Infinite dimensional Laplacian and spherical harmonics

By

Yasuo UMEMURA and Norio KÔNO

Summary

The purpose of the present paper is to discuss the properties of the Gaussian measure and the Laplacian operator in infinite dimensions as limits of finite dimensional analogues. In the limit of $n\to\infty$, the uniform measure on the n-dimensional sphere, the spherical Laplacian operator, and Gegenbauer polynomials (spherical harmonics) tend respectively to an infinite dimensional Gaussian measure, the infinite dimensional Laplacian operator, and Hermite polynomials. We also discuss the addition formula and integral representation formula of Hermite polynomials in this limit procedure.

Introduction

In the previous paper [6], one of the authors discussed the infinite dimensional Gaussian measure on the dual space of a nuclear space, for instance, on the space $(\mathcal{S}')$, regarding it as a limit measure of finite dimensional Gaussian measures. It can be also regarded as a limit measure of uniform measures on spheres, as was shown by Hida-Nomoto [3]: their measure is defined on the projective limit space Ω of the n-dimensional sphere Ω_n, but it is equivalent to our measure on $(\mathcal{S}')$, since there exists a measure-preserving one-to-one mapping from Ω into $(\mathcal{S}')$.

These two different interpretations reflect the fact that the infinite dimensional Gaussian measure is ergodic with respect to translations like the finite dimensional ones, and also ergodic with respect to

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rotations like the uniform measures on spheres.

Next we shall discuss the Laplacian operator. The Laplacian operator $\overline{\Delta}_n$ on Ω_n generates all rotationally invariant, symmetric operators on $L^2(\Omega_n, d\omega_n)$. On the other hand, the infinite dimensional Laplacian operator Δ_c defined in [7] also generates all rotationally invariant operators on $L^2(\mathcal{S}', \mu_c)$. (c.f. §1). Hence, a question arises: does $\overline{\Delta}_n$ tend to Δ_c in some sense? In this paper we shall answer it, namely we shall show that $\overline{\Delta}_n/n$ tends to Δ_c as $n\rightarrow\infty$.

P. Lévy defined the infinite dimensional Laplacian operator in [5] as the limit operator of $\overline{\Delta}_n/n$. Therefore Lévy's definition turns out to be just the same as ours given in [7].

This fact implies the convergency of the eigen values and the eigen functions of $\overline{\Delta}_n$ to those of Δ_c. Therefore Gegenbauer polynomials tend to Hermite polynomials. This has been known as a formula concerning special functions, for instance see [1]. However, we get here a new interpretation of this formula: finite dimensional spherical harmonics tend to infinite dimensional ones.

Similar new interpretations are possible also for the addition formula and the integral representation formula concerning Hermite polynomials. Especially, we shall show that the density function of the integral representation of the n-dimensional zonal spherical function converges to that of the integral representation of Hermite polynomial in the sense of $L^2(\mathcal{S}', \mu_c)$. The latter representation is equivalent with the Gauss transform.

Our standpoint will make clear some aspects of Hermite polynomials as infinite dimensional eigen functions.

We should also remark that the addition formula played an important role in S. Kakutani's discussions on Brownian motions [4].

In this paper, after preliminary discussions in §1, we shall show rather intuitively in §2 and more exactly in §3 that the projective limi tof the uniform measures on spheres is just the infinite dimensional Gaussian measure. In §4, we shall establish the relation between finite dimensional and infinite dimensional Laplacian operators, and then investigate how this relation reflects on their eigen functions; for zonal ones in §5 and more generally in §6. In this point of view, we interprete the integral representation formula in §7 and the addition formula in §8.

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§1. Preliminaries (For details, see [6] and [7]).

Let L be a real nuclear space, and H be its completion by a continuous Hilbertian norm $\| \ \|_H$. Then, we have a relation;

$$L \subset H \simeq H^* \subset L^*,$$

where H^* and L^* are the dual spaces of H and L respectively.

Now, define a function $\chi(\xi)$ on L as follows

$$\chi(\xi) = \exp\left[-\frac{c^2}{2}\|\xi\|_H^2\right]. \tag{1}$$

It satisfies the conditions of positive definiteness, continuity in the norm $\| \ \|_H$, and $\chi(0)=1$. So, by the theorem of Bochner-Minlos, $\chi(\xi)$ is the characteristic function of a probability measure μ_c on L^*:

$$\chi(\xi) = \int_{L^*} \exp[i\langle T, \xi\rangle] \, d\mu_c(T). \tag{2}$$

Here, $\langle T, \xi\rangle$ means the value at ξ of the linear functional T. μ_c is defined on the σ- algebra $\mathfrak{B}(L^*)$ which is generated by the family of all Borel cylinder sets of L^*. We call μ_c a Gaussian measure with variance c^2.

Definition 1. *An unitary operator u on H is called a rotation of L, if it satisfies;*

i) *u maps L onto L,*
ii) *u is homeomorphic on L.*

The whole of rotations of L forms a group, which we call the rotation group of L and denote it with $O(L)$. Identifying u with u^{-1*}, $O(L)$ is regarded as a transformation group on L^*.

Then, Gaussian measure μ_c is O-invariant, i.e. μ_c is invariant under any rotation of L. Moreover, it is O-ergodic, namely it can not be decomposed into the sum of two measures which are O-invariant and singular with each other.

Let $L^2(L^*, \mu_c)$ be the Hilbert space of all square integrable functions on L^* with respect to a Gaussian measure μ_c. Denote its inner product and its norm by $(\ ,\ )_c$ and $\|\ \|_c$ respectively.

Let $\xi_1, \cdots, \xi_n$ be a finite orthonormal system (F.O.N.S.) of L in the norm $\|\ \|_H$. We define the subspace $\mathfrak{A}_{\xi_1\cdots\xi_n}$ of $L^2(L^*, \mu_c)$ as follows:

$$\mathfrak{A}_{\xi_1\cdots\xi_n}=\Big\{F(T)\in L^2(L^*, \mu_c)\ |\ {}^\exists f(x_1, \cdots, x_n);\ \sqrt{\frac{d\mu_c}{dx}}(x_1, \cdots, x_n)f(x_1, \cdots, x_n)\in(\mathcal{S})_{R^n},\ F(T)=f(\langle T, \xi_1\rangle, \cdots, \langle T, \xi_n\rangle)\Big\},$$

where $\dfrac{d\mu_c}{dx}=\left(\dfrac{1}{\sqrt{2\pi}\,c}\right)^n\exp\left[-\dfrac{1}{2c^2}\sum_{k=1}^n x_k^2\right]$, and $(\mathcal{S})_{R^n}$ is the whole of rapidly decreasing C^∞-functions on R^n as defined by L. Schwartz.

Put $\mathfrak{A}=\bigcup_{\text{all F.O.N.S.}}\mathfrak{A}_{\xi_1\cdots\xi_n}$, then obviously $\mathfrak{A}$ is dense in $L^2(L^*, \mu_c)$.

Let $\Phi_{\xi_1\cdots\xi_n}$ be the mapping which maps $f(\langle T, \xi_1\rangle, \cdots, \langle T, \xi_n\rangle)$ to $f(x_1, \cdots, x_n)$.

Definition 2. *The infinite dimensional Laplacian operator Δ_c is an operator defined on $\mathfrak{A}$ as follows. If $F(T)\in\mathfrak{A}_{\xi_1\cdots\xi_n}$, then*

$$\Delta_c F(T)=\Phi_{\xi_1\cdots\xi_n}^{-1}\sum_{i=1}^n\left(\frac{\partial^2}{\partial x_i^2}-\frac{x_i}{c^2}\frac{\partial}{\partial x_i}\right)\Phi_{\xi_1\cdots\xi_n}F(T). \tag{3}$$

Since $\sum_{i=1}^n\left(\dfrac{\partial^2}{\partial x_i^2}-\dfrac{x_i}{c^2}\dfrac{\partial}{\partial x_i}\right)$ is rotationally invariant, the definition does not depend on the representation of $F(T)$.

Δ_c can be written symbolically as;

$$\Delta_c = \lim_{n\to\infty}\left[\Delta_n - \frac{r_n^2}{4c^4} + \frac{nI}{2c^2}\right],$$

where Δ_n is the usual n-dimensional Laplacian operator on $L^2(R^n, d^n x)$, $r_n^2 = \sum_{k=1}^n x_k^2$ and I is the identity operator.

For an example of a nuclear space, we have the space $(\mathcal{S})$, the whole of rapidly decreasing C^∞-functions on the real line. $(\mathcal{S})$ is nuclear because its topology is defined by countable Hilbertian norms;

$$\|\varphi\|_{pq} = \int_{-\infty}^{\infty} (1+x^2)^p \sum_{k=0}^{q} |\varphi^{(k)}(x)|^2 dx.$$

In the above discussions, considering the case that $L=(\mathcal{S})$ and $H=L^2(R^1)$, we see that we can define a Gaussian weasure μ_c on $(\mathcal{S}')$, and the infinite dimensional Laplacian operator Δ_c on $L^2(\mathcal{S}', \mu_c)$.

§2. A limit property of uniform measures on spheres

Let P_n be the uniform probability measure on an n-dimensional sphere Ω_n of radius $\sqrt{n+1}\,c$ and center at the origin.

$$\Omega_n : x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = (n+1)c^2.$$

For this P_n, consider the joint distribution $P_{n,m}$ $(n \geqq m)$ of $(x_1, x_2, \cdots, x_m)$, namely for $E \subset R^m$,

$$P_{n,m}(E) = P_n\Big(\{(x_1, \cdots, x_m, \cdots, x_{n+1}) \mid (x_1, \cdots, x_m) \in E\}\Big).$$

Proposition 1.

$$\lim_{n\to\infty} P_{n,m}(E) = \mu_{c,m}(E) \tag{4}$$

where $\mu_{c,m}$ is the m-dimensional Gaussian measure with variance c^2. The convergence is uniform for all Borel subsets E of R^m.

Proof Decompose R^{n+1} into the sum $R_1^m + R_2^{n-m+1}$, where

$$R_1^m = \{(x_1, x_2, \cdots, x_m, o, \cdots, o)\},$$

and
$$R_2^{n-m+1} = \{(o, \cdots, o, x_{m+1}, x_{m+2}, \cdots, x_{n+1})\}.$$

Let $x=(x_1, x_2, \cdots, x_{n+1})$ be a point on Ω_n, then the distance between x and R_1^m is $\sqrt{(n+1)c^2-(x_1^2+\cdots+x_m^2)}$, while the angle θ between the normal of Ω_n at x and the space R_2^{n-m+1} is given by

$$\frac{1}{\cos\theta}=\frac{\sqrt{n+1}\,c}{\sqrt{(n+1)c^2-(x_1^2+\cdots+x_m^2)}}.$$

Hence, clearly $P_{n,m}$ is given by

$$dP_{n,m}=\begin{cases}\dfrac{\sqrt{n+1}\,c}{\sqrt{(n+1)c^2-(x_1^2+\cdots+x_m^2)}}k_{n,m}\{(n+1)c^2-(x_1^2+\cdots+x_m^2)\}^{\frac{n-m}{2}} & \\ \qquad\times dx_1dx_2\cdots dx_m \quad \text{for } x_1^2+\cdots+x_m^2\leqq(n+1)c^2 & \\ 0 \qquad\qquad\qquad\qquad \text{for } x_1^2+\cdots+x_m^2>(n+1)c^2, & \end{cases}$$

where $k_{n,m}$ is the normalization constant. Putting $k_{n,m}((n+1)c^2)^{\frac{n-m}{2}}$ as $k'_{n,m}$, we have

$$dP_{n,m}=k'_{n,m}\left(1-\frac{x_1^2+\cdots+x_m^2}{(n+1)c^2}\right)_+^{\frac{n-m-1}{2}}dx_1\cdots dx_m, \tag{5}$$

where r_+ means $\mathrm{Max}(r,0)$. From this, we see that

$$\frac{dP_{n,m}}{k'_{n,m}dx_1\cdots dx_m}\xrightarrow[(n\to\infty)]{}\exp\left[-\frac{x_1^2+\cdots+x_m^2}{2c^2}\right].$$

It is easily seen that the convergence is uniform with respect to $(x_1,\cdots,x_m)$. This proves the proposition 1.

§3. The projective limit space Ω

In order to make the situation in Proposition 1 clearer, we shall construct the projective limit space Ω of Ω_n, and show that the limit measure P of P_n is isomorphic with an infinite dimensional Gaussian measure μ_c defined in §1.

First, for any $m<n$, we shall define a projection $f_{n,m}$ from Ω_n onto Ω_m as follows;

$$f_{n,m} \text{ maps } \omega^{(n)}=(x_1^{(n)},\cdots,x_{n+1}^{(n)})\in\Omega_n$$
$$\text{to} \quad \omega^{(m)}=(x_1^{(m)},\cdots,x_{m+1}^{(m)})\in\Omega_m,$$

where

(6) $$x_i^{(m)} = \frac{\sqrt{m+1}\, c x_i^{(n)}}{\sqrt{x_1^{(n)2} + \cdots + x_{m+1}^{(n)2}}} \quad \text{for } 1 \leqq i \leqq m+1$$

Then, $f_{n,m}$ is a measurable mapping defined almost everywhere on Ω_n. (The exceptional set is $\{\omega^{(n)} \in \Omega_n \mid x_1^{(n)} = \cdots = x_{m+1}^{(n)} = 0\}$).

(*Remark*; If we use the polar coordinates, $f_{n,m}$ is defined as a mapping which maps

$$(\theta_1, \cdots, \theta_m, \cdots, \theta_n) \text{ on } \Omega_n \text{ to } (\theta_1, \cdots, \theta_m) \text{ on } \Omega_m.$$

This expression was used in [3]).

The projection $f_{n,m}$ satisfies the following conditions;

i) $f_{l,n} = f_{l,m} \circ f_{m,n}$ for $l < m < n$

ii) $P_m(A) = P_n(f_{m,n}^{-1}(A))$
for $m < n$ and a Borel subset A of Ω_m.

Therefore, according to a theorem due to Bochner, we can construct the projective limit probability space $(\Omega, \mathfrak{B}, P)$. It satisfies the following properties:

P1) $\Omega \subset \prod_{n=1}^{\infty} \Omega_n$

P2) $f_m = f_{m,n} \circ f_n$ *for* $m < n$.

Here, f_n *is the restriction of* π_n *on* Ω, *where* π_n *is the projection from* $\prod_{n=1}^{\infty} \Omega_n$ *onto* Ω_n.

P3) $\mathfrak{B}$ *is generated by* $\bigcup_{n=1}^{\infty} f_n^{-1}(\mathfrak{B}_n)$, *where* $\mathfrak{B}_n$ *is the whole of Borel subsets of* Ω_n.

P4) $P(f_n^{-1}(A)) = P_n(A)$ *for* $A \in \mathfrak{B}_n$.

Since the coordinate $x_i^{(n)}$ $(1 \leqq i \leqq n+1)$ is a function on Ω_n, it can be regarded as a function on Ω. We denote this function by $X_i^{(n)}(\omega)$, then for any $\omega \in \Omega$,

(7) $$f_n(\omega) = (X_1^{(n)}(\omega), \cdots, X_{n+1}^{(n)}(\omega)) \in \Omega_n.$$

$X_i^{(n)}(\omega)$ is a measurable fucntion on Ω because of **P3**).

Lemma 1.

i) $\int_\Omega X_i^{(n)}(\omega)dP(\omega)=0, \quad \forall n, \ 1\leqq i\leqq n+1$

ii) $\int_\Omega X_i^{(n)}(\omega)X_j^{(m)}(\omega)dP(\omega)$

$$=\delta_{ij}c^2\sqrt{\frac{n+1}{m+1}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)}\frac{\Gamma\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

for $m\leqq n$, where Γ is Gamma function.

Proof From **P4**), we have

$$\int_\Omega X_i^{(n)}(\omega)dP(\omega)=\int_{\Omega_n}x_i^{(n)}dP_n(\omega^{(n)})=0$$

Similarly, using **P2**) also, we see that

$$\int_\Omega X_i^{(n)}(\omega)X_j^{(m)}(\omega)dP(\omega)$$

$$=\int_{\Omega_n}x_i^{(n)}\frac{\sqrt{m+1}\,cx_j^{(n)}}{\sqrt{x_1^{(n)2}+\cdots+x_{m+1}^{(n)2}}}dP_n(\omega^{(n)})$$

$$=\delta_{ij}\sqrt{m+1}\,c\int_{\Omega_n}\frac{x_1^{(n)2}}{\sqrt{x_1^{(n)2}+\cdots+x_{m+1}^{(n)2}}}dP_n(\omega^{(n)})$$

$$=\delta_{ij}\frac{c}{\sqrt{m+1}}\int_{\Omega_n}\sqrt{x_1^{(n)2}+\cdots+x_{m+1}^{(n)2}}\,dP_n(\omega^{(n)})$$

Substituting (5), we get

$$=\delta_{ij}c^2\sqrt{\frac{n+1}{m+1}}\int_0^1 t^{m+1}(1-t^2)^{\frac{n-m-2}{2}}dt\Big/\int_0^1 t^m(1-t^2)^{\frac{n-m-2}{2}}dt$$

$$=\delta_{ij}c^2\sqrt{\frac{n+1}{m+1}}B\left(\frac{m+2}{2},\frac{n-m}{2}\right)\Big/B\left(\frac{m+1}{2},\frac{n-m}{2}\right)$$

$$=\delta_{ij}c^2\sqrt{\frac{n+1}{m+1}}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)}\frac{\Gamma\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \qquad \text{(q.e.d.)}$$

The relation (5) is expressed as a formula concerning with P as follows.

For any $n \geqq m$, and for any Borel subset E of R^m, we have

$$P(\{\omega \mid (X_1^{(n)}(\omega), \cdots, X_m^{(n)}(\omega)) \in E\}) \tag{8}$$

$$= k'_{n,m} \int_E \left(1 - \frac{x_1^2 + \cdots + x_m^2}{(n+1)c^2}\right)_+^{\frac{n-m-1}{2}} dx_1 \cdots dx_m .$$

Lemma 2. $\{x_i^{(n)}(\omega);\ n = i,\ i+1,\ i+2,\ \cdots\cdots\}$ *forms a Cauchy sequence in* $L^2(\Omega, P)$.

Proof From the previous lemma, we get

$$\|X_i^{(n)} - X_i^{(m)}\|_P^2 = 2c^2 - 2c^2 \sqrt{\frac{n+1}{m+1}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{m+1}{2}\right)} \frac{\Gamma\left(\frac{m+2}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}$$

$$\to 0 \text{ as } n, m \to \infty,$$

because we have asymptotically

$$\frac{\Gamma\left(t + \frac{1}{2}\right)}{\Gamma(t)} \sim \sqrt{t}\,. \qquad \text{(q.e.d.)}$$

From lemma 2, $X_i^{(n)}$ converges to a function X_i in $L^2(\Omega, P)$. Then, $X_i(\omega)$ is defined for almost all ω.

Evidently,

$$\int_\Omega X_i(\omega)\, dP(\omega) = 0 \quad \text{and} \quad \int_\Omega X_i(\omega) X_j(\omega)\, dP(\omega) = \delta_{ij} c^2.$$

Next, we shall imbed Ω into the space L^*, the dual space of a nuclear space L. (c. f. §1).

Since L is nuclear, there exist a complete orthonormal system $\{\xi_k\}$ of L in the norm $\| \ \|_H$ and a square summable sequence $\{\lambda_k\}$ such that the norm:

$$p(\xi) = \sqrt{\sum_{k=1}^{\infty} \frac{(\xi, \xi_k)_H^2}{\lambda_k^2}} \tag{9}$$

is continuous in the topology of L.

Then, we see that *for P-almost all* $\omega \in \Omega$,

$$T_\omega(\xi) \equiv \sum_{k=1}^{\infty} (\xi, \xi_k)_H X_k(\omega) \tag{10}$$

is defined and continuous on L. Because

$$\sum_{k=1}^{\infty} |(\xi, \xi_k)_H| \, |X_k(\omega)| \leq p(\xi) q(\omega),$$

where $q(\omega) \equiv \sqrt{\sum_{k=1}^{\infty} \lambda_k^2 X_k(\omega)^2}$ is finite for P-almost all ω.

Consider the mapping $\Psi : \omega \in \Omega \to T_\omega \in L^*$. *$\Psi$ is one-to-one except on a suitable null-set of Ω,* because if $T_\omega(\xi) = T_{\omega'}(\xi)$ for any ξ, we have $X_k(\omega) = X_k(\omega')$ for any k. This implies $\omega = \omega'$, since for P-almost all ω, we get from (6)

$$X_i^{(m)}(\omega) = \frac{\sqrt{m+1}\, c X_i(\omega)}{\sqrt{X_1(\omega)^2 + \cdots + X_{m+1}(\omega)^2}}. \tag{11}$$

thus for any m we have $f_m(\omega) = f_m(\omega')$.

Next, we shall discuss the measurability. From P3), the probability measure P is defined on the smallest σ-algebra $\mathfrak{B}$ which makes all $X_i^{(n)}(\omega)$ measurable. From (11), this is equivalent to say $\mathfrak{B}$ is the smallest σ-algebra which makes all $X_i(\omega)$ measurable. Therefore, the image $\Psi(\mathfrak{B})$ is the smallest σ-algebra which makes all $\langle T, \xi \rangle$ measurable regarding ξ as a linear functional on L^*. In other words, $\Psi(\mathfrak{B})$ is the smallest σ-algebra which makes all Borel cylinder sets of L^* measurable. *This means that $\Psi(\mathfrak{B})$ is equal to $\mathfrak{B}(L^*)$ defined in* §1.

Finally, we shall show that *the measure P on Ω is mapped to a Gaussian measure μ_c on L^*.* Because of (8), we get

$$\begin{aligned} P[\{\omega \mid (X_1(\omega), \cdots, X_m(\omega)) \in E\}] \\ = \left(\frac{1}{\sqrt{2\pi}\, c}\right)^m \int_E \exp\left[-\frac{x_1^2 + \cdots + x_m^2}{2c^2}\right] dx_1 \cdots dx_m. \end{aligned} \tag{12}$$

So that if $\xi = \sum_{k=1}^{N} \alpha_k \xi_k$, we have from (10)

$$\begin{aligned} \int_\Omega \exp[i T_\omega(\xi)] \, dP(\omega) \\ = \exp\left[-\frac{c^2}{2} \sum_{k=1}^{N} (\xi, \xi_k)_H^2\right] = \exp\left[-\frac{c^2}{2} \|\xi\|_H^2\right]. \end{aligned}$$

Since the characteristic function of a measure on L^* must be continuous on L, we have for any $\xi \in L$,

$$\int_{\Omega} \exp[iT_{\omega}(\xi)]\,dP(\omega) = \exp\left[-\frac{c^2}{2}\|\xi\|_H^2\right]. \tag{13}$$

This means that P is mapped to μ_c, because of one-to-one correspondence between measures and characteristic functions. So far, we have proved:

Proposition 2. *The projective limit space* $(\Omega, \mathfrak{B}, P)$ *is isomorphic with* $(L^*, \mathfrak{B}(L^*), \mu_c)$.

Namely, there exists a measure-preserving one-to-one mapping from a suitable subset $\widetilde{\Omega}$ *of* Ω *onto a suitable subset* $\widetilde{L}^*$ *of* L^* *where* $P(\widetilde{\Omega}) = \mu_c(\widetilde{L}^*) = 1$.

Remark L^* is a vector space, while Ω is not.

§4. Laplacian operators

In this section, we shall prove that the Laplacian operator on Ω_n tends to the infinite dimensional Laplacian operator on L^*.

On $L^2(R^{n+1}, d^{n+1}x)$, the Laplacian operator is defined by $\Delta_{n+1} = \sum_{i=1}^{n+1} \frac{\partial^2}{\partial x_i^2}$. Using polar coordinates, it is expressed as follows;

$$\Delta_{n+1} = \frac{\partial^2}{\partial r^2} + \frac{n}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\overline{\Delta}_n. \tag{14}$$

Here $\overline{\Delta}_n$ is the Laplacian operator on the unit sphere;

$$\overline{\Delta}_1 = \frac{\partial^2}{\partial \theta_1^2} \tag{15}$$

$$\overline{\Delta}_n = \frac{\partial^2}{\partial \theta_n^2} + (n-1)\cot\theta_n \frac{\partial}{\partial \theta_n} + \frac{1}{\sin^2\theta_n}\overline{\Delta}_{n-1} \quad (n \geqq 2)$$

In order to show that the operator $\overline{\Delta}_n/(n+1)c^2$ tends to Δ_c in some sense, we shall first give a rough discussion, and later formulate it in an exacter way.

We shall operate $\overline{\Delta}_n/(n+1)c^2$ and Δ_c on a function $f(x_1, \cdots, x_m)$, $(m \leqq n)$. We have

$$\Delta_c f(x_1, \cdots, x_m) = \sum_{i=1}^{m}\left(\frac{\partial^2}{\partial x_i^2} - \frac{x_i}{c^2}\frac{\partial}{\partial x_i}\right) f(x_1, \cdots, x_m),$$

$$\frac{\overline{\Delta}_n}{(n+1)c^2} f(x_1, \cdots, x_m) = \frac{r^2}{(n+1)c^2}\left(\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2} - \frac{\partial^2}{\partial r^2} - \frac{n}{r}\frac{\partial}{\partial r}\right) f(x_1, \cdots, x_m).$$

Here, we substitute

$$\frac{\partial f}{\partial r} = \sum_{i=1}^{m}\frac{x_i}{r}\frac{\partial f}{\partial x_i}, \quad \frac{\partial^2 f}{\partial r^2} = \sum_{i,j=1}^{m}\frac{x_i x_j}{r^2}\frac{\partial^2 f}{\partial x_i \partial x_j},$$

then we get

$$\frac{\overline{\Delta}_n}{(n+1)c^2} f(x_1, \cdots, x_m)\Big|_{r=\sqrt{n+1}\,c} = \Big[\sum_{i=1}^{m}\frac{\partial^2}{\partial x_i^2} - \frac{1}{(n+1)c^2}\sum_{i,j=1}^{m} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j}$$

$$- \frac{n}{(n+1)c^2}\sum_{i=1}^{m} x_i \frac{\partial}{\partial x_i}\Big] f(x_1, \cdots, x_m),$$

thus in the limit of $n \to \infty$, we see that the right side converges to $\Delta_c f(x_1, \cdots, x_m)$.

Now, we shall give an exacter expression, regarding both $\overline{\Delta}_n$ and Δ_c as operators on $L^2(L^*, \mu_c)$.

For a fixed C. O. N. S. $\{\xi_k\}$ of L, the projective limit space Ω is imbedded into L^* as shown in §3. For this $\{\xi_k\}$, we consider the space $\mathfrak{A}_{\xi_1 \cdots \xi_n}$ following the definition in §1, and denote it by $\mathfrak{A}_n$. The union $\mathfrak{A}_\infty \equiv \bigcup_{n=1}^{\infty} \mathfrak{A}_n$ is a subspace of $\mathfrak{A}$ defined in §1. $\mathfrak{A}_\infty$ is also dense in $L^2(L^*, \mu_c)$.

On the other hand, consider an n-dimensional ball B_n of radius $\sqrt{n+1}\,c$ and center at origin.

$$B_n : x_1^2 + \cdots + x_n^2 \leq (n+1)c^2.$$

Then, the Hilbert space $L^2(B_n, P_{n,n})$, where the measure $P_{n,n}$ is defined by (5), is isomorphic with $L^2(\Omega_n^+, 2P_n)$, where $2P_n$ is the uniform probability measure on the hemisphere Ω_n^+.

$$\Omega_n^+ : x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = (n+1)c^2, \ x_{n+1} > 0.$$

$L^2(\Omega_n^+, 2P_n)$ is a subspace of $L^2(\Omega_n, P_n)$ which consists of all such functions that satisfy

$$f(x_1, \cdots, x_n, x_{n+1}) = f(x_1, \cdots, x_n, -x_{n+1}).$$

So, the Laplacian $\overline{\Delta}_n$ is defined on $L^2(\Omega_n^+, 2P_n)$, hence on $L^2(B_n, P_{n,n})$ also.

Define the mapping Q_n from $L^2(R^n, \mu_{c,n})$ into $L^2(B_n, P_{n,n})$ as follows;

Q_n maps any polynomial on R^n to its restriction on B_n.

Since a polynomial on B_n is uniquely extended on R^n, Q_n is one-to-one on the set $\mathfrak{P}_n$ of all polynomials of n-variables. Remark that $\mathfrak{P}_n$ is dense in $L^2(R^n, \mu_{c,n})$ and $Q_n\mathfrak{P}_n$ is dense in $L^2(B_n, P_{n,n})$.

Now, define an operator $\overline{\overline{\Delta}}_n$ on $\mathfrak{P}_n$ as follows;

$$\overline{\overline{\Delta}}_n = Q_n^{-1} \circ \overline{\Delta}_n \circ Q_n . \tag{16}$$

We shall express $\overline{\overline{\Delta}}_n$ as a differential operator on R^n.

On the hemisphere Ω_n^+:

$$x_1^2 + \cdots + x_n^2 + x_{n+1}^2 = (n+1)c^2, \quad x_{n+1} > 0,$$

$x_1, \cdots, x_n$ can be regarded as coordinates. Namely, $x_1, \cdots, x_n$ are independent variables on Ω_n^+, and $x_{n+1} = \sqrt{(n+1)c^2 - x_1^2 - \cdots - x_n^2}$ is a function of them. Using such coordinates, we have.

$$\overline{\Delta}_n = (n+1)c^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^{n} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - n \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} . \tag{17}$$

This expression is obtained in the following way: In the operator Δ_{n+1} we change variables from $(x_1, \cdots, x_n, x_{n+1})$ to $(\alpha_1, \cdots, \alpha_n, r)$, where $r = \sqrt{x_1^2 + \cdots + x_{n+1}^2}$ and $\alpha_i = x_i/r$, then subtract $\frac{\partial^2}{\partial r^2} + \frac{n}{r} \frac{\partial}{\partial r}$ from it, and finally put $r = \sqrt{n+1}\, c$.

So, for $f \in \mathfrak{P}_n$ we have

$$\overline{\overline{\Delta}}_n f(x_1, \cdots, x_n) = (n+1)c^2 \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2} - \sum_{i,j=1}^{n} x_i x_j \frac{\partial^2 f}{\partial x_i \partial x_j} - n \sum_{i=1}^{n} x_i \frac{\partial f}{\partial x_i} \tag{18}$$

Thus $\overline{\overline{\Delta}}_n$ is continuous in the topology of $(S)_{R^n}$. Namely, for a sequence

$\{f_k\} \subset \mathfrak{P}_n$, if $f_k(x_1, \cdots, x_n)\sqrt{\frac{d\mu_c}{dx}(x_1, \cdots, x_n)}$ tends to 0 in the topology of $(S)_{R^n}$, then $(\overline{\overline{\Delta}}_n f_k)\sqrt{\frac{d\mu_c}{dx}}$ tends also to 0 in $(S)_{R^n}$. Therefore, $\overline{\overline{\Delta}}_n$ can be extended continuously on $\mathfrak{A}_n$.

Consider the mapping $\Phi_{\xi_1 \cdots \xi_n}$ from $\mathfrak{A}_n$ into $L^2(R^n, \mu_{c,n})$ following the definition in §1, and denote it by Φ_n. Furthermore, put $\widetilde{\Delta}_n = \Phi_n^{-1} \circ \overline{\overline{\Delta}}_n \circ \Phi_n$. $\widetilde{\Delta}_n$ is an operator on $\mathfrak{A}_n$. Namely, for $F(T) = f(\langle T, \xi_1\rangle, \cdots, \langle T, \xi_n\rangle) \in \mathfrak{A}_n$,

$$\begin{aligned}(19) \qquad \widetilde{\Delta}_n F(T) = \Big[(n+1)c^2 \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=1}^{n} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} \\ - n \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \Big] f(x_1, \cdots, x_n) \Big|_{x_i = \langle T, \xi_i\rangle},\end{aligned}$$

where $|_{x_i = \langle T, \xi_i\rangle}$ means that after the calculation of the right hand side we replace x_i by $\langle T, \xi_i\rangle$.

Since $\mathfrak{A}_n \supset \mathfrak{A}_m$ for $n > m$, $\widetilde{\Delta}_n$ is defined also on $\mathfrak{A}_m$. So that on the space $\mathfrak{A}_\infty \equiv \bigcup_{m=1}^{\infty} \mathfrak{A}_m$, we can consider symbolically the limit of $\widetilde{\Delta}_n / (n+1)c^2$.

Now, we want to prove;

Proposition 3.

$$(20) \qquad \lim_{n\to\infty} \frac{\widetilde{\Delta}_n}{(n+1)c^2} = \Delta_c \ \textit{strongly on } \mathfrak{A}_\infty.$$

Namely, for any $F \in \mathfrak{A}_\infty$,

$$(20)' \qquad \lim_{n\to\infty} \frac{\widetilde{\Delta}_n}{(n+1)c^2} F = \Delta_c F \ \textit{in } L^2(L^*, \mu_c).$$

Proof Since $\mathfrak{A}_m$ can be regarded as a subspace of $L^2(R^m, \mu_{c,m})$, it is sufficient to show that

$$(21) \qquad \lim_{n\to\infty} \frac{\overline{\overline{\Delta}}_n}{(n+1)c^2} f(x_1, \cdots, x_m) = \sum_{i=1}^{m} \left(\frac{\partial^2}{\partial x_i^2} - \frac{x_i}{c^2} \frac{\partial}{\partial x_i} \right) f(x_1, \cdots, x_m)$$

in $L^2(R^m, \mu_{c,m})$.

If the function f does not depend on $x_{m+1} \cdots, x_n$, the sum in the right hand side of (18) is sufficient to take from 1 to m. Therefore, dividing the both hand sides of (18) by $(n+1)c^2$, and taking the limit of $n\to\infty$, the wanted relation (21) is easily obtained.

Moreover, it is clear that in (20)′, we can take the limit not only in the norm of $L^2(L^*, \mu_c)$, but also in pointwise way as functions on L^*. Indeed (20)′ holds in any meaning of the limit, provided that it makes the space $\mathfrak{A}_\infty$ a *topological* vector space.

§5. Zonal spherical functions

From the result of §4, we can expect that spherical functions on Ω_n tend to Hermite polynomials which are eigen functions of Δ_c. In this section, we shall assure this for zonal spherical functions.

On Ω_n, eigen functions of $\overline{\Delta}_n$ are called spherical functions. Especially, a spherical function dependent only on x_1 is called zonal. It is given by Gegenbauer polynomial as explained below.

From (18), $\overline{\overline{\Delta}}_n$ is written on $\mathfrak{P}_1$ as follows;

$$\overline{\overline{\Delta}}_n|\mathfrak{P}_1=\{(n+1)c^2-x_1^2\}\frac{d^2}{dx_1^2}-nx_1\frac{d}{dx_1} \tag{22}$$

Its eigen values are $-l(l+n-1)$, $l=0, 1, 2, \cdots$, and the corresponding eigen functions are

$$f_{l,n}(x_1)\equiv C_l^{\frac{n-1}{2}}\left(\frac{x_1}{\sqrt{n+1}\,c}\right) \tag{23}$$

Here, $C_l^p(x)$ is Gegenbauer polynomial. It is defined as the solution of

$$\left[(1-x^2)\frac{d^2}{dx^2}-(2p+1)x\frac{d}{dx}+l(l+2p)\right]C_l^p(x)=0, \tag{24}$$

which satisfies $C_l^p(1)=(2p+l-1)!/l!(2p-1)!$.

On the other hand, Δ_c is written on $\mathfrak{P}_1$ as follows;

$$\Delta_c|\mathfrak{P}_1=\frac{d^2}{dx_1^2}-\frac{x_1}{c^2}\frac{d}{dx_1}. \tag{25}$$

Its eigen values are $-l/c^2$, $l=0, 1, 2, \cdots$, and the corresponding eigen functions are

(26) $$f_l(x_1)=H_l\left(\frac{x_1}{\sqrt{2}\,c}\right).$$

Here, $H_l(x)$ is Hermite polynomial. It is defined as the solution of

(27) $$\left[\frac{d^2}{dx^2}-2x\frac{d}{dx}+2l\right]H_l(x)=0,$$

whose coefficient of x^l (=the highest degree) is 2^l.

Therefore, on $\mathfrak{P}_1$ the result in §4 means that

$$\left(1-\frac{x_1^2}{(n+1)c^2}\right)\frac{d^2}{dx_1^2}-\frac{nx_1}{(n+1)c^2}\frac{d}{dx_1}\to\frac{d^2}{dx_1^2}-\frac{x_1}{c^2}\frac{d}{dx_1}$$

strongly on $\mathfrak{P}_1$.

Corresponding to this, the eigen values $\lambda_{l,n}\equiv-\frac{l(l+n-1)}{(n+1)c^2}$ tend to $\lambda_l\equiv-\frac{l}{c^2}$, and the eigen functions $f_{l,n}(x_1)$ also converge to $f_l(x_1)$ except the normalization constants.

The latter result is easily verified by comparing the coefficients in the polynomial $f_l(x_1)$ whih those in $f_{l,n}(x_1)$. But, we can regard it as a direct result of the fact $\frac{\tilde{\Delta}_n}{(n+1)c^2}\to\Delta_c$.

More exactly speaking, let $\mathfrak{P}_{1,l}$ be the set of all polynomials of one variable with degrees at most l. $\mathfrak{P}_{1,l}$ is an $(l+1)$-dimensional vector space. Both the operators $\bar{\bar{\Delta}}_n$ and Δ_c keep $\mathfrak{P}_{1,l}$ invariant, so that they are represented as matrices on $\mathfrak{P}_{1,l}$.

Lemma 3. *If a $k\times k$-matrix A has k different eigen values, and if $k\times k$-matrices A_n tend to A, then for sufficiently large n. A_n also has k different eigen values. Moreover, eigen values and eigen vectors of A_n tend to those of A respectively.*

Proof is omitted.

Since Δ_c has $l+1$ different eigen values on $\mathfrak{P}_{1,l}$, we have the following proposition. The ratio of the normalization constants is determined by comparing the coefficients of x^l.

Proposition 4.

$$\lim_{n\to\infty}\frac{l!2^{l/2}}{n^{l/2}}C_l^{n/2}\left(\frac{x}{\sqrt{n}}\right)=H_l\left(\frac{x}{\sqrt{2}}\right). \tag{28}$$

Our new interpretation of this formula is that n-dimensional zonal spherical functions tend to infinite dimensional ones.

§6. Spherical functions of many variables.

In this section, generalizing the result in §5, we shall establish the relation between finite dimensional and infinite dimensional spherical functions which are not necessarily zonal.

As we have discussed in [7], the infinite dimensional Laplacian operator Δ_c on $L^2(L^*, \mu_c)$ has eigen values $-l/c^2$, $l=0,1,2,\cdots$. For the fixed C. O. N. S. $\{\xi_k\}$ of L, the eigen space $\mathfrak{H}_l$ is spanned by

$$F_{l_1 l_2\cdots l_k\cdots}(T)\equiv\prod_{k=1}^{\infty}H_{l_k}\left(\frac{\langle T,\xi_k\rangle}{\sqrt{2}c}\right), \tag{29}$$

where $l_1+l_2+\cdots+l_k+\cdots=l$. Consequently, except for $l=0$, each eigen value $-l/c^2$ has infinite multiplicity.

In order to resolve this multiplicity, let us consider the partial Laplacians $\Delta_c^{(R)}$. Let R be a finite dimensional subspace of L, and suppose that some elements of a C.O.N.S. $\{\xi_k\}$ of L, say $\xi_1, \xi_2, \cdots, \xi_\nu$, form a C.O.N.S. of R.

Then, for $F(T)\in\mathfrak{A}_{\xi_1\xi_2\cdots\xi_n}$ $(n>\nu)$, we put

$$\Delta_c^{(R)}F(T)=\Phi_{\xi_1\cdots\xi_n}^{-1}\sum_{i=\nu+1}^{n}\left(\frac{\partial^2}{\partial x_i^2}-\frac{x_i}{c_2}\frac{\partial}{\partial x_i}\right)\Phi_{\xi_1\cdots\xi_n}F(T). \tag{30}$$

$\Delta_c^{(R)}$ is defined consistently by (30), independent from the choice of a C.O.N.S. $\{\xi_k\}$ provided that its first ν members form a C.O.N.S. of R.

Now, we shall consider the fixed C.O.N.S. $\{\xi_k\}$ of L. We shall denote $\Delta_c^{(\nu)}$ instead of $\Delta_c^{(R)}$, when the subspace R is spanned by $\xi_1, \cdots, \xi_\nu$. Then, it is evident that

$$(31) \qquad \Delta_c^{(\nu)} F_{l_1 l_2 \cdots l_k \cdots}(T) = -\frac{l_{\nu+1}+l_{\nu+2}+\cdots}{c^2} F_{l_1 l_2 \cdots l_k \cdots}(T).$$

Therefore, $F_{l_1 l_2 \cdots l_k \cdots}(T)$ is a simultaneous eigen function of the family of operators $\Delta_c^{(\nu)}$, $\nu=0, 1, 2, \cdots$. (We put $\Delta_c^{(0)} \equiv \Delta_c$). Furthermore, each simultaneous eigen value is simple.

Namely, for any decreasing sequence $\{\lambda_\nu\}$ of non-negative integers which tend to zero;

$$(32) \qquad \lambda_0 \geqq \lambda_1 \geqq \lambda_2 \geqq \cdots \to 0, \quad \text{each } \lambda_\nu \text{ is an integer,}$$

there exists the *unique* simultaneous eigen function of $\Delta_c^{(\nu)}$ whose eigen value is $-\lambda_\nu/c^2$. It is given by $F_{l_1 l_2 \cdots l_k \cdots}(T)$ where $\lambda_\nu = l_{\nu+1}+l_{\nu+2}+\cdots$, or $l_k = \lambda_{k-1} - \lambda_k$.

Next, we shall consider the partial Laplacians on spheres. Usually, they are expressed in terms of polar coordinates, but here we shall adopt the coordinates used in §4.

Namely, we regard $x_1, \cdots, x_n$ as independent variables on Ω_n^+, and $x_{n+1} = \sqrt{(n+1)c^2 - x_1^2 - \cdots - x_n^2}$ as a function of them. Then, using this coordinates the partial Laplacians on the hemisphere Ω_n^+ is defined by:

$$(33) \qquad \overline{\Delta}_n^{(\nu)} = ((n+1)c^2 - x_1^2 - \cdots - x_\nu^2) \sum_{i=\nu+1}^{n} \frac{\partial^2}{\partial x_i^2} - \sum_{i,j=\nu+1}^{n} x_i x_j \frac{\partial^2}{\partial x_i \partial x_j} - (n-\nu) \sum_{i=\nu+1}^{n} x_i \frac{\partial}{\partial x_i}.$$

Remark that $\overline{\Delta}_n^{(0)} = \overline{\Delta}_n$.

It is well known that the simultaneous eigen function of $\overline{\Delta}_n^{(\nu)}$ $(\nu=0, 1, \cdots, n-1)$ is given by a product of Gegenbauer polynomials. Regarding $x_1, \cdots, x_n$ as independent variables on Ω_n^+, it is written as follows;

(34) $$Y_{n,l_1l_2\cdots l_n}(x_1,\cdots,x_n)\equiv\prod_{k=1}^{n}\{(n+1)c^2-x_1^2-\cdots-x_{k-1}^2\}^{l_k/2}$$
$$\times C_{l_k}^{\frac{1}{2}(n-k)+\lambda_k}\left(\frac{x_k}{\sqrt{(n+1)c^2-x_1^2-\cdots-x_{k-1}^2}}\right),$$
where $\lambda_k=l_{k+1}+\cdots+l_n$.

This function satisfies;

(35) $$\overline{\Delta}_n^{(\nu)}Y_{n,l_1l_2\cdots l_n}=-\lambda_\nu(\lambda_\nu+n-\nu-1)\,Y_{n,l_1l_2\cdots l_n}\,.$$

The set $\{Y_{n,l_1l_2\cdots l_n};\ l_k=0,1,2,\cdots\}$ forms a C.O.S. of $L^2(\Omega_n^+, 2P_n)$.

In the same way as in §4, the operators $\overline{\overline{\Delta}}_n^{(\nu)}$ are defined on $L^2(R^n,\mu_{c,n})$. $\overline{\overline{\Delta}}_n^{(\nu)}$ is also given by the right hand side of (33), so that regarding $Y_{n,l_1l_2\cdots l_n}$ as a polynomial on R^n, it is a simultaneous eigen function of $\overline{\overline{\Delta}}_n^{(\nu)}$. The operators $\widetilde{\Delta}_n^{(\nu)}$ are also defined on $L^2(L^*,\mu_c)$, and their simultaneous eigen function $F_{n,l_1l_2\cdots l_n}(T)$ is obtained by substituting $x_i=\langle T,\xi_i\rangle$ in the right hand side of (34).

It is easily seen that on the space $\mathfrak{A}_\infty$ defined in §4, the operator $\widetilde{\Delta}_n^{(\nu)}/(n+1)c^2$ tends strongly to $\Delta_c^{(\nu)}$ as $n\to\infty$. Corresponding to this fact, the former's eigen value $-\dfrac{\lambda_\nu(\lambda_\nu+n-\nu-1)}{(n+1)c^2}$ tends to the latter's eigen value $-\lambda_\nu/c^2$.

The space $\mathfrak{P}_{m,l}$, the whole of polynomials of m variables with degrees at most l, becomes a vector space. Its dimension is $d_{m,l}=\sum_{k=0}^{l}\binom{k+m-1}{k}$.

Regarding $\mathfrak{P}_{m,l}$ as a subspace of $\mathfrak{A}_m$, we see that on $\mathfrak{P}_{m,l}$ the set of the partial Laplacians $\Delta_c^{(0)},\Delta_c^{(1)},\cdots,\Delta_c^{(m-1)}$ has $d_{m,l}$-different sets of eigen values $\{-\lambda_\nu/c^2;\ 0\leq\nu\leq m-1\}$;

$$l\geqq\lambda_0\geqq\lambda_1\geqq\cdots\geqq\lambda_{m-1}\geqq 0.$$

So that, generalizing Lemma 3 in §5 to the case of simultaneous eigen values, we can show that a suitable spherical function tends to $F_{l_1l_2\cdots l_k\cdots}(T)$.

For $n\geqq m>\nu$, the operator $\overline{\overline{\Delta}}_n^{(\nu)}$ is defined on $\mathfrak{P}_{m,l}$, and in the limit of $n\to\infty$ it tends strongly to $\Delta_c^{(\nu)}$. Therefore, we get the following proposition. The ratio of the normalization constants is obtained by comparing the coefficients of $\prod_{k=1}^{\infty} x_k^{l_k}$.

Proposition 5.

$$\lim_{n\to\infty}\left[l_1!\,l_2!\cdots l_n!\frac{2^{l/2}}{(nc)^l}Y_{n,l_1 l_2\cdots l_n}(x_1,\cdots,x_n)\right] = \prod_{k=1}^{\infty} H_{l_k}\left(\frac{x_k}{\sqrt{2}\,c}\right), \tag{36}$$

where $l=l_1+l_2+\cdots+l_n+\cdots<\infty$.

Though this formula is obtained directly from Prop. 4, we have its new interpretation; the simultaneous eigen function of the partial Laplacians on the n-dimensional sphere tends to the infinite dimensional one.

Remark In general, for a sequence of C.O.N.S. $\{\xi_k^{(n)}\}$, even if $\lim_{n\to\infty}\{\xi_k^{(n)}\}=\xi_k$ strongly for any k, the limit O.N.S. $\{\xi_k\}$ is not always complete.

But in our case, a finite dimensional subspace $\mathfrak{P}_{m,l}$ is invariant both under $\overline{\overline{\Delta}}_n^{(\nu)}$ and $\Delta_c^{(\nu)}$, and $\bigcup_{m,l=1}^{\infty}\mathfrak{P}_{m,l}$ is dense in $L^2(L^*,\mu_c)$. Therefore, from the completeness of the system $\{Y_{n,l_1 l_2\cdots l_n}\}$ in $L^2(\Omega_n^+, 2P_n)$, we can conclude that the system $\left\{\prod_{k=1}^{\infty} H_{l_k}\left(\frac{\langle T,\xi_k\rangle}{\sqrt{2}\,c}\right);\ l_1+l_2+\cdots<\infty\right\}$ is complete in $L^2(L^*,\mu_c)$.

§7. Integral representation formula

For Gegenbauer polynomial, we have the following formula. (c.f. [2] page 177)

$$C_l^{n/2}(u)=\frac{\Gamma(n+l)}{2^{n-1}l!\,[\Gamma(n/2)]^2}\int_{-1}^{1}(u-it\sqrt{1-u^2})^l(1-t^2)^{\frac{n-2}{2}}dt \tag{37}$$

We can obtain this formula in a natural way, if we express the right side as an integral on the sphere Ω_n.

It is evident that if $a_0^2=a_1^2+\cdots+a_n^2$, the polynomial $f_{l,a_0\cdots a_n}(x_1,\cdots,x_{n+1})\equiv(a_0x_1-ia_1x_2-\cdots-ia_nx_{n+1})^l$ is homogeneous and harmonic on R^{n+1}. Thus, regarding it as a function on Ω_n, it is a spherical harmonics with degree l. Namely, it satisfies

$$\bar{\Delta}_n f_{l,a_0\cdots a_n} = -l(l+n-1) f_{l,a_0\cdots a_n}.$$

This is also true for any linear combination of $f_{l,a_0\cdots a_n}$.

Let $\omega'=(x_1', \cdots, x_n')$ be a point on Ω_{n-1}, then being $x_1'^2+\cdots+x_n'^2=nc^2$, the polynomial of $x_1, \cdots, x_n$ defined as

$$(38) \qquad \int_{\Omega_{n-1}} (\sqrt{n}\, c x_1 - i x_1' x_2 - \cdots - i x_n' x_{n+1})^l \, dm(\omega')$$

is a spherical function on Ω_n, where $m(\omega')$ is a measure on Ω_{n-1}.

Especially if we consider the uniform measure $P_{n-1}(\omega')$ on Ω_{n-1}, the integral (38) depends only on x_1 because of rotationally invariance of P_{n-1}, therefore the obtained polynomial becomes zonal. This must be a constant multiple of $C_l^{\frac{n-1}{2}}\left(\frac{x_1}{\sqrt{n+1}\, c}\right)$.

$$(39) \qquad C_l^{\frac{n-1}{2}}\left(\frac{x_1}{\sqrt{n+1}\, c}\right) \propto \int_{\Omega_{n-1}} (\sqrt{n}\, c x_1 - i x_1' x_2 - \cdots - i x_n' x_{n+1})^l \, dP_{n-1}(\omega')$$

$$\propto \int_{-\sqrt{n}\, c}^{\sqrt{n}\, c} (\sqrt{n}\, c x_1 - i t \sqrt{(n+1)c^2 - x_1^2})^l \left(1-\frac{t^2}{nc^2}\right)^{\frac{n-3}{2}} dt$$

Putting $x_1/\sqrt{n+1}\, c = u$ and replacing $n-1$ by n, we have

$$C_l^{n/2}(u) \propto \int_{-1}^{1} (u - i t \sqrt{1-u^2})^l (1-t^2)^{\frac{n-2}{2}} dt.$$

From this, comparing the value at $u=1$, we get (37).

Thus the integral representation formula (37) takes a more natural form by rewriting it into;

$$(39)' \qquad C_l^{\frac{n-1}{2}}\left(\frac{x_1}{\sqrt{n+1}\, c}\right) = \frac{\sqrt{\pi}\,\Gamma(n+l-1)}{n^{l/2}(n+1)^{l/2} c^{2l} 2^{n-2} l!\, \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}$$

$$\times \int_{\Omega_{n-1}} (\sqrt{n}\, c x_1 - i x_1' \sqrt{(n+1)c^2 - x_1^2})^l \, dP_{n-1}(\omega')$$

The right hand side is also written as an integral on Ω as follows:

$$(39)'' \qquad \frac{\sqrt{\pi}\,\Gamma(n+l-1)}{(n+1)^{l/2} c^l 2^{n-2} l!\, \Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-1}{2}\right)}$$

$$\times \int_{\Omega} \left(x_1 - i \frac{X_1^{(n+1)}(\omega)}{\sqrt{n}\, c} \sqrt{(n+1)c^2 - x_1^2}\right)^l dP(\omega).$$

Letting $n \to \infty$ in both sides of (39)′ we shall obtain the integral representation formula of Hermite polynomial. As we have shown in §5,

$$\frac{l!2^{l/2}}{n^{l/2}} C_l^{\frac{n-1}{2}}\left(\frac{x_1}{\sqrt{n+1}\,c}\right) \text{ tends to } H_l\left(\frac{x_1}{\sqrt{2}\,c}\right).$$

On the other hand, the integrand of (39)″ converges to $(x_1 - iX_1(\omega))^l$ in $L^2(\Omega, P)$. This comes from the fact that $(X_1^{(n)}(\omega))^k$ converges to $(X_1(\omega))^k$ in $L^2(\Omega, P)$ for any k. The latter is proved using a generalization of Lemma 1 in §3.

Lemma 1′.

$$\int_\Omega (X_1^{(n)}(\omega))^j (X_1^{(m)}(\omega))^k dP(\omega)$$

$$= c^{j+k}(n+1)^{j/2}(m+1)^{k/2} \frac{\Gamma\left(\frac{j+k+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{j+n+1}{2}\right)} \frac{\Gamma\left(\frac{j+m+1}{2}\right)}{\Gamma\left(\frac{j+k+m+1}{2}\right)}$$

for $m \leq n$ and $j+k =$ even. In the limit of $n, m \to \infty$, the right hand side tends to $1\cdot 3\cdot 5\cdots(j+k-1)c^{j+k}$.

Proof of Lemma 1′ is omitted.

By the isomorphism between $L^2(\Omega, P)$ and $L^2(L^*, \mu_c)$, the function $X_1(\omega)$ corresponds to $\langle T, \xi_1 \rangle$ therefore from (39)″ we have

$$H_l\left(\frac{x_1}{\sqrt{2}c}\right) = \frac{2^{l/2}}{c^l}\int_{L^*}(x_1 - i\langle T, \xi_1\rangle)^l d\mu_c(T). \tag{40}$$

We obtain so called Gauss transform rewriting (40) into the form;

$$\begin{aligned} H_l(u) &= \frac{2^{l/2}}{\sqrt{2\pi}\, c^{l+1}}\int_{-\infty}^{\infty}(\sqrt{2}\,cu - it)^l \exp\left(-\frac{t^2}{2c^2}\right)dt \\ &= \frac{2^l}{\sqrt{\pi}}\int_{-\infty}^{\infty}(u-it)^l \exp(-t^2)\,dt. \end{aligned}$$

So far we have proved;

Proposition 6. *The density of the integral representation of the zonal spherical function on n-dimensional sphere tends to that of Hermite Polynomial in the sense of $L^2(L^*, \mu_c)$.*

Remark Since a Gaussian measure is rotationally invariant, for any $\xi\in L$ such that $\|\xi\|=1$, we have;

$$(40)' \qquad H_l\left(\frac{x_1}{\sqrt{2}\,c}\right)=\frac{2^{l/2}}{c^l}\int_{L^*}(x_1-i\langle T,\xi\rangle)^l\,d\mu_c(T).$$

In other words

$$(40)'' \qquad H_l\left(\frac{\langle T,\xi\rangle}{\sqrt{2}\,c}\right)=\frac{2^{l/2}}{c^l}\int_{L^*}\langle T-iS,\xi\rangle^l\,d\mu_c(S).$$

§8. Addition formula

From (40)″, if we put $\xi=\xi_1\cos\theta+\xi_2\sin\theta$, we get

$$H_l\left(\frac{\langle T,\xi_1\rangle}{\sqrt{2}\,c}\cos\theta+\frac{\langle T,\xi_2\rangle}{\sqrt{2}\,c}\sin\theta\right)$$
$$=\frac{2^{l/2}}{c^l}\int_{L^*}(\cos\theta\langle T-iS,\xi_1\rangle+\sin\theta\langle T-iS,\xi_2\rangle)^l\,d\mu_c(S).$$

The integrand is written as

$$\sum_{k=0}^{l}\binom{l}{k}\cos^k\theta\,\sin^{l-k}\theta\langle T-iS,\xi_1\rangle^k\langle T-iS,\xi_2\rangle^{l-k}$$

Moreover the distributions of $\langle S,\xi_1\rangle$ and $\langle S,\xi_2\rangle$ are mutually independent. Thus after the integration we get

$$(41) \qquad H_l(x_1\cos\theta+x_2\sin\theta)=\sum_{k=0}^{l}\binom{l}{k}\cos^k\theta\,\sin^{l-k}\theta H_k(x_1)H_{l-k}(x_2).$$

This is the addition formula of Hermite polynomial. As we have seen here, it is obtained very easily if we regard Hermite polynomial as an *infinite dimensional* spherical function.

Putting

$$H_l(u,\sigma)=\frac{\sigma^{l/2}}{l!\,2^{l/2}}H_l\left(\frac{u}{\sqrt{2\sigma}}\right),$$

(41) is written as

$$H_l(u+v,\sigma+\tau)=\sum_{k=0}^{l}H_k(u,\sigma)H_{l-k}(v,\tau).$$

This is the formula of S. Kakutani [4] which plays an essential role in his paper.

Similarly, from the relation;

$$\prod_{k=1}^{m} H_{l_k}\left(\frac{\langle T, \xi_k\rangle}{\sqrt{2}\,c}\right)=\frac{2^{l/2}}{c^l}\int_{L^*}\prod_{k=1}^{m}\langle T-iS, \xi_k\rangle^{l_k}d\mu_c(S),$$

we shall have an addition formula of a product of Hermite polynomials. Since it is too complicated, we present the case $m=2$:

$$\begin{aligned}(42)\qquad & H_l(x_1\cos\theta+x_2\sin\theta)H_k(-x_1\sin\theta+x_2\cos\theta)\\ &=\sum_{j=0}^{l+k}\alpha_j H_j(x_1)H_{l+k-j}(x_2),\end{aligned}$$

where

$$\alpha_j=\sum_{i=\mathrm{Max}(0,\,j-k)}^{\mathrm{Min}(j,\,l)}\binom{l}{i}\binom{k}{j-i}(-1)^{j-i}\cos^{k-j+2i}\theta\,\sin^{j+l-2i}\theta.$$

Remark that this is not obtained directly from (41).

For Gegenbauer polynomial, we can not derive the addition formula directly from its integral representation. However, it is seen that the addition formula of Gegenbauer polynomial tends *term by term* to that of Hermite polynomial. The meaning of this fact is well understood if we regard the addition formula as a transformation of an orthogonal base in $L^2(L^*, \mu_c)$.

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