-\varepsilon)-1} ||(\lambda I+A_p)u||_{L_p}$$

for $u \in D(A_p)$ and $\lambda \geq 0$. Here ε is a sufficiently small positive number which can be taken equal to zero if $\beta+\frac{n}{p} \neq$ integer.

Proof: By the inequality of Sobolev, we have

(3. 7) $$||u||_{j,L_p} \leq C_1 ||u||_{2m,L_p}^{j/2m} ||u||_{L_p}^{1-j/2m} \qquad (0 \leq j \leq 2m).$$

By (3. 1) and (3. 2), it follows that

(3. 8) $$\begin{aligned} ||u||_{2m,L_p} &\leq C(||A_p u||_{L_p}+||u||_{L_p}) \\ &\leq C(||(\lambda I+A_p)u||_{L_p}+(\lambda+1)||u||_{L_p}) \\ &\leq C(1+M)||(\lambda I+A_p)u||_{L_p} \quad \text{for } u \in D(A_p). \end{aligned}$$

Thus (3. 5) follows from (3. 2), (3. 7) and (3. 8).

If we use instead of (3. 7) the interpolation theorem:

(3. 9) $$|u|_{j+\beta} \leq C_2 |u|_{2m-\beta-n/p-\varepsilon}^{(j+\beta+n/p)/(2m-\varepsilon)} ||u||_{L_p}^{1-(j+\beta+n/p)/(2m-\varepsilon)}$$

and a well-known inequality of Sobolev:

(3. 10) $$|u|_{2m-\beta-n/p-\varepsilon} \leq C_3 ||u||_{2m,L_p},$$

then we can obtain (3. 6) with the aid of (3. 2) and (3. 8).

Rewriting Lemma 3. 2, we obtain the following:

Lemma 3. 2′. *Let A_p, j, β and α be as in Lemma 3. 2. Then:*

(i) *For every $u \in L_p(G)$*

(3. 5′) $$||(\lambda I+A_p)^{-1}u||_{j,L_p} \leq C(\lambda+1)^{j/2m-1} ||u||_{L_p} \qquad (\lambda \geq 0).$$

---

(1) We often denote by C any constant independent of u. The content C in (3. 5) is not the same constant in (3. 1).

(ii) *For every* $u \in L_p(G)$

(3.6′) $\quad |(\lambda I+A_p)^{-1}u|_{j+\beta} \leq C'(\lambda+1)^{(j+\beta+n/p)/(2m-\varepsilon)-1}\|u\|_{L_p} \qquad (\lambda \geq 0)$.

Proof of Theorem 3.1: To prove (i) it is sufficient to show

(3.11) $\quad \|A_p^{-\alpha}v\|_{j,L_p} \leq C(j,\alpha)\|v\|_{L_p}$ for every $v \in L_p(G)$.

By the definition of $A_p^{-\alpha}$ (for $0<\alpha<1$) we have

$$A_p^{-\alpha}v = \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}(\lambda I+A_p)^{-1}v\,d\lambda\,.$$

Thus (3.5′) implies

$$\begin{aligned}\|A^{-\alpha}v\|_{j,L_p} &\leq \frac{\sin \pi\alpha}{\pi}\int_0^\infty \lambda^{-\alpha}\|(\lambda I+A_p)^{-1}v\|_{j,L_p}d\lambda \\ &\leq \frac{\sin \pi\alpha}{\pi}C\int_0^\infty \lambda^{-\alpha}(\lambda+1)^{j/2m-1}d\lambda\,\|v\|_{L_p} \\ &= C(j,\alpha)\|v\|_{L_p}\end{aligned}$$

The proof of (ii) is quite similar. (Let $\varepsilon>0$ be so small that $\frac{1}{2m-\varepsilon}\left(j+\beta+\frac{n}{p}\right)<\alpha$ holds.)

Now concerning the smoothness assumption of $(A, \{B_j\}, G)$, we make the following:

Definition 3.2. *A is said to have the smoothness of order* $k(\geq 0)$ *on a subdomain* G_0 *of* G, *if all the coefficients of* A *belong to* $C^k(G_0)$.

Definition 3.3. $(A, \{B_j\}, G)$ *is said to have the smoothness of order* k, *if* G *is of class* C^{2m+k}, *the coefficients of* A *are in* $C^k(\bar{G})$ *and the coefficients of* $\{B_j\}$ *in* C^{2m+k}.

Let us consider the regular elliptic boundary problem:

(3.12) $$\begin{cases} A(x\,;D)u(x)=f(x) & \text{in } G \\ B_j(x\,;D)u(x)=0 & \text{on } \partial G \quad (j=1,\cdots,n)\,. \end{cases}$$

Concerning the regularity property of the solution u of (3.12), we have the following important theorem due to Agmon-Douglis-Nirenberg [2] (see also Browder [4], [5] and [6]).

Lemma 3.3. *Let* $(A, \{B_j\}, G)$ *be regular and have the smoothness of order* k. *Then, if* $u \in D(A_p)$ *and* $f \in H_{k,L_p}(G)$ *satisfy* (3.12), u

belongs to $H_{2m+k,L_p}(G)$ *and the following estimate holds*:

$$\|u\|_{2m+k,L_p} \le C(\|f\|_{k,L_p} + \|u\|_{L_p}). \tag{3.13}$$

where C *is a constant independent of* u *and* f.
An easy consequence of Theorem 3.1 and Lemma 3.3 is the following:

Theorem 3.2. *Let* $(A, \{B_j\}, G)$ *be regular and have the smoothness of order* k. *Let* A_p *satisfy* (A.1) (ii). *Then, for* $0 < \frac{k}{2m} < \alpha$ $D(A_p^{1+\alpha}) \subset H_{2m+k,L_p}(G)$ *and the followingestimate holds*:

$$\|u\|_{2m+k,L_p} \le C\|A_p^{1+\alpha}u\|_{L_p} \quad \text{for} \quad u \in D(A_p^{1+\alpha}). \tag{3.14}$$

Now let us study some properties of A_p in the special case where $\{B_j\}_{j=1}^m$ is the Dirichlet boundary condition, that is, $B_j(x; D) = \left(\frac{\partial}{\partial \nu}\right)_x^{j-1} \equiv B_j^0(x; D)$ $(j=1, \cdots, m)$ where ν is the normal to ∂G at x. In what follows we assume on $(A, \{B_j^0\}, G)$ the following:

(A.4) *$A(x; D)$ is defined on some domain $G' \supset \bar{G}$ and if $n=2$, $A(x; D)$ satisfies the condition (R). The coefficients $a_\mu(x)$ of $A(x; D) = \sum a_\mu(x) D^\mu$ are in $C^{2m-|\mu|}(G')$. G is of class C^{4m}.*

The boundary value problem $(A, \{B_j^0\}, G)$ satisfying (A.4) is always regular and $(A', \{B_j^0\}, G)$ is also regular where A' is the formal adjoint of A. Suppose that A_p, the realization of $(A, \{B_j^0\}, G)$ in $L_p(G)$ $(1 < p < \infty)$, satisfies (A.1) (ii). Then, by the argument of Browder ([4], [5] and [6]), we can easily see that $A_p^* = A'_{p'}$ and $(A'_{p'})^* = A_p$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $A'_{p'}$ is the realization of $(A', \{B_j^0\}, G)$ in $L_{p'}(G)$. Moreover $A'_{p'}$ satisfies (A.1) (ii) and $A'^{\alpha}_{p'}$ can be defined. From the definition of A_p^α and $A'^{\alpha}_{p'}$, it follows that $(A_p^\alpha)^* = A'^{\alpha}_{p'}$ and $(A'^{\alpha}_{p'})^* = A_p^\alpha$.
Now let $0 < \alpha < \frac{k}{2m} \le 1$, $u \in C^k(G; \{B_j^0\}_{j<k})$ and $v \in D(A'_{p'})$. Then we have

$$\langle u, A'^{\alpha}_{p'} v\rangle = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} \langle u, A'_{p'}(\lambda I + A'_{p'})^{-1} v\rangle d\lambda,$$

where

$$\langle u, v\rangle = \int_G u(x)\overline{v(x)}\,dx.$$

Since u and $(\lambda I+A'_{p'})^{-1}v$ satisfy some conditions on ∂G, we can integrate by parts k-times in $\langle u, A'_{p'}(\lambda I+A'_{p'})^{-1}\rangle$ and we obtain

$$|\langle u, A'^{\alpha}_{p'}v\rangle| \leq C'\int_0^\infty \lambda^{\alpha-1}||u||_{k,L_p}||(\lambda I+A'_{p'})^{-1}v||_{2m-k,L_{p'}}d\lambda\,.$$

Using Lemma 3.2′, we have

$$|\langle u, A'^{\alpha}_{p'}v\rangle| \leq C||u||_{k,L_p}||v||_{L_{p'}} \tag{3.15}$$

where C is some constant independent of u and v. For any $v\in D(A'^{\alpha}_{p'})$ there exist $v_i\in D(A'_{p'})$ such that $s-\lim v_i=v$ and $s-\lim A'^{\alpha}_{p'}v_i=A'_{p'}v$. (For example, $-A'^{1/2}_{p'}$ generates an analytic semi-group.) Hence (3.15) holds for every $u\in C^k(G\,;\{B_j^0\}_{j<k})$ and $v\in D(A'^{\alpha}_{p'})$. This shows that $u\in D((A'^{\alpha}_{p'})^*)=D(A^{\alpha}_p)$ and

$$||A^{\alpha}_p u||_{L_p}\leq C||u||_{k,L_p}\,.$$

Summing up the above results, we have the following:

Theorem 3.3. *Let $(A, \{B_j^0\}, G)$ satisfy* (A.4) *and A_p, its realization in $L_p(G)$ $(1<p<\infty)$, satisfy* (A.1) (ii). *Then for $\alpha<\frac{k}{2m}\leq 1$, $H_{k,L_p}(G\,;\{B_j^0\}_{j<k})\subset D(A^{\alpha}_p)$ and the following estimate holds*:

$$||A^{\alpha}_p u||_{L_p}\leq C||u||_{k,L_p} \qquad \textit{for} \quad u\in H_{k,L_p}(G\,;\{B_j^0\}_{j<k})\,. \tag{3.16}$$

Theorem 3.3′. *Let $(A, \{B_j^0\}, G)$ satisfy* (A.4) *and have the smoothness of order k. Let A_p, its realization in $L_p(G)$ $(1<p<\infty)$, satisfy* (A.1) (ii). *Then for $\alpha<1+\frac{k}{2m}$ and $\varphi\in C_0^\infty(\bar{G})$, $\varphi\cdot H_{2m+k,L_p}(G)\subset D(A^{\alpha})$ and*

$$||A^{\alpha}_p(\varphi u)||_{L_p}\leq C||\varphi u||_{2m+k,L_p} \qquad \textit{for} \quad u\in H_{2m+k,L_p}(G)\,. \tag{3.16′}$$

Finally we state Agmon's theorem ([1]) which ensures the existence of a resolvent ray of the operator A_p having the same estimate as (A.1) (ii) along the ray.

Lemma 3.4. (Agmon [1]) *Let A_p be the realization of a regular system $(A, \{B_j\}, G)$ in $L_p(G)$ $(1<p<\infty)$ which satisfies the following*:

$$(-1)^m\frac{A_0(x\,;\xi)}{|A_0(x\,;\xi)|}\neq e^{i\theta} \tag{A.5 (i)}$$

for all real vector $\xi\neq 0$ and $x\in\bar{G}$.

(ii) *At any point* $x\in\partial G$ *let* ν *be the normal vector and* $\xi\neq 0$ *any tangential vector to* ∂G. *Denote by* $t_k^+(\xi\,;\lambda)$ *the* m *roots with positive imaginary parts of the polynomial in* t

$$(-1)^m A_0(x\,;\xi+t\nu)-\lambda \qquad (\arg\lambda=\theta)\,.$$

Then the polynomials (in t) $B_j'(x\,;\xi+t\nu)$ $(j=1,\cdots,m)$ *are linearly independent modulo the polonomial* $\prod_{k=1}^{m}(t-t_k^+(\xi\,;\lambda))$ *for any* λ *(with* $\arg\lambda=\theta$).

Then there exists $N\geq 0$ *such that* $L_{\theta,N}=\{\lambda\,;\ \arg\lambda=\theta,\ |\lambda|\geq N\}\subset\rho(A_p)$ *and*

$$\|(\lambda I-A_p)^{-1}\|\leq M/|\lambda| \qquad \text{for} \quad \lambda\in L_{\theta,N}\,.$$

§ 4. Existence of the local solution of semi-linear parabolic equations

Let G be a bounded domain in E_n and let $(A, \{B_j\}, G)$ be a regular elliptic boundary value problem. We consider the following initial value problem of a semi-linear parabolic partial differential equation:

(4.1)

$$\begin{cases} \dfrac{\partial u}{\partial t}=-A(x\,;D)u+F(t,x,D_x^\mu u) & (t,x)\in(0,T]\times G(|\mu|\leq 2m-1),\\ B_j(x\,;D)u=0 & (t,x)\in(0,T]\times\partial G \quad (j=1,\cdots,m)\,,\\ u|_{t=0}=u_0\,. \end{cases}$$

We use the notation

$$C_K=\{u\in C\,;\ |u|\leq K\}$$
$$C_K^r=C_K\times\cdots\times C_K \qquad (r\text{-times})\,.$$

We assume on F the following:

(A. 6) (i) $F=F(t,x,u,\cdots,u_\mu,\cdots)$ $(|\mu|\leq 2m-1)$ *is a complex-valued continnous function defined on* $(0,T]\times G\times C^r$, *where* r *is the number of* n*-tuple* μ *with* $|\mu|\leq 2m-1$.

(ii) *There exist non-negative and non-decreasing functions* f, g *and* h *defined on* $[0,\infty)$ *and a constant* γ *with* $0<\gamma<1$ *such that*

$|F(t, x, u)| \leq f(K)$ *for* $(t, x) \in (0, T] \times G$ *and* $u \in C_K^r$,

$|F(t, x, u) - F(t, x, v)| \leq g(K) \sum_{|\mu| \leq 2m-1} |u_\mu - v_\mu|$ *for* $(t, x) \in (0, T] \times G$ *and* $u, v \in C_K^r$.

$|F(t, x, u) - F(t', x, u)| \leq h(K) |t - t'|^\gamma$ *for* $t, t' \in (0, T]$ *and* $(x, u) \in G \times C_K^r$.

(A. 6) implies that if $u \in C^{2m-1}(G)$, then $F(t, x, D^\mu u) \in L_\infty(G) \subset L_p(G)$.

Now we show the existence of the unique local solution of (4. 1), by applying the argument of §2 and 3.
First we suppose that $(A, \{B_j\}, G)$ satisfies (A. 5) for any $\theta \in [\theta_0, 2\pi - \theta_0]$, where θ_0 is some constant with $0 < \theta_0 < \frac{\pi}{2}$. Let A_p be the realization of $(A, \{B_j\}, G)$ in $L_p(G)$ $(n < p < \infty)$. Then, by the argument of Agmon ([1]) (which was made to prove Lemma 3. 4), we can easily see that there exists a constant $N \geq 0$ such that $(A_p + NI)$ satisfies the condition (A. 2). We rewrite $A(x; D) + N$ as $A(x; D)$ and $F(t, x, u) + Nu$ as $F(t, x, u)$. Then A_p satisfies (A. 2) and F satisfies (A. 6) with trivial modifications of f, g and h. Let α be fixed with $\frac{1}{2m}\left(2m - 1 + \frac{n}{p}\right) < \alpha < 1$. Theorem 3. 1. (ii) implies $D(A_p^\alpha) \subset C^{2m-1}(G; \{B_j\})$ and

$$|u|_{2m-1} \leq C \| A_p^\alpha u \|_{L_p} \qquad \text{for} \quad u \in D(A_p^\alpha),$$

since $p > n$. Thus, modifying f, g and h in (A. 6) once more, we see that the condition (A. 3) holds in $X = L_p(G)$ for $A = A_p$ and such α. Now let us consider (4. 1) as an evolution equation in the Banach space $L_p(G)$ and apply the results of §2, Theorem 2. 2 and 2. 3. Then we have the local (strict) solution of (4. 1) in $[0, T_0]$ which is unique under the condition that $A_p^{\alpha'} u(t)$ is strongly bounded and continuous in $L_p(G)$, where α' is a constant with $\frac{1}{2m}\left(2m - 1 + \frac{n}{p}\right) < \alpha' < 1$. Thus we have the following:

Theorem 4. 1. *Let* $(A, \{B_j\}, G)$ *be regular and satisfy* (A. 5) *for any* $\theta \in [\theta_0, 2\pi - \theta_0]$ *with* $0 < \theta_0 < \frac{\pi}{2}$. *Let* F *satisfy* (A. 6). *Let* $n < p < \infty$ *and* A_p *be the realization of* $(A, \{B_j\}, G)$ *in* $L_p(G)$. *Then for any* $u_0 \in D(A_p^\alpha)$(1) *with* $\frac{1}{2m}\left(2m - 1 + \frac{n}{p}\right) < \alpha < 1$,

(1) We can assume without any loss of generality that A_p satisfies (A. 2).

(i) *there exists the local (strict) solution* $u(t)$ *of* (4.1) *in* $[0, T_0]$ *which is in* $D(A_p)=H_{2m,L_p}(G;\{B_j\})$ *for* $t\in(0, T_0]$ *and strongly Hölder continuous in* $H_{2m,L_p}(G)$ *with exponent* γ *as a function of* $t\in(0, T]$. T_0 *depends on* p, α, f *and* $\|A_p^\alpha u_0\|_{L_p}$.

(ii) *The (strict) solution* $u(t)$ *is unique under the condition that* $A_p^{\alpha'}u(t)$ *is strongly bounded and continuous in* $L_p(G)$, *where* $\frac{1}{2m}\left(2m-1+\frac{n}{p}\right)<\alpha'<1$.

Proof: We have only to show that $u(t)$ is strongly γ-Hölder continuous in $H_{2m,L_p}(G)$. But this follows from the Remark to Lemma 2.6 and Lemma 3.3 (and Lemma 2.3)(1) applied on the integral representation of $u(t)$ such as (2.2). Q.E.D.

We study the regularity property of the (strict) solution $u(t)$ of (4.1) in the following sections. But we state here some easy properties of $u(t)$. The relation $D(A_p^\alpha)\subset C^{2m-1}(G;\{B_j\})$ and Theorem 4.1 (i) imply that

(iii) $u(t)$ *satisfies the boundary couditions of* (4.1) *in the classical sense*. Moreover if we apply Lemma 5.1 to $u(t)$, we see that

(iv) $D_x^\mu u(t, x)$ *are Hölder continuous on* $[0, T_0]\times\bar{G}$ *for* $|\mu|\leq 2m-1$.

§ 5. Regularity in the interior

We consider in this section the regularity of the strict solution of (4.1) with respect to $(t, x)\in(t_1, t_2)\times G_0$. Here G_0 is a subdomain of G such that $\bar{G}_0\subset G$. We introduce some function spaces to state the results. Let $0<\gamma$, $\beta\leq 1$ and let $j\geq 1$ and $k\geq 0$ be integers. We denote by $C^{\gamma,\beta}(I, \bar{G})$ the class of functions $u(t, x)$ defined on $I\times\bar{G}$ for which there exist some constants K_1 and K_2 depending on u such that

$$|u(t, x)-u(t', x')|\leq K_1|t-t'|^\gamma+K_2|x-x'|^\beta \quad \text{for } (t, x), (t', x')\in I\times\bar{G}.$$

Here I is a closed interval. We denote by $C^{k+\gamma,2mk+j+\beta}(I, \bar{G})$ the class of functions $u(t, x)$ for which $D_t^{k'}D_x^\mu u(t, x)$ belong to $C^{\gamma,\beta}(I, \bar{G})$ for $k'\leq k$ and $|\mu|\leq 2m(k-k')+j$. Now let I be an open interval.

(1) and the argument in § 6.

We denote by $C^{k+\gamma,2mk+j+\beta}(I, G)$ the class of functions defined on $I\times G$ which belong to $C^{k+\gamma,2mk+j+\beta}(\bar{I}_0, \bar{G}_0)$ for every I_0 and G_0 such that $\bar{I}_0\subset I$ and $\bar{G}_0\subset G$. We also denote by $H^{\gamma}_{j,L_p}(I, G)$ the class of functions from I into $H_{j,L_p}(G)$ which are strongly γ-Hölder continuous, and by $H^{k+\gamma}_{2mk+j,L_p}(I, G)$ the class of functions $u(t)$ from I into $H_{j,L_p}(G)$ which are k-times strongly continuously differentiable and $\left(\frac{d}{dt}\right)^{k'}u(t)$ belongs to $H^{\gamma}_{2m(k-k')+j,L_p}(G)$ $(0\leq k'\leq k)$. We often use the above notations in somewhat different form such as $C^{\gamma,\beta}(I, \bar{G})$ where I is an open interval. The following lemmas are easy consequences of Sobolev's inequality.

Lemma 5.1. *Let G be a bounded domain of class C^2 in E_n. Let $n<p<\infty$ and $j\geq 1$. Then:*

(i) *If $u(t)\in H_{j,L_p}(I, G)$, $u(t, x)\in C^{\gamma,j-n/p}(I, G)$.*

(ii) *If $u(t)\in H^{k+\gamma}_{2mk+j,L_p}(I, G)$, $u(t, x)\in C^{k+\gamma,2mk+j-n/p}(I, G)$.*

Lemma 5.2. *Let $(A, \{B_j\}, G)$ be regular and have the smoothness of order 1. Let $n<p<\infty$. Then, if $u(t)\in D(A_p)\cap H^{\gamma}_{0,L_p}(I, G)$ and $A_p u(t)\in H^{\gamma}_{1,L_p}(I, G)$, $u(t, x)$ belongs to $C^{\gamma,2m+1-n/p}(I, \bar{G}; \{B_j\})$. In particular, for any q with $1<q<\infty$, $u(t)\in D(A_q)\cap H^{\gamma}_{2m,L_q}(I, G)$.*

On the smoothness of $F(t, x, u)$ of (4.1), we make the following:

Definition 5.1. *We say that $F(t, x, u)$ has the smoothness of order $(k+\gamma, 2mk+j+\delta)$ on (I_0, G_0), if $D_t^{k'}D^{\mu}_{(x,u)}F(t, x, u)$ is continuous in $I_0\times G_0\times C^r$ (or more precisely in $I_0\times G_0\times E_{2r}$) and belongs to $C^{\gamma,\delta}(I, C^r)$ as a function of (t, u) uniformly in $x\in G_0$ for $k'\leq k$ and $|\mu|\leq 2m(k-k')+j$. Here I_0 is a subinterval of $(0, T]$ and G_0 is a subdomain of G.*

Lemma 5.3. *Let $F(t, x, u)$ have the smoothness of order $(k+\gamma, 2mk+j+\delta)$ on (I_0, G_0). Let $n<p<\infty$. Then if $u(t)\in H^{k+\gamma'}_{2m(k+1)+j-1,L_p}(I_0, G_0)$, $F(t, u(t))$ belongs to $H^{k+\gamma''}_{2mk+j,L_p}(I_0, G_1)$ for every subdomain G_1 of G_0 such that $\bar{G}_1\subset G_0$. (We assume $j\geq 1$.)*

Now we state the regularity property of the strict solution of (4.1).

Theorem 5.1. *Let $(A, \{B_j\}, G)$ and F be as in Theorem 4.1. Let A be smooth of order $2mk+j$ on G_0 and F be smooth of order $(k+\gamma, 2mk+j+\delta)$ on (I_0, G_0). Then, if $u(t)$ is the strict solution of*

(4. 1) *(in* $[0, T]$*) in some* $L_p(G)$ *with* $n<p<\infty$*, there exists* γ' *with* $0<\gamma'<1$ *such that*

(i) $u(t)\in H^{k+1+\gamma'}_{2m(k+1)+j,L_p}(I_0^i, G_1)$ *for any q with* $1<q<\infty$ *and for every subdomian* G_1 *of* G_0 *such that* $\bar{G}_1\subset G_0$.

(ii) $u(t, x)\in C^{k+1+\gamma', 2m(k+1)+j-\varepsilon}(I_0^i, G_0)$ *for any* ε *with* $0<\varepsilon<1$.

(iii) *In particular, if* $I_0=(0, T]$ *and* $G_0=G$, $u(t, x)$ *is the classical solution of* (4. 1).

Proof: The strict solution $u(t)$ is in $D(A_p)\cap H^{\gamma}_{2m,L_p}((0, T], G)$. On account of Lemma 5. 1, it is sufficient to prove (i). Let G_1 be any subdomain of G_0 such that $\bar{G}_1\subset G_0$. We can take $\varphi\in C_0^\infty(G)$ and a subdomain G_2 of G_0 such that $\varphi=1$ on some neighbourhood of $\bar{G}_1$ and $\text{supp}(\varphi)\subset G_2$. We put

$$v(t) = \varphi u(t) \qquad (\text{for} \quad t\in I_0).$$

We take any closed subinterval $I=[t_1, t_2]$ of I_0.
Then it follows from (4. 1) that

$$(5.1)\qquad \begin{cases} \dfrac{dv}{dt} = -Av+w(t) & \text{for} \quad t\in[t_1, t_2], \\ v = 0 \quad \text{outside supp}(\varphi) & \text{for} \quad t\in[t_1, t_2], \\ v|_{t_1} = v(t_1), \end{cases}$$

where

$$w(t) = A(\varphi u(t))-\varphi Au(t)+\varphi F(t, u(t)).$$

Now let us prove (i) for $q=p$ by induction, that is, letting the proposition (i) for $q=p$ be denoted by $P(k, j)$, we will show that under the assumptions of Theorem 5. 1, $P(k, j-1)$ implies $P(k, j)$ and $P(k-1, 2m)$ implies $P(k, 0)$. First we prove the former. We may assume that G_2 is of class C^∞. We consider an elliptic boundary value problem $(A, \{B_j^0\}, G_2)$, which is regular and satisfies the same condition as $(A, \{B_j\}, G)$. Thus we can consider (5. 1) as a parabolic equation in G_2 associated to $(A, \{B_j^0\}, G_2)$. By the argument in § 4, we may assume that A, the realization of $(A, \{B_j^0\}, G_2)$ in $L_p(G_2)$, satisfies (A. 2). We also have by the assumption of induction that $v(t_1)\in D(A^{k+1})$ and $w(t)\in H^{k+\gamma''}_{2mk+j,L_p}(I, G_2)$. We will show that $v(t)\in H^{k+1+\gamma'''}_{2m(k+1)+j,L_p}(I^i, G_2)$. Then, since G_1 and I are arbitrary, we have

$P(k, j)$.

Since $w(t)$ has a compact support in G_2, it follows from Theorem 3.3′, that $w(t)\in D(A^{k+(j-1)/2m+\beta})$ for any β with $0<\beta<\frac{1}{2m}$ and $A^{k+(j-1)/2m+\beta}w(t)\in H^{\gamma''}_{0,L_p}(I, G_2)$. Since $v(t)\in D(A^{k+1})\cap H^{k+1+\gamma'}_{2m(k+1)+j-1,L_p}(I, G)$, if we multiply both sides of (5.1) by A^k, the reduced equality corresponds to the case $P(0, j-1)$. Let us prove that $P(0, j-1)$ implies $P(0, j)$. Then, concerning the original $v(t)$, we have that $A^{k+1}v(t)$ belongs to $H^{\gamma'''}_{j,L_p}(I, G_2)$ and by lemma 3.3, $v(t)\in H^{k+1+\gamma'''}_{2m(k+1)+j,L_p}(I, G_2)$. Thus we may assume $k=0$. Now using an integral representation of $v(t)$ (Theorem 2.3) and multiplying it by A, we have

$$Av(t) = e^{-(t-t_1)A}(Av(t_1))+\int_{t_1}^{t} Ae^{-(t-s)A}w(s)ds\,. \tag{5.2}$$

The first term of (5.2) belongs to $H_{j,L_p}(I^i, G_2)$ (Lemma 3.3.). Let $\varepsilon>0$ be sufficiently small. Decomposing the second term of (5.2) as in (2.4) in $[t_1+\varepsilon, t_2]=I_\varepsilon$, we have that I_0, I_1 and I_2 belong to $H^{\gamma''}_{j,L_p}(I_\varepsilon, G_2)$ (Lemma 2.6). Since $w(t)\in D(A^{(j-1)/2m+\beta})$ and $A^{(j-1)/2m+\beta}w(t)\in H^{\gamma''}_{0,L_p}(I_\varepsilon, G_2)$ for any β with $0<\beta<\frac{1}{2m}$, it follows from Lemma 2.6 that $A^{(j-1)/2m+\beta}I_3\in D(A^{\gamma'''})$ and $A^{(j-1)/2m+\beta+\gamma'''}I_3\in H^{\gamma''-\gamma'''}_{0,L_p}(I_\varepsilon, G_2)$ for any γ''' with $0<\gamma'''<\gamma''$. For such γ''' we can take such β as $\frac{1}{2m}-\gamma'''<\beta<\frac{1}{2m}$. Hence $I_3\in H^{\gamma''-\gamma'''}_{j,L_p}(I_\varepsilon, G_2)$ for any γ''' with $0<\gamma'''<\gamma''$ (Theorem 3.1, 3.2). Thus $Av(t)$ belongs to $H^{\gamma''-\gamma'''}_{j,L_p}(I_\varepsilon, G_2)$, which implies $v(t)\in H^{\gamma''-\gamma'''}_{2m+j,L_p}(I_\varepsilon, G_2)$ (Lemma 3.3). Thus we have $v(t)\in H^{1+\gamma''-\gamma'''}_{2m+j,L_p}(I^i_0, G_1)$, that is, $P(0, j)$ for $\gamma''-\gamma'''>0$, and the former half of the proof of (i) is nearly completed (since $P(0,0)$ follows from Theorem 4.1). In a similar way, we can prove that $P(k-1, 2m)$ implies $P(k, 0)$. Now applying Lemma 5.2 to the system $(A, \{B^0_j\}, G_2)$, we see that $v(t)\in D(A_q)$(1) $\cap H^{\gamma''-\gamma'''}_{2m}(I_0, G_2)$ for q with $1<q<\infty$. Thus repeating the above arguments for q, we complete the proof of (i).

§ 6. Regularity up to the boundary

This section is concerned with the regularity of the strict solution of (4.1) with respect to $(t, x)\in(t_1, t_2)\times\bar{G}$. Let $u(t)$ be the

(1) A_q is the realization of $(A, \{B^0_j\}, G_2)$ in $L_q(G_2)$.

strict solution of (4.1) in some $L_p(G)$ $(n<p<\infty)$. First we state a remark concerning the strong Hölder continuity of $Au(t)$[1] in $L_p(G)$. We may assume that A satisfies (A.2). It follows from Lemma 2.3 that $A^\alpha u(t)$ is $(1-\alpha)$-Hölder continuous. Hence $F(t, u(t))$ is γ_0-Hölder continuous with $\gamma_0=\min\{1-\alpha, \gamma\}$, where α is some constant such that $\left(2m-1+\frac{n}{p}\right)\Big/2m<\alpha<1$. Hence $Au(t)$ is γ_0-Hölder continuous (Remark to Lemma 2.6) and $u(t)$ is strongly continuously differentiable. By an interpolation relation

$$\|A^\alpha u\|_{L_p}\leq C\|Au\|_{L_p}^{\alpha}\|u\|_{L_p}^{1-\alpha},^{(2)}$$

we know that $A^\alpha u(t)$ is γ_1-Hölder continuous with $\gamma_1=\alpha\gamma_0+1-\alpha$. Repeating this argument, we know $A^\alpha u(t)$ is γ'-Hölder continuous with $\gamma\leq\gamma'<1$. Hence $F(t, u(t))$ is γ-Hölder continuous and $Au(t)$ is γ-Hölder continuous. Thus $u(t)\in H^{1+\gamma}_{2m,L_p}((0, T], G)$.

Suppose that $(A, \{B_j\}, G)$ has the smoothness of order k and $F(t, x, u)$ has the smoothness of order $(\gamma, k+1)$ on $(I, \bar{G})$, where I is an open subinterval of $(0, T]$ and $k/2m<\gamma<1$. Let $[t_1, t_2]\subset I$. Then, using an integral representation of $u(t)$, we have

$$u(t) = e^{-(t-t_1)A}u(t_1)+\int_{t_1}^{t} e^{-(t-s)A}w(s)ds \quad \text{for } t\in[t_1, t_2], \tag{6.1}$$

where $w(t)=F(t, u(t))$. Denoting by $v(t)$ the last term of (6.1) we have

$$\begin{aligned} Au(t) &= e^{-(t-t_1)A}Au(t_1)+\int_{t_1}^{t} Ae^{-(t-s)A}w(s)ds \\ &= e^{-(t-t_1)A}Au(t_1)+Ae^{-\varepsilon A}v(t-\varepsilon)+w(t) \\ &\quad -e^{-\varepsilon A}w(t)-\int_{t-\varepsilon}^{t} Ae^{-(t-s)A}\{w(t)-w(s)\}\,ds \end{aligned} \tag{6.2}$$

for $t\in[t_1+\varepsilon, t_2]$, where $\varepsilon>0$ is sufficiently small. The first two terms of the right hand side of (6.2) are in $H^{\gamma}_{2m,L_p}([t_1+\varepsilon, t_2], G)$ $=H^{\gamma}_{2m,L_p}$ (Lemma 2.6). The fourth term is also in it. The fifth term is in $H^{\gamma-\gamma_j}_{j,L_p}(1\leq j\leq k)$, where $j/2m<\gamma_j<\gamma$ (Lemma 2.6 and Theorem 3.1). Since $u(t)\in H^{\gamma}_{2m,L_p}$, $w(t)$ belongs to H^{γ}_{1,L_p} by the

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(1) In this section we denote by A the realigation of $(A, \{B_j\}, G)$ in $L_p(C)$.

(2) This inequality in due to sobolevskii [14].

smoothness of F (see Lemma 5.3). Thus $Au(t)$ $\left(\text{and } \frac{du}{dt} \text{ also}\right)$ belongs to $H^{\gamma-\gamma_1}_{1,L_p}$. Hence $u(t)\in H^{\gamma-\gamma_1}_{2m+1,L_p}$ (Lemma 3.3) and obviously $u(t)\in H^{1+\gamma-\gamma_1}_{2m+1,L_p}$. The same argument shows that $u(t)\in H^{1+\gamma-\gamma_j}_{2m+j,L_p}(I, G)$ for $1\leq j\leq k$, since $[t_1, t_2]\subset I$ and $\varepsilon>0$ are arbitrary.

Now suppose that $(A, \{B_j\}, G)$ has the smoothness of order $2m$ and $F(t, x, u)$ has that of order $(1+\gamma, 2m+1)$. Then, by the above arguments, we have $u(t)\in H^{1+1-\gamma'}_{4m-1,L_p}$ and $w(t)\in H^{1+\gamma''}_{2m,L_p}$, where $\gamma'=\gamma_{2m-1}$ and $\gamma''=\min(1-\gamma', \gamma)$. Hence, by Lemma 2.7, we have

$$(6.3)\qquad Au(t) = e^{-(t-t_1)A}Au(t_1)+w(t)-e^{-(t-t_1)A}w(t_1) - \int_{t_1}^{t} e^{-(t-s)A}w'(s)\,ds \qquad \text{for} \quad t\in(t_1, t_2]\,.$$

The right hand side of (6.3) is in $H^{1+\gamma''}_{2m,L_p}$. In fact, the first three terms are obviously in it, and the last term, which we denote by $v_1(t)$, is also in it, since $v_1(t)\in D(A)$ and $Av_1(t)$, $\frac{dv_1}{dt}\in H^{\gamma''}_{0,L_p}$ (Lemma 2.4 and 2.6). Thus $Au(t)\in H^{1+\gamma''}_{2m,L_p}$. Hence $u(t)\in H^{1+\gamma''}_{4m,L_p}$ (Lemma 3.3). Since A is a closed operator in $L_p(G)$, we have $u'(t)\in D(A)$ and $Au'(t)=(Au(t))'\in H^{\gamma''}_{0,L_p}$. Thus, differentiating both sides of (4.1) by t, we obtain the following strict equation:

$$(6.4)\qquad \frac{du'}{dt} = -Au'+\frac{\partial F}{\partial t}+\sum_{|\mu|\leq 2m-1}\frac{\partial F}{\partial u_\mu}\Big|_{u_\mu=D^\mu u}D^\mu u' \qquad t\in(t_1, t_2]\,,$$
$$u'|_{t=t_1} = u'(t_1)\,.$$

(Since $u(t)\in H^{1+\gamma''}_{4m,L_p}$ and $n<p<\infty$, $D^\mu u'(t)=D^\mu\frac{\partial u}{\partial t}=\frac{\partial}{\partial t}D^\mu u$ for $|\mu|\leq 2m-1$ (Lemma 5.1).) Thus $\frac{du'}{dt}\in H^{\gamma''}_{0,L_p}$ and $u(t)\in H^{2+\gamma''}_{4m,L_p}$.

Let us consider (6.4) as an evolution equation for unknown $u'(t)$. Then, we have $u'(t)\in H^{1+\gamma}_{2m,L_p}$, since $\frac{\partial F}{\partial t}$ and $\frac{\partial F}{\partial u_\mu}\big|_{u_\mu=D^\mu u}$ are in H^{γ}_{0,L_∞}. Hence $w'(t)\in H^{\gamma}_{0,L_p}$ and $v_1(t)\in H^{1+\gamma}_{2m,L_p}$ in (6.3). Since the first three terms of the right hand side of (6.3) are in $H^{1+\gamma}_{1,L_p}$, $Au(t)$ belongs to $H^{1+\gamma}_{1,L_p}$. Hence $u(t)\in H^{1+\gamma}_{2m+1,L_p}$. Proceeding in the same way, we have $u(t)\in H^{1+\gamma}_{4m,L_p}$. Finally from (6.4) we have $\frac{du'}{dt}\in H^{\gamma}_{0,L_p}$. Thus $u(t)\in H^{2+\gamma}_{4m,L_p}$.

Now, summing up above arguments in a general form and using Lemma 5.1. and 5.2, we obtain the following:

Theorem 6.1. *Let* $(A, \{B_j\}, G)$ *and* F *be as in Theorem* 4.1. *Let* $(A, \{B_j\}, G)$ *be smooth of order* $2mk+j$ *and* F *be smooth of order* $(k+\gamma, 2mk+j+1)$ *on* $(I, \bar{G})$, *where* I *is an open subinterval of* $[0, T]$ *and* $j/2m<\gamma<1$. *Then, if* $u(t)$ *is the strict solution of* (4.1) (*in* $[0, T]$) *in some* $L_p(G)$ *with* $n<p<\infty$,

(i) $u(t)\in H^{k+1+\gamma-\gamma_j}_{2m(k+1)+j, L^q}(I, G)$ *with* $1<q<\infty$ *and* $j/2m<\gamma_j<\gamma$.

(ii) $u(t,x)\in C^{k+1+\gamma-\gamma_j, 2m(k+1)+j-\varepsilon}(I, \bar{G})$ for any $\varepsilon>0$.

Remark 1 (existence of the global solution). Let us consider in what cases the global solution of (4.1) exists. For this purpose we return to Section 2 and ask for the conditions under which (2.2) has the global solution. First we assume the following conditions:

(1) $$\|e^{-tA}\|\leq Me^{-\delta t} \qquad (t\geq 0),$$

(2) $$\|F(t, u)\|\leq f(t, \|A^{\alpha}u\|)$$
$$f(t, a) = f_1(a)a+f_2(t, a),$$

where δ is a positive number and f_1, f_2 are non-negative, non-decreasing and continuous in $a\in[0, \infty)$.

We assume that we can take $T_0\in(0, T]$ such that

(3) $$Me^{-\delta T_0} = \kappa_0<1.$$

Putting

(4) $$M_{\alpha}\frac{1}{1-\alpha}T_0^{1-\alpha} = N$$

and

(5) $$M_{\alpha}\int_t^{t+T_0}(t+T_0-s)^{-\alpha}f_2(s, a)\,ds = M(t, a),$$

we assume that there exists $a>0$ such that

(6) $$Nf_1(a)+\frac{1}{a}M(t, a)\leq\kappa \qquad (t\in[0, T-T_0]),$$

where

(7) $$\frac{\kappa}{1-\kappa}\leq\frac{1-\kappa_0}{M}.$$

On account of (6) and (7) we have

(8) $$Ma_0+Nf_1(a)a+M(t, a)\leq a$$

and

$$Me^{-\delta T_0}a_0+Nf_1(a)a+M(t, a)\leq a_0\,, \tag{9}$$

where

$$\frac{a}{a_0}=\frac{M}{1-\kappa}\leq\frac{1-\kappa_0}{\kappa}\,. \tag{10}$$

Let us recollect the proof of Theorem 2.1. We assume that

$$||A^{\alpha}u_0||\leq a_0\,. \tag{11}$$

Then the arguments in the proof of Theorem 2.1 shows that

$$a_k(T_0)\leq a \qquad (k=0, 1, \cdots)\,.$$

In fact, we have $a_0(T_0)\leq Ma_0<a$, and (2.4) shows that "$a_k(T_0)\leq a$" implies "$a_{k+1}(T_0)\leq a$" on account of (8). Hence the mild solution $u(t)$ of (2.2) exists in $[0, T_0]$ and

$$||A^{\alpha}u||_{T_0}\leq a\,.$$

Hence on account of (9) we have

$$||A^{\alpha}u(T_0)||\leq a_0\,.$$

Thus, repeating above arguments, we know that the mild solution $u(t)$ of (2.2) exists in $[0, T]$.

The essential assumption in this argument is (6). It is sufficient for the existence of the global solution of (2.2) that (6) holds for $t=kT_0$ $(k=0, \cdots, 1, \cdots)$ and $T-T_0$. It is to be noted that (6) holds for sufficiently small $a>0$, if $f_1(0)=0$ and $f_2=0$.

We state another condition for the mild solution of (2.2) to exist in $[0, T_1]$. Put

$$M_{\alpha}\frac{1}{1-\alpha}T_1^{1-\alpha}=N_1 \tag{4'}$$

and

$$M_{\alpha}\int_t^{t+T_1}(t+T_1-s)^{-\alpha}f_2(s, a)\,ds=M(t, T_1, a)\,. \tag{5'}$$

If we assume that there exists $a>0$ such that

$$N_1f_1(a)+\frac{1}{a}M(0, T_1, a)\leq\kappa<1\,, \tag{6'}$$

then we have

$$(8') \qquad Ma_0 + N_1 f_1(a) a + M(0, T_1, a) \leq a\,,$$

where

$$(10') \qquad \frac{a}{a_0} = \frac{M}{1-\kappa} = M'\,.$$

Hence the mild solution $u(t)$ of (2.2) exists in $[0, T_1]$ for u_0 satisfying (11).

Let us assume that

$$(12) \qquad \begin{cases} f_1(a) = \text{const.} = f_1\,. \\ M(t, T_1, a) \leq m(t, T_1)(1+a)\,. \end{cases}$$

Then for sufficiently small T_1 and sufficiently large a we have

$$(6'') \qquad N_1 f_1 + \frac{1}{a} M(t, T_1, a) \leq \kappa < 1\,,$$

and hence

$$(8'') \qquad Ma_0 + N_1 f_1 a + M(t, T_1, a) \leq a\,.$$

In this case there is no restriction on the upper bound of a_0, and hence the global solution of (2.2) exists for any $u_0 \in D(A^\alpha)$.

Remark 2. Tanabe also proved Theorem 3.1 (independently of the author) and considered the problem (0.1). In his work A and B_j may depend on time t. Therefore his results corresponding to our Theorem 4.1 are more general than mine. However the author does not know the details and can not cite them here.

Our Theorem 6.1 can not be locallized on a part of G. Therefore it is not a complete result. If we establish a priori estimates for the problem (0.1), more complete results will be obtained. Recently such a priori estimates has been established by Tanabe and Kametaka (independently) in $L_2(G)$.

To consider the problem (0.1) in which A and $\{B_j\}$ depend on time t, we can make use of the results in [11], [14] and [15]. However Arima succeeded in constructing Green function for the parabolic boundary problem and obtained the estimates on it in her recent paper [17]. If we make use of her results, more complete results will be obtained on this problem.

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