A two points connection problem involving logarithmic polynomials

By

Mitsuhiko KOHNO

1. In the paper [1], [2], a two points connection problem between two sets of fundamental solutions for a system of ordinary differential equations

$$t\frac{dX}{dt} = (A+tB)X \tag{1. 1}$$

was studied under the assumptions that the eigenvalues of the diagonal matrix B satisfy the pentagonal condition, and that the matrix A has no congruent eigenvalues, or only one pair of congruent eigenvalues.

In this paper, we extend these results in the direction that the matrix A may have any sets of congruent eigenvalues. Although we will investigate the case where all the eigenvalues of the matrix A are congruent for the sake of simplicity, the method applies easily to the general case to yield the similar results.

A system of n linear ordinary differential equations of the form (1. 1) has a regular singular point at $t=0$, and an irregular singular point of rank one at $t=\infty$.

It is well known that one set of solutions of the system of differential equations (1. 1) has convergent power series expansions at the regular singular point, $t=0$, expressed by

$$X_j(t) = t^{\rho_j}\sum_{m=0}^{\infty} G_j(m)t^m \tag{1. 2}$$

where ρ_j $(j=1, 2, \cdots, n)$ are eigenvalues of the matrix A, when A has no congruent eigenvalues. In general, some of these expressions are replaced by polynomials in logarithmic function of t, with coefficients of the form (1. 2).

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On the other hand, at the irregular singular point $t=\infty$, there exists a set of formal solutions

$$X^{k}(t) \simeq \exp(\lambda_{k}t)t^{a_{kk}}\sum_{s=0}^{\infty} H^{k}(s)t^{-s} \tag{1.3}$$

which expresses a set of fundamental solutions asymptotically in an arbitrary sectorial neighbourhood of the infinity, with properly chosen width.

Now we will show the assumptions explicitly.

(1) The matrix B is a diagonal matrix:

$$B = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_n).$$

The eigenvalues $\lambda_k (k=1, 2, \cdots, n)$ satisfy the pentagonal condition, i.e,

$$|\lambda_j - \lambda_k| > |\lambda_k| > 0 \qquad (j \neq k). \tag{1.4}$$

(2) The eigenvalues ρ_j $(j=1, 2, \cdots, n)$ of the matrix A satisfy

$$\rho_j - \rho_1 = -m_j \qquad (j=2, 3, \cdots, n)$$

where m_j $(j=2, 3, \cdots, n)$ are positive integers and $m_{j+1} > m_j$.

(3) $a_{kk} - \rho_j \neq$ non-negative integers $(j, k=1, 2, \cdots, n)$ where a_{kk} are the diagonal elements of the matrix A.

According to the assumption (2), the set of fundamental solutions of the system (1. 1) at the origin can be written down as follows:

$$\begin{cases} X_1(t) = t^{\rho_1}\sum_{m=0}^{\infty} G_1(m)t^m \\ X_j(t) = \sum_{l=1}^{j} \dfrac{1}{(j-l)!}(\log t)^{j-l}\tilde{X}_l(t) \qquad (j=2, 3, \cdots, n). \end{cases} \tag{1.5}$$

where

$$\begin{cases} \tilde{X}_1(t) = X_1(t) \\ \tilde{X}_j(t) = t^{\rho_j}\sum_{m=0}^{\infty} \tilde{G}_j(m)t^m \qquad (j=2, 3, \cdots, n). \end{cases}$$

The purpose of this paper is to calculate the connection coefficients between two sets of solutions (1. 5) and (1. 3). And also we will show the simple method to derive the asymptotic forms of the convergent solutions near the entire neighbourhood of infinity.

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2. In this chapter we prove some lemmata for single inhomogeneous equations which are used and make important roles in the following chapter.

At first, we show the next lemma that was proved in the preceding paper [2]. The proof and the explanations in the preceding one were a little incomplete, so here we again prove it in detail.

Lemma 2. 1. *If* $\gamma(t)$ *is holomorphic and* $t^l\gamma(t)$ *is bounded for a real number* l *such that* $\mathrm{Re}(\alpha+l)>0$, *in the domain*:

$$\mathcal{D}^* = \left\{t\colon |t|\geqq t_0>0,\ |\arg t|\leqq\frac{3}{2}\pi-\eta\right\} - \{t\colon \mathrm{Re}\, t>-t_0,\ |\arg t|>\pi\} \tag{2. 1}$$

where η *is an arbitrary small positive real number, then a solution* $y(t)$, *which bounded in* $\mathcal{D}^*$, *of the equation*:

$$t\frac{dy}{dt} = (t+\alpha)y+\gamma(t) \tag{2. 2}$$

has the form

$$y(t) = O(t^{-l}) \qquad in\ \mathcal{D}^*\,. \tag{2. 3}$$

Proof. We prove the lemma only for t with non-negative argument, while the case for t with negative argument will be proved similarly.

The general solution of (2. 2) is easily obtained by quadrature.

$$y(t) = c\,e^t t^\alpha - \int_t^\infty e^{t-\tau}\left(\frac{t}{\tau}\right)^\alpha \gamma(\tau)\frac{d\tau}{\tau}.$$

Here the integral path P_t is taken as follows. For any t in $\mathcal{D}^*$ with non-negative argument, we determine t' such that

$$\begin{aligned} t' &= -\mathrm{Re}\, t && \left(\pi<\arg t\leqq\frac{3}{2}\pi-\eta\right)\\ &= |t| && (0\leqq\arg t\leqq\pi) \end{aligned}$$

and the path P_t consists of the following three parts:

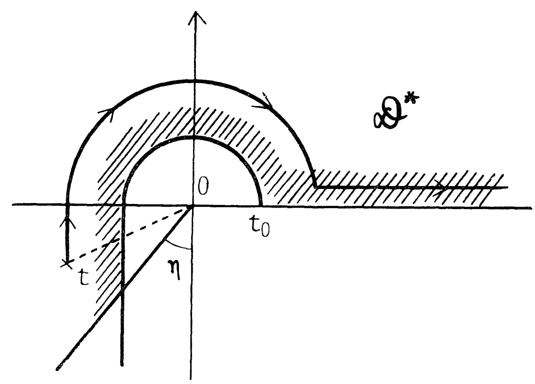
(i) straight line $\quad \tau=-t'+i\mu \quad (-\infty<\mu<0)$

(ii) semi-circle $\quad \tau=t'\exp(i(\pi-\mu)) \quad (0\leqq\mu<\pi)$

(iii) positive real axis $\quad \pi=\dfrac{t'}{\pi}\mu \qquad \pi\leqq\mu$.

(see below: Figure 1.)

We call this path P_t by "Friedrichs' path".



(Figure 1)

The integration is carried out along an arrow in Figure 1.

Because of the boundedness of $y(t)$ in $\mathcal{D}^*$, the integral constant c must be zero and then we only prove the boundedness of

$$-t^l\int_t^\infty e^{t-\tau}\left(\frac{t}{\tau}\right)^\alpha \gamma(t)\frac{d\tau}{\tau}=-\int_t^\infty e^{t-\tau}\left(\frac{t}{\tau}\right)^{\alpha+l}\tau^l\gamma(\tau)\frac{d\tau}{\tau}$$

in $\mathcal{D}^*$.

For the purpose of that, we estimate the three quantities of the integrand excepting $\tau^l\gamma(\tau)$ and obtain the table below.

	$\arg t>\pi$		
	(i)	(ii)	(iii)
	$\mathrm{Im}\, t\leqq\mu<0$	$0\leqq\mu<\pi$	$\pi\leqq\mu<\infty$
$\lvert e^{t-\tau}\rvert$	$=1$	$\leqq 1$	$\leqq e^{-t'\left(1+\frac{\mu}{\pi}\right)}$
$\left\lvert\left(\frac{t}{\tau}\right)^{\alpha+l}\right\rvert$	$\leqq\left(\frac{1}{\sin\eta}\right)^{\mathrm{Re}(\alpha+l)} e^{\theta_0\lvert\mathrm{Im}\,\alpha\rvert}$	$\leqq\left(\frac{1}{\sin\eta}\right)^{\mathrm{Re}(\alpha+l)} e^{\left(\frac{3}{2}\pi-\eta\right)\lvert\mathrm{Im}\,\alpha\rvert}$	$\leqq\left(\frac{\pi}{\sin\eta}\right)\frac{1}{\mu}$
$\left\lvert\frac{d\tau}{\tau}\right\rvert$	$\leqq\frac{d\mu}{t'}$	$=d\mu$	$=\frac{d\mu}{\mu}$

	$0<\arg t\leqq\pi$		$\arg t=0$
	(ii)	(iii)	(iii)
	$\pi-\arg t\leqq\mu<\pi$	$\pi\leqq\mu<\infty$	$\pi\leqq\mu<\infty$
$\lvert e^{t-\tau}\rvert$	$\leqq 1$	$\leqq e^{t'\left(1-\frac{\mu}{\pi}\right)}$	$\leqq e^{t'\left(1-\frac{\mu}{\pi}\right)}$
$\left\lvert\left(\frac{t}{\tau}\right)^{\alpha+l}\right\rvert$	$\leqq e^{\pi\lvert\mathrm{Im}\,\alpha\rvert}$	$\leqq\frac{\pi}{\mu}\,e^{\pi\lvert\mathrm{Im}\,\alpha\rvert}$	$=\frac{\pi}{\mu}$
$\left\lvert\frac{d\tau}{\tau}\right\rvert$	$=d\mu$	$=\frac{d\mu}{\mu}$	$=\frac{d\mu}{\mu}$

The estimation of the second and fourth columns are easy to derive and here we show the derivation of the estimate of the third column. As α is not always real number, if we denote the argument of (t/τ) by Θ, we get

$$\left|\left(\frac{t}{\tau}\right)^{\alpha+l}\right|=\left|\frac{t}{\tau}\right|^{\mathrm{Re}(\alpha+l)}e^{-\Theta(\mathrm{Im}\alpha)}\,.$$

For example, in the case of (i) the next inequality is derived.

$$1\leqq\left|\frac{t}{\tau}\right|\leqq\frac{|t|}{|t|\sin\eta}=\left(\frac{1}{\sin\eta}\right).$$

And if we set $\arg t=\varphi$, it follows that

$$\left(\frac{t}{\tau}\right)=\frac{te^{i\varphi}}{\mathrm{Re}\,t+i\mu}=\frac{|t|e^{i\varphi}(\mathrm{Re}\,t-i\mu)}{(\mathrm{Re}\,t)^2+\mu^2},\qquad(\mathrm{Im}\,t\leqq\mu<0)$$

$$\mathrm{Im}\left(\frac{t}{\tau}\right)=\frac{|t|(|t|\sin\varphi-\mu)\cos\varphi}{(\mathrm{Re}\,t)^2+\mu^2}\qquad(\mathrm{Re}\,t=|t|\cos\varphi)\,.$$

In the interval $|t|\sin\varphi\leqq\mu\leqq 0$, $\mathrm{Im}(t/\tau)$ is a monotone increasing function of μ, it follows that

$$0\leqq\mathrm{Im}\left(\frac{t}{\tau}\right)\leqq\left(\frac{\sin\varphi}{\cos\varphi}\right)\leqq\cot\eta\,.$$

$$\therefore\ 0\leqq\sin\Theta\leqq\frac{\cot\eta}{\left|\frac{t}{\tau}\right|}\leqq\cot\eta$$

$$0\leqq\Theta\leqq\theta_0\qquad(\theta_0=\sin^{-1}(\cot\eta))\,.$$

The estimate was derived and the remaining ones will be proved similarly.

From the above table the boundedness of the integral on the part (ii) is evident and the whole integral will be bounded if it is bounded independent of t on the part (i) and (iii) of P_t.

Now we get for $\arg t > \pi$,

$$\left|\int_{(\mathrm{i})}\right| \leqq \frac{K}{t'}\int_{\mathrm{Im}\,t}^{0} d\mu = \frac{K}{-|t|\cos\varphi}(-\mathrm{Im}\,t) = \frac{K\sin\varphi}{\cos\varphi} \leqq K\cot\eta$$

$$\left|\int_{(\mathrm{iii})}\right| \leqq K\int_{\pi}^{\infty} e^{-t'(1+(\mu/\pi))}\frac{d\mu}{\mu^2} \leqq Ke^{-t'}\int_{1}^{\infty} e^{-t'X}X^{-2}dX$$

$$\leqq K\frac{e^{-2t'}}{t'} \leqq K\frac{e^{-2t_0}}{t_0} \qquad (t' \geqq t_0)$$

and for $0 \leqq \arg t \leqq \pi$,

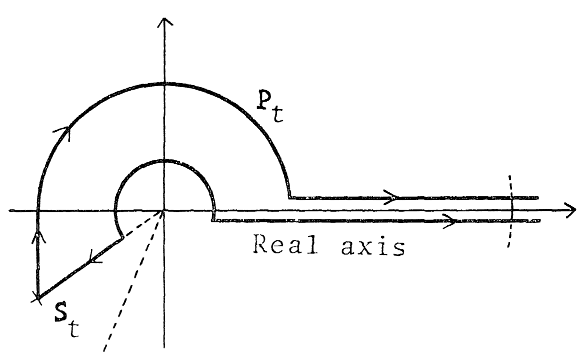
$$\left|\int_{(\mathrm{iii})}\right| \leqq K\int_{\pi}^{\infty} e^{t'(1-(\mu/\pi))}\frac{d\mu}{\mu^2} \leqq Ke^{t'}\int_{1}^{\infty} e^{-t'X}X^{-2}dX$$

$$\leqq Ke^{t'}\cdot\frac{e^{-t'}}{t'} = \frac{K}{t'} \leqq \frac{K}{t_0}.$$

This proves the lemma.

Remark. 1. If $\mathrm{Re}(\alpha) > 0$, the following integral along any paths which start from the origin and end at infinity is equal to the integral along the real axis. For example, we take the path that consists of the straight line S_t and the "Friedrichs' path" P_t as in Figure 2.

$$\int_{\substack{0\\(S_t+P_t)}}^{\infty} e^{-\tau}\tau^{\alpha-1}d\tau = \int_{\substack{0\\(\text{Real axis})}}^{\infty} e^{-\tau}\tau^{\alpha-1}d\tau$$



(Figure 2)

In fact we consider the contour which consists of (S_t+P_t), real axis and two arcs of circles around the origin with radii ε and R.

In the above contour $e^{-\tau}\tau^{\alpha-1}$is holomorphic and we obtain

$$\int_{(S_t+P_t)}+\int_{|\tau|=R}+\int_{|\tau|=\varepsilon}=\int_{(\text{Real axis})}$$

$$\left|\int_{|\tau|=R} e^{-\tau}\tau^{\alpha-1}d\tau\right| \leqq \int_{-\pi/2}^{\pi/2} e^{-R\cos\theta}R^{\mathrm{Re}(\alpha)}e^{-\theta\,\mathrm{Im}(\alpha)}d\theta \to 0 \qquad (R\to\infty)$$

$$\left|\int_{|\tau|=\varepsilon} e^{-\tau}\tau^{\alpha-1}d\tau\right| \leqq \int_{0}^{2\pi} \varepsilon^{\mathrm{Re}(\alpha)}e^{-\varepsilon\cos\theta}e^{-\theta\,\mathrm{Im}(\alpha)}d\theta \to 0 \qquad (\varepsilon\to 0)$$

q. e. d.

Next we consider a special system of linear differential equations, which, after some modifications, will be imbedded in the original system (1. 1). We will study the structures of the solutions and their coefficients in great detail, because the connection problem of the original system can be resolved into the interactions of these imbedded equations.

Definition 2. 1. *We denote the following domain by $D(\lambda)$*

$$D(\lambda) = \{t\colon \lambda t \in \mathscr{D}^*\}\ .$$

Definition 2. 2. *We denote the n by n lower cyclic matrix by Z*

$$Z=\begin{pmatrix} 0 & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{pmatrix}.$$

A vector, or a matrix which is a transpose of another, will be denoted by $*$ suffix. For example, A_* represents the transpose of A.

The column vector e_k stands for the unit vector with all the component zero except for the k-th. Thus we can write

$$Z=(0,\ e_1,\ e_2,\ \cdots,\ e_{n-1})_*$$

where 0 stands for the zero vector.

Proposition 2. 1. *If α is not a non-negative integer, the system of differential equations*

$$t\frac{dY(t)}{dt}=(\lambda t+\alpha-Z)Y(t)+e_1 \tag{2. 4}$$

has a unique holomorphic solution at the origin, and if we write this solution in the form

$$Y(t) = \sum_{m=0}^{\infty} \mathfrak{G}(m) t^m \tag{2.5}$$

the coefficients $\mathfrak{G}(m)$ $(m=0, 1, \cdots)$ *satisfy the system of recurrence formulae*

$$\begin{cases} (m+Z-\alpha)\mathfrak{G}(m) = \lambda \mathfrak{G}(m-1) \\ (Z-\alpha)\mathfrak{G}(0) = e_1 \end{cases} . \tag{2.6}$$

Proof. As the homogeneous part of the system has only two singular points: a regular singular point at the origin and an irregular singular point at infinity, there exists a unique holomorphic solution at the origin, since the characteristic exponents at the origin, α, is not a non-negative integer. In fact, if we expand this solution in the form (2. 5), the coefficients are determined by (2. 6), as is easily seen by the direct substitution. The matrix coefficients on the left hand side of (2. 6) can never be singular as long as the m takes non-negative integral value, and therefore we can determine the vectors $\mathfrak{G}(m)$ uniquely.

Lemma 2. 2. *Given any positive integer s, and t in $D(\lambda)$, we have*

$$Y(t) = e^{\lambda t} t^{\alpha - Z} C(\alpha, \lambda) - \sum_{l=1}^{s} \mathfrak{G}(-l) t^{-l} + O(t^{-s}) \tag{2.7}$$

where the constant vector $C(\alpha, \lambda)$ will be explicitly given in the proof below, and the matrix power of a scalar, t^A, is defined as follows.

$$t^A = \exp(A \log t) = \sum_{n=0}^{\infty} \frac{(\log t)^n A^n}{n!} .$$

Proof. First, suppose that $\mathrm{Re}(\alpha) < 0$.

We can solve the differential equations (2. 4) by quadrature, and by taking the form (2. 5) of the solution into account, we have

$$Y(t) = e^{\lambda t} t^{\alpha - Z} \int_0^t e^{-\lambda \tau} \tau^{Z - \alpha - 1} e_1 d\tau$$

where the path of the integration is the straight line $S_t(\lambda)$, which is, as we have indicated, equivalent to "the real axis $-P_t(\lambda)$". Here $S_t(\lambda)$ and $P_t(\lambda)$ are the straight line and "Friedrichs' path" respectively in the domain $D(\lambda)$. Hence, we have

$$Y(t) = e^{\lambda t} t^{\alpha - Z} \int_{\substack{0 \\ (\arg \lambda\tau = 0)}}^{\infty} e^{-\lambda\tau} \tau^{Z-\alpha-1} e_1 d\tau$$
$$- e^{\lambda t} t^{\alpha - Z} \int_{\substack{t \\ (P_t(\lambda))}}^{\infty} e^{-\lambda\tau} \tau^{Z-\alpha-1} e_1 d\tau \,.$$

Then the first term of the right hand side is easily calculated. In fact, the cyclic matrix has the property that if $p \geqq n$, $Z^p \equiv 0$. So we obtain

$$\tau^Z = \sum_{p=0}^{\infty} \frac{1}{p!} (\log \tau)^p Z^p = \sum_{p=0}^{n-1} \frac{1}{p!} (\log \tau)^p Z^p$$

and

$$C(\alpha, \lambda) = \int_{\substack{0 \\ (\arg \lambda\tau = 0)}}^{\infty} e^{-\lambda\tau} \tau^{Z-\alpha-1} e_1 d\tau$$
$$= \sum_{p=0}^{n-1} \int_{\substack{0 \\ (\arg \lambda\tau = 0)}}^{\infty} \frac{Z^p}{p!} e^{-\lambda\tau} (\log \tau)^p \tau^{-\alpha-1} d\tau \cdot e_1 \,.$$

We put

$$c_p = \int_0^{\infty} e^{-\lambda\tau} \tau^{-\alpha-1} (\log \tau)^p d\tau \qquad (p=0, 1, \cdots, n-1)\,,$$

then these constants c_p satisfies the following difference-differential equations:

$$\frac{\partial c_p}{\partial \alpha} = -\int_{\substack{0 \\ (\arg \lambda\tau = 0)}}^{\infty} e^{-\lambda\tau} \tau^{-\alpha-1} (\log \tau)^{p+1} d\tau = -c_{p+1}\,,$$

and especially,

$$c_0 = \int_{\substack{0 \\ (\arg \lambda\tau = 0)}}^{\infty} e^{-\lambda\tau} \tau^{-\alpha-1} d\tau = \lambda^{\alpha} \int_{\substack{0 \\ (\text{Real axis})}}^{\infty} e^{-t} t^{-\alpha-1} dt \,.$$
$$= \lambda^{\alpha} \Gamma(-\alpha)$$

If we differentiate c_0 in α, we can calculate c_1 explicitly and successively the remaining c_p too. c_p are expressed by $\Gamma, \Gamma', \cdots, \Gamma^{(p)}$ and $\log \lambda$.

Therefore

$$C(\alpha, \lambda) = \sum_{p=0}^{n-1} \frac{Z^p}{p!} c_p e_1 = \begin{pmatrix} c_0 \\ c_1 \\ \vdots \\ \dfrac{c_{n-1}}{(n-1)!} \end{pmatrix}.$$

So we obtain

$$Y(t) = e^{\lambda t}t^{\mathfrak{a}-Z}C(\alpha, \lambda) - e^{\lambda t}t^{\mathfrak{a}-Z}\int_{\substack{t \\ (P_t(\lambda))}}^{\infty} e^{-\lambda\tau}\tau^{Z-\mathfrak{a}-1}e_1 d\tau\,.$$

Now we put

$$E(t, s) = e^{\lambda t}t^{\mathfrak{a}-Z-s}\int_{\substack{t \\ (P_t(\lambda))}}^{\infty} e^{-\lambda\tau}\tau^{Z-\mathfrak{a}-1+s}e_1 d\tau\,.$$

If we integrate by parts, using the calculation of the differentiation of the matrix power of a scalar, we obtain the following recurrence relations

$$E(t, s) = \frac{1}{\lambda t}e_1 + \frac{1}{\lambda t}(Z-\alpha-1+s)E(t, s-1)$$

and by successive integrations by parts, and in accordance with (2. 6), it will be easily derived

$$E(t, 0) = \sum_{l=1}^{s} t^{-l}\mathfrak{S}(-l) + Y_s(t)\,.$$

In order to estimate the remainder term $Y_s(t)$, we consider the equations which are satisfied by $Y_s(t)$.

If we substitute $Y(t) = e^{\lambda t}t^{\mathfrak{a}-Z}C(\alpha, \lambda) - \sum_{l=1}^{s}\mathfrak{S}(-l)t^{-l} - Y_s(t)$ into (2. 4), we get these equations as follows:

$$\text{(2. 8)} \qquad t\frac{dY_s(t)}{dt} = (\lambda t+\alpha-Z)Y_s(t) + (s+\alpha-Z)\mathfrak{S}(-s)t^{-s} \qquad (t \in D(\lambda))\,.$$

So if we replace λt by t, we derive the equations of the same type as (2. 2):

$$\text{(2. 9)} \qquad t\frac{dY_s(t)}{dt} = (t+\alpha-Z)Y_s(t) + (s+\alpha-Z)\mathfrak{S}(-s)\lambda^s t^{-s} \qquad (t \in \mathscr{D}^*)\,.$$

Here we put the column vector $Y_s(t) = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$ and write down the system of equations (2. 9) in component-wise.

$$\text{(2. 10)} \qquad \begin{cases} t\dfrac{dy_1}{dt} = (t+\alpha)y_1 + e_{1*}(s+\alpha-Z)\mathfrak{S}(-s)\lambda^s t^{-s} \\ t\dfrac{dy_2}{dt} = (t+\alpha)y_2 - y_1 + e_{2*}(s+\alpha-Z)\mathfrak{S}(-s)\lambda^s t^{-s} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ t\dfrac{dy_n}{dt} = (t+\alpha)y_n - y_{n-1} + e_{n*}(s+\alpha-Z)\mathfrak{S}(-s)\lambda^s t^{-s}\,. \end{cases}$$

Whence, if we take a sufficiently large s such that $\mathrm{Re}(\alpha+s)>0$, we can apply the above lemma 2.1 to the first equation of (2.10) and repeatedly the remaining equations, and derive the estimates:

$$y_1(t)=O(t^{-s}),\ y_2(t)=O(t^{-s}),\cdots,y_n(t)=O(t^{-s}),\qquad (t\in\mathscr{D}^*).$$

It results that

$$Y_s(t)=O(t^{-s})\qquad (t\in D(\lambda)).$$

This proves the lemma 2.2 for $\mathrm{Re}(\alpha)<0$.

In order to prove the lemma for arbitrary, we use the notations as used in [2], to clarify the dependence of $Y(t)$ on the parameter α.

$$Y(t,\alpha)=\sum_{m=0}^{\infty}\mathfrak{G}(m,\alpha)t^m.$$

The vectorial coefficients $\mathfrak{G}(m,\alpha+1)$ satisfy the difference equations

$$(m-\alpha-1+Z)\mathfrak{G}(m,\alpha+1)=\lambda\mathfrak{G}(m-1,\alpha+1)$$

which are the same equations for $\mathfrak{G}(m-1,\alpha)$. Thus there is a constant matrix C independent of m, but dependent on α, such that

$$\mathfrak{G}(m,\alpha+1)=C\mathfrak{G}(m-1,\alpha) \tag{2.11}$$

for all integral values of m. Now we can get the followwing recurrence relation between $Y(t,\alpha)$ and $Y(t,\alpha+1)$:

$$\begin{aligned}Y(t,\alpha+1)&=\sum_{m=0}^{\infty}\mathfrak{G}(m,\alpha+1)t^m=\sum_{m=0}^{\infty}C\mathfrak{G}(m-1,\alpha)t^m\\&=\sum_{m=1}^{\infty}C\mathfrak{G}(m-1,\alpha)t^m+C\mathfrak{G}(-1,\alpha)\\&=Ct\sum_{m=0}^{\infty}\mathfrak{G}(m,\alpha)t^m+C\mathfrak{G}(-1,\alpha)\\&=CtY(t,\alpha)+C\mathfrak{G}(-1,\alpha)=CtY(t,\alpha)+\mathfrak{G}(0,\alpha+1).\end{aligned}$$

The constant matrix C is determined at $m=0$, from

$$\begin{cases}(Z-\alpha-1)\mathfrak{G}(0,\alpha+1)=e_1\\(Z-\alpha)\mathfrak{G}(0,\alpha)=e_1\end{cases}.$$

Namely, from $e_1=\lambda\mathfrak{G}(-1,\alpha)$

$$(Z-\alpha-1)\mathfrak{G}(0,\alpha+1)=\lambda\mathfrak{G}(-1,\alpha).$$
$$\therefore\quad C=(Z-\alpha-1)^{-1}\lambda$$

From these relations, we can deduce the asymptotic expansions for $\alpha+1$.

$$
\begin{aligned}
Y(t, \alpha+1) &= Ct\{e^{\lambda t}t^{\alpha-Z}C(\alpha, \lambda)-\sum_{l=1}^{s+1}\mathfrak{G}(-l, \alpha)t^{-l}+O(t^{-s-1})\}+\mathfrak{G}(0, \alpha+1)\\
&= e^{\lambda t}t^{\alpha+1-Z}C\cdot C(\alpha, \lambda)-\sum_{l=1}^{s}C\mathfrak{G}(-l-1, \alpha)t^{-l}\\
&\quad +\mathfrak{G}(0, \alpha+1)-C\mathfrak{G}(-1, \alpha)+O(t^{-s}).\\
&= e^{\lambda t}t^{\alpha+1-Z}C(\alpha+1, \lambda)-\sum_{l=1}^{s}\mathfrak{G}(-l, \alpha+1)t^{-l}+O(t^{-s}).
\end{aligned}
$$

Here we used the relations (2. 11). And the lemma will be proved by repeating this process.

Next we give the corollary of lemma 2. 2., as we need it in the latter parts of this paper.

Corollary of lemma 2. 2. For any t in $D(\lambda)$, we have for any positive integer σ

$$
\sum_{m=0}^{\infty}\mathfrak{G}(m+s)t^{m} = e^{\lambda t}t^{\alpha-s-Z}C(\alpha, \lambda)-\sum_{l=1}^{\sigma}\mathfrak{G}(s-l)t^{-l}+O(t^{-\sigma}).
$$

Proof.

$$
\begin{aligned}
\sum_{m=0}^{\infty}\mathfrak{G}(m+s)t^{m} &= t^{-s}\sum_{m=0}^{\infty}\mathfrak{G}(m+s)t^{m+s}\\
&= t^{-s}\{\sum_{m=0}^{\infty}\mathfrak{G}(m)t^{m}-\sum_{m=0}^{s-1}\mathfrak{G}(m)t^{m}\}\\
&= t^{-s}\{e^{\lambda t}t^{\alpha-Z}C(\alpha, \lambda)-\sum_{l=1}^{\sigma-s}\mathfrak{G}(-l)t^{-l}+O(t^{-\sigma+s})\}-\sum_{m=0}^{s-1}\mathfrak{G}(m)t^{m-s}\\
&= e^{\lambda t}t^{\alpha-s-Z}C(\alpha, \lambda)-\{\sum_{l=1}^{\sigma-s}\mathfrak{G}(-l)t^{-l-s}+\sum_{m=0}^{s-1}\mathfrak{G}(m)t^{m-s}\}+O(t^{-\sigma})\\
&= e^{\lambda t}t^{\alpha-s-Z}C(\alpha, \lambda)-\sum_{l=1}^{\sigma}\mathfrak{G}(s-l)t^{-l}+O(t^{-\sigma})
\end{aligned}
$$

q. e. d.

3. Now we will investigate the relations among the components and the behavior near the infinity of the vectorrial coefficients $\mathfrak{G}(m)$.

We put the column vectors $\mathfrak{G}(m)$ as follows

$$
\mathfrak{G}(m) = \begin{pmatrix} g_1(m) \\ \vdots \\ g_n(m) \end{pmatrix}
$$

and substitute them into the difference equations (2. 6):

$$\begin{pmatrix} m-\alpha & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & m-\alpha \end{pmatrix} \begin{pmatrix} g_1(m) \\ \vdots \\ g_n(m) \end{pmatrix} = \lambda \begin{pmatrix} g_1(m-1) \\ \vdots \\ g_n(m-1) \end{pmatrix}$$

$$\begin{pmatrix} -\alpha & & & 0 \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ 0 & & 1 & -\alpha \end{pmatrix} \begin{pmatrix} g_1(0) \\ \vdots \\ g_n(0) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Then we obtain the relations of the components $g_1(m), \cdots, g_n(m)$.

$$(3.1) \qquad \begin{cases} (m-\alpha)g_j(m) = \lambda g_j(m-1) - g_{j-1}(m) & (j=1, 2, \cdots, n) \\ g_0(m) \equiv 0 \end{cases}$$

$$(3.2) \qquad g_1(0) = -\frac{1}{\alpha}, \; g_2(0) = -\frac{1}{\alpha^2}, \cdots, g_n(0) = -\frac{1}{\alpha^n}.$$

Here we define new vectors $\tilde{\mathfrak{G}}(m)$ as follows.

$$(3.3) \qquad \mathfrak{G}(m) = \frac{\lambda^m}{\Gamma(m-\alpha+1)} \tilde{\mathfrak{G}}(m)$$

Substituing $\mathfrak{G}(m)$ into the difference equations (2. 6) again, we have

$$(m+Z-\alpha)\tilde{\mathfrak{G}}(m) = (m-\alpha)\tilde{\mathfrak{G}}(m-1).$$

Therefore

$$\tilde{\mathfrak{G}}(m) - \tilde{\mathfrak{G}}(m-1) = -\frac{Z}{m-\alpha} \tilde{\mathfrak{G}}(m).$$

We write out the above relations in componentwise, attaching the components of $\tilde{\mathfrak{G}}(m)$ with the "wave" symbol.

$$(3.4) \qquad \begin{cases} \tilde{g}_1(m) - \tilde{g}_1(m-1) = 0 \\ \tilde{g}_j(m) - \tilde{g}_j(m-1) = -\dfrac{1}{m-\alpha} \tilde{g}_{j-1}(m) & (j=2, 3, \cdots, n). \end{cases}$$

From these relations, we get easily

$$(3.5) \qquad \begin{cases} \tilde{g}_1(m) = \Gamma(-\alpha) \\ \tilde{g}_j(m) = \sum\limits_{p=0}^{m} \{\psi(p-\alpha) - \psi(p+1-\alpha)\} \tilde{g}_{j-1}(p) & (j=2, 3, \cdots, n) \end{cases}$$

where $\psi(z)$ denotes the Gauss's "psi-function".

In particular,

$$g_1(m) = \frac{\lambda^m \Gamma(-\alpha)}{\Gamma(m-\alpha+1)}.$$

Next we replace the positive real integer m in the above difference equations (2. 6) and (3. 1) by the complex variable w.

(3. 6) $$(w+Z-\alpha)\mathfrak{G}(w) = \lambda\mathfrak{G}(w-1)$$

(3. 7) $$(w-\alpha)g_j(w) = \lambda g_j(w-1) - g_{j-1}(w).$$

It is easy to see that

$$g_1(w) = \frac{\lambda^w \Gamma(-\alpha)}{\Gamma(w-\alpha+1)}.$$

Here we investigate the other solutions $g_j(w)$ $(j=2, 3, \cdots, n)$ of the difference equations (3. 7).

Proposition 3. 1. *If we define* $\varphi_j(w) = \frac{1}{(j-1)!}\frac{d^{j-1}}{dw^{j-1}}\{g_1(w)\}$ $(j=1, 2, \cdots, n)$ *and* $\varphi_0(w) \equiv 0)$ *then we have*

$$(w-\alpha)\varphi_j(w) - \lambda\varphi_j(w-1) = -\varphi_{j-1}(w).$$

Proof. For $j=1$, this is trivial from the property of the gamma-function.

Suppose the proposition for $j=1, 2, \cdots, p$. Then we have

$$(w-\alpha)\varphi_p(w) - \lambda\varphi_p(w-1) = -\varphi_{p-1}(w).$$

Differentiating both sides of the above relations, we obtain

$$(w-\alpha)\frac{d}{dw}(\varphi_p(w)) - \lambda\frac{d}{dw}(\varphi_p(w-1)) + \varphi_p(w) = -\frac{d}{dw}(\varphi_{p-1}(w)).$$

Here the right hand side

$$\frac{d}{dw}(\varphi_{p-1}(w)) = (p-1)\varphi_p(w)$$

and so we obtain

$$(w-\alpha)p\cdot\varphi_{p+1}(w) - \lambda p\varphi_{p+1}(w) = -p\varphi_p(w).$$

Hence

$$(w-\alpha)\varphi_{p+1}(w) - \lambda\varphi_{p+1}(w) = -\varphi_p(w).$$

The proposition 3. 1 was proved by induction.

Proposition 3. 2. *The following matrix is a fundamental matrix solution of the system* (3. 6)

$$\begin{pmatrix} \varphi_1(w) & & & & 0 \\ \varphi_2(w) & \varphi_1(w) & & & \\ \varphi_3(w) & \varphi_2(w) & \varphi_1(w) & & \\ \vdots & \vdots & & \ddots & \\ \varphi_n(w) & \varphi_{n-1}(w) & \cdots\cdots & & \varphi_1(w) \end{pmatrix}.$$

Proof. By the proposition 3. 1, it is easy to see that each column vector of the above matrix is a solution of the system (3. 6).

To show that each column vector is linearly independent of the others, we calculate the determinant, which is equal to $\{g_1(w)\}^n$ and cannot be zero.

Proposition 3. 3. *Multiplying the fundamental matrix solution of the proposition* 3. 2 *by the constant column vector from the left hand side, we can get the solution of the system* (3. 6) *with the initial conditions* (3. 2). *The constant column vector is determined from the initial conditions.*

$$\mathfrak{G}(w) = \begin{pmatrix} \varphi_1(w) & & & & 0 \\ \varphi_2(w) & \varphi_1(w) & & & \\ \varphi_3(w) & \varphi_2(w) & \varphi_1(w) & & \\ \vdots & \vdots & & \ddots & \\ \varphi_n(w) & \varphi_{n-1}(w) & \cdots\cdots & & \varphi_1(w) \end{pmatrix} \begin{pmatrix} 1 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \end{pmatrix}. \tag{3. 8}$$

Proposition 3. 4. *If h_0 is a large positive number, then for* $\operatorname{Re} w \geqq h_0$, $\varphi_j(w)$ $(j=1, 2, \cdots, n)$ *have the following asymptotic forms.*

$$\varphi_j(w) \simeq \frac{\lambda^w \Gamma(-\alpha)}{\Gamma(w-\alpha+1)} (\log w)^{j-1} \left\{ c + O\left(\frac{1}{w}\right) \right\} \qquad \left(|\arg w| \leqq \frac{\pi}{2} \right) \tag{3. 9}$$

where c is a constant.

Proof. It is well known that for $|\arg t| < \pi$,

$$\frac{1}{\Gamma(t+\alpha)} \simeq \left(\frac{e}{t}\right)^t t^{(1/2)-\alpha} \left\{ \sum_{l=0}^{s-1} b_l t^{-l} + O(t^{-s}) \right\}.$$

So we have

$$\varphi_1(w) = g_1(w) = \frac{\lambda^w \Gamma(-\alpha)}{\Gamma(w-\alpha+1)} \simeq \lambda^w \left(\frac{e}{w}\right)^w w^{\alpha-(1/2)} \left\{ c + O\left(\frac{1}{w}\right) \right\} \tag{3. 10}$$

for $|\arg w| < \pi$.

If we differentiate both sides of (3. 10), we get

$$\varphi_2(w) = \frac{d}{dw}\varphi_1(w) \cong \lambda^w\left(\frac{e}{w}\right)^w w^{\alpha-(1/2)}(\log\lambda+2+\log w)\left\{c+O\left(\frac{1}{w}\right)\right\}$$

for $|\arg w|<\pi-\delta$. Here δ is a small positive number.

$$\begin{aligned}\varphi_2(w) &\cong \lambda^w\left(\frac{e}{w}\right)^w w^{\alpha-(1/2)}\log w\left\{1+\frac{2+\log\lambda}{\log w}\right\}\left\{c+O\left(\frac{1}{w}\right)\right\} \\ &\cong \frac{\lambda^w\Gamma(-\alpha)}{\Gamma(w-\alpha+1)}\log w\left\{c+O\left(\frac{1}{w}\right)\right\}\end{aligned}$$

Repeating this process, we get

$$\begin{aligned}\varphi_j(w) &\cong \lambda^w\left(\frac{e}{w}\right)^w w^{\alpha-(1/2)}(\log w+\log\lambda+2)^{j-1}\left\{c+O\left(\frac{1}{w}\right)\right\} \\ &\cong \lambda^w\left(\frac{e}{w}\right)^w w^{\alpha-(1/2)}(\log w)^{j-1}\left\{c+O\left(\frac{1}{w}\right)\right\} \\ &\cong \frac{\lambda^w\Gamma(-\alpha)}{\Gamma(w-\alpha+1)}(\log w)^{j-1}\left\{c+O\left(\frac{1}{w}\right)\right\}\end{aligned}$$

for $|\arg w|<\pi-\delta'$.

Lemma 3. 1. *For* $\operatorname{Re} w\geqq h_0$, *where* h_0 *is a large positive number,* $g_j(w)$ $(j=1, 2, \cdots, n)$ *have the following asymptotic forms.*

$$g_j(w) \cong \frac{\lambda^w\Gamma(-\alpha)}{\Gamma(w-\alpha+1)}(\log w)^{j-1}\left\{c+O\left(\frac{1}{w}\right)\right\} \qquad \left(|\arg w|\leqq\frac{\pi}{2}\right) \tag{3. 11}$$

Proof. By (3. 8), $g_j(w)$ $(j=1, 2, \cdots, n)$ are represented as follows

$$g_j(w) = \varphi_j(w)+d_1\varphi_{j-1}(w)+\cdots+d_{j-1}\varphi_1(w)\,.$$

According to the asymptotic forms of $\varphi_j(w)$, we have

$$\begin{aligned}g_j(w) &\cong \frac{\lambda^w\Gamma(-\alpha)}{\Gamma(w-\alpha+1)}\{(\log w)^{j-1}+\cdots+(\log w)+1\}\left\{c+O\left(\frac{1}{w}\right)\right\} \\ &\cong \frac{\lambda^w\Gamma(-\alpha)}{\Gamma(w-\alpha+1)}(\log w)^{j-1}\left\{c+O\left(\frac{1}{w}\right)\right\}.\end{aligned}$$

Here we define $\tilde{g}_j(w)=\dfrac{g_j(w)}{g_1(w)}$, for which we remark the following corollary.

Corollary of lemma 3. 1. For a complex variable w and the positive integers σ and s, we obtain the following relations.

$$\tilde{g}_j(w+s)-\tilde{g}_j(w+\sigma)=\sum_{p=\sigma+1}^{s}\{\psi(w-\alpha+p)-\psi(w-\alpha+p+1)\}\tilde{g}_{j-1}(w+p)$$

and $\tilde{g}_j(w)$ have the following asymptotic forms.

$$\tilde{g}_j(w)\cong(\log w)^{j-1}\left\{c+O\left(\frac{1}{w}\right)\right\}\qquad\left(|\arg w|\leqq\frac{\pi}{2}\right)\tag{3. 12}$$

Now we prove some lemmas which we need in the latter parts of this paper.

Lemma. 3. 2. *If we define* h_0 *by*

$$h_0=\max\left\{1,\frac{3}{2}+\mathrm{Re}\,\alpha\right\}\tag{3. 13}$$

and $q_{1,\sigma}(w,w_0:s)$ *as follows.*

$$q_{1,\sigma}(w,w_0:s)=\frac{g_1(w_0+\sigma)}{g_1(w+\sigma)}\left\{\frac{g_1(w+s)}{g_1(w_0+s)}-\frac{g_1(w+s-1)}{g_1(w_0+s-1)}\right\}\tag{3. 14}$$

Here w_0 is such a integer as

$$w_0=-\sigma-\left[\frac{3}{4}+|\alpha|\right]\tag{3. 15}$$

where $[z]$ denotes the so-called Gauss' symbol and means the largest integer which does not exceed z.

Then the series

$$Q_{1,\sigma}(w,w_0)=\sum_{s=\sigma}^{\infty}q_{1,\sigma}(w,w_0:s)\tag{3. 16}$$

are absolutely convergent and uniformly bounded in the right half-plane

$$\mathrm{Re}\,w\geqq h_0-\sigma\,.\tag{3. 17}$$

Proof. At first we estimate the absolute values of each term (3. 14)

$$\begin{aligned}q_{1,\sigma}(w,w_0:s)&=\frac{\Gamma(w-\alpha+\sigma+1)}{\Gamma(w_0-\alpha+\sigma+1)}\left\{\frac{\Gamma(w_0-\alpha+s+1)}{\Gamma(w-\alpha+s+1)}-\frac{\Gamma(w_0-\alpha+s)}{\Gamma(w-\alpha+s)}\right\}\\&=\frac{\Gamma(w-\alpha+\sigma+1)\Gamma(w_0-\alpha+s)}{\Gamma(w_0-\alpha+\sigma+1)\Gamma(w-\alpha+s)}\left\{-1+\frac{w_0-\alpha+s}{w-\alpha+s}\right\}\\&=\frac{(w_0-\alpha+s-1)\cdots(w_0-\alpha+\sigma+1)}{(w-\alpha+s-1)\cdots(w-\alpha+\sigma+1)}\times\frac{w_0-w}{w-\alpha+s}\end{aligned}$$

Here we have the following estimates for sufficiently large value of p by using (3. 13) and (3. 15)

$$
(3.18)\quad \begin{cases} |w-\alpha+p| \geqq \mathrm{Re}\,(w-\alpha+p) \geqq h_0-\sigma-\mathrm{Re}\,\alpha+p \geqq \dfrac{3}{2}+(p-\sigma) \\ |w_0-\alpha+p| \leqq |w_0+p|+|\alpha| \leqq p-\sigma-\left[\dfrac{3}{4}+|\alpha|\right]+|\alpha| \\ \qquad \leqq (p-\sigma)+\dfrac{1}{4}, \end{cases}
$$

and so we obtain

$$
\begin{aligned}
|q_{1,\sigma}(w,\, {}_0w: s)| &\leqq \frac{\left(s-\sigma-\frac{3}{4}\right)\cdots\left(\frac{5}{4}\right)}{\left(s-\sigma+\frac{1}{2}\right)\cdots\left(\frac{5}{2}\right)}\left\{1+\frac{\left(s-\sigma+\frac{1}{4}\right)}{\left(s-\sigma+\frac{3}{2}\right)}\right\} \\
&= \frac{\Gamma\left(s-\sigma+\frac{1}{4}\right)\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(s-\sigma+\frac{3}{2}\right)\Gamma\left(\frac{5}{4}\right)}\left\{1+\frac{\left(s-\sigma+\frac{1}{4}\right)}{\left(s-\sigma+\frac{3}{2}\right)}\right\} \\
&\leqq K(s-\sigma)^{-(3/2)+(1/4)} = K(s-\sigma)^{-(5/4)}. \\
&\qquad (K\text{: constant})
\end{aligned}
$$

In the last part of the above calculation, we used the relations

$$
\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \cong z^{\alpha-\beta}\left\{1+O\left(\frac{1}{z}\right)\right\}
$$

Hence

$$
|Q_{1,\sigma}(w,\, w_0)| \leqq \sum_{s=\sigma}^{\infty} |q_{1,\sigma}(w,\, w_0: s)| \leqq K \sum_{s=\sigma}^{\infty} (s-\sigma)^{-5/4}
$$

Since the series of the right hand side is convergent, we can conclude the validity of the lemma 3. 2.

Lemma 3. 3. *For $j=2, 3, \cdots, n$, we define $R_{j,\sigma}(w)$ as follows*:

$$
(3.19)\qquad R_{j,\sigma}(w) = \frac{1}{g_1(w+\sigma-1)} \sum_{s=\sigma}^{\infty} \frac{1}{w-\alpha+s} \tilde{g}_{j-1}(w+s) g_1(w+s)
$$

Then $R_{j,\sigma}(w)$ are absolutely convergent and uniformly bounded in the right half-plane (3. 17).

Proof. Using the estimates of (3. 12) and (3. 18), we try to estimate the absolute value of the $(s-\sigma+1)$-th term of (3. 19) which has the following form:

$$\lambda^{s-\sigma-1}\frac{\tilde{g}_{j-1}(w+s)}{w-\alpha+s}\times\frac{1}{(w+s-\alpha)\cdots(w+\sigma-\alpha)}$$

Then we have in the right half-plane (3.17) for $j=2$,

$$\begin{aligned}&|\lambda|^{s-\sigma-1}\frac{1}{|w-\alpha+s|}\left|\frac{1}{(w+s-\alpha)\cdots(w+\sigma-\alpha)}\right|\\ &\leqq|\lambda|^{s-\sigma-1}\frac{1}{\left(s-\sigma+\frac{3}{2}\right)}\times\frac{1}{\left(s-\sigma+\frac{3}{2}\right)\cdots\left(\frac{3}{2}\right)}\\ &=|\lambda|^{s-\sigma-1}\frac{1}{\left(s-\sigma+\frac{3}{2}\right)}\times\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(s-\sigma+\frac{5}{2}\right)}\end{aligned}$$

and for $j=3, 4, \cdots, n$,

$$\begin{aligned}&|\lambda|^{s-\sigma-1}\frac{|\log(w+s)|^{j-2}}{|w-\alpha+s|}\times\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(s-\sigma+\frac{5}{2}\right)}\\ &\leqq|\lambda|^{s-\sigma-1}\left\{\left|\frac{(w+s)^{\delta}}{w-\alpha+s}\right|\right\}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(s-\sigma+\frac{5}{2}\right)}\\ &\leqq K|\lambda|^{s-\sigma-1}\frac{\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(s-\sigma+\frac{5}{2}\right)}\end{aligned}$$

Here δ is a positive number such as $0<\delta<1$.

Hence

$$|R_{j,\sigma}(w)|\leqq\begin{cases}\displaystyle\sum_{s=\sigma}^{\infty}\frac{|\lambda|^{s-\sigma-1}\Gamma\left(\frac{3}{2}\right)}{\left(s-\sigma+\frac{3}{2}\right)\Gamma\left(s-\sigma+\frac{5}{2}\right)} & \text{for } j=2\\ \displaystyle K\sum_{s=\sigma}^{\infty}\frac{|\lambda|^{s-\sigma-1}\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(s-\sigma+\frac{5}{2}\right)} & \text{for } j=3, 4, \cdots, n\,.\end{cases}$$

The series of the right hand side are convergent. q. e. d.

Remark 2. In the latter parts of this paper, we will use the results obtained until now, in the form where λ and α are replaced by λ_k and $(a_{kk}-\rho_1)$ respectively, with upper index k.

Here we will show the new notations which are used in the latter parts.

We denote $D(\lambda_k)$ by, $\mathscr{D}_k$ i.e.,

$$\mathscr{D}_k = \{t: \lambda_k t \in \mathscr{D}^*\} \tag{3.20}$$

and the total intersection by $\mathscr{D}$, i.e.,

$$\mathscr{D} = \bigcap_{k=1}^{n} \mathscr{D}_k . \tag{3.21}$$

The constant column vectors C^k are used instead of $C(a_{kk}-\rho_1, \lambda_k)$.

We use the same character h_0 to define

$$h_0 = \max\left\{1, \frac{3}{2}+\mathrm{Re}\,(a_{kk}-\rho_1);\ k=1, 2, \cdots, n\right\}. \tag{3.22}$$

4. We now come back to the differential equations (1. 1).

In the eigenvalues of the matrix A, ρ_1 has the maximum of the real part by the assumptions (2). So the solution which corresponds to the characteristic exponent ρ_1 has no logarithmic functions and the other solutions have the logarithmic polynomials.

We write those solutions as follows.

$$X_1(t) = t^{\rho_1}\sum_{m=0}^{\infty} G_1(m)t^m \tag{4.1}$$

$$X_j(t) = \sum_{l=1}^{j}\frac{1}{(j-l)!}(\log t)^{j-l}\tilde{X}_l(t) \qquad (j=2, 3, \cdots, n) \tag{4.2}$$

where $\tilde{X}_1(t)=X_1(t)$, and $\tilde{X}_j(t)$ $(j=2, 3, \cdots, n)$ have the convergent power serise with the characteristic exponents ρ_j respectively.

Indeed, we substitute $X_j(t)$ into the differential equations (1. 1).

$$t\frac{dX_j(t)}{dt} = (A+tB)X_j(t)$$

The left hand side is

$$\begin{aligned} t\frac{dX_j}{dt} &= \sum_{l=1}^{j}\frac{1}{(j-l)!}(\log t)^{j-l}\cdot\left(t\frac{d\tilde{X}_l}{dt}\right)+\sum_{l=1}^{j}\frac{1}{(j-l-1)!}(\log t)^{j-l-1}\tilde{X}_l \\ &= \sum_{l=2}^{j}\frac{1}{(j-l)!}(\log t)^{j-l}\left\{t\frac{d\tilde{X}_l}{dt}+\tilde{X}_{l-1}\right\}+\frac{1}{(j-1)!}(\log t)^{j-1}t\frac{d\tilde{X}_1}{dt} \end{aligned}$$

and the right hand side is

$$(A+tB)X_j = \sum_{l=1}^{j} \frac{1}{(j-l)!}(\log t)^{j-l}(A+tB)\tilde{X}_l$$

The coefficients of $(\log t)^{j-l}$ in the both sides are equal, so we get the following differential equations

$$(4.\ 3) \qquad \begin{cases} t\dfrac{d\tilde{X}_1}{dt} = (A+tB)\tilde{X}_1 \\ t\dfrac{d\tilde{X}_l}{dt} = (A+tB)\tilde{X}_l - \tilde{X}_{l-1} \qquad (l=2, 3, \cdots, n) \end{cases}$$

Hence $\tilde{X}_j(t)$ $(j=1, 2, \cdots, n)$ are the solutions which satisfy the above differential equations (4. 3) and have the convergent power series with the characteristic exponents ρ_j around the origin. Here we suppose that $\tilde{X}_j(t)$ $(j=2, 3, \cdots, n)$ have the following forms:

$$(4.\ 4) \qquad \tilde{X}_j(t) = t^{\rho_j}\sum_{m=0}^{\infty}\tilde{G}_j(m)t^m \qquad (j=2, 3, \cdots, n)$$

Proposition. 4. 1. *The coefficient vectors* $G_1(m)$ *and* $\tilde{G}_j(m)$ $(j=2, 3, \cdots, n)$ *satisfy the following difference equations respectively.*

$$(4.\ 5) \qquad (\rho_1+m-A)G_1(m)=BG_1(m-1)$$

with the initial condition

$$(4.\ 6) \qquad (\rho_1-A)G_1(0) = 0$$

and

$$(4.\ 7) \quad (\rho_j+m-A)\tilde{G}_j(m) = B\tilde{G}_j(m-1)-\tilde{G}_{j-1}(m-\rho_{j-1}+\rho_j) \quad (m\geqq\rho_{j-1}-\rho_j)$$

$$(4.\ 8) \quad (\rho_j+m-A)\tilde{G}_j(m) = B\tilde{G}_j(m-1) \qquad (1\leqq m<\rho_{j-1}-\rho_j)$$

with initial conditions

$$(4.\ 9) \qquad (\rho_j-A)\tilde{G}_j(0) = 0 \qquad (j=2, 3, \cdots, n)$$

Proof. We easily obtain by substituting (4. 1) and (4. 4) into (4. 3).

Here it should be remarked that if s is positive integers, $G_1(-s)$ and $\tilde{G}_j(-s)$ are always zero vectors which are derived from the initial conditions. So we need not devide the difference equations for $\tilde{G}_j(m)$ into two cases (4. 7) and (4. 8), since $\tilde{G}_{j-1}(m-\rho_{j-1}+\rho_j)\equiv 0$ when $1\leqq m<\rho_{j-1}-\rho_j$.

In the neighbourhood of the infiinty, there exist the formal solutions which have the forms (1. 3). Next we remark about the coefficients of them.

Proposition. 4. 2. *All eigenvalues of the matrix B are distinct by the pentagonal condition* (1.4), *so the coefficient vectors $H^k(s)$ of the formal solutions* (1. 3) *satisfy the following difference equations*

$$(4.10)\quad (a_{kk}-s-A)H^k(s) = (B-\lambda_k)H^k(s+1) \qquad (k=1, 2, \cdots, n)$$

with the initial conditions

$$(4.11)\qquad (B-\lambda_k)H^k(0) = 0 \qquad (k=1, 2, \cdots, n)$$

Now we define the matrices $\mathcal{F}^k(\varepsilon, m)$ $(k=1, 2, \cdots, n)$.

Definition. 4. 1. *We define the formal power series in a complex ε by*

$$(4.12)\qquad \mathcal{F}^k(\varepsilon, m) = \sum_{s=0}^{\infty} H^k(s)\mathfrak{G}^k_*(s+m)\varepsilon^{m+s+\rho_1-a_{kk}}$$

where we denote the transposed vectors by $*$ suffix.

For these matrices $\mathcal{F}^k(\varepsilon, w)$, we carry out the formal calculations.

$$\varepsilon\frac{d\mathcal{F}^k(\varepsilon, w)}{d\varepsilon} = \sum_{s=0}^{\infty} H^k(s)\mathfrak{G}^k_*(s+m)(m+s+\rho_1-a_{kk})\varepsilon^{m+s+\rho_1-a_{kk}}$$

From the relations (2. 6), we have $\mathfrak{G}^k_*(m)(m+\rho_1-a_{kk}+Z_*)=\lambda_k\mathfrak{G}^k_*(m-1)$ Hence we get

$$\begin{aligned}(4.13)\quad \varepsilon\frac{d\mathcal{F}^k(\varepsilon, w)}{d\varepsilon} &= \sum_{s=0}^{\infty} H^k(s)\{\lambda_k\mathfrak{G}^k_*(m+s-1)-\mathfrak{G}^k_*(m+s)Z_*\}\varepsilon^{m+s+\rho_1-a_{kk}} \\ &= \varepsilon\lambda_k\sum_{s=0}^{\infty} H^k(s)\mathfrak{G}^k_*(m-1+s)\varepsilon^{m-1+s+\rho_1-a_{kk}} \\ &\quad -\sum_{s=0}^{\infty} H^k(s)\mathfrak{G}^k_*(m+s)\varepsilon^{m+s+\rho_1-a_{kk}}\cdot Z_* \\ &= \varepsilon\lambda_k\mathcal{F}^k(\varepsilon, m-1)-\mathcal{F}^k(\varepsilon, m)\cdot Z_*\end{aligned}$$

On the other hand, if we use the relations (4. 10) and (4. 11), we get

$$\begin{aligned}(4.14)\quad \varepsilon\frac{d\mathcal{F}^k(\varepsilon, m)}{d\varepsilon} &= \sum_{s=0}^{\infty}(s-a_{kk}+A+m-\rho_1+A)H^k(s)\mathfrak{G}^k_*(s+m)\varepsilon^{m+s+\rho_1-a_{kk}} \\ &= \sum_{s=0}^{\infty}(\lambda_k-B)H^k(s+1)\mathfrak{G}^k_*(s+m)\varepsilon^{m+s+\rho_1-a_{kk}} \\ &\quad +(m+\rho_1-A)\sum_{s=0}^{\infty} H^k(s)\mathfrak{G}^k_*(s+m)\varepsilon^{m+s+\rho_1-a_{kk}}\end{aligned}$$

$$= \sum_{s=0}^{\infty} (\lambda_k - B) H^k(s) \mathfrak{G}_*^k(s+m-1) \varepsilon^{m-1+s+\rho_1-a_{kk}}$$
$$-(\lambda_k - B) H^k(0) \mathfrak{G}_*^k(m-1) \varepsilon^{m-1+\rho_1-a_{kk}}$$
$$+(m+\rho_1-A) \sum_{s=0}^{\infty} H^k(s) \mathfrak{G}_*^k(s+m) \varepsilon^{m+s+\rho_1-a_{kk}}$$
$$= (\lambda_k - B)\mathcal{F}^k(\varepsilon, m-1) + (m+\rho_1-A)\mathcal{F}^k(\varepsilon, m).$$

Eliminating $\mathcal{F}^k(\varepsilon, m-1)$ from the differential equations (4. 13) and (4.14), we obtain the differential equations

$$\left\{\varepsilon - \left(\frac{\lambda_k - B}{\lambda_k}\right)\right\} \frac{d\mathcal{F}^k(\varepsilon, m)}{d\varepsilon} \tag{4. 15}$$
$$= (m+\rho_1-A)\mathcal{F}^k(\varepsilon, m) + \frac{1}{\varepsilon\lambda_k}(\lambda_k - B)\mathcal{F}^k(\varepsilon, m)\cdot Z_*.$$

If we eliminate $\varepsilon \dfrac{d\mathcal{F}^k(\varepsilon, m)}{d\varepsilon}$ from the differential equations (4. 13) and (4. 14), we have

$$(m+\rho_1-A)\mathcal{F}^k(\varepsilon, m) = (\varepsilon\lambda_k - \lambda_k + B)\mathcal{F}^k(\varepsilon, m-1) - \mathcal{F}^k(\varepsilon, m)\cdot Z_*. \tag{4. 16}$$

As for the matrices $\mathcal{F}^k(\varepsilon, m)$, we obtain the next lemma.

Lemma 4. 1. *The matrices $\mathcal{F}^k(\varepsilon, m)$ converge uniformly on the closed unit disk $|\varepsilon| \leqq 1$ of the complex ε-plane because of the pentagonal condition* (1. 4).

So in particular, $\mathcal{F}^k(1, m)$ are the convergent power series and then satisfy the difference equations

$$(m+\rho_1-A)\mathcal{F}^k(1, m) = B\mathcal{F}^k(1, m-1) - \mathcal{F}^k(1, m)\cdot Z_*. \tag{4. 17}$$

Proof. We showed the matrices $\mathcal{F}^k(\varepsilon, m)$ satisfy the differential equations (4. 15).

If we can prove that $\mathcal{F}^k(\varepsilon, m)$ have the convergent power series at the origin, $\varepsilon = 0$, the formal calculations we carried out are justified and we obtain the convergent power series (4. 12).

We put

$$\mathcal{F}^k(\varepsilon, m) = \{F_1^k(\varepsilon, m),\ F_2^k(\varepsilon, m), \cdots, F_n^k(\varepsilon, m)\} \tag{4. 18}$$

where $F_1^k(\varepsilon, m)$, $F_2^k(\varepsilon, m), \cdots, F_n^k(\varepsilon, m)$ are the column vectors.

We write out the differential equations (4. 15) by the column vectors, then we have

$$(4.19)\quad \begin{cases} \left\{\varepsilon-\left(\dfrac{\lambda_k-B}{\lambda_k}\right)\right\}\dfrac{dF_1^k(\varepsilon,m)}{d\varepsilon}=(m+\rho_1-A)F_1^k(\varepsilon,m) \\ \left\{\varepsilon-\left(\dfrac{\lambda_k-B}{\lambda_k}\right)\right\}\dfrac{dF_2^k(\varepsilon,m)}{d\varepsilon}=(m+\rho_1-A)F_2^k(\varepsilon,m) \\ \qquad\qquad\qquad\qquad\qquad\qquad +\dfrac{1}{\varepsilon\lambda_k}(\lambda_k-B)F_1^k(\varepsilon,m) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \left\{\varepsilon-\left(\dfrac{\lambda_k-B}{\lambda_k}\right)\right\}\dfrac{dF_n^k(\varepsilon,m)}{d\varepsilon}=(m+\rho_1-A)F_n^k(\varepsilon,m) \\ \qquad\qquad\qquad\qquad\qquad\qquad +\dfrac{1}{\varepsilon\lambda_k}(\lambda_k-B)F_{n-1}^k(\varepsilon,m)\,. \end{cases}$$

All the vectorial differential equations (4. 19) have only n regular singular points at $\dfrac{\lambda_k-\lambda_i}{\lambda_k}$ $(i=1, 2, \cdots, n)$ in the finite ε-plane. In particular, $\varepsilon=0$ (in the case when $i=k$) is also a regular singular point.

So according to the fundamental theorem of the ordinary differential equations, it is easy to see that the formal power series $F_1^k(\varepsilon, m)$, and repeatedly $F_2^k(\varepsilon, m), \cdots, F_n^k(\varepsilon, m)$ converge uniformly in the neighbourhood of the origin and become the solutions of (4. 19).

Since the nearest singular points to the origin are some of $\dfrac{\lambda_k-\lambda_i}{\lambda_k}$ $(i\neq k)$, the radius of convergence is less than the minimum of the absolute values of $\dfrac{\lambda_k-\lambda_i}{\lambda_k}$ $(i\neq k)$, all of which are larger than 1 by the pentagonal condition.

Therefore we can conclude that the formal power series $\mathscr{F}^k(\varepsilon, m)$ defined by (4. 12) converge uniformly on the closed unit disk $|\varepsilon|\leqq 1$ and are the solutions of the differential equations (4. 15).

Here if we substitute $\varepsilon=1$ in both sides of (4. 16), we get (4. 17).

Definition. 4. 2. *For a positive integer σ, we define the vector valued functions $P_{j,\sigma}^k(w)$ $(j=1, \cdots, n,\ k=1, 2, \cdots, n)$ of a complex variable w as follows.*

For $j=1$,

$$(4.20)\qquad P_{1,\sigma}^k(w)=\frac{1}{g_1^k(w+\sigma)}\sum_{s=\sigma}^{\infty}H^k(s)g_1^k(w+s)$$

and for $j=2, 3, \cdots, n$,

$$P^k_{j,\sigma}(w) = \frac{1}{g^k_1(w+\sigma-1)} \sum_{s=\sigma}^{\infty} H^k(s) g^k_j(w+s) \tag{4.21}$$

So it follows that for $k=1, 2, \cdots, n$,

$$F^k_1(1, w) = \sum_{s=0}^{\sigma-1} H^k(s) g^k_1(w+s) + g^k_1(w+\sigma) P^k_{1,\sigma}(w) \tag{4.22}$$

and

$$F^k_j(1, w) = \sum_{s=0}^{\sigma-1} H^k(s) g^k_j(w+s) + g^k_1(w+\sigma-1) P^k_{j,\sigma}(w) \qquad (j=2, 3, \cdots, n) \tag{4.23}$$

Lemma. 4. 2. *We define h by,*

$$h = \max\left\{1, \frac{3}{2} - \mathrm{Re}\,(\rho_j - a_{kk});\ \ 1 \leqq j \leqq n,\ \ 1 \leqq k \leqq n\right\}. \tag{4.24}$$

Then $P^k_{j,\sigma}(w)$ are holomorphic and bounded in the right half-plane

$$\mathrm{Re}\, w \geqq h - \sigma\,. \tag{4.25}$$

Proof: At first we prove the lemma for $j=1$.

By the assumption (2), it is easy to see that $h \geqq h_0$.

Applying Abel's transformation to (4. 20), we have

$$\begin{aligned}
P^k_{1,\sigma}(w) &= \frac{1}{g^k_1(w_0+\sigma)} \sum_{s=\sigma}^{\infty} H^k(s) g^k_1(w_0+s) \frac{g^k_1(w_0+\sigma)}{g^k_1(w+\sigma)} \frac{g^k_1(w+s)}{g^k_1(w_0+s)} \\
&= \frac{1}{g^k_1(w_0+\sigma)} \sum_{s=\sigma}^{\infty} H^k(s) g^k_1(w_0+s) \\
&\quad + \frac{1}{g^k_1(w_0+\sigma)} \sum_{s=\sigma+1}^{\infty} H^k(s) g^k_1(w_0+s) \sum_{p=\sigma+1}^{\infty} q^k_{1,\sigma}(w, w_0; p) \\
&= P^k_{1,\sigma}(w_0) + \frac{1}{g^k_1(w_0+\sigma)} \sum_{p=\sigma+1}^{\infty} q^k_{1,\sigma}(w, w_0; p) \sum_{s=p}^{\infty} H^k(s) g^k_1(w_0+s) \\
&= P^k_{1,\sigma}(w_0) + \frac{1}{g^k_1(w_0+\sigma)} \sum_{p=\sigma+1}^{\infty} [q^k_{1,\sigma}(w, w_0; p) g^k_1(w_0+p) P^k_{1,p}(w_0)]
\end{aligned}$$

where w_0 denote the integer defined by (3. 13).

Here $g^k_1(w_0+p)\, P^k_{1,p}(w_0)$ are bounded, since they are the remainders of the convergent power series by the lemma 4. 1, and $\sum_{p=\sigma+1}^{\infty} q^k_{1,\sigma}(w, w_0; p)$ are absolutely convergent and uniformly bounded by the lemma 3. 2, which also guarantee the validity of Abel's transformation carried out in the above calculation.

Hence we can conclude that $P^k_{1,\sigma}(w)$ are holomorphic and bounded in the right half-plane (4. 25).

For $j=2, 3, \cdots, n$, we have

$$\sum_{s=\sigma}^{\infty} H^k(s)g^k_j(w+s) = \sum_{s=\sigma}^{\infty} H^k(s)g^k_1(w+s)\tilde{g}^k_j(w+s)$$

$$= \sum_{s=\sigma}^{\infty} H^k(s)g^k_1(w+s)[\tilde{g}^k_j(w+\sigma-1)$$

$$+\sum_{p=\sigma}^{s} \{\psi(w+p-a_{kk}+\rho_1)-\psi(w+p+1-a_{kk}+\rho_1)\}\tilde{g}_{j-1}(w+p)]$$

$$= \tilde{g}^k_j(w+\sigma-1)\sum_{s=\sigma}^{\infty} H^k(s)g^k_1(w+s)$$

$$+\sum_{p=\sigma}^{\infty} \{\psi(w+p-a_{kk}+\rho_1)-\psi(w+p+1-a_{kk}+\rho_1)\}$$

$$\times \tilde{g}^k_{j-1}(w+p)\sum_{s=p}^{\infty} H^k(s)g^k_1(w+s)$$

$$= \tilde{g}^k_j(w+\sigma-1)g^k_1(w+\sigma)P^k_{1,\sigma}(w)-g^k_j(w+\sigma-1)$$

$$\times\sum_{p=\sigma}^{\infty}\left\{\frac{1}{w+p-a_{kk}+\rho_1}\,\frac{\tilde{g}^k_{j-1}(w+p)g^k_1(w+p)}{g^k_1(w+\sigma-1)}\right\}P^k_{1,p}(w)\,.$$

So we have

$$P^k_{j,\sigma}(w) = \frac{\tilde{g}^k_j(w+\sigma-1)g^k_1(w+\sigma)}{g^k_1(w+\sigma-1)}P^k_{1,\sigma}(w)$$

$$-\sum_{p=\sigma}^{\infty}\left\{\frac{1}{w+p-a_{kk}+\rho_1}\,\frac{\tilde{g}^k_{j-1}(w+p)g^k_1(w+p)}{g^k_1(w+\sigma-1)}\right\}P^k_{1,p}(w)\,.$$

Here $P^k_{1,p}(w)$ are holomorphic and bounded by the above proof. $\dfrac{\tilde{g}^k_j(w+\sigma-1)g^k_1(w+\sigma)}{g^k_1(w+\sigma-1)} \simeq \dfrac{\{\log(w+\sigma-1)\}^{j-1}}{(w+\sigma-a_{kk}+\rho_1)}$ are also bounded and holomorphic, and $\sum_{p=\sigma}^{\infty}\left\{\dfrac{1}{w+p-a_{kk}+\rho_1}\times\dfrac{\tilde{g}^k_{j-1}(w+p)g^k_1(w+p)}{g^k_1(w+\sigma-1)}\right\}$ are absolutely convergent and uniformly bounded by the lemma 3. 3.

So we can conclude that $P^k_{j,\sigma}(w)$ $(j=1, 2, \cdots, n)$ are also holomorphic and uniformly bounded.

Lemma 4. 3. *For $j=1$, the set of $\{F^k_1(1, m)\colon k=1, 2, \cdots, n\}$ constitutes an independent set of solutions of the difference equations* (4. 5) *for all non-negative integers m.*

Proof. According to the lemma 4. 2, for sufficiently large w such that $\operatorname{Re} w \geqq h-1$, we have

$$F_1^k(1,\, w) = H^k(0)g_1^k(w) + g_1^k(w+1)P_{1,1}^k(w)$$
$$= H^k(0)g_1^k(w)\left\{1+O\left(\frac{1}{w}\right)\right\}.$$

Now we calculate the Casorati-determinant (Wronskian) for sufficiently large w.

$$C(w) = \det(F_1^1(w),\ F_1^2(w), \cdots, F_1^n(w))$$
$$\cong \det(H^1(0),\ H^2(0), \cdots, H^n(0))g_1^1(w) \cdots g_1^n(w)$$

Here the vectors $H^k(0)$ $(k=1, 2, \cdots, n)$ are the eigenvectors of the matrix B which correspond to the distinct eigenvalues λ_k respectively, so the Casorati-determinant $C(w)$ can not be zero for large positive real numbers because of the properties of $g_1^k(w)$.

On the other hand, using the difference equations (4. 17), we have

$$\det(\rho_1 + w - A)\cdot C(w) = \det B\cdot C(w-1)$$

from which we can deduce $C(w-1)\neq 0$ from $C(w)\neq 0$, if $\det(\rho_1+w-A)\neq 0$. But $\det(\rho_1+w-A)$ cannot be zero except for $w=-m_2,\ -m_3, \cdots, -m_n$.

Hence for positive integers w, $C(w)$ are not always zero.

We proceed to some of main theorems which determine the so-called "Stokes' multipliers".

Theorem 4. 1. *There exists a set of scalar constants* $\{T_1^k;\, k=1, 2, \cdots, n\}$ *such that*

$$G_1(m) = \sum_{k=1}^{n} T_1^k F_1^k(1,\, m) \tag{4. 26}$$

becomes the solution of the difference equation (4. 5) *with the initial condition* (4. 6).

Proof. Since $F_1^k(1,\, m)$ satisfy

$$(m+\rho_1 - A)F_1^k(1,\, m) = BF_1^k(1,\, m-1)$$

by (4. 17), $G_1(m)$ also become the solution of the same difference equation.

We can determine the constants T_1^k, using the initial vectors $G_1(0)$, by

$$G_1(0) = \sum_{k=1}^{n} T_1^k F_1^k(1,\, 0)\,,$$

for $F_1^k(1, 0)$ $(k=1, 2, \cdots, n)$ are the independent vectors and det $\{F_1^k(1, 0)\}$ are not zero.

Lemma 4. 4. *Using the "Stokes' multipliers" T_1^k, we can obtain a special solution $\tilde{G}_2^*(m)$ of the difference equations,*

$$(\rho_2+m-A)\tilde{G}_2(m) = B\tilde{G}_2(m-1)-G_1(m-\rho_1+\rho_2)\,.$$

Proof. By (4. 17), the vectors $F_2^k(1, m)$ satisfy

$$(\rho_1+m-m_2-A)F_2^k(1, m-m_2)=BF_2^k(1, m-m_2-1)-F_1^k(1, m-m_2)\,.$$

Multiplying T_1^k to both sides of the above relations, and summing them up over k, we have, because of $\rho_1-m_2=\rho_2$,

$$(\rho_2+m-A)\sum_{k=1}^{n} T_1^k F_2^k(1, m-m_2)=B\sum_{k=1}^{n} T_1^k F_2^k(1, m-m_2-1)-G_1(1, m-m_2)\,.$$

Hence we put

$$\tilde{G}_2^*(m) = \sum_{k=1}^{n} T_1^k F_1^k(1, m-m_2)\,, \tag{4. 27}$$

then $\tilde{G}_2^*(m)$ satisfies the above difference equation.

Theorem 4. 2. *There exists a set of scalar constants $\{T_2^k;\, k=1, 2, \cdots, n\}$ such that*

$$\tilde{G}_2(m) = \tilde{G}_2^*(m)+\sum_{k=1}^{n} T_2^k F_1^k(1, m-m_2) \tag{4. 28}$$

becomes the solution of the difference equation (4. 7) *with the initial condition* (4. 9) *for* $j=2$.

They are determined by

$$\tilde{G}_2(m_2) = \tilde{G}_2^*(m_2)+\sum_{k=1}^{n} T_2^k F_1^k(1, 0)\,. \tag{4. 29}$$

Proof. We can obtain the general solution of the inhomogeneous equations by adding the general solutions of the homogeneous equations to a special solution. The homogeneous equation of the form

$$(\rho_2+m-A)\tilde{G}_2(m) = B\tilde{G}_2(m-1)$$
$$(\rho_1+m-m_2-A)\tilde{G}_2(m) = B\tilde{G}_2(m-1)\,,$$

therefore $F_1^k(1, m-m_2)$ become the solutions of the above difference equations. From the independence of the vectors $F_1^k(1, 0)$, it is easy to determine the "Stokes' multipliers" T_2^k uniquely.

Theorem 4. 3. *There exist the "Stokes' multipliers"*

$$\{T_j^k\colon k = 1, 2, \cdots, n, \quad j=2, 3, \cdots, n\}.$$

And using them, we can represent the solutions of the difference equations (4. 7) (4. 8) as follows

$$\tilde{G}_p(m) = \sum_{j=1}^{p} \sum_{k=1}^{n} T_{p+1-j}^k F_j^k(m-m_p). \tag{4. 30}$$

We have a special solution of the inhomogeneous equations (4. 7) (4. 8) for $j=p$ such as the following forms

$$\tilde{G}_p^*(m) = \sum_{j=2}^{p} \sum_{k=1}^{n} T_{p+1-j}^k F_j^k(m-m_p) \qquad (p=2, 3, \cdots, n). \tag{4. 31}$$

Proof. We prove the theorem 4. 3 by induction. We already proved it for $p=2$ by the theorem 4. 2.

Now supposing that it is right for p, we will prove that it is also right for $p+1$.

The vectors $F_j^k(m)$ $(j=2, 3, \cdots, n)$ satisfy

$$\begin{aligned}(\rho_1+m-m_{p+1}-A)&F_j^k(1, m-m_{p+1})\\ &= BF_j^k(1, m-1-m_{p+1})-F_{j-1}^k(1, m-m_{p+1}).\end{aligned}$$

Multiplying T_{p+2-j}^k to both sides of the above relations, and summing them up from $k=1$ to $k=n$ and from $j=2$ to $j=p+1$, we have

$$(\rho_{p+1}+m-A)\tilde{G}_{p+1}^*(m) = B\tilde{G}_{p+1}^*(m-1)-\sum_{j=2}^{p+1} \sum_{k=1}^{n} T_{p+2-j}^k F_{j-1}^k(1, m-m_{p+1}).$$

The last term

$$\begin{aligned}&\sum_{j=2}^{p+1} \sum_{k=1}^{n} T_{p+2-j}^k F_{j-1}^k(1, m-m_p+m_p-m_{p+1})\\ &= \sum_{j=1}^{p} \sum_{k=1}^{n} T_{p+1-j}^k F_j^k(1, m-m_p+\rho_{p+1}-\rho_p)\\ &= \tilde{G}_p(m-\rho_p+\rho_{p+1})\end{aligned}$$

Hence $\tilde{G}_{p+1}^*(m)$ becomes a special solution of the difference equation (4. 7) for $p+1$. On the other hand, $F_1^k(1, m-m_{p+1})$ are the solutions of the homogeneous part of the equation.

So we have the solution $\tilde{G}_{p+1}(m)$ as follows

$$\tilde{G}_{p+1}(m) = \tilde{G}_{p+1}^*(m)+\sum_{k=1}^{n} T_{p+1}^k F_1^k(1, m-m_{p+1}).$$

Here the "Stokes' multipliers" T^k_{p+1} ($k=1, 2, \cdots, n$) are determined by

$$\tilde{G}_{p+1}(m_{p+1}) = \tilde{G}^*_{p+1}(m_{p+1}) + \sum_{k=1}^{n} T^k_{p+1} F^k_1(1, 0). \tag{4. 32}$$

5. Now we proceed to the main part of this paper. We investigate the behaviors of the convergent solutions near infinity.

Lemma 5. 1. *(E.M. Wright) If $\varphi(w)$ is holomorphic and bounded in the right half-plane*

$$\operatorname{Re} w \geqq h' > 0 \tag{5. 1}$$

and

$$h' > \frac{3}{2} - \operatorname{Re}\beta, \tag{5. 2}$$

then we have

$$\sum_{m=[h']+1}^{\infty} \frac{\varphi(m)}{\Gamma(m+\beta)} z^m = O(e^z z^{1-\beta}) + O(z^{h'}) \tag{5. 3}$$

as z tends to infinity in the sector

$$|\arg z| \leqq \frac{3}{2}\pi . \tag{5. 4}$$

Lemma. 5. 2. *If $\sigma > h^* = [h]+1$, where h is defined by* (4. 24), *we have*

$$\sum_{m=0}^{\infty} g^k_1(m+\sigma) P^k_{1,\sigma}(m) t^m = O(e^{\lambda_k t} t^{a_{kk}-\rho_1-\sigma}) + O(t^{h-\sigma}) - \sum_{l=1}^{\sigma-h^*} g^k_1(\sigma-l) P^k_{1,\sigma}(-l) t^{-l} \tag{5. 5}$$

and for $j=2, 3, \cdots, n$,

$$\sum_{m=0}^{\infty} g^k_1(m+\sigma-1) P^k_{j,\sigma}(m) t^m = O(e^{\lambda_k t} t^{a_{kk}-\rho_1+1-\sigma}) + O(t^{h-\sigma}) - \sum_{l=1}^{\sigma-h^*} g^k_1(\sigma-1-l) P^k_{j,\sigma}(-l) t^{-l} \tag{5. 6}$$

in the sector

$$|\arg \lambda_k t| \leqq \frac{3}{2}\pi . \tag{5. 7}$$

Proof. Here we prove the lemma only for $j=2, 3, \cdots, n$. At first we decompose the left hand side of (5. 6) so that we can apply the above Wright's lemma.

$$\sum_{m=0}^{\infty} g_1^k(m+\sigma-1)P_{j,\sigma}^k(m)t^m$$

$$= \sum_{m=h^*-\sigma}^{\infty} g_1^k(m+\sigma-1)P_{j,\sigma}^k(m)t^m - \sum_{l=1}^{\sigma-h^*} g_1^k(\sigma-1-l)P_{j,\sigma}^k(-l)t^{-l}$$

$$= \sum_{m=h^*}^{\infty} g_1^k(m-1)P_{j,\sigma}^k(m-\sigma)t^{m-\sigma} - \sum_{l=1}^{\sigma-h^*} g_1^k(\sigma-1-l)P_{j,\sigma}^k(-l)t^{-l}$$

$$= t^{-\sigma}\sum_{m=h^*}^{\infty} \frac{\lambda_k^{-1}\Gamma(\rho_1-a_{kk})P_{j,\sigma}^k(m-\sigma)}{\Gamma(m+\rho_1-a_{kk})}(\lambda_k t)^m - \sum_{l=1}^{\sigma-h^*} g_1^k(\sigma-1-l)P_{j,\sigma}^k(-l)t^{-l}$$

Each component of $P_{j,\sigma}^k(w-\sigma)$ is holomorphic and uniformly bounded in $\mathrm{Re}\, w \geqq h$ by the lemma 4. 2, and the condition (5. 2) of the Wright's lemma is satisfied by the definition of h, (4. 24).

So applying the Wright's lemma 5. 1 to the first term of the right hand side, we have

$$t^{-\sigma}\sum_{m=h^*}^{\infty} \frac{\lambda_k^{-1}\Gamma(\rho_1-a_{kk})P_{j,\sigma}^k(m-\sigma)}{\Gamma(m+\rho_1-a_{kk})}(\lambda_k t)^m = t^{-\sigma}\{O(e^{\lambda_k t}t^{1-\rho_1+a_{kk}})+O(t^h)\}$$
$$= O(e^{\lambda_k t}t^{a_{kk}-\rho_1+1-\sigma})+O(t^{h-\sigma}).$$

This proves the lemma 5. 2.

Here if we define the matrix $\mathcal{P}_\sigma^k(w)$ as follows

$$(5.8)\quad \mathcal{P}_\sigma^k(w) = \left\{\frac{1}{g_1^k(w+\sigma)}\sum_{s=\sigma}^{\infty} H^k(s)g_1^k(w+s),\right.$$
$$\left.\frac{1}{g_1^k(w+\sigma-1)}\sum_{s=\sigma}^{\infty} H^k(s)g_2^k(w+s),\cdots,\frac{1}{g_1^k(w+\sigma-1)}\sum_{s=\sigma}^{\infty} H^k(s)g_n^k(w+s)\right\},$$

then we can rewrite the result of the lemma 5. 2 by the following simple form.

$$(5.9)\qquad \sum_{m=0}^{\infty} \mathcal{P}_\sigma^k(m)\begin{pmatrix} g_1^k(m+\sigma) & & 0 \\ & g_1^k(m+\sigma-1) & \\ 0 & & \ddots \\ & & & g_1^k(m+\sigma-1)\end{pmatrix} t^m$$

$$= \sum_{m=0}^{\infty}\sum_{s=\sigma}^{\infty} H^k(s)\mathfrak{G}_*^k(m+s)t^m$$

$$= O(e^{\lambda_k t}t^{a_{kk}-\rho_1-\sigma})(\vec{1}, \vec{t}, \cdots, \vec{t}) + O(t^{h-\sigma})(\vec{1}, \vec{1}, \cdots, \vec{1})$$

$$-\sum_{l=1}^{\sigma-h^*} t^{-l}\mathcal{P}_\sigma^k(-l)\begin{pmatrix} g_1^k(\sigma-l) & & 0 \\ & g_1^k(\sigma-1-l) & \\ 0 & & \ddots \\ & & & g_1^k(\sigma-1-l)\end{pmatrix}$$

where $*$ suffix is used to indicate that the matrix or the vector is transposed, and $\rightarrow$ denotes the column vector.

Lemma. 5. 3. *In the sector* $\mathscr{D}_k$ *of* (3. 20), *we have*

$$\sum_{m=0}^{\infty}\mathscr{F}^k(1,m)t^m = e^{\lambda_k t}t^{a_{kk}-\rho_1}\sum_{s=0}^{\sigma-1}H^k(s)C_*^k t^{-Z_*}t^{-s} + O(e^{\lambda_k t}t^{a_{kk}-\rho_1-\sigma})(\vec{1},\vec{t},\vec{t},\cdots,\vec{t}) + O(t^{h^*-\sigma})(\vec{1},\cdots,\vec{1}) - \sum_{l=1}^{\sigma-h^*}\mathscr{F}^k(1,-l)t^{-l} \tag{5. 10}$$

Proof. By the definition 4. 1, we have

$$\begin{aligned}\sum_{m=0}^{\infty}\mathscr{F}^k(1,m)t^m &= \sum_{m=0}^{\infty}\Big[\sum_{s=0}^{\sigma-1}H^k(s)\mathfrak{G}_*^k(s+m)+\sum_{s=\sigma}^{\infty}H^k(s)\mathfrak{G}_*^k(s+m)\Big]t^m \\ &= \sum_{s=0}^{\sigma-1}H^k(s)\Big[\sum_{m=0}^{\infty}\mathfrak{G}_*^k(s+m)t^m\Big]+\sum_{m=0}^{\infty}\Big[\sum_{s=\sigma}^{\infty}H^k(s)\mathfrak{G}_*^k(s+m)\Big]t^m .\end{aligned}$$

Applying the corollary of lemma 2. 2 to the first term and the lemma 5. 2 to the second term, we obtain

$$\begin{aligned}\sum_{m=0}^{\infty}\mathscr{F}^k(1,m)t^m &= \sum_{s=0}^{\sigma-1}H^k(s)\Big\{e^{\lambda_k t}t^{a_{kk}-\rho_1-s}C_*^k t^{-Z_*} \\ &\quad -\sum_{l=1}^{\sigma-h^*}\mathfrak{G}_*^k(s-l)t^{-l}+O(t^{h^*-\sigma})(\vec{1},\vec{1},\cdots,\vec{1})\Big\} \\ &\quad +O(e^{\lambda_k t}t^{a_{kk}-\rho_1-\sigma})(\vec{1},\vec{t},\cdots,\vec{t})+O(t^{h^*-\sigma})(\vec{1},\cdots,\vec{1}) \\ &\quad -\sum_{l=1}^{\sigma-h^*}t^{-l}\mathscr{P}_\sigma^k(-l)\begin{pmatrix} g_1^k(\sigma-l) & & 0 \\ & g_1^k(\sigma-1-l) & \\ 0 & & \ddots \\ & & g_1^k(\sigma-1-l)\end{pmatrix} \\ &= e^{\lambda_k t}t^{a_{kk}-\rho_1}\sum_{s=0}^{\sigma-1}H^k(s)C_*^k t^{-Z_*}t^{-s} \\ &\quad +O(e^{\lambda_k t}t^{a_{kk}-\rho_1-\sigma})(\vec{1},\vec{t},\cdots,\vec{t})+O(t^{h^*-\sigma})(\vec{1},\cdots,\vec{1}) \\ &\quad -\sum_{l=1}^{\sigma-h^*}\Big\{\sum_{s=0}^{\sigma-1}H^k(s)\mathfrak{G}_*^k(s-l)t^{-l}+\sum_{s=\sigma}^{\infty}H^k(s)\mathfrak{G}_*^k(s-l)t^{-l}\Big\} \\ &= e^{\lambda_k t}t^{a_{kk}-\rho_1}\sum_{s=0}^{\sigma-1}H^k(s)C_*^k t^{-Z_*}t^{-s}+O(e^{\lambda_k t}t^{a_{kk}-\rho_1-\sigma})(\vec{1},\vec{t},\cdots,\vec{t}) \\ &\quad +O(t^{h^*-\sigma})(\vec{1},\cdots,\vec{1})-\sum_{l=1}^{\sigma-h^*}\mathscr{F}^k(1,-l)t^{-l} .\end{aligned}$$

Lemma, 5. 4. *For* t *in* $\mathscr{D}_k$, *we have*

(5. 11)
$$
\begin{aligned}
t^{\rho_p}\sum_{m=0}^{\infty}\mathcal{F}^k(1,\, m-m_p)t^m
&= e^{\lambda_k t}t^{a_{kk}}\sum_{s=0}^{\sigma-1}H^k(s)C_*^k t^{-Z_*}t^{-s} \\
&\quad + O(e^{\lambda_k t}t^{a_{kk}-\sigma})(\vec{1},\, \vec{t}, \cdots, \vec{t}) + O(t^{h^*-\sigma+\rho_1})(\vec{1}, \cdots, \vec{1}) \\
&\quad - \sum_{l=1}^{\sigma-h^*-m_p}\mathcal{F}^k(1,\, -m_p-l)t^{-l+\rho_p}
\end{aligned}
$$

Proof.

$$
\begin{aligned}
\sum_{m=0}^{\infty}\mathcal{F}^k(1,\, m-m_p)t^m &= \sum_{m=0}^{m_p-1}\mathcal{F}^k(1,\, m-m_p)t^m + t^{m_p}\sum_{m=0}^{\infty}\mathcal{F}^k(1,\, m)t^m \\
&= e^{\lambda_k t}t^{a_{kk}-\rho_1+m_p}\sum_{s=0}^{\sigma-1}H^k(s)C_*^k t^{-Z_*}t^{-s} \\
&\quad + O(e^{\lambda_k t}t^{a_{kk}-\rho_1+m_p-\sigma})(\vec{1},\, \vec{t}, \cdots, \vec{t}) + O(t^{h^*-\sigma+m_p})(\vec{1}, \cdots, \vec{1}) \\
&\quad - \sum_{l=1}^{\sigma-h^*}\mathcal{F}^k(1,\, -l)t^{-l+m_p} + \sum_{m=0}^{m_p-1}\mathcal{F}^k(1,\, m-m_p)t^m
\end{aligned}
$$

The last two terms are calculated as follows

$$
\begin{aligned}
&-\sum_{l=1}^{\sigma-h^*}\mathcal{F}^k(1,\, -l)t^{-l+m_p} + \sum_{l=1}^{m_p}\mathcal{F}^k(1,\, -l)t^{-l+m_p} \\
&= -\sum_{l=m_p+1}^{\sigma-h^*}\mathcal{F}^k(1,\, -l)t^{-l+m_p} \\
&= -\sum_{l=1}^{\sigma-h^*-m_p}\mathcal{F}^k(1,\, -l-m_p)t^{-l}\,.
\end{aligned}
$$

Hence, multiplying t^{ρ_p} and using the relations $\rho_1-\rho_p=m_p$, we can obtain (5. 11).

Theorem. 5. 1. *For t in $\mathcal{D}=\bigcap_{k=1}^{n}\mathcal{D}_k$, we have*

$$
\begin{aligned}
\tilde{X}_p(t) &= t^{\rho_p}\sum_{m=0}^{\infty}\tilde{G}_p(m)t^m \\
&= \sum_{k=1}^{n}\{e^{\lambda_k t}t^{a_{kk}}\sum_{s=0}^{\sigma-1}H^k(s)C_*^k t^{-Z_*}t^{-s} + O(e^{\lambda_k t}t^{a_{kk}-\sigma})(\vec{1},\, \vec{t}, \cdots, \vec{t}) \\
&\quad + O(e^{h^*-\sigma+\rho_1})\}\begin{pmatrix} T_p^k \\ T_{p-1}^k \\ \vdots \\ T_1^k \\ 0 \\ \vdots \\ 0 \end{pmatrix} - \sum_{l=1}^{\sigma-h^*-m_p}\tilde{G}_p(-l)t^{-l+\rho_p}\,.
\end{aligned}
$$

Proof. In the theorem 4. 3, we obtained the relations between $\tilde{G}_p(m)$ and $F^k_j(m)$. We can rewrite them as follows

$$\tilde{G}_p(m)=\sum_{k=1}^{n}\mathcal{F}^k(1,\, m-m_p)\begin{pmatrix}T^k_p\\ T^k_{p-1}\\ \vdots\\ T^k_1\\ 0\\ \vdots\\ 0\end{pmatrix}. \tag{5. 13}$$

Multiplying t^{ρ_p+m} to both sides of the above relations (5. 13) and summing them up over m, we have

$$\tilde{X}_p(t)=t^{\rho_p}\sum_{m=0}^{\infty}\tilde{G}_p(m)t^m=\sum_{k=1}^{n}\sum_{m=0}^{\infty}t^{\rho_p}\mathcal{F}^k(1,\, m-m_p)t^m\begin{pmatrix}T^k_p\\ T^k_{p-1}\\ \vdots\\ T^k_1\\ 0\\ \vdots\\ 0\end{pmatrix}$$

and by the lemma 5. 4, we obtain

$$\tilde{X}_p(t)=\sum_{k=1}^{n}\{e^{\lambda_k t}t^{a_{kk}}\sum_{s=0}^{\sigma-1}H^k(s)C^k_*t^{-Z_*}t^{-s}+O(e^{\lambda_k t}t^{a_{kk}-\sigma})(\vec{1},\, \vec{t},\cdots,\vec{t})$$
$$+O(t^{h^*-\sigma+\rho_1})\}\begin{pmatrix}T^k_p\\ \vdots\\ T^k_1\\ 0\\ \vdots\\ 0\end{pmatrix}-\sum_{k=1}^{n}\sum_{l=1}^{\sigma-h^*-m_p}\mathcal{F}^k(1,\, -l-m_p)t^{-l+\rho_p}\begin{pmatrix}T^k_p\\ \vdots\\ T^k_1\\ 0\\ \vdots\\ 0\end{pmatrix}.$$

The last term can be calculated as follows

$$-\sum_{l=1}^{\sigma-h^*-m_p}\sum_{k=1}^{n}\mathcal{F}^k(1,\, -l-m_p)\begin{pmatrix}T^k_p\\ T^k_{p-1}\\ \vdots\\ T^k_1\\ 0\\ \vdots\\ 0\end{pmatrix}t^{-l+\rho_p}=-\sum_{l=1}^{\sigma-h^*-m_p}\tilde{G}_p(-l)t^{-l+\rho_p}.$$

This proves the theorem 5. 1.

Corollary of theorem 5. 1. If, in the sector $\mathscr{D}$, there is at least one index k, for which Re $(\lambda_k t)$ is non-negative, then we have

$$\tilde{X}_\rho(t) \simeq \sum_{k=1}^{n} e^{\lambda_k t} t^{a_{kk}} H^k(0) C_*^k t^{-Z_*} \begin{pmatrix} T_p^k \\ T_{p-1}^k \\ \vdots \\ T_1^k \\ 0 \\ \vdots \\ 0 \end{pmatrix}. \tag{5. 14}$$

Proof. If in the theorem 5. 1, σ is a sufficiently large positive integer, we have

$$\sum_{l=1}^{\sigma-h^*-m_p} \tilde{G}_p(-l) t^{-l+\rho_p} \equiv 0 .$$

For, when l is a positive integer, $\tilde{G}^\rho(-l) \equiv 0$ which are derived from the initial conditions (4. 6) and (4. 9). And we can imbed the other terms into the right hand side of (5. 14).

Here we remark that $H^k(0) = \begin{array}{c} \\ k\left\{\begin{array}{c} \\ \\ \\ \end{array}\right. \\ \\ \\ \\ \\ \end{array} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

Now we will show the main theorems in this paper.

On account of simplicity, we assume that there exist at least one index k, for which Re $(\lambda_k t)$ is non-negative in the sector $\mathscr{D}$.

Theorem. 5. 2. *We have*

$$\{\tilde{X}_1(t), \tilde{X}_2(t), \cdots, \tilde{X}_n(t)\} = \sum_{k=1}^{n} e^{\lambda_k t} t^{a_{kk}} H^k(0) C_*^k t^{-Z_*} \begin{pmatrix} T_1^k & T_2^k & \cdots\cdots & T_n^k \\ & T_1^k & \ddots & \vdots \\ & & \ddots & T_2^k \\ 0 & & & T_1^k \end{pmatrix} \tag{5. 15}$$

in the sector $\mathscr{D}$.

Theorem. 5. 3. *For t in the sector $\mathscr{D}$, we have*

$$\text{(5. 16)} \qquad \{X_1(t), X_2(t), \cdots, X_n(t)\} = \sum_{k=1}^{n} e^{\lambda_k t} t^{a_{kk}} H^k(0) C_*^k \begin{pmatrix} T_1^k & T_2^k & \cdots\cdots & T_n^k \\ & T_1^k & \ddots & \vdots \\ & & \ddots & T_2^k \\ 0 & & & T_1^k \end{pmatrix}$$

Proof. We can rewrite the relations (4. 1) (4. 2) between $X_j(t)$ and $\tilde{X}_j(t)$ in the matrix form.

$$\{X_1(t), X_2(t), \cdots, X_n(t)\}$$

$$= \{\tilde{X}_1(t), \tilde{X}_2(t), \cdots, \tilde{X}_n(t)\} \begin{pmatrix} 1 & \frac{(\log t)}{1!} & \frac{(\log t)^2}{2!} & \cdots & \frac{(\log t)^{n-1}}{(n-1)!} \\ & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \frac{(\log t)^2}{2!} \\ & & & \ddots & \frac{\log t}{1!} \\ 0 & & & & 1 \end{pmatrix}$$

$$= \{\tilde{X}_1(t), \tilde{X}_2(t), \cdots, \tilde{X}_n(t)\}\, t^{Z_*}.$$

So we have, by the theorem 5. 2,

$$\{X_1(t), X_2(t), \cdots, X_n(t)\} = \sum_{k=1}^{n} e^{\lambda_k t} t^{a_{kk}} H^k(0) C_*^k t^{-Z_*} \begin{pmatrix} T_1^k & T_2^k & \cdots & T_n^k \\ & \ddots & \ddots & \vdots \\ & & \ddots & T_2^k \\ 0 & & \ddots & T_1^k \end{pmatrix} t^{Z_*}.$$

Here we can decompose the matrix of the "Stokes' multipliers" as follows

$$\begin{pmatrix} T_1^k & T_2^k & \cdots & T_n^k \\ & \ddots & \ddots & \vdots \\ & & \ddots & T_2^k \\ 0 & & \ddots & T_1^k \end{pmatrix} = T_1^k I + T_2^k Z_* + T_3^k Z_*^2 + \cdots + T_n^k Z_*^{n-1}$$

where I denotes the identity matrix.

Hence we can exchange the matrix of the "Stokes' multipliers" for the matrix t^{Z_*}. Since $t^{-Z_*} \cdot t^{Z_*} = I$, we obtain (5. 16).

The nature of the matrix solutions of the differential equations (1. 1) near infinity are represented by the very simple forms in the last theorem 5. 3, and it is not difficult to represent the matrix solutions in componentwise.

In order that we can analyze the solutions of the differential equations more easily, we are now investigating the matrix functions, for instance, matrix Γ-functions and matrix ψ-functions on which we will give some results in the subsequent paper.

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