On the uniqueness in Cauchy's problem for elliptic equations

By

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1. It is well known that the uniqueness in Cauchy problem holds for the second order elliptic equation ([1], [4], etc.). On the other hand Mergelyan [9] showed in 1956 that the harmonic function vanishes identically if the Cauthy data tends rapidly to zero at a point along the initial surface. This fact has been extended immediately for the second order elliptic epuation by Landis [7] and Lavrentév [8]. In this note we shall try to extend their results to more general elliptic equations.

Let Γ be a smooth Cauchy initial surface in a N dimensional space R^N and Ω be a domain such that the boundary of Ω contains Γ. And we consider an elliptic operator L of order m defined in $\bar{\Omega}$. Then we shall say that the strong Cauchy uniqueness property holds for L and Γ, if any solution u satisfying two following conditions vanishes identically in Ω:

(i) $u \in C^m(\Omega)$ and $Lu = 0$ in Ω.

(ii) At a point P_0 in the interior of Γ, the Cauchy data of u tends to zero along Γ in such a way that

$$(1.1) \qquad \frac{\partial^{|\alpha|} u(P)}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = o(\exp(-|P - P_0|^{-\delta}))$$

$$(P \to P_0, P \in \Gamma) \quad |\alpha| \leq m - 1,$$

where δ is a positive number depending only on L and Γ.

First we consider the case where L is of second order $(m = 1)$

$$(1.2) \qquad L = a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + b_i \frac{\partial}{\partial x_i} + c,$$

in which $a_{ij} \in C^2(\bar{\Omega})$ and b_i, $c \in C^1(\bar{\Omega})$. In this case Landis and Lavrentév proved our strong Cauchy uniqueness property. In particular, the latter showed that the exponent δ in (1.1) can be taken as an arbitrary number larger than 2, if Γ is a portion of a spherical surface. Their method could be said to give an explicit estimate expressing a relation between the solution and the Cauchy data. This relation is called "well behaved" in the sense of F. John [6].

Now does the strong Cauchy uniqueness property hold for higher order elliptic equations? We can show this property if the dimension $N=2$ and if the elliptic operator has distinct characteristic roots. This is an extension of a result of Carleman [4]. In particular, for the second order elliptic operator in (1.2) the regularity assumptions on the coefficients can be relaxed in such a way

$$a_{ij} \in C^1(\bar{\Omega}), \ b_i \quad \text{and} \quad c \in L^\infty(\bar{\Omega}),$$

where the dimension $N=2$ and $L^\infty(\bar{\Omega})$ means a class of bounded measurable functions in Ω. The method used in this note consists in establishing an energy integral estimates containing an adequate weight function.

From now on we consider in the 2 dimensional space with the coordinate (x, y) and with the metric norm $r(=(x^2+y^2)^{1/2})$. Let Γ be a smooth curve containing the origin in its interior and let Ω be a domain whose boundary contains Γ. First we treat the solutions $u_p(p=1, \cdots, m)$ of the elliptic system

$$(1.3) \qquad \frac{\partial u_p}{\partial x} + \sum_{q=1}^m a_{p,q}(x, y) \frac{\partial u_q}{\partial y} + \sum_{q=1}^m b_{p,q}(x, y) u_q = 0,$$

where $a_{p,q} \in C^1(\bar{\Omega})$, $b_{p,q} \in L^\infty(\bar{\Omega})$ and $u_p \in C^1(\bar{\Omega})$. We assume that all characteristic roots of λ_p of the determinant

$$(1.4) \qquad\qquad |a_{p,q} - \lambda \delta_{p,q}| = 0$$

are distinct in $\bar{\Omega}$ and all imaginary parts of λ_p are not zero in $\bar{\Omega}$. This system was considered by Carleman [4].

Theorem 1. *There is a positive number δ such that if the solu-*

tions u_p of (1.3) *satisfies*

$$u_p = o(\exp(-r^{-2\delta})) \quad (r \to 0) \quad p = 1, \cdots, m$$

along Γ, then $u \equiv 0$ in Ω, where δ depends only on $\{a_{p,q}\}$ and Γ.

Next we consider the single elliptic equation of order m

$$(1.5) \qquad Lu \equiv \sum_{i+j \leq m} a_{ij} \frac{\partial^{i+j} u}{\partial x^i \partial y^j} = 0.$$

In the same manner we assume that the characteristic roots of L are distinct in $\bar{\Omega}$ and $a_{ij}(i+j=m)$ are in $C^1(\bar{\Omega})$, and the other coefficients are in $L^\infty(\bar{\Omega})$.

Theorem 2. *Let u be a solution of* (1.5) *in Ω and let u be in $C^m(\bar{\Omega})$. Then there is a positive number δ depending only on L and Γ. And if u satisfies*

$$\frac{\partial^{i+j} u}{\partial x^i \partial y^j} = o(\exp(-r^{-2\delta})) \quad (r \to 0) \quad i+j \leq m-1$$

along Γ, then $u \equiv 0$ in Ω. In particular when $m=2$ and the coefficients of the principal part of L are real, the number δ can be taken as an arbitrary number larger than 1 under the assumption that Γ is a circular arc.

The proof of these theorems will be given in the last section. The author wishes to express his heartly thanks to Prof. S. Mizohata for his kind suggestions and encouragement.

2. First we prepare a mean value property for solutions of the elliptic equation. We consider in $r \leq R_0 (<1)$ the first order elliptic equation

$$(2.1) \qquad Lu \equiv u_x + \lambda(x,y) u_y = f(x,y),$$

where u, $\lambda \in C^1$ in $r \leq R_0$ and the imaginary part of λ is not zero in $r \leq R_0$. We put $M(R) = \max_{r \leq R}(1, |\lambda|, |\lambda_x|, |\lambda_y|)$ and $m(R) = \max_{r \leq R}(1, |u|, |f|)$. And we denote the origin simply by 0. Then we have

Lemma 1. *When $\lambda(0) = i$ and $1 < p < 2$, the solution of* (2.1) *satisfies*

(2.2) $|u(0)| \leq CR^{(2/p)-2} m(R)^{(2/p)-1} M(R)$

$$\left\{ \left(\iint_{r \leq R} |u|^2 dxdy \right)^{1-(1/p)} + \left(\iint_{r \leq R} |f|^2 dxdy \right)^{1-(1/p)} \right\},$$

where C is a constant independent of u, λ and R.

Proof. From now on we denote by $C_1, C_2, \cdots$ the constants independent of u, λ and R.

We take the C^∞ function $\phi(r)$:

$$\phi(r) = \begin{cases} 1 & in \quad r \leq R/2 \\ 0 & in \quad r > R \end{cases} \quad and \quad |\phi_x|, \ |\phi_y| \leq C_1 R^{-1}.$$

Set $v = \phi u$. Then we see

(2.3) $Lv = L\phi \cdot u + \phi f$

and

(2.4) $v_x + iv_y = Lv + (\lambda(0) - \lambda) v_y.$

Since v has compact carrier, we have from (2.4)

(2.5) $u(0) = -\dfrac{1}{2\pi} \iint_{r \leq R} \dfrac{Lv + (\lambda(0) - \lambda) v_y}{x + iy} dxdy.$

And we see by (2.3)

(2.6) $\left| \iint_{r \leq R} \dfrac{Lv}{x + iy} dxdy \right| \leq C_2 \left(R^{-1} M(R) \iint_{r \leq R} \dfrac{|u|}{r} dxdy \right.$

$$\left. + \iint_{r \leq R} \dfrac{|f|}{r} dxdy \right).$$

By Green's formula we get

$$\iint_{r \leq R} \dfrac{\lambda(0) - \lambda}{x + iy} v_y dxdy = \iint_{r \leq R} \dfrac{\lambda_y v}{x + iy} dxdy$$

$$+ \iint_{r \leq R} \dfrac{i(\lambda(0) - \lambda)}{(x + iy)^2} v dxdy.$$

Hence we have

(2.7) $\left| \iint_{r \leq R} \dfrac{\lambda(0) - \lambda}{x + iy} v_y dxdy \right| \leq C_3 M(R) \iint_{r \leq R} \dfrac{|u|}{r} dxdy.$

Combining (2.5), (2.6) and (2.7), we obtain

(2.8) $|u(0)| \leqq C_4 R^{-1} M(R) \iint_{r \leqq R} \frac{1}{r} (|u| + |f|) dxdy.$

Applying Hölder's inequality we see

$$\iint_{r \leqq R} \frac{1}{r} \frac{|u|}{m(R)} dxdy$$

$$\leqq \left(\iint_{r \leqq R} r^{-p} dxdy \right)^{1/p} \left(\iint_{r \leqq R} \left(\frac{|u|}{m(R)} \right)^q dxdy \right)^{1/q},$$

where $\frac{1}{p} + \frac{1}{q} = 1$. Since $1 < p < 2$, $2 < q$ and $|u| \leqq m(R)$, it holds

$$\iint_{r \leqq R} \frac{1}{r} \frac{|u|}{m(R)} dxdy \leqq \frac{1}{2-p} R^{2-p/p} \left(\iint_{r \leqq R} \left(\frac{|u|}{m(R)} \right)^2 dxdy \right)^{1/q}.$$

That is,

(2.9) $\iint_{r \leqq R} \frac{|u|}{r} dxdy \leqq \frac{1}{2-p} R^{(2/p)-1} m(R)^{(2/p)-1}$

$$\times \left(\iint_{r \leqq R} |u|^2 dxdy \right)^{1-(1/p)}.$$

Similarly we have

(2.10) $\iint_{r \leqq R} \frac{|f|}{r} dxdy \leqq \frac{1}{2-p} R^{(2/p)-1} m(R)^{(2/p)-1}$

$$\times \left(\iint_{r \leqq R} |f|^2 dxdy \right)^{1-(1/p)}.$$

Thus combining (2.8), (2.9) and (2.10), we have obtained the inequality (2.2).

Now we want to eliminate the assumption $\lambda(0) = i$ from Lemma 1. We put $\lambda = \lambda_1 + i\lambda_2$. And we transform the coordinate by

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\lambda_1(0)/\lambda_2(0) & 1/\lambda_2(0) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Then the operator L in (2.1) is transformed into

(2.11) $L'u \equiv u_{x'} + (-\lambda_1(0) + \lambda_1(x,y) + i\lambda_2(x,y)) u_{y'}/\lambda_2(0).$

If we denote the new metric norm by $\tilde{r}$, we see easily the following metric equivalent relation

(2.12) $B^{-1}r \leqq \tilde{r} \leqq Br,$

where B is a positive constant depending only on $\lambda_1(0)$, $\lambda_2(0)$.

Applying Lemma 1 to the operator L' in (2.11) we have

(2.13) $|u(0)| \leqq CR^{(2/p)-2} \widetilde{m}(R)^{(2/p)-1} \widetilde{M}(R) \left\{ \left(\iint_{\tilde{r} \leqq R} |u|^2 dx' dy' \right)^{1-(1/p)} \right.$

$$+ \left. \left(\iint_{\tilde{r} \leqq R} |f|^2 dx' dy' \right)^{1-(1/p)} \right\},$$

where $\widetilde{M}(R) = \max_{\tilde{r} \leqq R}(1, |\lambda|, |\lambda_{x'}|, |\lambda_{y'}|)$ and $\widetilde{m}(R) = \max_{\tilde{r} \leqq R}(1, |u|, |f|)$. We

note that $\lambda_{y'} = \lambda_2(0)\lambda_y$ and $\lambda_{x'} = \lambda_x + \lambda_1(0)\lambda_y$. Then using (2.12), we get

$$\widetilde{M}(R) \leqq M(BR) \max(1 + |\lambda_1(0)|, |\lambda_2(0)|).$$

Similarly using the relation (2.12) for (2.13), we obtain

Proposition 1. *For the solution of* (2.1) *it holds that if* $1 < p < 2$,

(2.14) $|u(0)| \leqq kR^{(2/p)-2} \widetilde{m}(BR)^{(2/p)-1} M(BR) \left\{ \left(\iint_{r \leqq BR} |u|^2 dx dy \right)^{1-(1/p)} \right.$

$$+ \left. \left(\iint_{r \leqq Br} |f|^2 dx dy \right)^{1-(1/p)} \right\},$$

where $k = C\lambda_2(0)^{(1/p)-1} \max(1 + |\lambda_1(0)|, |\lambda_2(0)|)$ *and* C *is a constant independent of* u, λ, R.

3. In this section we see how the behavior of the solution of (1.3) (or (1.5)) is influenced by the Cauchy data. Let us denote by S_d an open disc with the center $(d/2, 0)$ and with the radius $d/2$. We put $\Omega_h = \{0 < x < h\} \cap S_1$, $\Gamma_h = \{0 \leqq x \leqq h\} \cap \partial S_1$ and $l_h = \{x = h\} \cap S_1$.

Lemma 2. *If* $u \in C^1(\bar{\Omega}_a)$ *and* $u = o(\exp(-r^{-2\delta-\varepsilon}))$ $(r \to 0)$ *along* Γ_a *for some positive numbers* δ, ε, *then there is a function* v *such that*

(i) $v \in C^0(\bar{\Omega}_a)$ *and* $v = u$ *on* Γ_a

(ii) $v \in C^1(\bar{\Omega}_a - \{0\})$ *and*

$$\int_{l_h} |v|^2 dy, \int_{l_h} |v_y|^2 dy \quad \text{and} \quad \int_{l_h} |v_x|^2 dy$$

$$= o(\exp(-h^{-\delta})) \quad (h \to 0).$$

Proof. We take a C^∞ function $\varphi(x)$ such that

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1, \quad \text{and the carrier of } \varphi \subset \{|x| \leqq 1\}.$$

If we set

$$f_s(x) = \int_{-2s}^{2s} \varphi((x-y)/s)\frac{1}{s}\,dy,$$

we see that

(3.1) $$f_s(x) \in C^\infty(-\infty, \infty),$$

(3.2) $$f_s(x) = 1 \quad \text{for} \quad |x| \leqq s \quad \text{and} \quad f_s(x) = 0 \quad \text{for} \quad |x| \geqq 3s,$$

(3.3) $$|f_s(x)| \leqq 1 \quad \text{and} \quad |df_s(x)/dx| \leqq c/s \quad (c \text{ is independent of } s).$$

We put $k(x) = \exp(-x^{-\delta-(\varepsilon/2)})$ and write a curve Γ' such that

$$\Gamma': \quad \begin{aligned} x^2 + (y+k(x))^2 &= x \quad \text{for} \quad y > 0, \\ x^2 + (y-k(x))^2 &= x \quad \text{for} \quad y < 0. \end{aligned}$$

The curve Γ' is smooth and tangent to S_1 from the interior. We define a function $v(x,y)$ by

$$v(x,y) = \begin{cases} u(x, \sqrt{x-x^2})f_{k(x)/3}(y - \sqrt{x-x^2}) & \text{for} \quad y > 0, \\ u(x, -\sqrt{x-x^2})f_{k(x)/3}(y + \sqrt{x+x^2}) & \text{for} \quad y < 0. \end{cases}$$

We see from (3.1), (3.2) that this function satisfies (i).

We have from (3.3)

$$|v_y| \leqq \begin{cases} c\,|u(x, \sqrt{x-x^2})|/k(x), \\ c\,|u(x, -\sqrt{x-x^2})|/k(x). \end{cases}$$

Let us note that

$$u = o(\exp(-h^{-\delta-(\varepsilon/2)})) \quad (h \to 0) \quad \text{on} \quad \Gamma_a.$$

Hence $u(x, \sqrt{x-x^2})/k(x) = o(1) \ (x \to 0)$. Thus we get

(3.4) $$\int_{l_h} |v^2|\,dy, \int_{l_h} |v_y|^2\,dy = o(\exp(-h^{-\delta})) \quad (h \to 0).$$

A computation shows for $y > 0$

$$v_x = \left\{ u_x(x, \sqrt{x-x^2}) + \frac{1}{2} u_y(x, \sqrt{x-x^2})\frac{1-2x}{\sqrt{x-x^2}} \right\}.$$

$$f_{k(x)/3}(y-\sqrt{x-x^2})$$

$$+\frac{2k'(x)}{k(x)}u(x,\sqrt{x-x^2})\left\{\varphi\left(\frac{3}{k(x)}(y-\sqrt{x-x^2}-\frac{2}{3}k(x))\right)\right.$$

$$\left.+\varphi\left(\frac{3}{k(x)}(y-\sqrt{x-x^2}+\frac{2}{3}k(x))\right)\right\}$$

$$+u(x,\sqrt{x-x^2})\int_{-\frac{2}{3}k(x)}^{\frac{2}{3}k(x)}\left\{\varphi\left((y-\sqrt{x-x^2}-z)\frac{3}{k(x)}\right)\frac{3}{k(x)}\right\}_x dz.$$

while,

$$\left\{\varphi\left(\frac{3}{k(x)}(y-\sqrt{x-x^2}-z)\right)\frac{3}{k(x)}\right\}_x$$

$$=-\frac{3k'(x)}{k(x)^2}\varphi\left(\frac{3}{k(x)}(y-\sqrt{x-x^2}-z)\right)$$

$$-\frac{9}{k(x)^3}\left\{\frac{k(x)}{2}\frac{1-2x}{\sqrt{x-x^2}}+k'(x)(y-\sqrt{x-x^2}-z)\right\}$$

$$\times\varphi'\left((y-\sqrt{x-x^2}-z)\frac{3}{k(x)}\right).$$

Hence we have for $y>0$

$$|v_x|\leq \text{const.}\left\{|u_x(x,\sqrt{x-x^2})|+\frac{1}{\sqrt{x}}|u_y(x,\sqrt{x-x^2})|\right.$$

$$\left.+|u(x,\sqrt{x-x^2})|\left(\frac{|k'(x)|}{k(x)}+\frac{1}{\sqrt{x}}\frac{1}{k(x)}+\frac{|k'(x)|}{k(x)^2}\right)\right).$$

Similarly we have the same inequality for $y<0$. Thus we obtain

$$(3.5) \qquad \int_{l_h}|v_x|^2 dy = o(\exp(-h^{-\delta})) \quad (h\to 0).$$

Combining (3.4) and (3.5), we have completed the proof.

Now we consider in $\overline{\Omega}_a$ the m-th order elliptic equation (1.5). We can assume that the coefficient $a_{m,0}$ equals identically 1 in $\overline{\Omega}_a$. Let us set

$$v_j = \partial^{m-1}u/\partial x^{j-1}\partial y^{m-j} \quad j=1,\cdots,m,$$

then we see

$$(3.6)\qquad \frac{\partial}{\partial x}\begin{pmatrix}v_1\\ \vdots\\ v_m\end{pmatrix}+\begin{pmatrix}0 & -1 & \cdot & & & 0\\ \vdots & & 0 & \ddots & & \\ & & & & \ddots & -1\\ a_{0,m} & \cdots & & a_{m-2,2} & a_{m-1,1}\end{pmatrix}\frac{\partial}{\partial y}\begin{pmatrix}v_1\\ \vdots\\ v_m\end{pmatrix}=\begin{pmatrix}0\\ \vdots\\ 0\\ \mathcal{B}u\end{pmatrix}$$

where $\mathcal{B}$ is a differential operator of order $m-2$ whose coefficients are in $L^{\infty}(\overline{\Omega}_a)$. We write (3.6) in the following form:

$$v_x + H v_y = g.$$

Set

$$N = \begin{pmatrix} 1 \cdots\cdots 1 \\ \lambda_1 \cdots\cdots \lambda_m \\ \lambda_1^{m-1} \cdots \lambda_m^{m-1} \end{pmatrix}, \quad D = \begin{pmatrix} -\lambda_1 & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & \cdot \; -\lambda_m \end{pmatrix},$$

where λ_j are characteristic roots of (1.5), and $w = N^{-1}v$, then we see easily

(3.7) $$w_x + D w_y = N^{-1}(g - N_x w - H N_y w).$$

From now on we denote $\int_{l_h} |u(h, y)|^2 dy$ by $\|u(h)\|^2$ and $\int_{l_h} u(h, y) \overline{v(h, y)} dy$ by $[u(h), v(h)]$.

Proposition 2. *Let u be a solution of the elliptic equation (1.5) in Ω_a and u be in $C^m(\overline{\Omega}_a)$. If for some $\varepsilon > 0$, $\delta > 1$*

$$\frac{\partial^{i+j} u}{\partial x^i \partial y^j} = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r \to 0, i+j \leq m-1) \quad \text{on} \quad \Gamma_a,$$

then we have[1]

(3.8) $$\int_0^h \left\| \frac{\partial^{i+j} u}{\partial x^i \partial y^j} \right\|^2 dx = o(\exp(-h^{-\delta})) \quad (h \to 0, i+j \leq m-1).$$

Proof. We reduce the equation (1.5) into the form (3.7). And we put

$$M_i \equiv \frac{\partial}{\partial x} - \lambda_i(x, y) \frac{\partial}{\partial y}, \quad i = 1, \cdots, m,$$

where $\lambda_i = \lambda_{i1} + i\lambda_{i2}$ and $\lambda_{i2} \neq 0$ in $\overline{\Omega}_a$.

If $\psi(x)$ is a function of one variable and $\psi(x) \in C^1[a, b]$, then it holds

$$\int_a^b |\psi(x)|^2 dx \leq 2(b-a)\left((b-a)\int_a^b |\psi'(x)|^2 dx + |\psi(a)|^2\right),$$

using successively this inequality, we have for the operator $\mathcal{B}$ in (3.6)

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1) When all characteristic roots of (1.5) are real in $\overline{\Omega}_a$ (hyperbolic), this proposition holds also.

$$\|(\mathcal{B}u)(h)\| \leq \mathrm{const.}\left\{\sum_{j=1}^{m}\|w_j(h)\| + \sum_{i+j\leq m-2}\left|\frac{\partial^{i+j}u}{\partial x^i \partial y^j}(P_h)\right|\right\},$$

where $P_h \in \Gamma_a$ and $P_h = (h, \sqrt{h-h^2})$. Hence we see

(3.9) $$\|M_i w_i(h)\|^2 \leq \mathrm{const.}\left(\sum_{j=1}^{m}\|w_j(h)\|^2 + \exp(-h^{-\delta-(\varepsilon/2)})\right).$$
$$i=1,\cdots,m.$$

From Lemma 2 there are functions $\widetilde{w}_i \in C^0(\overline{\Omega}_a)$ $(i=1,\cdots,m)$ such that

(3.10) $\widetilde{w}_i = w_i$ on Γ_a and $\widetilde{w}_i \in C^1(\overline{\Omega}_a - \{0\})$,

(3.11) $\|\widetilde{w}_i(h)\|^2$, $\|\widetilde{w}_{iy}(h)\|^2$ and $\|\widetilde{w}_{ix}(h)\|^2$
$$= o(\exp(-h^{-\delta-(\varepsilon/3)})) \quad (h\to 0).$$

Now we put $z_i = w_i - \widetilde{w}_i$ and $\varphi_n(x) = \left(x + \dfrac{1}{n}\right)^{-n}$. And let us denote by $c_i(i=1,2,\cdots)$ the positive constants independent of n and h. We have by [3] and [10]

$$\int_\delta^h \varphi_n(x)^2 \left\|\frac{\partial z_i}{\partial x} - \lambda_i \frac{\partial z_i}{\partial y}\right\|^2 dx$$

$$\geq \frac{c_1}{n}\left(1 - c_1\left(h + \frac{1}{n}\right)\right)\int_\delta^h \varphi'_n(x)^2 \|z_i\|^2 dx - I_n(\delta, h)$$

$$\left(if\ \left(h + \frac{1}{n}\right) < c_1^{-1}\right),$$

where $$|I_n(\delta, h)| \leq n\varphi_n(\delta)^2\|z_i(\delta)\|^2 + \varphi_n(\delta)^2 |[z_i(\delta), \lambda_{i2}z_{iy}(\delta)]|$$
$$+ \varphi_n(h)^2 |[z_i(h), \lambda_{i2}z_{iy}(h)]|.$$

We see from (3.11) that $\|z_i(\delta)\| \to 0$, $\|z_{iy}(\delta)\| \to 0$ as $\delta \to 0$. Thus,

(3.12) $$\int_0^h \varphi_n(x)^2 \left\|\frac{\partial z_i}{\partial x} - \lambda_i \frac{\partial z_i}{\partial y}\right\|^2 dx$$

$$\geq \frac{c_1}{n}\left(1 - c_1\left(h + \frac{1}{n}\right)\right)\int_0^h \varphi'_n(x)^2 \|z_i\|^2 dx$$

$$- \varphi_n(h)^2 |[z_i(h), \lambda_{i2}z_{iy}(h)]|.$$

Since $\varphi_n(h)^2 |[z_i(h), \lambda_{i2}z_{iy}(h)]| \leq c_2\left(h + \dfrac{1}{n}\right)^{-2n}$, taking $h + \dfrac{1}{n} < \dfrac{1}{2c_1}$ in

(3.12), we get

$$c_2\left(h+\frac{1}{n}\right)^{-2n}+\int_0^h \varphi_n(x)^2\|M_i z_i\|^2 dx \geqq \frac{c_1}{2n}\int_0^h \varphi'_n(x)^2\|z_i\|^2 dx.$$

We substitute $z_i=w_i-\widetilde{w}_i$ into the inequality. Then we see

$$c_2\left(h+\frac{1}{n}\right)^{-2n}+2\int_0^h \varphi_n(x)^2\|M_i w_i\|^2 dx+2\int_0^h \varphi_n(x)^2\|M_i\widetilde{w}_i\|^2 dx$$

$$+\frac{c_1}{2n}\int_0^h \varphi'_n(x)^2\|\widetilde{w}_i\|^2 dx \geqq \frac{c_1}{4n}\int_0^h \varphi'_n(x)^2\|w_i\|^2 dx.$$

From (3.9) and (3.11) we have

$$c_3 n^{2n+2}\left(\frac{h}{2}+\frac{1}{n}\right)^{2n+2}\exp\left(-h^{-\delta-(\varepsilon/3)}\right)$$

$$+c_4 n^{-1}\left(h+\frac{1}{n}\right)^{-2n}\left(\frac{h}{2}+\frac{1}{n}\right)^{2n+2} \geqq \sum_{i=1}^m \int_0^{h/2}\|w_i\|^2 dx.$$

Let us take $h+\dfrac{1}{n}$ sufficiently small and nh sufficiently large. As an easy computation shows, in order to prove

$$(3.13) \qquad \int_0^{h/2}\|w_i\|^2 dx = o\left(\exp\left(-\left(\frac{h}{2}\right)^{-\delta}\right)\right) \quad (h\to 0,\ i=1,\cdots,m),$$

it is sufficient to show that we can choose n in such a way that

$$n^{2n+2}\leqq\exp\left(\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1}\right),$$

$$(3.14)$$

$$\left(\frac{hn+2}{2hn+2}\right)^{2n}\leqq\exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right) \quad \varepsilon_1,\ \varepsilon'>0,$$

where ε_1 is a given number and ε' will be determined later. The inequalities (3.14) are verified if the following inequalities are satisfied

$$n^{1+\bar{\varepsilon}}\leqq\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1} \qquad \bar{\varepsilon}>0,$$

$$(3.15)$$

$$2n\log\frac{2}{3}\leqq-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}.$$

Here we can take $\bar{\varepsilon}$ arbitrarily small if n is sufficiently large. If we

take $\bar{\varepsilon}$ sufficiently small, there is a positive number ε' such that

$$\frac{1}{2\log\dfrac{3}{2}}\left(\frac{2}{h}\right)^{\delta+\varepsilon'} < \left(\frac{2}{h}\right)^{(\delta+\varepsilon_1)/(1+\varepsilon)}.$$
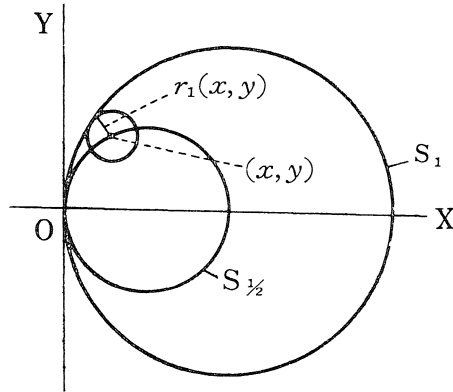
Let us take n in such a way that

$$\frac{1}{2\log\dfrac{3}{2}}\left(\frac{2}{h}\right)^{\delta+\varepsilon'} < n < \left(\frac{2}{h}\right)^{(\delta+\varepsilon_1)/(1+\varepsilon)}.$$

Then (3.15) is satisfied and $nh\to\infty$ as $h\to 0$. Thus we have proved (3.13).

Now we take an open disk $S_{1/2}$ in S_1. For the point (x,y) in $S_{1/2}$ we denote by $r_1(x,y)$ the radius of a circle tangent to S_1 whose center is (x,y). It is easily seen that there is a constant c such that

$$(3.16) \qquad\qquad c^{-1}x \leq r_1(x,y) \leq cx.$$

Proposition 3.   *Let u be a solution of the elliptic equation (1.5) in Ω_a and u be in $C^m(\bar{\Omega}_a)$. Then if for some $\varepsilon > 0$, $\delta > 1$*

$$\frac{\partial^{i+j}u}{\partial x^i \partial y^j} = o(\exp(-r^{-2\delta-\varepsilon})) \quad (r\to 0, i+j\leq m-1) \quad \text{on} \quad \Gamma_a,$$

we have in $S_{1/2}\cap\Omega_a$

$$(3.17) \qquad \frac{\partial^{i+j}u}{\partial x^i\partial y^j} = o(\exp(-r^{-\delta})) \quad (r\to 0, i+j\leq m-1).$$

Proof. We reduce the equation (1.5) into the form (3.7). We set $-2\delta-\varepsilon=-2\left(\delta+\dfrac{\varepsilon}{3}\right)-\dfrac{\varepsilon}{3}$. And we regard $\delta+\dfrac{\varepsilon}{3}$ as new δ. Then by Proposition 2 we have

$$\int_0^h \left\|\frac{\partial^{i+j}u}{\partial x^i \partial y^j}\right\|^2 dx = o(\exp(-h^{-\delta-(\varepsilon/3)}))$$
$$(h\to 0, i+j\leq m-1).$$

This means that

$$(3.18) \qquad \int_0^h \|w_i\|^2 dx = o(\exp(-h^{-\delta-(\varepsilon/3)})) \quad (h\to 0),$$

and for $\mathscr{B}u$ in (3.6)

$$(3.19) \qquad \int_0^h \|\mathscr{B}u\|^2 dx = o(\exp(-h^{-\delta-(\varepsilon/3)})) \quad (h\to 0).$$

We apply Proposition 1 for the equation (3.7) in a disk with center (x,y) and with radius $r_1(x,y)$. In (2.14) we can take B and k as fixed constant in $S_{1/2}$. And from the behavior of u at the origin it is assumed that $\widetilde{m}(BR)\leq 1$. Putting $p=3/2$ in (2.14), we have from (2.14) and (3.7)

$$|w_i(x^{(0)},y^{(0)})|$$
$$\leq CR^{-2/3}\left\{\left(\iint_{\sqrt{(x-x^{(0)})^2+(y-y^{(0)})^2}\leq BR} |w_i|^2 dxdy\right)^{1/3}\right.$$
$$+\left(\iint_{\sqrt{(x-x^{(0)})^2+(y-y^{(0)})^2}\leq BR} \sum_{j=1}^{m} |w_i|^2 dxdy\right.$$
$$\left.\left.+\iint_{\sqrt{(x-x^{(0)})^2+(y-y^{(0)})^2}\leq BR} |\mathscr{B}u|^2 dxdy\right)^{1/3}\right\},$$

where $(x^{(0)},y^{(0)})\in S_{1/2}$, $BR=r_1(x^{(0)},y^{(0)})$ and C is a constant independent of (x,y). Thus we get

$$|w_i(x,y)|\leq C\tilde{r}(x,y)^{-2/3}\left\{\left(\int_0^{x+r_1(x,y)} \|w_i\|^2 dt\right)^{1/3}\right.$$
$$\left.+\left(\int_0^{x+r_1(x,y)}\sum_{j=1}^{m}\|w_j\|^2 dt+\int_0^{x+r_1(x,y)} \|\mathscr{B}u\|^2 dt\right)^{1/3}\right\},$$

combining (3.18) and (3.19), we obtain

$$|w_i(x,y)|\leq \text{const.}(c^{-1}x)^{-2/3}\exp(-(cx+x)^{-\delta-(\varepsilon/3)})$$

$$\leq \text{const.} \exp(-x^{-\delta-(\varepsilon/4)}).$$

Hence we have

(3.20) $\left|\dfrac{\partial^{m-1}u}{\partial x^i \partial y^j}\right| \leq \text{const.} \exp(-r^{-\delta-(\varepsilon/4)}) \quad i+j\leq m-1.$

Let us denote by $l_{(x,y)}$ the line connecting the origin and the point (x,y). Then using the mean value theorem on $l_{(x,y)}$ we see

$$\left|\dfrac{\partial^{m-2}u}{\partial x^i \partial y^j}(x,y)\right| \leq r \sum_{i+j\leq m-1}\left|\dfrac{\partial^{m-1}u}{\partial x^i \partial y^j}(x^{(i)},y^{(j)})\right|,$$

where $(x^{(i)},y^{(j)})\in l_{(x,y)}$. Hence from (3.20) we get for $\partial^{m-2}u/\partial x^i \partial y^j$ the same inequality as in (3.20). Repeating this process we have obtaind (3.17).

We put for $d>0$ $\Omega_{h,d}=\{0<x<h\}\cap S_d$ and $\Gamma_{h,d}=\{0\leq x\leq h\}\cup \partial S_d$. Then the following corollary holds. This proof is reduced to Proposition 2 by an adequate coordinate transformation.

Corollary 1. *Let u be a solution of the elliptic equation* (1.5) *in $\Omega_{a,d}$ and u be in $C^m(\bar{\Omega}_{a,d})$. Then if for some $\varepsilon>0$, $\delta>1$*

$$\dfrac{\partial^{i+j}u}{\partial x^i \partial y^j}=o(\exp(-r^{-2\delta-\varepsilon})) \quad (r\to 0, i+j\leq m-1) \quad \text{on} \quad \Gamma_{a,d},$$

we have in $\Omega_{a,d/2}$

$$\dfrac{\partial^{i+j}u}{\partial x^i \partial y^j}=o(\exp(-r^{-\delta})) \quad (r\to 0, \quad i+j\leq m-1).$$

In the case of elliptic system (1.3) the reduced equation (3.7) has $g\equiv 0$. Thus we have in particular.

Corollary 2. *Let $\{u_p\}$ be solutions of the elliptic system* (1.3) *in $\Omega_{a,d}$ and $\{u_p\}$ be in $C^1(\bar{\Omega}_{a,d})$. Then if for some $\varepsilon>0$, $\delta>1$*

$$u_p=o(\exp(-r^{-2\delta-\varepsilon})) \quad (r\to 0, 1\leq p\leq m) \quad \text{on} \quad \Gamma_{a,d},$$

we have in $\Omega_{a,d/2}$

$$u_p=o(\exp(-r^{-\delta})) \quad (r\to 0, 1\leq p\leq m).$$

4. We consider the equation (1.3) and (1.5) in $\bar{\Omega}_{a,d}$.

Proposition 4. *There is a constant δ such that if u is in $C^m(\overline{\Omega}_{a,d})$ and is a solution of* (1.5) *in $\Omega_{a,d}$ satisfying*

$$(4.1) \qquad \frac{\partial^{i+j}u}{\partial x^i y^j} = o(\exp(-r^{-\delta})) \quad (r \to 0, i+j \leq m-1) \quad in \quad \Omega_{a,d}$$

then u vanishes identically in $\Omega_{a,d}$. Here δ is independent of u.

Before proving this proposition we prepare some lemmas. We consider the next transformation from (x, y)-plane to (θ, ρ)-plane

$$(4.2) \qquad \rho = r^2/x, \quad \theta = \tan^{-1}(y/x).$$

Let us denote by R_d an open rectangular in (θ, ρ)-plane such as

$$-\pi/2 < \theta < \pi/2, \ 0 < \rho < d.$$

Then the transformation (4.2) maps S_d onto R_d. Further this transformation and its inverse are C^∞. We see

$$(4.3) \qquad \begin{pmatrix} \theta_x & \theta_y \\ \rho_x & \rho_y \end{pmatrix} = \begin{pmatrix} -\dfrac{1}{\rho}\tan\theta & 1/\rho \\ 1-\tan^2\theta & 2\tan\theta \end{pmatrix}.$$

We eliminate the part $\rho = d$ from the boundary of R_d and denote the remainder by ∂R_d. And let us put $\overline{R}_d = R_d + \partial R_d$. For the function $f(x, y)$ in $\overline{S}_d$ we define a function $\widetilde{f}(\theta, \rho)$ in $\overline{R}_d$ by

$$\widetilde{f}(\theta, \rho) = \begin{cases} f(\theta, \rho) & \text{for} \quad (\theta, \rho) \in \partial R_d, \\ f(0, 0) & \text{for} \quad (\theta, \rho) \in \partial R_d. \end{cases}$$

Lemma 3. *If $f(x, y) \in C^1(\overline{S}_d)$, then $\widetilde{f}(\theta, \rho) \in C^1(\overline{R}_d)$.*

Since it is easily verified, we omit the proof. From now on we denote $\widetilde{f}$ by f for simplicity. We consider the next equation in S_d

$$(4.4) \qquad u_x + \lambda u_y \equiv F,$$

where $u \in C^1(\overline{S}_d)$, $\lambda \in C^1(\overline{S}_d)$ and the imaginary part of $\lambda \neq 0$ in $\overline{S}_d$. Now by (4.3) the equation is transformed into

$$(4.5) \qquad (\rho_x + \lambda\rho_y)u_\rho + (\theta_x + \lambda\theta_y)u_\theta \equiv F.$$

Lemma 4. *If we set $f(\theta, \rho) = |\rho_x + \lambda\rho_y|^2 \cos^4\theta$, then we see $f \in C^1(\overline{R}_d)$ and there is a positive constant m such that $f > m$ in $\overline{R}_d$.*

Proof. Since $f=|\cos^2\theta+2\lambda\sin\theta\cos\theta-\sin^2\theta|^2$, there is a required m. And we see by Lemma 3 $f\in C^1(\overline{R_d})$.

Now the equation (4.5) is reduced to

(4.6) $u_\rho+f^{-1}\cos^4\theta\,(\rho_x+\bar{\lambda}\rho_y)\,(\theta_x+\lambda\theta_y)\,u_\theta\equiv f^{-1}\cos^4\theta\,(\rho_x+\bar{\lambda}\rho_y)\,F.$

Setting this equation into the form

(4.7) $u_\rho+\dfrac{1}{\rho}(Q+iP)u_\theta\equiv G,$

we have from (4.3)

(4.8)
$P(\theta,\rho)=\lambda_2 f^{-1}\cos^2\theta$

$Q(\theta,\rho)=f^{-1}\cos\theta\,\{\sin\theta\,(\sin^2\theta-\cos^2\theta)$

$\qquad\qquad +\lambda_1(\cos^3\theta-3\cos\theta\sin^2\theta)+2|\lambda|^2\cos^2\theta\sin\theta\},$

where $\lambda=\lambda_1+i\lambda_2$. By Lemma 3 and Lemma 4, we see $P,Q\in C^1(\overline{R_d})$. Let us set

(4.9) $\alpha(\theta,\rho)=Q_\theta-\dfrac{QP_\theta}{P}-1.$

Then we have

Lemma 5. *QP_θ/P is continuous in $\overline{R_d}$. And in particular when $\lambda(0)=\pm i$, $\alpha(\theta,\rho)\to 0$ uniformly in θ as $\rho\to 0$.*

Proof. As we easily see from (4.8), $QP_\theta=f^{-1}\cos^2\theta\times$ (continuous function). Then, in view of $|\lambda_2|\geq c\,(>0)$, the first assertion follows. If we note $\lambda(\theta,0)=i$, $\lambda_\theta(\theta,0)=0$ and that $f(\theta,0)=1$, $f_\theta(\theta,0)=0$, we have $\alpha(\theta,0)=0$. Since $\alpha(\theta,\rho)$ is continuous in $\overline{R_d}$, we have completed the proof.

From now on we denote by $\|\ \ \|$ a L^2 norm with respect to $\theta(|\theta|\leq\pi/2)$. And we define a weight function by

(4.10) $\phi_n(\rho)=\exp(n\rho^{-\delta}).$

Proposition 5. *Suppose that $\lambda\in C^1(\overline{S_d})$ and the imaginary part of $\lambda\neq 0$ in $\overline{S_d}$. Then there is a positive number δ depending only on λ. And if $u\in C^1(\overline{S_d})$ and satisfies for some positive ε*

$u=o(\exp(-r^{-\delta-\varepsilon}))\quad (r\to 0)\quad in\quad S_d,$

then we have for sufficiently small h

(4. 11)
$$\int_0^h \phi_n^2 \left\| u_\rho + \frac{1}{\rho}(Q+iP)u_\theta \right\|^2 d\rho$$

$$\geq c_2 n \int_0^h \frac{\phi_n^2}{\rho^{\delta+2}} \|u\|^2 d\rho - \frac{c_1}{h} \exp(2nh^{-\delta}),$$

where $c_1, c_2, \cdots$ are positive constants independent of n.

Proof. We set

(4. 12)
$$\phi u_\rho + \frac{1}{\rho}\phi(Q+iP)u_\theta$$

$$= \left[(\phi u)' + \frac{1}{\rho}\phi Q u_\theta \right] + \left[\frac{1}{\rho}\phi i P u_\theta - \phi' u \right].$$

Here we omit $\phi_n(\rho)$ by ϕ and $d/d\rho$ by ' (prime). In order to estimate the L^2 norm of the left side of (4.11) it is necessary to estimate following four terms.

(4. 13)
$$-\int_0^h \big([(\phi u)', \phi' u] + [\phi' u, (\phi u)'] \big) d\rho$$

$$\geq \int_0^h (\phi\phi'' - \phi'^2) \|u\|^2 d\rho$$

$$= \int_0^h \frac{n\delta(\delta+1)}{\rho^{\delta+2}} \phi^2 \|u\|^2 d\rho.$$

(4. 14)
$$-\int_0^h \left(\left[\frac{1}{\rho}\phi Q u_\theta, \phi' u \right] + \left[\phi' u, \frac{1}{\rho}\phi Q u_\theta \right] \right) d\rho$$

$$= \int_0^h \frac{\phi\phi'}{\rho} [Q_\theta u, u] d\rho.$$

(4. 15)
$$\int_0^h \left(\left[(\phi u)', \frac{1}{\rho}\phi i P u_\theta \right] + \left[\frac{1}{\rho}\phi i P u_\theta, (\phi u)' \right] \right) d\rho$$

$$= \left[\phi u, \frac{1}{\rho}\phi i P u_\theta \right](h) + \int_0^h \left[\phi u, \frac{i}{\rho^2}\phi P u_\theta \right] d\rho$$

$$- \int_0^h \left[\phi u, \frac{1}{\rho}\phi i P' u_\theta \right] d\rho$$

$$+ \int_0^h \left[-\frac{1}{\rho}P_\theta \phi u, (\phi u)' + \frac{1}{\rho}\phi Q u_\theta \right] d\rho$$

$$+ \int_0^h \left[\frac{1}{\rho}\phi i P_\theta u, \frac{1}{\rho}\phi Q u_\theta \right] d\rho.$$

(4.16) $\int_0^h \left(\left[\frac{1}{\rho}\phi Q u_\theta, \frac{1}{\rho}\phi i P u_\theta \right] + \left[\frac{1}{\rho}\phi i P u_\theta, \frac{1}{\rho}\phi Q u_\theta \right] \right) d\rho = 0.$

We put the right side of (4.15) into $\sum\limits_{j=1}^{5} I_j$. Then we see

$$I_2 = \int_0^h \left[\frac{\phi u}{\rho}, \frac{1}{\rho}\phi i P u_\theta - \phi' u \right] d\rho + \int_0^h \frac{\phi\phi'}{\rho}[u, u] d\rho,$$

$$I_3 = -\int_0^h \left[\phi u, \frac{P'}{P}\left(\frac{1}{\rho}\phi i P u_\theta - \phi' u \right) \right] d\rho + \int_0^h \phi\phi' \left[\frac{P'}{P} u, u \right] d\rho,$$

$$I_5 = -\int_0^h \left[\frac{1}{\rho}\phi u, \frac{P_\theta Q}{P}\left(\frac{1}{\rho}\phi i P u_\theta - \phi' u \right) \right] d\rho$$

$$\qquad -\int_0^h \frac{\phi\phi'}{\rho}\left[\frac{P_\theta Q}{P} u, u \right] d\rho.$$

Here we note $P_\theta Q/P$ is bounded by Lemma 5. Applying Cauchy's inequality to each term we get

(4.17) $(4.14) + (4.15) \geq \int \frac{\phi\phi'}{\rho} \left[\left(Q_\theta + 1 - \frac{P_\theta Q}{P} \right) u, u \right] d\rho$

$$\qquad -\frac{c_1}{h}\exp(2nh^{-\delta}) - c_2 \int_0^h \frac{\phi^2}{\rho^2}\|u\|^2 d\rho$$

$$\qquad -c_3 \int_0^h \phi^2 \|u^2\| d\rho - c_4 \int_0^h |\phi\phi'|\,\|u\|^2 d\rho$$

$$\qquad -\int_0^h \left\{ \left\| (\phi u)' + \frac{1}{\rho}\phi Q u_\theta \right\|^2 + \left\| \frac{1}{\rho}\phi P u_\theta - \phi' u \right\|^2 \right\} d\rho.$$

We put $M = \max \left\| Q_\theta + 1 - \dfrac{P_\theta Q}{P} \right\|$. Then combining (4.13) and (4.17) we obtain

$$\int_0^h \phi^2 \left\| u_\rho + \frac{1}{\rho}(Q + iP)u_\theta \right\|^2 d\rho$$

$$\geq n\delta(\delta + 1 - M)\int_0^h \frac{\phi^2}{\rho^{\delta+2}}\|u\|^2 d\rho - \frac{c_1}{h}\exp(2nh^{-\delta})$$

$$\qquad -c_2 \int_0^h \frac{\phi^2}{\rho^2}\|u\|^2 d\rho - c_3 n\delta \int_0^h \frac{\phi^2}{\rho^{\delta+1}}\|u\|^2 d\rho.$$

Thus taking $\delta > M - 1$ and h sufficiently small, we have completed the proof.

   Corollary 3. *If we assume* $\lambda(0) = \pm i$ *in Proposition* 5, *then we*

can take δ as an arbitrary number larger than 1.

Proof. We see from Lemma 5 $Q_\theta + 1 - \dfrac{QP_\theta}{P} = 2 + \alpha(\theta, \rho) \ (\alpha(\theta, \rho) \to 0$ uniformly in θ as $\rho \to 0$). Thus combining (4.13) and (4.17) we have

$$\int_0^h \phi^2 \left\| u_\rho + \frac{1}{\rho}(Q + iP) u_\theta \right\|^2 d\rho$$

$$\geq \int_0^h \frac{n\delta(\delta+1)}{\rho^{\delta+2}} \phi^2 \|u\|^2 d\rho - \int_0^h \frac{|\phi\phi'|}{\rho} [(2 + \alpha(\theta, \rho)) u, u] d\rho$$

$$- c_2 \int_0^h \frac{\phi^2}{\rho^2} \|u\|^2 d\rho - c_3 \int_0^h \phi^2 \|u\|^2 d\rho$$

$$- c_4 \int_0^h |\phi\phi'| \, \|u\|^2 d\rho - \frac{c_1}{h} \exp(2nh^{-\delta}).$$

Since $\alpha(\theta, \rho) \to 0$ uniformly in θ as $\rho \to 0$, we see

$$\int_0^h \frac{n\delta(\delta+1)}{\rho^{\delta+2}} \phi^2 \|u\|^2 d\rho - \int_0^h \frac{|\phi\phi'|}{\rho} [(2 + \alpha(\theta, \rho)) u, u] d\rho$$

$$\geq \int_0^h \frac{n\delta}{\rho^{\delta+2}} (\delta - 1 - \alpha(\rho)) \phi^2 \|u\|^2 d\rho,$$

where $\alpha(\rho) \to 0$ as $\rho \to 0$. Hence we can take δ as an arbitrary number larger than 1, if h is taken sufficiently small.

Remark 1. The decomposition of (4.12) is due to Mizohata [10].

Remark 2. I. S. Bernstein [2] proved the uniqueness in Cauchy problem for the elliptic operator with distinct characteristic roots without dimensional condition. He treated the problem by polar coordinate system (r, θ). We used the above coordinate system (ρ, θ) instead of the polar coordinate system.

5. First we prove Theorem 2. By an adequate coordinate transformation we can take $\Omega = \Omega_{a,d}$ and $\Gamma = \Gamma_{a,d}$. The exponent δ will be determined later. We have from Corollary 1

$$\frac{\partial^{i+j} u}{\partial x^i \partial y^j} = o(\exp(-r^{-\delta+\varepsilon})) \quad (r \to 0) \quad \text{in} \quad \Omega_{a, d/2}$$

$$i + j \leq m - 1,$$

where ε can be taken as an arbitrary small positive number. By proposition 5 there is a positive constant δ_0 such that if $\delta_0 < \delta - \varepsilon$,

$$\sum_{j=1}^{m} \int_0^h \phi_n(\rho)^2 \left\| w_{j\rho} + \frac{1}{\rho}(Q_j + iP_j)w_{j\theta} \right\|^2 d\rho$$

$$\geqq -\frac{c_1}{h}\exp(2nh^{-\delta_0}) + c_2 n \sum_{j=1}^{m} \int_0^h \frac{\phi_n(\rho)^2}{\rho^{\delta_0+2}}\|w_j\|^2 d\rho,$$

where $\phi_n(\rho) = \exp(n\rho^{-\delta_0})$. Hence we see from (3.7)

(5.1)
$$\int_0^h \phi_n^2 \|\mathscr{B}u\|^2 d\rho + \frac{c_1}{h}\exp(2nh^{-\delta_0})$$

$$\geqq c_3 n \sum_{j=1}^{m} \int_0^h \frac{\phi_n^2}{\rho^{\delta_0+2}}\|w_j\|^2 d\rho.$$

By Poincaré's inequality we have

$$\|u\| \leqq \text{const.}\|u_\theta\|.$$

On the other hand

$$u_\theta = u_x(\rho\cos^2\theta)_\theta + u_y(\rho\cos\theta\sin\theta)_\theta.$$

Hence

$$\|u\| \leqq \text{const.}(\|u_x\| + \|u_y\|).$$

Using successively this inequality, we get

(5.2)
$$\|\mathscr{B}u\| \leqq \text{cost.}\sum_{j=1}^{m}\|w_j\|.$$

Combining (5.1) and (5.2), we obtain

$$c_4(hn)^{-1}\exp(2nh^{-\delta_0}) \geqq \sum_{j=1}^{m}\int_0^h \frac{\phi_n^2}{\rho^{\delta_0+2}}\|w_j\|^2 d\rho.$$

Therefore when we take $\delta > \delta_0 + \varepsilon$, we see that it is contradict if $\sum_{j=1}^{m}\|w_j\|^2 \not\equiv 0$ in the neighbourhood of $\rho = 0$. By classical procedure we see $w_j \equiv 0$ in Ω. Thus the first assertion of Theorem 2 has been proved.

Next we consider the case where $m = 2$ and the coefficients a_{ij} of the principal part are real. Let us set $\delta = 1 + \varepsilon(\varepsilon > 0)$. Then we have similarly

$$\frac{\partial^{i+j}u}{\partial x^i \partial y^j} = o(\exp(-r^{-1-\varepsilon'})) \quad (r \to 0, \varepsilon' > 0)\, i+j \leqq 1.$$

We can assume $a_{ij}(0) = \delta_{ij}$ by a coordinate transformation. Thus we can apply Corollary 3 in place of Proposition 5, and we can follow the same process as in the proof of the first assertion. Hence we have finished the proof of Theorem 2. If the equation (1.3) is reduced to the equation (3.7), we see that $g = 0$. Therefore the proof of Theorem 1 is contained in that of Theorem 2 as a special case.

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