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On some fields of meromorphic functions on fibers

By

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§1. Introduction

1.1. In this paper we consider the extension problem of meromorphic functions on fibers of complex analytic fiber spaces to neighborhoods of the fibers.

Let $X \xrightarrow{\pi} Y$ be a complex analytic fiber space, where X and Y are normal and connected complex spaces and π is a proper holomorphic mapping of X onto Y with irreducible fibers. We denote by K_t the meromorphic function field of a fiber $X_t := \pi^{-1}(t)$, and by K'_t the subfield of K_t consisting of all elements of K_t which can be extended to some neighborhoods of X_t. By [6] or [9], the field K_t is isomorphic to a finite algebraic extension of a rational function field.

We discuss here the following problem.

Let $f_1, \cdots, f_l$ be meromorphic functions on X and g be a meromorphic function on a fiber X_t which is dependent on $f_{1,t}, \cdots f_{l,t}$, where $f_{i,t} (i=1, \cdots, l)$ is the analytic restriction of f_i to X_t. Then, can we extend the function g to a meromorphic function on some neighborhood of X_t?

We can answer this problem as follows.

(I) *The complement of the set $\{t \in Y |$ any meromorphic function on X_t which is dependent on $f_{1,t}, \cdots, f_{l,t}$ can be extended to some neighborhoods of $X_t\}$ is nowhere dense in Y.*

The proof of this theorem is essentially due to the *Stein factorization* of a proper holomorphic mapping. This notion (or the

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notion of *complex base*) is useful to research *dependency* of holomorphic or meromorphic mappings (for example, see [5], [6], [8], [9]).

Using (I) we obtain:

(II) *The set* $\{t\in Y \mid K'_t$ *is not algebraically closed in* $K_t\}$ *is nowhere dense in* Y.

Furthermore, by a similar method to the proof of (I) we have:

(III) *If the transcendence degree of* K'_t *over the complex number field* $\boldsymbol{C}$ *is equal to the (complex) dimension of the fiber* X_t, *then* $K'_t=K_t$.

1.2. In this paper, we assume all complex spaces to be *reduced*, and we denote the complex projective space of dimension m by $\boldsymbol{P}_m$, and the Osgood space of dimension l by $\boldsymbol{P}^l$.

We recall here the concepts of *rank* and of *degeneracy* of mappings.

Let σ: $M\rightarrow N$ be a holomorphic mapping of an irreducible complex space M to a complex space N. We define the *local rank* of σ at a point x of M by $\dim_x M-\dim_x \sigma^{-1}(\sigma(x))$ and denote it by $r_x(\sigma)$. Further we define the *rank* of σ by $\sup_{x\in M} r_x(\sigma)$ and denote it by $r(\sigma)$.

Now, if $r_x(\sigma)\neq r(\sigma)$ for a point x of M, we call this point x *a point of degeneracy* of σ. By R. Remmert [8], the set of all points of degeneracy is an analytic subset, and any holomorphic mapping without points of degeneracy (we say such a mapping is *non-degenerated* or is *of constant rank*) to a normal complex space whose dimension is equal to the rank of the mapping is an open mapping.

§2. Some remarks on fiber spaces and meromorphic mappings

2.1. Let X and Y be complex spaces and $\{X_i\}$ be the set of irreducible components of X.

Now let f be a correspondence between X and Y. We denote the graph of f by G and the natural projections of G to X and Y by $\check{f}$ and $\hat{f}$ respectively. Conforming to [9], we call the correspondence f to be a *meromorphic mapping* of X to Y if the following conditions are satisfied;

(a) there is a dense open set of X on which f defines a holomorphic mapping to Y,

(b) the graph G is an analytic subset of $X \times Y$, and $f^{-1}(X_i)$ is an irreducible component of G for each X_i,

(c) the projection $\check{f}$ is proper.

Let f be a meromorphic mapping of X to Y. We call a point x of X a singular point of f if f is not holomorphic at x, and call f to be ***proper*** (resp. ***surjective***) if $\hat{f}$ is proper (resp. surjective). Further we define the ***rank*** of f by $r(\hat{f})$ and denote it by $r(f)$. Moreover we say that a meromorphic mapping f of X to Y is ***bimeromorphic*** if the correspondence f defines a meromorphic mapping of Y to X.

Next, we recall some fundamental properties of meromorphic mappings.

(i) The set of all singular points of a meromorphic mapping is an analytic subset.

(ii) A meromorphic mapping of a certain complex space X to the complex projective space $\boldsymbol{P}^1$ which maps X not constantly to ∞ is nothing but a meromorphic function in the usual sense.

(iii) Let X, Y and Z be complex spaces and f and g be meromorphic mappings of X to Y and of Y to Z respectively. We define naturally a correspondence between X and Z such that a point x of X corresponds to the subset $g(f(x))$ of Z. If there is a dense open set U of X on which the above correspondence between X and Z is single-valued, then we can define naturally one meromorphic mapping h of X to Z such that $h(x)=g(f(x))$ for $x \in U$. We denote it by $g \circ f$. In particular, if X is a subspace of Y and f is the inclusion map, we denote $g \circ f$ by $g \| X$.

(iv) Let $X, Y_1, \cdots, Y_l$ be complex spaces and f_i be a meromorphic mapping of X to $Y_i (i=1, \cdots, l)$. Then we can naturally define one meromorphic mapping of X to the product space $Y_1 \times \cdots \times Y_l$. We denote it by $f_1 \times \cdots \times f_l$.

(v) Let X and Y be irreducible complex spaces of the same dimension and f be a proper and surjective meromorphic mapping of X to Y. Then there is a thin analytic subset N of Y such that f is holomorphic on $X-\check{f}(\hat{f}^{-1}(N))$ and the map $f\|(X-\check{f}(\hat{f}^{-1}(N))$ is a proper holomorphic covering map of $X-\check{f}(\hat{f}^{-1}(N))$ to $Y-N$. We call such a meromorphic mapping to be a *meromorphic covering*.

Next, we recall the notion of *dependency* of meromorphic mappings.

Let X, Y and Z be complex spaces and f and g be meromorphic mappings of X to Y and of X to Z respectively. Then we say that *g depends on f* if $r(f\times g)=r(f)$. Further let $f_1, \cdots, f_l$ be meromorphic functions on X. Then we say that the system $\{f_1, \cdots, f_l\}$ is *independent* if $r(f_1\times\cdots\times f_l)=l$.

2. 2. Let X and Y be complex spaces and π be a proper holomorphic mapping of X to Y. We denote the set of all connected components of all fibers of the map π by X'. By [1] we can define on the set X' a topology and a complex structure which have the following properties;

(a) the natural maps π_1: $X\to X'$ and π_2: $X'\to Y$ are holomorphic.

(b) an arbitrary map h of X' to a complex space Z such that $h\circ\pi_1$ is holomorphic is holomorphic.

We call this sequence $X\xrightarrow{\pi_1}X'\xrightarrow{\pi_2}Y$ the *Stein factorization* of π.

Proposition 1. *Let X be a compact irreducible complex space, and $f_1, \cdots, f_l$ be meromorphic functions on X. We put $F=f_1\times\cdots\times f_l$, $G=$the graph of F. Let $\widetilde{G}\xrightarrow{\mu}G$ be the normalization of G and $\widetilde{G}\xrightarrow{h_1}H\xrightarrow{h_2}\boldsymbol{P}^l$ be the Stein factorization of the proper holomorphic mapping $\hat{F}\circ\mu$, where $\hat{F}$ is the natural projection of G to $\boldsymbol{P}^l$.*

Then, for any meromorphic function g on X dependent on F, there is a meromorphic function g' on H such that $g=g'\circ h_1\circ\mu^{-1}\circ\check{F}^{-1}$.

Proof. Since X and $\widetilde{G}$ are bimeromorphically equivalent, we may assume that X is normal and connected and F is holomorphic on X. Under these assumptions we may identify the sequence $\widetilde{G}\xrightarrow{h_1}H\xrightarrow{h_2}\boldsymbol{P}^l$ with the Stein factorization $X\xrightarrow{h_1}X'\xrightarrow{h_2}\boldsymbol{P}^l$ of the proper holomorphic mapping F.

Let $S(g)$ be the singular set of g. Since X is compact and g depends on F, there is a polynomial $P_s(X_1, \cdots, X_l)X^s + \cdots + P_0(X_1, \cdots, X_l)$, where $s>0$, such that $P_s(f_1, \cdots, f_l)g^s + \cdots + P_0(f_1, \cdots, f_l) \equiv 0$ on X and $P_s(f_1, \cdots, f_l) \not\equiv 0$ on X (see [9], p. 864). Now we take a point $z=(z_1, \cdots, z_l)$ of $F(X)$ such that $z_i \neq \infty$ for all i and $P_s(z_1, \cdots, z_l) \neq 0$. Let x be a point of $F^{-1}(z)$. Then $g(x)$ is a finite set in $\boldsymbol{P}^1$ since $P_s(z_1, \cdots, z_l) \neq 0$. This fact and the normality of X yield the holomorphy of g at x (see [9], Prop. 3. 1. 3). Hence $F(S(g)) \neq F(X)$.

Since $F(X)$ is an irreducible complex space, $F(S(g))$ is a thin analytic subset of $F(X)$ and so $F^{-1}(F(S(g)))$ is a thin analytic set of X. We put $X_0 = X - F^{-1}(F(E) \cup F(S(g)))$, where E is the set of degeneracy of F. ($F^{-1}(F(E))$ is thin in X.) We denote the Stein factorization of the proper holomorphic mapping $F|X_0$ by $X_0 \to X_0' \to F(X_0)$. Then we may consider that $X_0' = h_1(X_0) (\subset X')$. For a point x of X_0, $r_x(F) = r_x(F \times g)$ because $r_x(F) \leq r_x(F \times g) \leq r(F \times g) = r(F) = r_x(F)$. Hence g is constant along each connected component of $F^{-1}(z)$ for any z of $F(X_0)$ (see [8], p. 300). Therefore we obtain a holomorphic function g_0' on X_0' such that $g \| X_0 = g_0' \circ h_1$.

Put $G(g)$ = the graph of g, and G' = the Image of $G(g)$ by the map $h_1 \times 1$ of $X \times \boldsymbol{P}^1$ to $X' \times \boldsymbol{P}^1$. Then G' gives a moromorphic function g' on X' such that $g' \| X_0' = g_0'$.

Remark. Proposition 1 can be generalized as follows:

Let π: $X \to Y$ be a proper holomorphic mapping, where X is irreducible, and $f_1, \cdots, f_l$ be meromorphic functions on X. We put $\sigma = f_1 \times \cdots \times f_l \times \pi$, G = the graph of σ, and $\hat{\sigma}$, $\check{\sigma}$ = the natural projections of G to $\boldsymbol{P}^l \times Y$ and to X. Let $\widetilde{G} \xrightarrow{\mu} G$ be the normalization of G and $\widetilde{G} \xrightarrow{h_1} H \xrightarrow{h_2} \boldsymbol{P}^l \times Y$ be the Stein factorization of $\hat{\sigma} \circ \mu$.

Then, for any meromorphic function g on X dependent on σ, there is a meromorphic function g' on H such that $g = g' \circ h_1 \circ \mu^{-1} \circ \check{\sigma}^{-1}$.

Proposition 2. *Let V be an irreducible analytic subspace of $\boldsymbol{P}_m$. Then any element of the field $K(V)$ of all meromorphic functions on V is the restriction of a rational function of $\boldsymbol{P}_m$.*

Furthermore let $\{f_1, \cdots, f_l\}$ *be a transcendence base of* $K(V)$ *over the complex number field* $\mathbf{C}$. *Then the degree of* $K(V)$ *over the field* $\mathbf{C}(f_1, \cdots, f_l)$ *is equal to the number of sheet of the meromorphic covering map* $F\colon V \to \mathbf{P}^l$, *where* $F = f_1 \times \cdots \times f_l$.

Proof. Let $K_R(V)$ be the subfield of $K(V)$ consisting of all elements of $K(V)$ which can be extended to a rational function of $\mathbf{P}_m$. Then the transcendence degrees of $K(V)$ and $K_R(V)$ over $\mathbf{C}$ are equal to the dimension of V. Let $\{f_1, \cdots, f_l\}$ be a transcendence base of $K_R(V)$ over $\mathbf{C}$ and F be the meromorphic mapping $f_1 \times \cdots \times f_l$ of V to $\mathbf{P}^l$. Then F is a meromorphic covering map, because dim $V = l$ $=$ dim $\mathbf{P}^l$ and the system $\{f_1, \cdots, f_l\}$ is independent. So there is an analytic subset N of $\mathbf{P}^l$ such that F is holomorphic on $V - F^{-1}(N)$ and $F \| (V - F^{-1}(N))$ is a proper unramified holomorphic covering map to $\mathbf{P}^l - N$. We put $b =$ the number of sheet of $F \| (V - F^{-1}(N))$, $d = [K(V)\colon \mathbf{C}(f_1, \cdots, f_l)]$ and $d' = [K_R(V)\colon \mathbf{C}(f_1, \cdots, f_l)]$. Then clearly $b \geq d \geq d'$, because any element f of $K(V)$ satisfies; $f^b + H_{b-1} f^{b-1} + \cdots + H_0 = 0$, where $H_i (i = 0, 1, \cdots, b-1)$ is a suitable rational function of $\mathbf{P}^l$ which is considered as an element of $\mathbf{C}(f_1, \cdots, f_l)$.

On the other hand, we can find an element g of $K_R(V)$ whose degree over $\mathbf{C}(f_1, \cdots f_l)$ is not smaller than b. In fact, fix a point p of $\mathbf{P}^l - N$, and put $F^{-1}(p) = \{p_1, \cdots, p_b\}$. Then we can easily find two linear forms $w_1 = a_0 z_0 + \cdots + a_m z_m$, $w_2 = b_0 z_0 + \cdots + b_m z_m$, for a system of homogeneous coordinate $\{z_0, \cdots, z_m\}$ of $\mathbf{P}_m$, such that $w_1(p_i) \neq 0$ for all i, and $\frac{w_2(p_i)}{w_1(p_i)} \neq \frac{w_2(p_j)}{w_1(p_j)}$ (for $i \neq j$). Now we put $\tilde{\alpha} = \frac{w_2}{w_1}$ and $\alpha = \tilde{\alpha} \| V$. Then it can be easily proved that the degree of α over $\mathbf{C}(f, \cdots, f_l)$ is not smaller than b.

Theorem. (H. Grauert and R. Remmert, [2], [4]). *Let* X *aud* Y *be complex spaces and* σ *be a proper holomorphic mapping of* X *to the product space* $\mathbf{P}_m \times Y$ *with discrete fibers. Let* U *be a relatively compact Stein open set of* Y. *We put* $X_U = \sigma^{-1}(\mathbf{P}_m \times U)$.

Then, there is a natural number N *and a biholomorphic mapping* ω *of* X_U *to an analytic subspace of the product space*

$\boldsymbol{P}_m \times U \times \boldsymbol{P}_N$ *such that* $\sigma | X_U = p \circ \omega$, *where* p *is the natural projection of* $\boldsymbol{P}_m \times Y \times \boldsymbol{P}_N$ *to* $\boldsymbol{P}_m \times Y$.

Proposition 3. *Let* $\pi: X \to Y$ *be a proper holomorphic mapping of a normal complex space* X *onto a complex space* Y. *Then the set* $\{t \in Y |$ *the space* $\pi^{-1}(t)$ *is not locally irreducible*$\}$ *is nowhere dense in* Y.

The proof of this proposition is essentially due to W. Thimm [11]. We prove this in the next section.

§3. Proof of Proposition 3

To prove our proposition we use local descriptions of the normal complex space X. Therefore we start by setting the following notations. We put;

$$T = \{(t_1, \cdots, t_n) \in \boldsymbol{C}^n | \quad |t_i| < \tau_i; \ i = 1, \cdots, n\},$$

$$Z_m = \{z_1, \cdots, z_m) \in \mathbf{C}^m | \quad |z_j| < \zeta_j; \ j = 1, \cdots, m\},$$

$D_m = T \times Z_m$, $p =$ the natural projection of D_m to T,

$Z_{m,t} = p^{-1}(t)$, where t is a point of T.

Now let A be an analytic set of D_m and T_0 be the set $\{t \in T | Z_{m,t} \cap A = Z_{m,t}\}$. We consider the following condition (*) for a point x of D_m with respect to A:

(*) *The point* $p(x)$ *does not belong to* T_0 *and there is a fundamental system of neighborhoods* $\{U_i\}$ *of the point* x *which satisfies the following condition* (*C*);

(C) *for a curve* C *in* U_i *such that* $C \cap A = \phi$ *and* $p(C(0)) = p(C(1)) = p(x)$, *there is a deformation of the curve* C *to a curve in* $U_i \cap Z_{m, p(x)}$ *through the space* $U_i - A$, *with the end points* $C(0)$ *and* $C(1)$ *fixed.*

Lemma 3.1. *Let* M *be a connected normal complex space and* r *be a proper holomorphic covering map of* M *to* D_m *which is unramified over* $D_m - A$. *If, for a point* x *of* M, *the point* $r(x)$

satisfies the condition $(*)$ *with respect to* A, *then* x *is an irreducible point of the fiber* $(p\circ r)^{-1}((p\circ r)(x))$.

Proof. Suppose that $r(x)$ satisfies the condition $(*)$ with respect to A. We put $M_x=(p\circ r)^{-1}(p\circ r)(x)$. Then $M_x\cap r^{-1}(A)$ is a thin analytic set of M_x, and $M_x-r^{-1}(A)$ is non-singular. Hence x is an irreducible point of M_x if and only if there is a fundamental system of neighborhoods $\{U'_k\}$ of the point x in the space M_x such that $U'_k-r^{-1}(A)$ is connected.

Take a connected neighborhood V of x in the space M such that $V\cap M_x$ is sufficiently small and,

(a) the open set $r(V)$ satisfies the condition (C) with respect to A at $r(x)$,

(b) the mapping $r|V: V\to r(V)$ is proper.

We put $U'=V\cap M_x$. Then from the above (a) and (b) $U'-r^{-1}(A)$ is connected. In fact, let x_1 and x_2 be points of $U'-r^{-1}(A)$. Since V is connected and normal, we can connect x_1 to x_2 by a curve $\widetilde{C}$ in $V-r^{-1}(A)$. We put $C=r(\widetilde{C})$. By (a), C can be deformed to a curve in $Z_{m,p(r(x))}\cap(r(V)-A)$ through the space $r(V)-A$, fixing the end points. On the other hand, the map $r|V$ is a proper unramified covering over $r(V)-A$. Hence we can deform $\widetilde{C}$ to a curve of $U'-r^{-1}(A)$ through the space $V-r^{-1}(A)$, by lifting the deformation of the curve C. Hence $U'-r^{-1}(A)$ is connected.

Lemma 3.2. *We put;* $Z_{m-1}=\{(z_1,\cdots,z_{m-1})\in\boldsymbol{C}^{m-1}\,|\,|z_j|<\zeta_j; j=1,\cdots,m-1\}$, *and* $D_{m-1}=T\times Z_{m-1}$ *and* $q=$ *the natural projection of* D_m *to* D_{m-1}.

Suppose that $q|A$ *is a proper holomorphic covering map onto* D_{m-1} *and it is unramified over* $D_{m-1}-B$, *where* B *is a thin analytic set of* D_{m-1}.

Then, for a point x *of* D_m, *if* $q(x)$ *satisfies the condition* $(*)$ *with respect to* B *then* x *also satisfies the condition* $(*)$ *with respect to* A.

Proof. Let W be a neighborhood of x. Then we can find a neighborhood U of x having the following properties;

(a) $U \subset W$,

(b) U is of the form $q(U) \times D$, where D is a disk of $\boldsymbol{C}^1$,

(c) $q(U)$ satisfies the condition (C) at $q(x)$ with respect to B,

and

(d) $q|A \cap U$: $A \cap U \rightarrow q(U)$ is proper.

Then we can prove that the open set U satisfies the condition (C) at x with respect to A by the same methods as in [11]. We give only an outline of the proof.

Let C be a curve in $U-A$ with the end points $C(0)$ and $C(1)$ such that $p(C(0))=p(C(1))=p(x)$. Without loss of generality, we may assume that $q(C(0))$ and $q(C(1))$ do not belong to B, because $Z_{m-1,p(x)} \cap B \neq Z_{m-1,p(x)}$ by above (c) and so we can replace the end points by two suitable points in $U \cap Z_{m,p(x)}-(A \cup q^{-1}(B))$ which are connected to $C(0)$ and $C(1)$ by arcs in $U \cap Z_{m,p(x)}-A$ respectively. Moreover we may assume that $q(C)$ is disjoint with B, because the curve C can be deformed, fixing the end points, to a curve which is sufficiently near to C and whose projection to $q(U)$ is disjoint with B (see [11], §2). Under these assumptions, $q(C)$ can be deformed by the above property (c) to a curve of $q(U) \cap q(Z_{m,p(x)})$ though the space $q(U)-B$ with the end points fixed. On the other hand, since $q|A$ is proper and unramified over $D_{m-1}-B$, we can construct a deformation of C in $U-A$ with the desired properties lying above the deformation of $q(C)$ (see [10], §2 and [11], §2).

Lemma 3.3. *We suppose that A is purely 1-codimentional in D_m, and put $D_m^* = \{x \in D_m \mid x$ satisfies the condition $(*)$ with respect to $A\}$.*

Then $p(K_m - D_m^*)$ is nowhere dense in T for any relatively compact subset K_m of D_m.

Proof. We prove the lemma by induction on m. If $m=0$, it is trivial. So we suppose that $m>0$ and that the result holds for $m-1$.

We denote the ε-neighborhood of the set T_0 by $T_0(\varepsilon)$. Then $p(K_m - D_m^*)$ is nowhere dense in T if and only if it is nowhere dense in $T - T_0(\varepsilon)$ for any positive number ε. We put $K_m(\varepsilon) = K_m - p^{-1}(T_0(\varepsilon))$.

Now let x be a point of D_m. If $x \notin A$, take a neighborhood $U_m(x)$ of x such that $U_m(x) \cap A = \phi$. Then any point of $U_m(x)$ satisfies the condition (*) with respect to A. Next we suppose $x \in A - p^{-1}(T_0)$. Then, since A is purely codimensional 1, we can find a neighborhood $V_m(x)$ of x satisfying the following properties:

(a) $V_m(x)$ is the product of two polycylinders $T(x)$ and $Y_m(x)$, where $T(x)$ and $Y_m(x)$ are defined as follows;

$$T(x) = \{(t'_1, \cdots, t'_n) \in \boldsymbol{C}^n \mid \; |t_i| < \tau'_i;\; i=1, \cdots, n\},$$
$$Y_m(x) = \{(y_1, \cdots, y_m) \in \boldsymbol{C}^m \mid \; |y_j| < \eta_j;\; j=1, \cdots, m\},$$

where $t'_i = t_i - t_i(p(x))$ and $y_j = \sum_{k=1}^{m} c_{jk} z_k + d_j$ such that $y_j(x) = 0$ (for any j) and the matrix (c_{jk}) is non-singular, and τ'_i and η_j are suitable positive numbers.

(b) Let $Y_{m-1}(x) = \{(y_1, \cdots, y_{m-1}) \in \boldsymbol{C}^{m-1} \mid \; |y_j| < \eta_j;\; j=1, \cdots, m-1\}$ and $V_{m-1}(x) = T(x) \times Y_{m-1}(x)$ and $q =$ the natural projection of $V_m(x)$ to $V_{m-1}(x)$. In this situation, $q \mid V_m(x) \cap A$ is a proper covering map and unramified over $V_{m-1}(x) - B$, where B is an analytic subset of $V_{m-1}(x)$ purely of codimension 1.

We denote the natural projection of $V_{m-1}(x)$ to $T(x)$ by p_{m-1}, and the set $\{s \in V_{m-1}(x) \mid s$ satisfies the condition (*) with respect to $B\}$ by $V_{m-1}^*(x)$. Let now $U_{m-1}(x)$ be an arbitrarily fixed relatively compact open neighborhood of $q(x)$ in $V_{m-1}(x)$. Then, by the hypothesis of induction, $p_{m-1}(U_{m-1}(x) - V_{m-1}^*(x))$ is nowhere dense in $T(x)$. Hence, by Lemma 3. 2, $p(U_m(x) - D_m^*)$ is nowhere dense in $T(x)$, where $U_m(x)$ is the set $q^{-1}(U_{m-1}(x))$.

For each point x of $K_m(\varepsilon)$ we take such an open neighborhood $U_m(x)$ mentioned above. Since $K_m(\varepsilon)$ is compact, it is covered by a finite system of such neighborhoods $U_m(x_k)$ and hence $p(K_m(\varepsilon) - D_m^*)$

is nowhere dense in T.

Proof of Proposition 3. We may assume that Y is non-singular and π is of constant rank. For, our assertion is of local character about Y and the π-image of the set of degeneracy of π is a thin analytic set in Y. Moreover we may assume; $Y=\{(t_1, \cdots, t_n)\in \mathbf{C}^n \mid |t_i|<\tau_i; i=1, \cdots, n\}$. Then, for each point x of X, we can find a connected open neighborhood $U(x)$ such that there is a proper holomorphic covering map r of $U(x)$ to D_m, where D_m is a polycylinder which is obtained by replacing t_i by $t_i-t_i(\pi(x))$ in D_m of the beginning of this section.

Let A be a purely one codimensional analytic set in D_m such that r is unramified over D_m-A. Further let W be a relatively compact open set of D_m containing $r(x)$ and $V(x)$ be the open set $r^{-1}(W)\cap U(x)$. Then, by Lemma 3. 3, $p(W-D_m^*)$ is nowhere dense in $T(\subset Y)$ and hence $X_t\cap V(x)$ is locally irreducible by Lemma 3. 1 for any point t of $p(W)-p(W-D_m^*)$.

For each point x of X, we take such a neighborhood $V(x)$. Let Q be a relatively compact open set of Y. Then the set $\pi^{-1}(\overline{Q})$ is compact and so it is covered by a finite system of open sets $V(x_k)$. Hence the set $\{t\in Y \mid X_t$ is not locally irreducible$\}$ is nowhere dense in Y.

§4. Meromorphic function fields on fibers

In this section, we consider a fiber space $X\xrightarrow{\pi}Y$, where X and Y are complex spaces and π is a proper surjective holomorphic mapping. We put $\dim Y=n$ and $\dim X=m+n$. Furthermore we assume;

(a) X and Y are normal and connected,

(b) π is of constant rank, n,

(c) for every $t\in Y$, the fiber X_t is irreducible.

These assumptions imply,

(d) $\pi^{-1}(U)$ is connected for any connected open set U of Y.

From now on, we use occasionally a notation h_t instead of $h\| X_t$, where h is a meromorphic mapping of X to a certain complex space and t is a point of Y such that $h\|X_t$ is defined.

Lemma 4. 1. *Let $f_1, \cdots, f_l$ be meromorphic functions on X. We put $F=f_1\times\cdots\times f_l$ and $S(F)=$the singular set of F. Then the set $\{t\in Y|X_t\not\subset S(F)\}$ is a dense open subset of Y.*

Let t be a point of Y such that $X_t\not\subset S(F)$. We suppose that $\{f_{1,t}, \cdots, f_{l,t}\}$ is independent. Then there is an open neighborhood U of t such that $f_{i,t'}(i=1, \cdots, l)$ is defined and $\{f_{1,t'}, \cdots, f_{l,t'}\}$ is independent for any t' of U. (In this case, $r(F\times\pi)=n+l$ and *$(F\times\pi)(X)=\boldsymbol{P}^l\times Y$).*

Proof. The first assertion is trivial.

Suppose that $\{f_{1,t}, \cdots, f_{l,t}\}$ is independent. We can find a point x of X_t such that $x\notin S(F)$ and $r_x(F_t)=r(F_t)=l$ (here we consider F as a holomorphic mapping on a neighborhood of x). Then $r_x(F\times\pi)=r(F\times\pi)=n+l$ because $\dim_x(F\times\pi)^{-1}((F\times\pi)(x))=\dim_x F_t^{-1}(F_t(x))=m-r_x(F_t)=m-l$, and so $(F\times\pi)(X)=\boldsymbol{P}^l\times Y$.

Take a neighborhood Q of x such that $Q\cap S(F)=\phi$ and $r_{x'}(F\times\pi)=n+l$ for any point x' of Q. Put $U=\pi(Q)$. Since π is of constant rank, U is an open set and clearly has our desired properties.

Theorem I. *Let t_0 be a point of Y and $f_1, \cdots, f_l$ be meromorphic functions on X such that f_{i,t_0} is defined for any i and the system $\{f_{1,t_0}, \cdots, f_{l,t_0}\}$ is independent. We put $F=f_1\times\cdots\cdots\times f_l$, $\sigma=F\times\pi$, $G=$the graph of σ, and $G_{t_0}=$the graph of F_{t_0}, and we denote the normalization of G by $\widetilde{G}\xrightarrow{\mu}G$.*

We suppose that;

(I) *the complex space $\widetilde{G}|X_{t_0}$ ($=$the restriction of $\widetilde{G}$ over X_{t_0}) is locally irreducible.*

Then there is an open neighborhood U of t_0 such that any meromorphic function defined on X_{t_0} which is dependent on F_{t_0} can be extended to a meromorphic function on $\pi^{-1}(U)$.

Proof. Since X is normal, every fiber of the map $\widetilde{G}\to X$ is connected, and X_t is irreducible by the assumption. Hence $\widetilde{G}|X_t$ is connected for any t of Y. On the other hand, $G|X_{t_0}$ is locally irreducible by the assumption (I). Therefore $\widetilde{G}|X_{t_0}$ is irreducible, and so $G|X_{t_0}$ is also irreducible and hence $G|X_{t_0}=G_{t_0}$. By these facts the space $\widetilde{G}|X_{t_0}$ is homeomorphic and bimeromorphic to the normalization $\widetilde{G}_{t_0}$ of G_{t_0}.

Let $\widetilde{G}\xrightarrow{h_1}H\xrightarrow{h_2}\boldsymbol{P}^l\times Y$ be the Stein factorization of the proper holomorphic mapping $\hat{\sigma}\circ\mu$, where $\hat{\sigma}$ is the natural projection of G to $\boldsymbol{P}^l\times Y$, and $\widetilde{G}_{t_0}\xrightarrow{h_{1,t_0}}H_{t_0}\xrightarrow{h_{2,t_0}}\boldsymbol{P}^l$ be the Stein factorization $\hat{F}_{t_0}\circ\mu_{t_0}$, where μ_{t_0} is the normalization map $\widetilde{G}_{t_0}\xrightarrow{\mu_{t_0}}G_{t_0}$ and $\hat{F}_{t_0}$ is the natural projection of G_{t_0} to $\boldsymbol{P}^l$. From above, $h_2^{-1}(\boldsymbol{P}^l\times t_0)$ is also naturally homeomorphic and bimeromorpic to H_{t_0}, so we may identify H_{t_0} with $h_2^{-1}(\boldsymbol{P}^l\times t_0)$.

By proposition 1, we can find a meromorphic function g' on H_{t_0} such that $g=g'\circ h_{1,t_0}\circ\mu_{t_0}^{-1}\circ\check{F}_{t_0}^{-1}$, where $\check{F}_{t_0}$ is the natural projection of G_{t_0} to X_{t_0}.

On the other hand, the map h_2: $H\rightarrow\boldsymbol{P}^l\times Y$ is proper (and surjective) with discrete fibers. Hence, by Theorem of §2, there is a neighborhood U of t_0 and a biholomorphic mapping ω of $h_2^{-1}(\boldsymbol{P}^l\times U)$ to an analytic subspace L_U of $\boldsymbol{P}^l\times U\times\boldsymbol{P}_N$.

We put $g''=g'\circ(\omega\|H_{t_0})^{-1}$ on L_{t_0}. By proposition 2, there is a rational function g''' on $\boldsymbol{P}^l\times\boldsymbol{P}_N(\cong\boldsymbol{P}^l\times t\times\boldsymbol{P}_N)$ such that $g'''\|L_{t_0}=g''$. Further we put $\widetilde{g}'''=g'''\circ\tau$, where τ is the natural projection of $\boldsymbol{P}^l\times U\times\boldsymbol{P}_N$ to $\boldsymbol{P}^l\times\boldsymbol{P}_N$, and put $\widetilde{g}''=\widetilde{g}'''\|L_U$. Finally we set $\widetilde{g}=\widetilde{g}''\circ\omega\circ h_1\circ\mu^{-1}\circ\check{\sigma}^{-1}$ on $\pi^{-1}(U)$. Then $\widetilde{g}$ is a meromorphic function with $g=\widetilde{g}\|X_{t_0}$.

Remark. By the construction of $\widetilde{g}$, it is easily shown that *there is a polynomial $P(t)(X_0, X_1, \cdots, X_l)$ with holomorphic functions on U as coefficients (if necessary, replace U with a smaller neighborhood of t_0) such that $P(t)(g, f_1, \cdots, f_l)\equiv 0$ on $\pi^{-1}(U)$ and $P(t_0)(X_0, X_1, \cdots, X_l)\neq 0$.*

We use the following notations:

$K_t=$ the field of all meromorphic functions on the fiber X_t,

$K'_t=$ the subfield of K_t consisting of all elements of K_t, which can be extended to neighborhoods of Y_t,

$Y(k)=\{t\in Y|$ there is a neighborhood U of t in Y such that the transcendence degree of $K'_{t_1}=k$ for any t_1 of $U\}$, and

$Y'=Y(0)\cup Y(1)\cup\cdots\cup Y(m)$.

Corollary. *Let $f_1,\cdots,f_l$ be meromorphic functions on X such that $f_{i,t}$ is defined for any i and the system $\{f_{1,t},\cdots,f_{l,t}\}$ is independent for any t of Y. We set $K''_t(f)=$ the algebraic closure of the field $\boldsymbol{C}(f_{1,t},\cdots,f_{l,t})$ in K_t. Then the set $\{t\in Y|K''_t(f)\not\subset K'_t\}$ is nowhere dense in Y.*

Proof. By Proposition 3 there is a nowhere dense set Y_0 of Y such that any point of $Y-Y_0$ satisfies the condition (I) of Theorem I. Hence our assertion is proved by Theorem I.

Theorem II. *The set $Y_1=\{t\in Y|K'_t$ is not algebraically closed in $K_t\}$ is nowhere dense in Y.*

Proof. The assertion is of local character about Y, and Y' is a dense open set of Y. So we may assume that $Y=Y(k)$ and there are k meromorphic functions on X such as in the above corollary. Hence Y_1 is nowhere dense in Y by corollary of Theorem I.

Lastly we discuss the case $Y=Y(m)$ (where m is the dimension of fibers).

Lemma 4.2. *Let Z be a complex space and W be a compact irreducible analytic subspace of Z of dimension m, and $f_1,\cdots,f_m$ be meromorphic functions on Z such that $f_i\|W$ is defined for any i and $\{f_1\|W,\cdots,f_m\|W\}$ is independent. We put $F=f_1\times\cdots\times f_m$, $F_0=F\|W$, $G=$ the graph of F, $G_0=$ the graph of F_0, and $G_1=G|W$ (the restriction of G over W).*

Now λ be the natural projection of G_1 to $\boldsymbol{P}^m$ and $G_1\xrightarrow{\lambda_1}G'_1\xrightarrow{\lambda_2}\boldsymbol{P}^m$ be the Stein factorization of λ. Then the holomorphic mapping $\lambda_1|G_0$: $G_0\to\lambda_1(G_0)$ is bimeromorphic and $\lambda_1(G_0)$ is an irreducible component of G'_1.

Proof. The graph G_0 of F_0 is an irreducible component of G_1. Let G_2 be the union of all the irreducible components of G_1 which are distinct from G_0. Since $\{f_1\| W, \cdots, f_m\| W\}$ is independent and dim $W=m$, the proper holomorphic mapping $\lambda | G_0$: $G_0 \rightarrow \boldsymbol{P}^m$ is surjective and of rank m. From this, it follows that $\lambda(G_0 \cap G_2) \neq \boldsymbol{P}^m$, for $\lambda(G_0 \cap G_2) = \boldsymbol{P}^m$ implies $G_t \subset G_2$. Hence our assertion is proved.

Theorem III. *If the transcendence degree of the field* K_t' *is equal to the (complex) dimension of the fiber, then* $K_t' = K_t$.

Proof. By Lemma 4. 1, we may assume that $Y = Y(m)$ and that there are m meromorphic functions $f_1, \cdots, f_m$ on X such that $f_{i,t}$ is defined $(i=1, \cdots, m)$ and the system $\{f_{1,t}, \cdots, f_{m,t}\}$ is independent. We put $F = f_1 \times \cdots \times f_m$, and $G =$ the graph of $F \times \pi$ and $G_t =$ the graph of F_t, and denote the Stein factorization of $\widehat{F \times \pi}$ by $G \rightarrow G' \rightarrow \boldsymbol{P}^m \times Y$. Then, by Lemma 4. 2, G_t is bimeromorphically equivalent to an irreducible component of $G' | X_t$. Hence we can prove this theorem similarly to Theorem I.

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