Orthogonality of generalized eigenfunctions in Weyl's expansion theorem

By

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§ 0. Introduction

Let us consider an ordinary differential equation of the second order

$$\frac{d^2u}{dx^2} - q(x)u(x) + \lambda u(x) = 0\,, \qquad (0 \leqq x < \infty)\,. \tag{0.1}$$

Here $q(x)$ is a real-valued function which is locally summable and $x=0$ is a regular point of the equation. Let $\varphi(x:\lambda)$ be the solution of the equation which satisfies

$$\varphi(0;\lambda) = \sin\theta\,,\ \frac{\partial\varphi}{\partial x}(0;\lambda) = \cos\theta\,,$$

where θ is a real constant.

H. Weyl proved that there exists a spectral measure $\rho_0(\lambda)$ which satisfies the following two conditions (A) and (B) (Weyl [14]):

(A) The transformation

$$\mathcal{F}_{\rho_0}:\ f(x) \to \int_0^\infty f(x)\varphi(x;\lambda)\,dx$$

from $L_2(0, \infty; dx)$ into $L_2(-\infty, +\infty; d\rho_0)$ is isometric, namely

$$\int_{-\infty}^{+\infty} |\mathcal{F}_{\rho_0}f(\lambda)|^2 d\rho_0(\lambda) = \int_0^\infty |f(x)|^2 dx\,. \tag{0.2}$$

(B) $\mathcal{F}_{\rho_0}$ transforms $L_2(0, \infty; dx)$ onto $L_2(-\infty, +\infty; d\rho_0)$.

Such a spectral measure is unique(1) and is called the Weyl spectral measure.

Let us call a spectral measure $\rho(\lambda)$ which only satisfies (A) a pseudo-spectral measure.

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I. M. Gelfand and B. M. Levitan succeeded in determining the potential function $q(x)$ from a given spectral measure $\rho(\lambda)$. In general, however, $\rho(\lambda)$ turns out to be a pseudo-spectral measure for the equation with the $q(x)$ thus determined. They gave a sufficient condition for $\rho(\lambda)$ to be the Weyl spectral measure. In particular, if $\rho(\lambda)$ vanishes for large $-\lambda$, $\rho(\lambda)$ will be the Weyl spectral measure (Gelfand-Levitan [4]).

Also M. G. Krein gave a sufficientt and necessary condition for a pseudo-spectral measure to be the Weyl spectral measure (Krein [7]). Previous to them V. A. Marčenko proved that $q(x)$ is uniquely determined by $\rho(\lambda)$ (Marčenko [9]).

We are interested in the condition which $q(x)$ should satisfy in order that every pseudo-spectral measure of the equation be the Weyl spectral measure.

We shall prove in § 1 that, in the limit-point case at infinity, a pseudo-spectral measure is the Weyl spectral measure and that, in the limit-circle case at infinity, there always exists a pseudo-spectral measure which is not the Weyl spectral measure.

Combining this theorem with the result of Gelfand-Levitan, we see that the spectrum in the limit-circle case at infinity is always unbounded below (§ 2).

K. O. Friedrichs stated that the following theorem was a result of Rellich (Friedrichs [3]):

Let $q(x)$ be continuous and negative on $[0, \infty]$. Then, if $q(x) \to -\infty$ as $x \to \infty$ and if

$$\int^{\infty} |q(x)|^{-1/2} dx < \infty , \tag{0. 3}$$

the spectrum is discrete and unbounded below.

Also D. B. Sears and E. C. Titchmarsh obtained the following result: if the inequality (0. 3) holds, the equation is of the limit-circle type at infinity and hence the spectrum is discrete (Sears-Titchmarsh [11], Titchmarsh [13]). Our theorem gave a proof of lower unboundedness of the spectrum.

Proposition 3 in § 2 is due to Y. Saitō.

(1) In the limit-circle case at infinity we set a boundary condition at infinity.

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§ 1. Properties of the pseudo-spectral measure

Let $\mathcal{D}_0$ be the class of all functions $f(x)$ which satisfy the following four conditions (i)..(iv):

(i) $f(x)$ has a compact carrier.

(ii) $f(x)$ is expressed in the form

$$f(x) = \int_0^x dy \int_0^y g(z)\, dz + c$$

where $g(z)$ is a locally summable function and c is a constant.

(iii) $-g(x)+q(x)f(x)$ belongs to $L_2(0, \infty; dx)$.

(iv) $f(0) \cos \theta - f'(0) \sin \theta = 0$.

We shall define a symmetric operator L_0 with the domain $\mathcal{D}_0$ by

$$L_0 f(x) = -g(x)+q(x)f(x) . \tag{1. 1}$$

Let us define a pseudo-spectral measure as a spectral measure $\rho(\lambda)$ which satisfies the following identity for every $f(x)$ in the class C_0 of all continuous functions each of which has a compact carrier,

$$\int_0^\infty |f(x)|^2 dx = \int_{+\infty}^{+\infty} \left| \int_0^\infty f(x) \varphi(x; \lambda)\, dx \right|^2 d\rho(\lambda) . \tag{1. 2}$$

Let us denote $\int_0^\infty f(x) \varphi(x;\lambda)\, dx$ by $\mathcal{F}_0 f(\lambda)$ for $f(x)$ in C_0. Then we have an isometric transformation $\mathcal{F}_0$ from C_0 into $L_2(-\infty, +\infty; d\rho)$. Since C_0 is dense in $L_2(0, \infty; dx)$, we can extend $\mathcal{F}_0$ to the isometric transformation $\mathcal{F}_\rho$ which is defined on $L_2(0, \infty; dx)$. $\mathcal{F}_\rho$ is expressed for every function $h(x)$ in $L_2(0, \infty; dx)$ in the form

$$\mathcal{F}_\rho h(\lambda) = \underset{N\to\infty}{\text{l. i. m.}} \int_0^N h(x) \varphi(x; \lambda)\, dx . \tag{1. 3}$$

We shall define a bounded linear transformation $\mathcal{F}_\rho^*$ from $L_2(-\infty,$

$+\infty; d\rho)$ onto $L_2(0, \infty; dx)$ by

$$\langle \mathcal{F}_\rho^* \xi, f\rangle = \langle \xi, \mathcal{F}_\rho f\rangle_\rho . \tag{1.4}$$

Here $\langle , \rangle$ and $\langle , \rangle_\rho$ are the inner products in $L_2(0, \infty; dx)$ and $L_2(-\infty, +\infty; d\rho)$, respectively.

We have $\|\mathcal{F}_\rho^* \xi\| \leqq \|\xi\|_\rho$.

Putting $\xi = \mathcal{F}_\rho h$ in (1.4), we have

$$\langle \mathcal{F}_\rho^* \mathcal{F}_\rho h, f\rangle = \langle \mathcal{F}_\rho h, \mathcal{F}_\rho f\rangle_\rho = \langle h, f\rangle .$$

Hence it follows that $\mathcal{F}_\rho^* \mathcal{F}_\rho$ is the identity operator.

Proposition 1. *Let $\rho(\lambda)$ be a pseudo-spectral measure. Then $\mathcal{F}_\rho^*$ is expressed in the form*

$$\mathcal{F}_\rho^* \xi(x) = \int_{-\infty}^{+\infty} \xi(\lambda) \varphi(x, \lambda) d\rho(\lambda) \tag{1.5}$$

for every continuous function $\xi(\lambda)$ which has a compact carrier.

Proof. By the definition we have

$$\int_0^\infty \mathcal{F}_\rho^* \xi(x) \overline{f(x)} dx = \int_{-\infty}^{+\infty} \xi(\lambda) \overline{\mathcal{F}_\rho f(\lambda)} d\rho(\lambda) .$$

Suppose that $f(x)$ belongs to C_0. Then we get

$$\int_{-\infty}^{+\infty} \xi(\lambda) \overline{\mathcal{F}_\rho f(\lambda)} d\rho(\lambda) = \int_{-\infty}^{+\infty} \xi(\lambda) d\rho(\lambda) \int_0^\infty \overline{f(x)} \varphi(x; \lambda) dx$$
$$= \int_0^\infty \overline{f(x)} dx \int_{-\infty}^{+\infty} \xi(\lambda) \varphi(x; \lambda) d\rho(\lambda) .$$

Hence we obtain

$$\int_0^\infty \overline{f(x)} dx \left\{ \mathcal{F}_\rho^* \xi(\lambda) - \int_{-\infty}^{+\infty} \xi(\lambda) \varphi(x; \lambda) d\rho(\lambda) \right\} = 0 .$$

Since this identity holds for every $f(x)$ in C_0, we have the proposition.

Corollary. *Let $\rho(\lambda)$ be a pseudo-spectral measure. Then for every $h(x)$ in $L_2(0, \infty; dx)$ we have*

$$h(x) = \underset{N \to \infty}{\text{l. i. m.}} \int_{-N}^{N} \mathcal{F}_\rho h(\lambda) \varphi(x; \lambda) d\rho(\lambda) . \tag{1.6}$$

The following three conditions are equivalent:

(i) $\mathcal{F}_\rho \mathcal{F}_\rho^* = \text{identity}$.

(ii) $\mathcal{F}_\rho^*$ is injective.

(iii) $\mathcal{F}_\rho$ is surjective.

Theorem 1. *In the limit-circle case at infinity there always exists a pseudo-spectral measure which is not the Weyl spectral measure. In the limit-point case at infinity any pseudo-spectral measure $\rho(\lambda)$ is the Weyl spectral measure.*

Proof. In the limit-circle case at infinity we can set two different boundary conditions at infinity. According to these boundary conditions we have two different Plancherel's identities

$$||h||^2 = \sum_{n=1}^{\infty} |\langle h, \varphi(\cdot\,;\lambda_n)\rangle|^2 c_n ,$$

$$||h||^2 = \sum_{m=1}^{\infty} |\langle h, \varphi(\cdot\,;\lambda'_m)\rangle|^2 c'_m .$$

Hence we have

$$||h||^2 = \sum_{n=1}^{\infty} |\langle h, \varphi(\cdot\,;\lambda_n)\rangle|^2 \frac{c_n}{2} + \sum_{m=1}^{\infty} |\langle h, \varphi(\cdot\,;\lambda'_m)\rangle|^2 \frac{c'_m}{2} .$$

Let $\rho(\lambda)$ be the measure defined by

$$\int_\Delta d\rho(\lambda) = \sum_{\lambda_n \in \Delta} \frac{c_n}{2} + \sum_{\lambda'_m \in \Delta} \frac{c'_m}{2} .$$

Then $\rho(\lambda)$ is a pseudo-spectral measure.

However, no function $h(x)$ in $L_2(0, \infty; dx)$ can satisfy $\langle h, \varphi(\cdot\,;\lambda_1)\rangle = 1$ and $\langle h, \varphi(\cdot\,;\lambda'_m)\rangle = 0$ (for every m) simultaneously, because $\{\varphi(x;\lambda'_m)\}_{m=1}^{\infty}$ is a complete system. Hence $\rho(\lambda)$ is not the Weyl spectral measure.

In the limit-point case at infinity the operator L_0 is essentially self-adjoint. The class $\mathcal{R}_\mu$ of all functions which can be expressed in the form $h(x) = (\mu - L_0) f(x)$ with $f(x)$ in $\mathcal{D}_0$ is dense in $L_2(0, \infty; dx)$ for every complex number μ that is not real.

Let $\rho(\lambda)$ be a pseudo-spectral measure. Then for $h(x)$ in $\mathcal{R}_\mu$ we get

$$\mathcal{F}_\rho h(\lambda) = \mathcal{F}_\rho(\mu - L_0)f(\lambda) = (\mu - \lambda)\mathcal{F}_\rho f(\lambda)\,.$$

Put $\boldsymbol{L}_\mu = (\mu - \bar{L}_0)^{-1}$. Then, by the identity $\boldsymbol{L}_\mu h(x) = f(x)$ we have

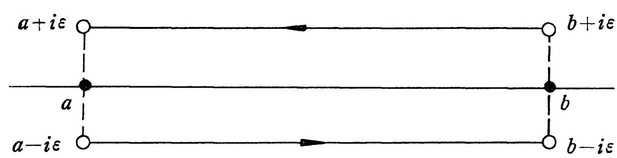
$$\langle \boldsymbol{L}_\mu h, h\rangle = \int_{-\infty}^{+\infty} \frac{|\mathcal{F}_\rho h(\lambda)|^2}{\mu - \lambda} d\rho(\lambda)\,. \tag{1.7}$$

This identity holds for every $h(x)$ in $L_2(0, \infty; dx)$, because $\mathcal{R}_\mu$ is dense in $L_2(0, \infty; dx)$.

For every finite interval $\Delta = (a, b)$, let us define the bounded symmetric operator E_Δ by

$$\langle E_\Delta h, h\rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_{c_\varepsilon} \langle \boldsymbol{L}_\mu h, h\rangle d\mu\,, \tag{1.8}$$

where c_ε is the following path of integration.



Then E_Δ proves to be a projection in $L_2(0, \infty; dx)$.[(1)]

Let P_Δ denote the projection defined by

$$P_\Delta \xi(\lambda) = \begin{cases} \xi(\lambda)\,, & \lambda \in \Delta \\ 0\,, & \lambda \notin \Delta\,. \end{cases}$$

First we shall prove the identity

$$\mathcal{F}_\rho E_\Delta = P_\Delta \mathcal{F}_\rho\,. \tag{1.9}$$

For every $f(x)$ in C_0 we have by (1.7) and (1.8)

$$\langle E_\Delta f, f\rangle = \int_\Delta |\mathcal{F}_\rho f(\lambda)|^2 d\rho(\lambda)\,. \tag{1.10}$$

Writing this as

$$\langle E_\Delta f, f\rangle = \langle P_\Delta \mathcal{F}_\rho f, \mathcal{F}_\rho f\rangle_\rho\,,$$

we get

$$\langle E_\Delta f_1, f_2\rangle = \langle P_\Delta \mathcal{F}_\rho f_1, \mathcal{F}_\rho f_2\rangle_\rho \tag{1.11}$$

(1) Stone [12], Theorem 5.6.

for every pair of $f_1(x)$ and $f_2(x)$ in C_0.

For $f(x)$ in C_0, put $\xi(\lambda)=\mathcal{F}_\rho E_\Delta f(\lambda)-P_\Delta \mathcal{F}_\rho f(\lambda)$. Then we get by (1. 11)

$$\begin{aligned}\langle \xi, \mathcal{F}_\rho f_1\rangle_\rho &= \langle \mathcal{F}_\rho E_\Delta f - P_\Delta \mathcal{F}_\rho f, \mathcal{F}_\rho f_1\rangle_\rho \\ &= \langle E_\Delta f, f_1\rangle - \langle P_\Delta \mathcal{F}_\rho f, \mathcal{F}_\rho f_1\rangle_\rho = 0 .\end{aligned}$$

Hence the identity

$$||P_\Delta \mathcal{F}_\rho f||_\rho^2 = ||\mathcal{F}_\rho E_\Delta f||_\rho^2 + ||\xi||_\rho^2 \tag{1. 12}$$

holds, because $\xi(\lambda)$ and $\mathcal{F}_\rho E_\Delta f(\lambda)$ are othhogonal. Since E_Δ is a projection, we have

$$\langle E_\Delta f, f\rangle = ||E_\Delta f||^2 = ||\mathcal{F}_\rho E_\Delta f||_\rho^2 . \tag{1. 13}$$

On the other hand, since P_Δ is a projection, we have by (1. 10)

$$\langle E_\Delta f, f\rangle = \langle P_\Delta \mathcal{F}_\rho f, \mathcal{F}_\rho f\rangle_\rho = ||P_\Delta \mathcal{F}_\rho f||_\rho^2 . \tag{1. 14}$$

Comparing (1. 13) with (1. 14), we get

$$||\mathcal{F}_\rho E_\Delta f||_\rho^2 = ||P_\Delta \mathcal{F}_\rho f||_\rho^2 .$$

Hence we obtain from (1. 12) the identity

$$||\xi||_\rho^2 = 0 .$$

Since C_0 is dense in $L_2(0, \infty; dx)$, the identity (1. 9) is proved.

Suppose that a function $\eta(\lambda)$ in $L_2(-\infty, +\infty; d\rho)$ satisfies $\langle \eta, \mathcal{F}_\rho f\rangle_\rho=0$ for every $f(x)$ in $L_2(0, \infty; dx)$. We prove that $\eta=0$ in $L_2(-\infty, +\infty; d\rho)$. It follows from (1. 9) that $P_\Delta \mathcal{F}_\rho f(\lambda)$ belongs to the image of $\mathcal{F}_\rho$ for every $f(x)$ in C_0. Hence for every finite interval Δ we have

$$\langle \eta, P_\Delta \mathcal{F}_\rho f\rangle_\rho = \int_\Delta \eta(\lambda)\overline{\mathcal{F}_\rho f(\lambda)}\,d\rho(\lambda) = 0 .$$

Since the inequality

$$\int_{-\infty}^{+\infty} |\eta(\lambda)\mathcal{F}_\rho f(\lambda)|\,d\rho(\lambda) < \infty$$

holds, we obtain

$$\int_{-\infty}^{+\infty} |\eta(\lambda)\mathcal{F}_\rho f(\lambda)|\,d\rho(\lambda) = 0 .$$

For every λ_0 in $(-\infty, +\infty)$ we can take such a function $f_0(x)$ in C_0 that

$\mathscr{F}_\rho f_0(\lambda_0) \neq 0$. Since $\mathscr{F}_\rho f_0(\lambda)$ is a continuous function of λ, $\eta(\lambda)$ vanishes on a neighbourhood of λ_0. Hence we have $\eta = 0$ in $L_2(-\infty, +\infty; d\rho)$.

Remark 1. K. Yosida constructed a pseudo-spectral measure with the classical method of Hilbert (Yosida [15], Hilbert [5]).

Remark 2. The second part of our theorem can be proved by the method E. A. Coddington and N. Levinson used to prove Weyl's theorem in the limit-point case at infinity. (Coddington-Levinson [1], 239–242). See also Ikebe [6].

Remark 3. If we assume the existence of the Weyl spectral measure, we can prove the second part of our theorem more easily using Lemma 2 in Saitō [10] § 1. See Remark in Saitō [10] § 1.

§ 2. Pseudo-spectral measures in the limit-circle case at infinity

The spectrum in the limit-circle case at infinity is discrete (Weyl [14]). Now we shall prove

Theorem 2. *The spectrum in the limit-circle case at infinity is always unbounded below.*

Proof. We parametrize boundary conditions at infinity with α which varies over $[0, 2\pi)$. Let $\rho_\alpha(\lambda)$ be the Weyl spectral measure under the boundary condition parametrized with α.

First we shall prove that there exists at most one value of α for which $\rho_\alpha(\lambda)$ vanishes for large $-\lambda$. Suppose in contrary that there exist two such values α_1 and α_2. Then we could construct from $\rho_{\alpha 1}(\lambda)$ and $\rho_{\alpha 2}(\lambda)$ a pseudo-spectral measure $\rho(\lambda)$which is not the Weyl spectral measure and vanishes for large $-\lambda$. But this is impossible, because by a result of Gelfand-Levitan (Gelfand-Levitan [4], Levitan [8]) every pseudo-spectral measure that vanishes for large $-\lambda$ is the Weyl spectral measure.

If there exists a parameter α_0 such that$\rho_{\alpha 0}(\lambda)$ vanishes for large $-\lambda$, we have for every $f(x)$ in $\mathscr{D}_0$

$$\langle L_0 f, f \rangle \geqq c\|f\|^2,$$

where c is a real constant. Then every self-adjoint extension of L_0 becomes lower bounded.(1) Hence every $\rho_\alpha(\lambda)$ vanishes for large $-\lambda$. This is a contradiction.

Corollary. *For every pseudo-spectral measure $\rho(\lambda)$ in the limit-circle case at infinity, the support of the measure $d\rho$ is unbounded below.*

Remark 1. E. C. Titchmarsh and D. B. Sears proved the following theorem:

Let (a) $q(x) \leqq 0, \quad q'(x) < 0,$

$$q(x) \to -\infty \quad \text{as} \quad x \to \infty ,$$

$$q'(x) = O\{|q(x)|^c\} , \quad \left(0 < c < \frac{3}{2}\right)$$

and let (b) $q''(x)$ be ultimately of one sign. Then, if

$$\int^\infty |q(x)|^{-1/2} dx < \infty , \tag{2. 1}$$

the equation is of the limit-circle type at infinity and hence the spectrum is discrete (Titchmarsh [13], Theorem 5. 11., Dunford-Schwartz [2], 1448., Sears-Titchmarsh [11]).

Combining our theorem with Sears-Titchmarsh's we have the following result:

Under the conditions (a) and (b), if the inequality (2. 1) holds, then the spectrum is discrete and unbounded below.(2)

Remark 2. By Weyl's classification theorem, Theorem 2 is equivalent to the following: if the L_0 in § 1 is bounded below, then the equation is of the limit-point type at infinity. In higher-dimensional cases, it is known that, if the operator $L_0 \equiv -\Delta + V(x)$ is lower bounded, then it is essentially self-adjoint. However, in this case it is assumed that $V(x)$ is locally square summable, while in our case $q(x)$ is assumed to be locally summable. See Saitō [10] § 2.

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(1) Dunford-Schwartz [2], 1454.

(2) To prove this fact only, it is unnecessary to use Theorem 2. The lower unboundedness of the spectrum in the case where $q(x) \to -\infty$ as $x \to \infty$, can be proved by an elementary method. This remark was given by K. Asano.

The following proposition explains why a spectral measure which only satisfies (A) in §0 is called here a pseudo-spectral measure.

Let $\mathscr{R}(\rho)$ be the range of $\mathscr{F}_\rho$. Then $\mathscr{R}(\rho)$ is the Hilbert space. If both $\xi(\lambda)$ and $\lambda\xi(\lambda)$ belong to $\mathscr{R}(\rho)$, we set $\Lambda_\rho\xi(\lambda)=\lambda\xi(\lambda)$. The image of $\mathscr{D}_0$ by $\mathscr{F}_\rho$ is contained in the domain of Λ_ρ, because for every $f(x)$ in $\mathscr{D}_0$ we have

$$\lambda\mathscr{F}_\rho f(\lambda) = \mathscr{F}_\rho L_0 f(\lambda)\,.$$

Hence Λ_ρ is a closed symmetric operator in $\mathscr{R}(\rho)$.

Proposition 2. *Let $\rho(\lambda)$ be a pseudo-spectral measure which is not the Weyl spectral measure. Then we have*

$$\mathscr{F}_\rho^{-1}\Lambda_\rho\mathscr{F}_\rho = \bar{L}_0\,.$$

Here $\mathscr{F}_\rho^{-1}$ is the restriction of $\mathscr{F}_\rho^*$ on $\mathscr{R}(\rho)$.

Proof. We prove the following: if $L\equiv\mathscr{F}_\rho^{-1}\Lambda_\rho\mathscr{F}_\rho$ is self-adjoint, then $\rho(\lambda)$ is the Weyl spectral measure.

By virtue of a theorem of Stone (Stone [12], Theorem 10.17.) there corresponds a boundary condition at infinity to the self-adjoint L, because L is an extension of L_0. Hence we have a complete orthogonal system of eignefunctions of L, $\{\varphi(x;\lambda_n)\}_{n=1}^{\infty}$. By the definition we get

$$\lambda_n\mathscr{F}_\rho\varphi_n(\lambda) = \lambda\mathscr{F}_\rho\varphi_n(\lambda)\,,$$

where $\varphi_n(x)=\varphi(x;\lambda_n)$.

Since $\mathscr{F}_\rho\varphi_n(\lambda)=\langle\varphi_n,\ \varphi(\cdot\,;\lambda)\rangle$ vanishes only on λ_m $(m\neq n)$, we have $\int_I d\rho=0$ for every I that does not contain an eigenvalue of L.

Writing $\int_{\{\lambda_m\}} d\rho$ as c_m, we have $c_n=\|\varphi_n\|^{-2}$, because

$$\begin{aligned}\|\varphi_n\|^2 &= \int_{-\infty}^{+\infty}|\mathscr{F}_\rho\varphi_n(\lambda)|^2 d\rho(\lambda)\\ &= \sum_{m=1}^{\infty}|\mathscr{F}_\rho\varphi_n(\lambda_m)|^2 c_m = |\mathscr{F}_\rho\varphi_n(\lambda_n)|^2 c_n\\ &= \|\varphi_n\|^4 c_n\,.\end{aligned}$$

Hence $\mathscr{F}_\rho$ is surjective and $\mathscr{R}(\rho)=L_2(-\infty,\ +\infty;d\rho)$.

Remark. V. A. Marčenko proved that for every increasing function

$\rho(\lambda)$ there exists at most one equation of type (0. 1) which has $\rho(\lambda)$ as a pseudo-spectral measure (Marčenko [9]). Hence if $\rho(\lambda)$ is a pseudo-spectral measure which is not the Weyl spectral measure of an equation of type (0. 1), then $\rho(\lambda)$ can not be the Weyl spectral measure of any equation of type (0. 1).

Let $\sigma(\alpha)$ be a measure on $[0, 2\pi)$ with $\int_0^{2\pi} d\sigma = 1$. Then we have the following identity for $f(x)$ in C_0

$$||f||^2 = \int_0^{2\pi} d\sigma(\alpha) \int_{-\infty}^{+\infty} |\mathcal{F}_0 f(\lambda)|^2 d\rho_\alpha(\lambda) . \tag{2. 2}$$

Hence we obtain a pseudo-spectral measure $\pi_\sigma(\lambda)$ defined by

$$\int_\Delta d\pi_\sigma(\lambda) = \int_0^{2\pi} d\sigma(\alpha) \int_\Delta d\rho_\alpha(\lambda) . \tag{2. 3}$$

As to the inverse of this statement we only have the following

Proposition 3.(1) *Let* $\{\varphi_m\}_{m=1}^\infty$ *and* $\{\psi_n\}_{n=1}^\infty$ *be the complete orthonormal systems of eigenfunctions which correspond to different boundary conditions at infinity. If we have the identity for every* $h(x)$ *in* $L_2(0, \infty; dx)$

$$||h||^2 = \sum_{m=1}^\infty |\langle h, \varphi_m \rangle|^2 a_m + \sum_{n=1}^\infty |\langle h, \psi_n \rangle|^2 b_n , \tag{2. 4}$$

where a_m *and* b_n *are non-negative numbers, then we have* $a_m = a$, $b_n = 1 - a$, *for all m and n.*

Proof. By (2. 4) we have

$$\begin{aligned} \langle f, h \rangle = & \sum_{m=1}^\infty \langle f, \varphi_m \rangle \langle \varphi_m, h \rangle a_m \\ & + \sum_{n=1}^\infty \langle f, \psi_n \rangle \langle \psi_n, h \rangle b_n \end{aligned} \tag{2. 5}$$

for every pair of $f(x)$ and $h(x)$ in $L_2(0, \infty; dx)$. Put $f(x) = \varphi_k(x)$ and $h(x) = \psi_l(x)$ in (2. 5). Then we obtain

$$\langle \varphi_k, \psi_l \rangle = a_k \langle \varphi_k, \psi_l \rangle + b_l \langle \varphi_k, \psi_l \rangle . \tag{2. 6}$$

Since $\langle \varphi_k, \psi_l \rangle$ does not vanish for any k and l, we get from (2. 6), $1 = a_k$

(1) This proposition is due to Y. Saitō.

$+b_l$. Putting $k=1$, we have $1=a_1+b_l$, and putting $l=1$, we have $1=a_k+b_1$. This proves our proposition.

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