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Cartan-Kuranishi's prolongation of differential systems combined with that of Lagrange and Jacobi

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§0. Introduction

There exist differential systems which can not be prolonged to an involutive system by Kuranishi's standard prolongation, although they have a solution. A simple example was given by Kuranishi himself in [8]. *We shall construct here a partial prolongation, combining Cartan-Kuranishi's standard prolongation and Lagrange-Jacobi's classical prolongation. Applying our partial prolongation, we can prolong any differential system to an involutive system if it has a solution.*

Let $(M, N; \pi)$ be a real analytic fibered manifold and let $J^l(M, N; \pi)$ be the space of l-jets. A subsheaf of ideals (which is locally finitely generated) in $\mathcal{O}(J^l)$, the sheaf of germs of real analytic functions defined on $J^l(M, N; \pi)$, is called a system of differential equations of order l on N. We consider here systems of differential equations of the most general type which may be *non-linear* and overdetermined in general.

Roughly speaking, a differential system is said to be involutive, if its general solution can be obtained by solving successively equations of Cauchy-Kowalevsky's type.

As to exterior differential systems, E. Cartan characterized involutive systems in [2]. M. Kuranishi constructed the standard prolongation of exterior differential systems in [8]. He gave in [8] and [9] a necessary and sufficient condition under which a system can be

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prolonged to an involutive system by the standard prolongation.

Lagrange considered systems of linear differential equations of the first order with one unknown function. He showed that every such system can be prolonged either to an involutive system of the first order or to an incompatible system (see [4]). Generalizing the method of Lagrange, Jacobi proved that every system of non-linear differential equations of the first order with one unknown function can be prolonged either to an involutive system of the first order or to an incompatible system (see [4]).

E. Cartan showed in [3] that any exterior differential system with two independent variables can be prolonged to an involutive system if it has a solution. He also conjectured in [3] that any exterior differential system with more than two independent variables can be prolonged to an involutive system if it has a solution.

J. A. Schouten and W. v. d. Kulk obtained in [14] the theorem of prolongation on exterior differential systems of the special type.

H. H. Johnson treated in [7] certain types of differential systems which are prolonged to an involutive system by Kuranishi's standard prolongation.

Recently M. Kuranishi characterized involutive systems of differential equations in [10]. He gave a clear proof of his prolongation theorem on systems of differential equations also in [10].

We shall compare Kuranishi's prolongation theorem with the classical theorem of Lagrange and Jacobi. Let us consider a system of differential equations Φ of the first order with one unknown function. Then it can be proved that, if Φ is not involutive, it can not be prolonged to an involutive system by the standard prolongation. Hence Kuranishi's prolongation theorem does not contain Lagrange-Jacobi's theorem as a special case.

Roughly speaking, we say that a system of differential equations Φ of order l is *quasi-involutive*, if $p\Phi$, the prolongation of Φ, contains a system $\{\phi_1, \cdots, \phi_r\}$ of functions defined on $J^l(M, N; \pi)$ with the following property: Φ is involutive if and only if Φ contains all $\phi_i(1\leq$

$i \leqq r$).

From this point of view, it follows from Kuranishi's prolongation theorem that every system of differential equations can be prolonged to a quasi-involutive system of higher order by the standard prolongation, if it has a solution.

Generalizing the method of Lagrange and Jacobi, we shall define *the prolongation of the same order* for every system of differential equations of order l. Let $p_0\Phi$ be the set of all functions defined on $J^l(M, N; \pi)$ that are contained in $p\Phi$. Then $p_0\Phi$ is a subsheaf of ideals in $\mathcal{O}(J^l)$, which contains Φ. We call $p_0\Phi$ the prolongation of the same order of Φ.

Also we shall call the subsheaf of ideals $\bigcup_{n=1}^{\infty} p_0^n\Phi$ in $\mathcal{O}(J^l)$ *the p-closure* of Φ. Here $p_0^n\Phi$ is defined by $p_0^n\Phi = p_0(p_0^{n-1}\Phi)$ inductively. By this definition the theorem of Lagrange and Jacobi can be expressed in the following form: if Φ is a system of differential equations of the first order with one unknown function, then the p-closure of Φ is either involutive or incompatible. The success of Lagrange and Jacobi results from the fact that every system is quasi-involutive in their case.

The algebraic aspect of Kuranishi's prolongation theorem was described in a purely algebraic theorem by V. W. Guillemin, I. M. Singer and S. Sternberg in [5] and [15]. The theorem was conjectured first by them and was proved by J.-P. Serre (see Appendix in [5]). In his proof, J.-P. Serre clarified the relation between vanishing of Spencer's cohomologies and involutiveness, applying a theorem on commutative algebra in [1].

We shall combine the prolongation of Cartan and Kuranishi with that of Lagrange and Jacobi in the following way. For a given system of differential equations Φ of order l, let Ψ_0 be the p-closure of Φ. For every integer n we inductively define Ψ_n as the p-closure of $p\Psi_{n-1}$. Then we have the sequence of systems $\{\Psi_0, \Psi_1, \Psi_2, \cdots\cdots\}$ with the following property: for every n, $p_0\Psi_n$ and $p\Psi_n$ are contained in Ψ_n and Ψ_{n+1} respectively.

We say that a system Ψ of differential equations is *p-closed*, when Ψ contains $p_0\Psi$. Then every system Ψ_n above constructed for Φ is p-closed. We shall prove in Theorem 1 that a system Ψ is involutive if and only if Ψ is p-closed and quasi-involutive.

Applying the theorem of Kuranishi, Guillemin, Singer, Sternberg and Serre, we can prove that the system Ψ_n above constructed for Φ is quasi-involutive for sufficiently large n, provided it is compatible. Since every Ψ_n is p-closed, we see in Theorem 2 that Ψ_n is involutive for such sufficiently large n. Hence we can prolong every system Φ either to an involutive system or to an incompatible system.

As to prolongation of G-structures, N. Tanaka recently constructed in [19] the partial prolongation. He gave an application, proving finiteness of the automorphism groups of certain G-structures which are not of finite type. Our construction of the partial prolongation was motivated by Tanaka's construction of his prolongaiton.

As to prolongation of systems of *linear* differential equations, fruitful results are being obtained by D. C. Spencer, M. Kuranishi, D. G. Quillen, W. J. Sweeney, C. Buttin, H. Goldschmidt and others. A part of their results has been published (see [11], [13], [16], [17], [18]).

Their results are being obtained in the category of infinite differentiability. However, we discuss the problem here in the category of real analyticity. Also we discuss here local existence of solutions.

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§1. Systems of differential equations

Let $\mathcal{O}(J^l)$ be the sheaf of germs of real analytic functions defined on $J^l(M, N; \pi)$ and let Φ be a subsheaf of ideals in $\mathcal{O}(J^l)$. Here we say that Φ is a system of differential equations of order l. We do not assume that Φ is locally finitely generated. The following Proposition 1 explains what happens if we do not assume that Φ is locally finitely generated.

A point X in $J^l(M, N; \pi)$ is called an integral point of Φ, if every φ in Φ_X vanishes at X. Let $\bar{\Phi}$ be the set of all φ in $\mathcal{O}(J^l)$ such that there exists a domain $\mathcal{U}_0$ with the property that φ is defined over $\mathcal{U}_0$ and vanishes on $\mathcal{U}_0 \cap I\Phi$. Here $I\Phi$ is the set of all integral points of Φ. Then $\bar{\Phi}$ is a subsheaf of ideals in $\mathcal{O}(J^l)$, which contains Φ. We have the identity $I\Phi = I\bar{\Phi}$. Hence the system $\bar{\Phi}$ has the following property: for an arbitrary domain $\mathcal{U}$, any section over $\mathcal{U}$ which vanishes on $\mathcal{U} \cap I\bar{\Phi}$ belongs to $\bar{\Phi}$.

Proposition 1. *For any domain $\mathcal{U}_0$ in $J^l(M, N; \pi)$ containing an integral point of $\bar{\Phi}$, there exists in $\mathcal{U}_0$ an integral point X_0 around which $\bar{\Phi}$ is finitely generated on $I\bar{\Phi}$.*

Proof. For every point X on $\mathcal{U}_0 \cap I\bar{\Phi}$, we define $s(X)$ by

$$s(X) = \dim \{d\varphi;\ \varphi \in \bar{\Phi}_X\}.$$

We assume that the function $s(X)$ attains the maximum s at X_0. We can take a coordinate system $(\varphi_1, \cdots, \varphi_s, u_1, \cdots, u_t)$ around X_0 choosing φ_i in $\bar{\Phi}_{X_0} (1 \leqq i \leqq s)$. Let X_1 be an integral point which belongs to a sufficiently small neighbourhood of X_0. For every φ in $\bar{\Phi}_{X_1}$, there exists a function ψ defined on a neighbourhood $\mathcal{U}_1$ of X_1 which satisfies the congruence

$$\varphi(\varphi_1, \cdots, \varphi_s, u_1, \cdots, u_t)$$
$$\equiv \psi(u_1, \cdots, u_t) \mod(\varphi_1, \cdots, \varphi_s).$$

Since the function ψ belongs to $\bar{\Phi}_{X_1}$, we have the identity $d\psi = 0$ at every point on $\mathcal{U}_1 \cap I\bar{\Phi}$. Hence every derivative $\frac{\partial \psi}{\partial u_k}$ vanishes on $\mathcal{U}_1 \cap I\bar{\Phi}$ and belongs to $\bar{\Phi}_{X_1}$.

It follows also that every derivative $\frac{\partial^m \psi}{\partial u_{k_1} \cdots \partial u_{k_m}}$ of higher order belongs to $\bar{\Phi}_{X_1}$ and vanishes at X_1. Since the function ψ is real analytic at X_1, it vanishes identically. Hence we have the congruence $\varphi \equiv 0 \mod(\varphi_1, \cdots \varphi_s)$ for every φ in $\bar{\Phi}_{X_1}$. This proves the proposition.

An integral point X is called an ordinary integral point, if $\Phi = 0$

is a regular local equation of $I\Phi$ around X.

Let ρ_{l-1}^l be the projection from $J^l(M, N; \pi)$ onto $J^{l-1}(M, N; \pi)$ defined by

$$\rho_{l-1}^l(j_x^l(f)) = j_x^{l-1}(f).$$

Then for every pair of X and $\overline{X} = \rho_{l-1}^l X$, we have the injection i from $\mathcal{O}_{\overline{X}}(J^{l-1})$ into $\mathcal{O}_X(J^l)$ defined by

$$i\varphi = \varphi \circ \rho_{l-1}^l.$$

We identify $\mathcal{O}_{\overline{X}}(J^{l-1})$ and its image by i in $\mathcal{O}_X(J^l)$.

Let τ be the natural projection from $T_X^*(J^l)$ onto $T_X^*(J^l)/\mathrm{Im}(\pi\circ\rho_0^l)^*$. We say that Φ is compatible at X_0 if Φ_{X_0} is generated by such functions $\varphi_1, \cdots, \varphi_s$ that $\tau d\varphi_1, \cdots, \tau d\varphi_s$ are independent at X_0 and if X_0 is an integral point of Φ.

A real analytic mapping f from a domain U in N to M which satisfies $\pi\circ f = \text{identity}$ is called a solution of Φ, if $j_x^l(f)$ is an integral point of Φ for every point x in U.

Proposition 2. *Let X_0 be an ordinary integral point of Φ. If a solution f passes through X_0, then Φ is compatible at X_0.*

Proof. We choose such functions $\varphi_1, \cdots, \varphi_s$ in Φ_{X_0} that $\tau d\varphi_1, \cdots, \tau d\varphi_s$ are indepeudent at X_0, where $s = \dim\{\tau d\varphi;\ \varphi \in \Phi_{X_0}\}$.

We take a coordinate system $(\varphi_1, \cdots, \varphi_s, u_1, \cdots, u_r, x_1, \cdots, x_n)$ around X_0 with the property that $\tau d\varphi_1, \cdots, \tau\varphi_s, \tau du_1, \cdots, \tau du_r$ are independent at X_0 and every x_i belongs to $\mathcal{O}(N)$. For every φ in Φ_{X_0} there exists a function ψ which satisfies the congruence

$$\varphi(\varphi_i, u_j, x_k) \equiv \psi(u_j, x_k) \mod (\varphi_1, \cdots, \varphi_s).$$

Since $\tau d\psi = 0$ at X_0, for every $h(1 \leqq h \leqq r)$ we have

$$\frac{\partial\psi}{\partial u_h}(u_j^0, x_k^0) = 0$$

at X_0. On the other hand we have the identity

$$\psi(u_j(j_x^l(f)), x_k) = 0,$$

from which for every $i(1 \leqq i \leqq n)$ we have

$$\frac{\partial\psi}{\partial u_h}(u_j^0, x_k^0)\frac{\partial u_h(j_x^l(f))}{\partial x_i}+\frac{\partial\psi}{\partial x_i}(u_j^0, x_k^0)=0 \tag{1}$$

at X_0. Hence for every $i(1\leqq i\leqq n)$ we get

$$\frac{\partial\psi}{\partial x_i}(u_j^0, x_k^0)=0,$$

and we have the identity $d\psi=0$ at X_0. Since X_0 is an ordinary integral point, Φ_{X_0} is generated by $\varphi_1, \cdots, \varphi_s$.

§2. Prolongation of systems of differential equations

Let Φ be a system of differential equations of order l. M. Kuranishi defined in [10] the prolongation of Φ in the following way. Let ξ be a vector field on N. To every function φ in $\mathcal{O}(J^l)$ we associate the function φ_ξ in $\mathcal{O}(J^{l+1})$ defined by

$$\varphi_\xi(j_x^{l+1}(f))=\xi(\varphi(j_x^l(f))).$$

Let $p\Phi$ be the subsheaf of ideals in $\mathcal{O}(J^{l+1})$ generated by Φ and φ_ξ, where φ and ξ vary over all elements of Φ and all vector fields on N respectively. M. Kuranishi called $p\Phi$ the prolongation of Φ. Let $(x_i, y_\alpha, p_\alpha^i, \cdots, p_\alpha^{i_1\cdots i_l})$ be a coordinate system around X, where $p_\alpha^{i_1\cdots i_k}(j_x^l(f))=\dfrac{\partial^k f_\alpha}{\partial x_{i_1}\cdots\partial x_{i_k}}$. If Φ_X is generated by $\varphi_k(1\leqq k\leqq r)$, then $(p\Phi)_{\tilde{X}}$ is generated by Φ and $\partial_\sharp^i\varphi_k(1\leqq k\leqq r, 1\leqq i\leqq \dim N)$ for every $\tilde{X}$ in $(\rho_l^{l+1})^{-1}X$. Here $\partial_\sharp^i\varphi_k$ is the function defined by

$$\partial_\sharp^i\varphi_k=\frac{\partial\varphi_k}{\partial x_i}+\frac{\partial\varphi_k}{\partial y_\alpha}p_\alpha^i+\cdots+\frac{\partial\varphi_k}{\partial p_\alpha{}^{i_1\cdots i_l}}p_\alpha{}^{i_1\cdots i_l i}.$$

Generalizing the method of Lagrange and Jacobi, we define $p_0\Phi$, the prolongation of the same order of Φ, as the sheaf associated to the presheaf $\{\mathcal{U}\to\mathcal{G}(\mathcal{U})\}$. Here $\mathcal{G}(\mathcal{U})$ is the set of all elements of $\Gamma(\mathcal{U})$ that are contained in the $\Gamma(\tilde{\mathcal{U}})$-module generated by Φ and $\partial_\sharp\Phi$, where $\Gamma(\mathcal{U})$ and $\Gamma(\tilde{\mathcal{U}})$ are the rings of all sections over $\mathcal{U}$ and $(\rho_l^{l+1})^{-1}\mathcal{U}$ respectively. Then $p_0\Phi$ is a subsheaf of ideals in $\mathcal{O}(J^l)$, which contains Φ.

We say that Φ is p-closed at X, when Φ_X contains $(p_0\Phi)_X$.

Following [10], we define the subspace $C_X(\Phi)$ of $Q_X(J^l)$ for every integral point X by

$$C_X(\Phi)=\{\mathscr{X}\in Q_X(J^l);\ \mathscr{X}(\varphi)=0 \quad \text{for all} \quad \varphi\in\Phi_X\},$$

where $Q_X(J^l)=\mathrm{Ker}(d\rho^l_{l-1})$. Also we define $p(C_X(\Phi))$, the prolongation of $C_X(\Phi)$, by

$$p(C_X(\Phi))=C_X(\Phi)\otimes T^*_x(N)\cap Q_X(M)\otimes S^{l+1}(T^*_x(N)),$$

where $Q_X(M)=\mathrm{Ker}(d\pi)$. Then we have the identity

$$C_{\tilde{X}}(p\Phi)=p(C_X(\Phi))$$

for every $\widetilde{X}$ in $(\rho^{l+1}_l)^{-1}X\cap I(p\Phi)$.

Proposition 3. *Let X_0 be an ordinary integral point of Φ which satisfies the following two conditions* (i) *and* (ii):

(i) *Φ is p-closed at X_0.*

(ii) $\dim p(C_X(\Phi))=$ constant *on a neighbourhood of X_0 in $I\Phi$.*

Then there exists such a neighbourhood $\mathcal{U}_0$ of X_0 that $(\widetilde{\mathcal{V}}_0, \mathcal{V}_0;\ \rho^{l+1}_l)$ forms a fibered manifold, where $\widetilde{\mathcal{V}}_0$ and $\mathcal{V}_0$ are $(\rho^{l+1}_l)^{-1}\mathcal{U}_0 \cap I(p\Phi)$ and $\mathcal{U}_0\cap I\Phi$ respectively.

Proof. We take a system of generators $\{\varphi_1, \cdots, \varphi_s\}$ of Φ_{X_0}, where $s=\mathrm{codim}\ I\Phi$.

Let $\{\tilde{\varphi}_k;\ 1\leq k\leq ns\}$ be an arrangement of $\{\partial^r_\#\varphi_i;\ 1\leq r\leq n, 1\leq i\leq s\}$, where $n=\dim N$. And let $\{\tilde{p}_h;\ 1\leq h\leq m\binom{l+n}{l+1}\}$ be an arrangement of $\{p^{I_{l+1}}_\alpha;\ 1\leq\alpha\leq m,\ I_{l+1}=(i_1, \cdots, i_{l+1}),\ 1\leq i_1, \cdots, i_{l+1}\leq n\}$, where $m=\dim M-n$. Then we have the identity

$$m\binom{l+n}{l+1}=\dim p(C_{X_0}(\Phi))+\mathrm{rank}\left(\frac{\partial\tilde{\varphi}_i}{\partial\tilde{p}_h}\right)_{X_0}.$$

We denote by T the rank of the matrix $\left(\frac{\partial\tilde{\varphi}_i}{\partial\tilde{p}_h}\right)_{X_0}$ and assume that we have the inequality

$$\frac{D(\tilde{\varphi}_1, \cdots, \tilde{\varphi}_T)}{D(\tilde{p}_1, \cdots, \tilde{p}_T)}\neq 0$$

at X_0.

Every $\tilde{\varphi}_i$ can be expressed in the form

$$\tilde{\varphi}_i=\sum_{j=1}^{K} C_i^j \tilde{p}_j+\psi_i, \quad \left(K=m\binom{l+n}{l+1}\right),$$

where C_i^j and ψ_i belong to $\mathcal{O}_{X_0}(J^l)$. By the assumption we have the inequality

$$\det(C_i^j(X_0))_{i,j=1,2,\cdots,T}\neq 0.$$

For every $i(\geqq T+1)$ we can solve the following system of linear equations with unknown functions $B_i^j(1\leqq j\leqq T)$ uniquely:

$$\sum_{j=1}^{T} B_i^j C_j^k = C_i^k \quad (1\leqq k\leqq T).$$

The solutions B_i^j belong to $\mathcal{O}_{X_0}(J^l)$. Then every $\tilde{\varphi}_i$ $(i=t+1)$ is expressed in the form

$$\tilde{\varphi}_i=\sum_{j=1}^{T} B_i^j \tilde{\varphi}_j+\sum_{k=T+1}^{K} A_i^k \tilde{p}_k+\phi_i.$$

Here A_i^k and ϕ_i are the following functions:

$$A_i^k=C_i^k-\sum_{i=1}^{T} B_i^j C_j^k \quad (T+1\leqq k\leqq K)$$

and

$$\phi_i=\psi_i-\sum_{j=1}^{T} B_i^j \psi_j.$$

The functions A_i^k and ϕ_i belong to $\mathcal{O}_{X_0}(J^l)$, $(T+1\leqq i, k)$.

It follows from the assumption (ii) that every A_i^k vanishes on a neighbourhood of X_0 in $I\Phi$ and that every A_i^k belongs to Φ_{X_0}. Hence we see that every ϕ_i belongs to $(p_0\Phi)_{X_0}$ by the definition of $p_0\Phi$. It follows from assumption (i) that every ϕ_i belongs to Φ_{X_0}.

Let $\mathcal{U}_0$ be a sufficiently small neighbourhood of X_0. Then for every $\tilde{X}$ in $(\rho_l^{l+1})^{-1}\mathcal{U}_0$, $(p\Phi)_{\tilde{X}}$ is generated by $\varphi_1, \cdots, \varphi_s$ and $\tilde{\varphi}_1, \cdots, \tilde{\varphi}_T$. This proves the proposition, because $\sigma_l^{l+1}d\tilde{\varphi}_1, \cdots, \sigma_l^{l+1}d\tilde{\varphi}_T$ are independent at $\tilde{X}$. Here σ_l^{l+1} is the projection from $T_{\tilde{X}}^*(J^{l+1})$ onto $T_{\tilde{X}}^*(J^{l+1})/\mathrm{Im}(\rho_l^{l+1})^*$.

Remark. *Let X_0 be an ordinary integral point of Φ which only satisfies the condition* (ii) *in Proposition 3. Let us consider the system Φ_0 generated by Φ and ϕ_i $(T+1\leqq i\leqq ns)$. Then we have the identity $p_0\Phi=\Phi_0$ on $\mathcal{U}_0$, if $\Phi_0=0$ is a regular local equation of*

$I\Phi_0$ around X_0.

§3. Involutive systems of differential equations

In [10] M. Kuranishi defined involutiveness by the following

Definition 1. *We say that Φ is involutive at X_0, when the following three conditions* (i)~(iii) *are satisfied:*

(i) *X_0 is an ordinary integral point of Φ.*

(ii) *There exists a domain $\tilde{\mathcal{U}}_0$ in $J^{l+1}(M, N; \pi)$ such that $\tilde{\mathcal{U}}_0 \cap I(p\Phi)$ is a submanifold $\tilde{\mathcal{V}}_0$ which forms a fibered manifold $(\tilde{\mathcal{V}}_0, \mathcal{V}_0; \rho_l^{l+1})$ with a neighbourhood $\mathcal{V}_0$ of X_0 in $I\Phi$.*

(iii) *$C_{X_0}(\Phi)$ is an involutive subspace of $Q_{X_0}(J^l)$.*

If Φ is involutive at X_0, then Φ is compatible at X_0. To prove this fact, let f be a mapping from N to M which satisfies the identities $\pi\circ f =$ indentity and $j_{x_0}^{l+1}(f) = \tilde{X}_0$. Here $\tilde{X}_0$ is an integral point of $p\Phi$ which satisfies $\rho_l^{l+1}\tilde{X}_0 = X_0$. We see that such f exists by the condition (ii) in Def. 1. We replace the f in the proof of Prop. 2 by the f thus taken. Then we have the identity (1) by the definition of $\partial_\tau^i \psi$. This proves our assertion.

An involutive system Φ is completely integrable at X_0, if and only if the identity $C_{X_0}(\Phi) = \{0\}$ holds. Let Φ be a system of the first order which is completely integrable at every point in M. Then for every point z in M, there exists the global solution which passes through z (see [10]). T. Nagano treated in [12] completely integrable systems with singularities.

We define quasi-involutiveness by the following

Definition 2. *We say that Φ is quasi-involutive at X_0, when the following three conditions* (i)~(iii) *are satisfied:*

(i) *X_0 is an ordinary integral point of Φ.*

(ii) $\dim p(C_X(\Phi)) =$ constant *on a neighbourhood of X_0 in $I\Phi$.*

(iii) *$C_{X_0}(\Phi)$ is an involutive subspace of $Q_{X_0}(J^l)$.*

If Φ is quasi-involutive at X_0, then there exists a system $\{\phi_i; 1 \leqq i \leqq r\}$ of functions in $p_0\Phi$ with the following property: Φ is involutive at X_0 if and only if Φ contains all $\phi_i (1 \leqq i \leqq r)$. This fact can be proved by the method used in the proof of Prop. 3.

We have a necessary and sufficient condition for a system to be involutive in the following

Theorem 1. *Let Φ be a system of differential equations of order l. Then in order that Φ be involutive at X_0, it is necessary and sufficient that Φ is p-closed and quasi-involutive at X_0.*

Proof. Let Φ be p-closed and quasi-involutive at X_0. Then the conditions (i) and (iii) in Def. 1 are satisfied by the definition of quasi-involutiveness. The condition (ii) in Def. 1 is satisfied by Prop. 3.

The necessity of the conditions will be proved later.

M. Kuranishi showed in [10] that his definition of involutiveness coincides with the classical notion of involutiveness, proving the following theorem:

Let Φ be a system of differential equations of the first order which is involutive at X_0. Then Φ is generated by the following functions $\varphi^i_\alpha (0 \leqq i \leqq n,\ 1 \leqq \alpha \leqq \kappa_i)$ in a neighbourhood $\mathcal{U}_0$ of X_0 and $p\Phi$ is generated by Φ and $\partial^r_\sharp \varphi^i_\alpha (1 \leqq i \leqq n,\ 1 \leqq \alpha \leqq \kappa_i,\ 1 \leqq r \leqq i)$ in the domain $(\rho^2_1)^{-1}\mathcal{U}_0$:

$$(2)\qquad \begin{cases} \varphi^0_\alpha = y_\alpha - \psi^0_\alpha(x, q^0), \\ \varphi^i_\alpha = p^i_\alpha - \psi^i_\alpha(x, y, p^1, \cdots, p^{i-1}, q^i), \\ (0 \leqq \kappa_0 \leqq \cdots \leqq \kappa_n \leqq m), \end{cases}$$

where

$$x = (x_1, \cdots, x_n),\quad \text{independent variables}$$

$$y = (y_1, \cdots, y_m),\quad \text{dependent variables}$$

$$p^i = (p^i_1, \cdots, p^i_m),\quad \text{derivatives with respect to } x_i$$

and

$$q^i = (p^i_{\kappa_i+1}, \cdots, p^i_m),\quad q^0 = (y_{\kappa_0+1}, \cdots, y_m).$$

Then we can construct the general solution of Φ around X_0 by solving successively equations of Cauchy-Kowalevsky's type.

For a system Φ of higher order, we construct from Φ the system of differential equations of the first order $\hat{\Phi}$ in the natural way.

For example, let Φ be a system of the second order generated by

$$\varphi_k(x, y;\, z;\, p, q;\, r, s, t), \quad (1 \leqq k \leqq r),$$

where

$x,\ y$; independent valiables

z ; dependent valiable

$$p=\frac{\partial z}{\partial x}, \quad q=\frac{\partial z}{\partial y}, \quad r=\frac{\partial^2 z}{\partial x^2}, \quad s=\frac{\partial^2 z}{\partial x \partial y}, \quad t=\frac{\partial^2 z}{\partial y^2}.$$

Then $\hat{\Phi}$ is the system of the first order generated by

$$\varphi_k\left(x, y;\, z, p, q;\, \frac{\partial p}{\partial x}, \frac{\partial q}{\partial x}, \frac{\partial q}{\partial y}\right), \quad (1 \leqq k \leqq r)$$

and

$$\frac{\partial z}{\partial x}-p, \quad \frac{\partial z}{\partial y}-q, \quad \frac{\partial p}{\partial y}-\frac{\partial q}{\partial x}.$$

In general $\hat{\Phi}$ has the following properties: integration of $\hat{\Phi}$ is equivalent to that of Φ and, $\hat{\Phi}$ is involutive if and only if Φ is involutive. Hence in the general case Kuranishi's definition of involutiveness coincides with the classical notion of involutiveness.

Also $\hat{\Phi}$ has the following properties (i) and (ii):

(i) $\hat{\Phi}$ is p-closed if and only if Φ is p-closed.

(ii) $\hat{\Phi}$ is quasi-involutive if and only if Φ is quasi-involutive.

Now we shall prove the necessity in Th. 1. In the case where Φ is of the first order, we can prove the necessity by Kuranishi's theorem above stated. Hence, in the general case, we see that the conditions in Th. 1 are necessary by the remark above mentioned.

In the case where Φ is a system of the first order with one unknown function, the subspace $C_X(\Phi)$ is always involutive (see [10]). Hence in this case our theorem is equivalent to the classical theorem of Lagrange and Jacobi (see [4], [6], [10]).

§4. The prolongation theorem

M. Kuranishi obtained in [9] and [10] the following prolongation theorem:

For every non-negative integer n, let Φ_n be a system of differential equations of order $l+n$. We assume that, $p\Phi_n \subset \Phi_{n+1}$ for every

n and that, there exist for every n an ordinary integral point X_0^n of Φ_n and its neighbourhood $\mathcal{V}_n$ in $I\Phi_n$ which satisfy the following two conditions (i) *and* (ii):

(i) $\rho_{l+n}^{l+n+1} X_0^{n+1} = X_0^n$,

(ii) $(\mathcal{V}_0^{n+1}, \mathcal{V}_0^n; \rho_{l+n}^{l+n+1})$ *forms a fibered manifold.*

Then Φ_n is involutive at X_0^n for sufficiently large n.

In his new proof he applied the following

Lemma. (Kuranishi-Guillemin-Singer-Sternberg-Serre). *Let E and F be vector spaces over real numbers. For every non-negative integer n, let A_n be a linear subspace of $E \otimes S^{l+n}(F^*)$. We assume that A_n is contained in $p(A_{n-1})$ for every $n(\geqq 1)$. Then for sufficiently large n, we have the identity $p(A_{n-1}) = A_n$ and A_n is an involutive subspace.*

Let Φ be a system of differential equations of order l. We define $p_0^n \Phi$ inductively by $p_0^n \Phi = p_0(p_0^{n-1}\Phi)$ for every $n \geqq 1$. We call the subsheaf $\bigcup_{n=1}^{\infty} p_0^n \Phi$ the p-closure of Φ. Then the p-closure of Φ is a system of differential equations of order l which is p-closed at every point in $J^l(M, N; \pi)$. By the definition we see that integration of Φ is equivalent to integration of the p-closure of Φ.

Theorem 2. *Let Φ be a system of differential equations of order l. We define Ψ_n inductively as the p-closure of $p\Psi_{n-1}$ for every integer n, where Ψ_0 is the p-closure of Φ. We assume that there exists an ordinary integral point X_0^n of Ψ_n for every n with the following two properties* (i) *and* (ii):

(i) $\rho_{l+n-1}^{l+n} X_0^n = X_0^{n-1}$ $(n \geqq 1)$,

(ii) $\dim p(C_X(\Psi_n)) = \text{constant}$ *on a neighbourhood of X_0^n in $I\Psi_n$.*

Then Ψ_n is involutive at X_0^n for sufficiently large n.

Proof. Since $p\Psi_{n-1}$ is contained in Ψ_n, it follows from the Lemma that $C_{X_0^n}(\Psi_n)$ is involutive for sufficiently large n. By Theorem 1 we see that the system Ψ_n is involutive at X_0^n for such n, because Ψ_n is p-closed at every point.

Corollary. *Let Φ be a system of differential equations of order l. We assume that the system Φ has a solution f defined on a domain U in N and that, there exists a point x_0 in U with the following properties* (i) *and* (ii):

(i) $X_0^n = j_{x_0}^{l+n}(f)$ *is an ordinary integral point of* Ψ_n *for every* n.

(ii) $\dim p(C_X(\Psi_n)) = \text{constant}$ *on a neighbourhood of* X_0^n *in* $I\Psi_n$ *for every* n.

Here Ψ_n *is the system of differential equations of order* $l+n$ *defined for* Φ *in Theorem 2. Then* Ψ_n *is involutive at* X_0^n *for sufficiently large* n.

Example 1. (given by Kuranishi in [8]). Let Φ be the system generated by the following functions φ_1, φ_2 and φ_3:

$$\varphi_1 = u_1 + x_2 \frac{\partial u_1}{\partial x_2},$$

$$\varphi_2 = u_2 + x_1 \frac{\partial u_2}{\partial x_1},$$

$$\varphi_3 = \frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}.$$

The general solution of Φ is given by

$$u_1 = \frac{c}{x_1^2 x_2}, \quad u_2 = \frac{c}{x_1 x_2^2},$$

where c is a constant. The system Φ can not be prolonged to an involutive system by the standard prolongation. For the system Φ we see that Ψ_0 is generated by Φ and the following functions φ_4 and φ_5:

$$\varphi_4 = \partial_{\#}^1(x_1 u_1 - x_2 u_2) = u_1 + x_1 \frac{\partial u_1}{\partial x_1} - x_2 \frac{\partial u_2}{\partial x_1},$$

$$\varphi_5 = \partial_{\#}^2(x_1 u_1 - x_2 u_2) = x_1 \frac{\partial u_1}{\partial x_2} - u_2 - x_2 \frac{\partial u_2}{\partial x_2}.$$

The system Ψ_0 is involutive at every integral point X for which $x_1 \neq 0$ and $x_2 \neq 0$.

Example 2. Let Φ be the system generated by $\varphi_1 = y - p^2$, where

$p=\frac{dy}{dx}$. Then $\Psi_0=\Phi$, and $p\Psi_0$ is generated by φ_1 and $\varphi_2=p\left(p'-\frac{1}{2}\right)$, where $p'=\frac{d^2y}{dx^2}$. We have

$$(\Psi_1)_X=\begin{cases}\{\varphi_1,\varphi_2\}, & \left(p'=\frac{1}{2}\right)\\ \{y,p,p'\}, & \left(p'\neq\frac{1}{2}\right).\end{cases}$$

Hence Ψ_1 is involutive at $(x, 0, 0, 0)$.

By the standard prolongation the solution $y=0$ remains a singular solution of $p^n\Phi$ for every n.

Example 3. (Clairaut's equation). Let Φ be the system generated by $\varphi_1=y-px-f(p)$. Then $p\Phi$ is generated by φ_1 and $\varphi_2=p'(x-f'(p))$. We have

$$(\Psi_1)_X=\begin{cases}\{\varphi_1,p'\}, & (x-f'(p)\neq 0)\\ \{\varphi_1, x-f'(p), \ 1-f''(p)p'\}, & (p'\neq 0)\\ \{\varphi_1,\varphi_2\}, & (p'=x-f'(p)=0).\end{cases}$$

Hence Clairaut's singular solution is a regular solution of Ψ_1.

Example 4. Let Φ be the system generated by

$$\varphi_1=z-px-qy+\frac{1}{2}(p^2+q^2),$$

where $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$. Then $p\Phi$ is generated by φ_1, φ_2 and φ_3:

$$\varphi_2=r(x-p)+s(y-q),$$
$$\varphi_3=s(x-p)+t(y-q),$$

where $r=\frac{\partial^2 z}{\partial x^2}$, $s=\frac{\partial^2 z}{\partial x\partial y}$ and $t=\frac{\partial^2 z}{\partial y^2}$.

We have

$$(\Psi_1)_X=\{\varphi_1, p-x, q-y, r-1, s, t-1\}$$

at X for which $rt-s^2=0$. Hence Lagrange's singular solution is a regular solution of Ψ_1.

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