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Control problems of contingent equation

By

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0. Introduction

In this paper we shall prove the existences of solutions for the Carathéodory type contingent equation and, using this existence theorem, we shall consider a control problem for the contingent equation.

Furthermore we shall extend the existence theorem of optimal control that was considered in [3] to the case of the contingent equation.

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1. Notations and definitions

The notations used in this paper are the followings.

Let X be a metric space. The distance between two points $x, y \in X$ is denoted by $\mathrm{dist}(x, y)$. The distance between a point $x \in X$ and a set $A \subset X$ is defined by $\mathrm{dist}(x, A) = \inf\{\mathrm{dist}(x, y); y \in A\}$. For $\delta > 0$, the δ-neighborhood of a set $A \subset X$ is denoted by

$$U(A, \delta) = \{x \in X; \mathrm{dist}(x, A) < \delta\}.$$

For two compact sets $A, B \subset X$, the distance between A and B is denoted by $\mathrm{Dist}(A, B)$, where $\mathrm{Dist}(A, B) = \inf\{\delta > 0;\ U(A, \delta) \supset B, U(B, \delta) \supset A\}$. This (Hausdorff) distance makes the set of compact sets into a metric space.

Definition 1. *A compact-set (in X) valued function $F(t)$ defined on a topological space T, is said to be upper (resp. lower) semi-*

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continuous at $t_0(t_0 \in T)$, *if for every* $\varepsilon > 0$ *we can find some neighborhood of* t_0, *say* V, *such that* $U(F(t_0), \varepsilon) \supset F(t)$ (*resp.* $U(F(t), \varepsilon) \supset F(t_0)$) *for all* $t \in V$. *When* $F(t)$ *is upper (resp. lower) semi-continuous at every point of* T, $F(t)$ *is said to be upper (resp. lower) semi-continuous on* T. *A function* $F(t)$ *is said to be continuous at* t_0 (*resp. on* T) *when* $F(t)$ *is upper and lower semi-continuous at* t_0 (*resp. on* T).

Definition 2. *If a compact-set (in* X*) valued function* $F(t)$ *defined on a measurable space* E *is such that, for every compact* C *of* X, *the set* $\{t \in E;\ F(t) \subset C\}$ *is measurable, then* F *is said to be measurable on* E.

Definition 3. *For a sequence of subsets (in* X*)* $\{A_n\}$ $(n=1, 2, \cdots)$ *we define*

$$\lim_{n\to\infty} \inf A_n = \{x \in X; \lim_{n\to\infty} \operatorname{dist}(x, A_n) = 0\}$$

$$\lim_{n\to\infty} \sup A_n = \{x \in X; \varliminf_{n\to\infty} \operatorname{dist}(x, A_n) = 0\}$$

and

$$\lim_{n\to\infty} A_n = \lim_{n\to\infty} \inf A_n = \lim_{n\to\infty} \sup A_n,$$

when $\lim_{n\to\infty} \inf A_n = \lim_{n\to\infty} \sup A_n$.

It is known [1] that these sets are closed.

For a set A in X we denote by clA the closure of A.

We denote by R^m an m-dimensional Euclidean space with the usual norm $|x|$ for each $x \in R^m$, and by I the compact interval $[t_0, t_0+a]$ in R^1.

Let $F(t)$ be a compact and convex set valued measurable function (*in* R^m) defined on a measurable set E.

We denote by $|F|(t)$ a scalar function $\sup\{\operatorname{dist}(0, x); x \in F(t)\}$. If $|F|(t)$ is integrable on E, then the Lebesgue integral $\int_E F(t)dt$ has been defined in [2]. In this case we say that F is integrable.

2. Propositions

In [3] we have proved the following Propositions.

Proposition 1. *Let* $\{F_n(t)\}$ $(n=1, 2, \cdots,)$ *be a sequence of compact-set (in* R^m*) valued functions defined and measurable on* E *and monotone decreasing in* n.

Then $\bigcap_{n=1}^{\infty} F_n(t)$ *is measurable.*

Proposition 2. *Let* $\{F_n(t)\}$ $(n=1, 2, \cdots)$ *be a sequence of compact-set (in* R^m*) valued functions defined and measurable on* E *and* $F(t)\subset C(t)$ $(n=1, 2, \cdots)$ *for some compact-set (in* R^m*) valued function* $C(t)$. *Then cl* $\bigcup_{n=1}^{\infty} F_n(t)$ *is measurable.*

Proposition 3. *Let* $F(t)$ *be a compact-set (in* R^m*) valued function defined on* E. *Suppose that* $\text{meas}(E)<\infty$.

$F(t)$ *is measurable on* E *if and only if, for every real positive* ε, *there exists a compact set* E' *in* E *such that* $\text{meas}(E-E')<\varepsilon$ *and such that* $F(t)$ *is upper semi-continuous on* E'.

Remark. In [2] it has been proved that the continuity of F in this sense follows from the measurability of F.

Proposition 4. *Let* $F(t)$ *be a compact-set (in* R^m*) valued function defined and measurable on* E. *Suppose that* $\text{meas}(E)<\infty$. *Then there exists a measurable function* $f(t)$ *on* E *such that* $f(t)\in F(t)$ *for each* $t\in E$.

Proposition 5. *Let* $F(t, x)$ *be a compact-set (in* R^m*) valued function defined on* $I\times R^m$ *and measurable in* t *for each fixed* $x\in R^m$ *and upper semi-continuous in* x *for each fixed* $t\in I$.

Then $F(t, x(t))$ *is measurable in* t *for each continuous function* $x(t)\in R^m$.

Remark. Proposition 5 also holds if I is replaced by a compact set.

Further we can prove the following Propositions.

Proposition 6. *Let* $F(t, x)$ *be a compact-set (in* R^m*) valued function defined on* $I\times R^m$ *and measurable in* t *for each fixed* $x\in R^m$ *and upper semi-continuous in* x *for each fixed* $t\in I$.

Then $F(t, x(t))$ is measurable in t for each measurable function $x(t)\in R^m$.

Proof. Since $x(t)$ is measurable on I, for every real positive ε we can find a compact set J in I such that $x(t)$ is continuous on J. $F(t, x(t))$ is measurable on J.

Hence $F(t, x(t))$ is measurable on I.

Proposition 7. *Let $F(t, u)$ be a compact-set (in R^m) valued function defined on $I\times R^r$ and measurable in t for each fixed $u\in R^r$ and continuous in u for each fixed $t\in I$.*

Then for every compact set U (in R^r) $F(t, U)$ is a compact-set (in R^m) valued function and measurable in t.

Proof. For each fixed $t\in I, F(t, U)$ is a compact set in R^m. Indeed, let $\{x_n\}$ be a sequence of points in $F(t, U)$. For each n we can select $u_n\in U$ such that $x_n\in F(t, u_n)$. Since U is a compact set, we can assume that $\{u_n\}$ converges to $u\in U$. From the continuity of $F(t, u)$ in u, there is a subsequence of $\{x_n\}$ which converges to some $x\in F(t, u)\subset F(t, U)$.

The measurability of $F(t, U)$ follows from the following relation; $F(t, U)=cl\bigcup_{i=1}^{\infty}F(t, u_i)$, where $\{u_i\}$ is a dense subset of U.

Proposition 8. *Let $F(t, u)$ be a compact-set (in R^m) valued function defined on $I\times R^r$ and measurable in t for each fixed $u\in R^r$ and continuous in u for each fixed $t\in I$.*

Then $F(t, Q(t))$ is measurable in t for each measurable compact-set valued function $Q(t)\subset R^r$.

Proof. We first prove Proposition when $Q(t)$ is continuous. We denote a subdivision of I by D: $t_0<t_1<\cdots<t_k=t_0+a$, and $\delta(D)=\max_{0\leq i\leq k-1}(t_{i+1}-t_i)$. For $t_i\leqq t<t_{i+1}$ we define $Q(t;D)=Q(t_i)$. Let $\{D_n\}$ $(n=1, 2, \cdots)$ be a sequence of subdivisions of I such that each division point of D_n belongs to that of D_{n+1} and $\{\delta(D_n)\}$ tends to zero as $n\to\infty$. Also we define $F_n(t)=F(t, Q(t;D_n))$, and

$$I'(C)=\bigcup_{N=1}^{\infty}\bigcap_{n=N}^{\infty}I(F_n\subset\subset C\},$$

where $I(F_n \subset\subset C)$ denotes the set $\{t \in I; F_n(t) \subset\subset C\}$. $I(F_n \subset\subset C)$ is measurable for every n, and hence $I'(C)$ is measurable.

From the continuity of $F(t, x)$ and $Q(t)$, we can show that $I(F \subset\subset C) \subset I'(C)$ and $I(F \subset\subset C') \supset I'(C)$ for every compact set $C'(\supset \supset C)$. Let $C_n(n=1, 2, \cdots)$ be the compact set $cl\, U(C, \frac{1}{n})$. From the relations stated above, it follows $I(F \subset\subset C_{n+1}) \subset I'(C_{n+1}) \subset I(F \subset\subset C_n)$. Since $I(F \subset C) = \bigcap_{n=1}^{\infty} I(F \subset C_n) = \bigcap_{n=1}^{\infty} I'(C_n)$, $F(t, Q(t))$ is measurable on I.

When $Q(t)$ is measurable on I, for every $\varepsilon > 0$ there is a compact set J in I such that $\text{meas}(I - J) < \varepsilon$, and $Q(t)$ is continuous on J.

Therefore $F(t, Q(t))$ is measurable on J and hence $F(t, Q(t))$ is measurable on I.

Propositon 9. *Let $Q(x)$ be a compact-set (in R^r) valued function defined on R^m and upper semi-continuous in $x \in R^m$.*

Let $F(x, u)$ be a compact-set (in R^m) valued function defined on $R^m \times R^r$ and be upper semi-continuous in (x, u).

Then $R(x) = F(x, Q(x))$ is upper semi-continuous in x.

Proof. Take any $x_0 \in R^m$, and $\varepsilon > 0$. According to the upper semi-continuity of $Q(x)$, $F(x, X) \subset U(R(x_0), \varepsilon)$ whenever $\text{Dist}(X, Q(x_0)) < \delta$ and $|x - x_0| < \delta$ for some $\delta (> 0)$. We can take $Q(x_0) \cup Q(x)$ as X if x is sufficiently near x_0. Consequently the relation

$$R(x) = F(x, Q(x)) \subset F(x, Q(x_0) \cup Q(x)) \subset U(R(x_0), \varepsilon)$$

holds since $\text{Dist}\,(Q(x_0) \cup Q(x), Q(x_0)) < \delta$ holds for every x sufficiently near x_0.

Proposition 10. *Let $F(t, u)$ be a compact-set (in R^m) valued function defined on $I \times R^r$ and measurable in t for each fixed $u \in R^r$ and continuous in u for each fixed $t \in I$.*

Let $Q(t)$ be a compact-set (in R^r) valued function defined and bounded on I and measurable in t. Let $y(t)$ be a measurable function (in R^m) on I.

If $\{u; F(t, u) \ni y(t),\ u \in Q(t)\}$ is empty nowhere on I, then the *compact-set valued function* $F(t)$ *(in* R^r*) defined as*

$$F(t)=\{u;F(t,u)\ni y(t),u\in Q(t)\}$$

is compact-set valued function and measurable on I.

Proof. From the compactness of $Q(t)$ and upper semi-continuity of $F(t,u)$ in u, $F(t)$ can be verified to be compact for each $t\in I$.

Take a denumerable set of points $\{u_i\}$ $(i=1,2,\cdots)$ which is dense in R^r, and a monotone decreasing sequence $\{\varepsilon_j\}\downarrow 0$ $(j=1,2,\cdots)$. Denote the following compact set by $F_{ij}(t)$.

$$F_{ij}(t)=\{u\in\{u_1\cdots u_i\};\operatorname{dist}(y(t),F(t,u))<\varepsilon_j,\ u\in U(Q(t),\varepsilon_j)\}.$$

$F_{ij}(t)$ is measurable on I, and the relation

$$F(t)=\bigcap_{j=1}^{\infty} cl\bigcup_{i=1}^{\infty}F_{ij}(t)$$

shows that $F(t)$ is measurable on I.

Proposition 11. *Let* $\{F_n(t)\}$ $(n=1,2,\cdots)$ *be a sequence of compact-set (in* R^m*) valued functions defined and measurable on* E. *Suppose that there exists a compact-set (in* R^m*) valued function* $F_0(t)$ *such that* $F_n(t)\subset F_0(t)$ $(n=1,2,\cdots)$ *on* E. *Then* $F(t)=\lim_{n\to\infty}\sup F_n(t)$ *is measurable.*

Proof. Since $F_n(t)\subset F_0(t)$ on E, $\lim_{n\to\infty}\sup F_n(t)$ exists and is a compact set in R^m. $F(t)$ is measurable since $F(t)$ can be expressed as follows.

$$F(t)=\bigcap_{N=1}^{\infty} cl\bigcup_{n=N}^{\infty}F_n(t)$$

on E.

Proposition 12. *Let* $\{F_n(t)\}$ $(n=1,2,\cdots)$ *be a sequence of compact and convex set (in* R^m*) valued functions defined and integrable on* E, *and suppose that there is an integrable function* $F_0(t)$ *(which is a compact and convex set valued function) such that* $F_n(t)\subset F_0(t)$ $(n=1,2,\cdots)$ *on* E, *then* $\lim_{n\to\infty}\sup F_n(t)$ *is integrable and*

$$\lim_{n\to\infty}\sup\int_E F_n(t)dt\subset\int_E\lim_{n\to\infty}\sup F_n(t)\,dt \text{ holds.}$$

Proof. By Proposition 2.1 [1] $\limsup_{n\to\infty} F_n(t)$ exists for each $t\in E$. Similarly $\limsup_{n\to\infty}\int_E F_n(t)dt$ exists, since

$$\int_E F_n(t)dt\subset\int_E F_0(t)dt.$$

Let x be any point in $\limsup_{n\to\infty}\int_E F_n(t)dt$. Then there is a subsequence $\{F_{n'}(t)\}$ such that

$$\lim_{n\to\infty} \text{dist}\ (x, \int_E F_{n'}(t)\ dt)=0$$

holds. By Proposition 1.3 [1] we can select a further subsequence $\{F_{n''}(t)\}$ such that $\lim_{n\to\infty} F_{n''}(t)=F(t)$ exists for each $t\in E$. By Proposition 3.2 [1] we conclude that

$$\lim_{n\to\infty} \text{Dist}\ (F_{n''}(t), F(t))=0.$$

Since

$$\begin{aligned}&\text{dist}\ (x, \int_E F(t)dt)\\ &\leqq\lim_{n\to\infty} \text{dist}\left(x, \int_E F_{n''}(t)\ dt\right)+\lim_{n\to\infty} \text{Dist}\left(\int_E F_{n''}(t)dt, \int_E F(t)dt\right),\end{aligned}$$

and

$$\lim_{n\to\infty} \text{Dist}\left(\int_E F_{n''}(t)dt, \int_E F(t)dt\right)=0$$

[2], then

$$\text{dist}\left(x, \int_E F(t)dt\right)=0$$

holds.

Hence

$$x\in\int_E \lim_{n\to\infty} F_{n''}(t)dt\subset\int_E \limsup_{n\to\infty} F_n(t)dt.$$

Remark. Proposition 12 also holds if the conditions $F_n(t)\subset F_0(t)$ are replaced by the following conditions; $|F_n(t)|\leqq M(t)$ on E for some integrable scalar function $M(t)$.

3. Existence theorem for contingent equation

Theorem 1. *Let the compact and convex set (in R^m) valued function $F(t, x)$ be defined on a parallelepiped R(in $R^1 \times R^m$): $t_0 \leqq t \leqq t_0+a$, $|x-x_0| \leqq b$, and measurable in t for each fixed x, and upper semi-continuous in x for each fixed t. Let there exist a scalar function $M(t)$, integrable on $I=[t_0, t_0+a]$, such that*

$$|F(t, x)| \leqq M(t), \int_{t_0}^{t_0+a} M(t)\,dt \leqq b$$

for all $(t, x) \in R$.

Then there is an absolutely continuous function $x(t)$ such that

$$dx(t)/dt \in F(t, x(t))$$

for almost all t in I, and $x(t_0)=x_0$.

Proof. Let D be a subdivision of $I: t_0 < t_1 < \cdots < t_k = t_0+a$. We denote $\max_{0 \leqq i \leqq k-1} (t_{i+1}-t_i)$ by $\delta(D)$. Since $F(t, x_0)$ is measurable on $[t_0, t_1]$, we can select a measurable function $f_0(t)$ such that $f_0(t) \in F(t, x_0)$ for each $t \in [t_0, t_1]$.

For $t_0 \leqq t \leqq t_1$ we define

$$x(t; D) = x_0 + \int_{t_0}^{t} f_0(t)\,dt,$$

and put

$$x_1 = x_0 + \int_{t_0}^{t_1} f_0(t)\,dt.$$

Then

$$|x_1 - x_0| \leqq \int_{t_0}^{t_1} M(t)\,dt \leqq b$$

holds.

We define inductively $\{x_i\}$ and $\{f_i(t)\}$ $(i=0, 1, \cdots n-1)$ as follows. Suppose that we have defined x_i such that

$$|x_i - x_0| \leqq \int_{t_0}^{t_i} M(t)\,dt,$$

and then for $t_i \leqq t \leqq t_{i+1}$ we define

$$x(t; D) = x_i + \int_{t_i}^{t} f_i(t)\,dt$$

and put $x_{i+1}=x(t_{i+1};D)$, where $f_i(t)$ is a measurable function such that $f_i(t)\in F(t, x_i)$ for each $t\in[t_i, t_{i+1}]$. Then

$$|x_{i+1}-x_0|\leqq\int_{t_0}^{t_{i+1}}M(t)dt$$

holds.

The function $x(t;D)$ has thus been defined for all $t\in I$. By defining $y(t;D)=x_i$ for $t\in[t_i, t_{i+1})$, $0\leqq i\leqq k-2$, and $y(t;D)=x_{k-1}$ for $t\in[t_{k-1},t_k]$

$$x(t;D)\in x(\tau;D)+\int_\tau^t F(t, y(t;D))dt$$

holds for all $t\in[\tau, t_0+a]$.

Hence $x(t;D)$ is absolutely continuous, and

$$|x'(t;D)|\leqq M(t),\ x(0;D)=x_0,$$

hold independently of the choice of D.

Let $\{D_n\}$ $(n=1, 2, \cdots)$ be a sequence of subdivisions of I such that $\{\delta(D_n)\}$ tends to zero as $n\to\infty$. Since $\{x(t;D_n)\}$ is equi-continuous on I and satisfies the same initial condition, $\{x(t;D_n)\}$ is a normal family. Hence we can select a subsequence of $\{x(t;D_n)\}$ (without changing the notation) which converges to a function $x(t)$ uniformly on I, and

$$|x'(t)|\leqq M(t),\ x(0)=x_0$$

hold.

From the equi-continuity of $\{x(t;D_n)\}$ and the construction of $\{y(t;D_n)\}$, we conclude that $\{y(t;D_n)\}$ also converges to $x(t)$ uniformly on I.

From the relation

$$x(t;D_n)\in x(\tau, D_n)+\int_\tau^t F(t, y(t;D_n))dt$$

and the upper semi-continuity of $F(t, x)$ in x and Proposition 12

$$\begin{aligned}x(t)\in x(\tau)&+\limsup_{n\to\infty}\int_\tau^t F(t, y(t;D_n))dt\\ &\subset x(\tau)+\int_\tau^t \limsup_{n\to\infty} F(t, y(t;D_n))dt\\ &\subset x(\tau)+\int_\tau^t F(t, x(t))dt.\end{aligned}$$

Then

$$dx(t)/dt \in F(t, x(t))$$

holds for almost all t in I and $x(t_0) = x_0$.

Theorem 2. *Let $F(t, x)$ be a compact and convex set (in R^m) valued function defined on $I \times R^m$, and be measurable in t for each fixed $x \in R^m$ and upper semi-continuous in x for each fixed $t \in I$. Suppose that $x \cdot y \leqq C(|x|^2 + 1)$ $(C > 0)$ holds for every y such that $y \in F(t, x)$, where the dot denotes the scalar product, and that $F(t, x)$ carries every bounded set in $I \times R^m$ into a bounded set in R^m.*

Then for every $x_0 \in R^m$ there exists an absolutely continuous vector function $x(t)$ such that

$$dx(t)/dt \in F(t, x(t))$$

for almost all t in I, and $x(t_0) = x_0$.

Proof. We first prove this theorem under the assumption that $F(t,x)$ is bounded. Similarly as in the proof of Theorem 1 we define $\{x(t; D_n)\}$, and $\{y(t; D_n)\}$ such that

$$x(t; D_n) \in x(\tau, D) + \int_\tau^t F(t, y(t; D_n))dt.$$

Since $F(t, x)$ is bounded, $\{x(t; D_n)\}$ is a normal family and then $\{x(t; D_n)\}$ with $\{y(t; D_n)\}$ can be assumed to converge to a function $x(t)$ uniformly on I. From Proposition 12 we conclude that

$$x(t) \in x(\tau) + \int_\tau^t F(t, x(t))dt.$$

Hence

$$dx(t)/dt \in F(t, x(t))$$

for almost all $t \in I$, and $x(t_0) = x_0$.

Next we denote $(|x_0|^2 + 1)\exp(2Ca) - 1$ by $H^2 (H > 0)$.

By taking a sufficiently large C, we can assume that $H > 1$. Also we define

$$\overline{F}(t,x) = \begin{cases} F(t, x), & |x| \leqq H \\ F(t, Hx/|x|), & |x| > H. \end{cases}$$

$\overline{F}(t, x)$ defined as above can be verified to be measurable in t for each fixed $x \in R^m$ and upper semi-continuous in x for each fixed $t \in I$, and is bounded on $I \times R^m$. Hence there exists an absolutely continuous function $x(t)$ such that

$$dx(t)/dt \in \overline{F}(t, x(t))$$

for almost all $t \in I$ and $x(t_0) = x_0$.

For $\overline{F}(t, x)$ the same relation as $F(t, x)$, i.e., $x \cdot y \leqq C(|x|^2 + 1)$ for every $y \in \overline{F}(t, x)$, holds. From this condition we can conclude that all solutions of

$$dx/dt \in \overline{F}(t, x) \text{ and } x(t_0) = x_0$$

satisfy $|x(t)| \leqq H$ on I.

Indeed if $z(t) = |x(t)|^2 + 1$, then $dz(t)/dt \leqq 2Cz(t)$, hence $z(t) \leqq (|x_0|^2 + 1) \exp(2Ca)$, *i.e.* $|x(t)| \leqq H$. In $|x| \leqq H$, $\overline{F}(t, x)$ and $F(t,x)$ coincide. Hence a solution $x(t)$ for

$$dx/dt \in \overline{F}(t, x), \quad x(t_0) = x_0$$

is also that for

$$dx/dt \in F(t,x), \quad x(t_0) = x_0.$$

Consequently we have proved the existence of solutions.

Theorem 3. *Let $F(t, x)$ satisfy the condition in Theorem 2.*

Then for every compact set K in R^m the collection of all solutions $x(t)$ of the contingent equation such that $x(t_0) \in K$ is compact in the topology of the uniform convergence.

Proof. Let $\{x_n(t)\}$ $(n = 1, 2, \cdots)$ be a sequence of solutions. We must show that there exists a subsequence which converges uniformly to a solution. Since

$$dx_n(t)/dt \in F(t, x_n(t))$$

for almost all $t \in I$, it follows that

$$x_n(t) - x_n(\tau) \in \int_\tau^t F(t, x_n(t))dt.$$

On the other hand $\{x_n(t)\}$ is uniformly bounded and equi-continuous on I. Thus there exists a subsequence (without changing the notation) which converges uniformly to some function $x(t)$, and $x(t_0) \in K$ holds.

Further, since all the $x_n(t)$ satisfy the same Lipschitz condition, their limit $x(t)$ satisfies the same Lipschitz condition.

Hence $x(t)$ is absolutely continuous.

By Proposition 12,

$$\begin{aligned} x(t)-x(\tau) &\in \limsup_{n\to\infty} \int_\tau^t F(t, x_n(t))dt \\ &\subset \int_\tau^t \limsup_{n\to\infty} F(t, x_n(t))dt \\ &\subset \int_\tau^t F(t, x(t))dt \end{aligned}$$

holds for all $t\in[\tau, t_0+a]$.

Since $x(t)$ is absolutely continuous,

$$dx(t)/dt \in F(t, x(t))$$

for almost all $t\in I$. This completes the proof of the theorem.

4. Existence of optimal control

In this chapter we shall consider the control problem for the contingent equation and prove the existence of optimal control.

We shall make the following assumptions.

1) $F(t, x, u)$ is a compact-set (in R^m) valued function defined in $I\times R^m\times R^r$.

2) $F(t, x, u)$ is measurable in t for each fixed $(x, u)\in R^m\times R^r$, and continuous in (x, u) for each fixed $t\in I$.

3) $F(t, x, u)$ carries every bounded set in $I\times R^m\times R^r$ into a bounded set in R^m.

4) $Q(t, x)$ is a compact-set (in R^r) valued function defined in $I\times R^m$ and measurable in t for each fixed $x\in R^m$, and upper semi-continuous in x for each fixed $t\in I$.

5) $Q(t, x)$ carries every bounded set in $I\times R^m$ into a bounded set in R^r.

6) $R(t, x)=F(t, x, Q(t, x))=\{y; y\in F(t, x, u),\ u\in Q(t, x)\}$ is a compact and convex set (in R^m) for each $(t, x)\in I\times R^m$.

7) For every t and x and $u\in Q(t, x)$, $x\cdot y\leqq C(|x|^2+1)$ holds for every y such that $y\in F(t, x, u)$, where the dot denotes the scalar product.

8) K is a compact set in R^m. $K(t)$ is a compact-set (in R^m) valued function defined in I, and upper semi-continuous in t.

9) $f(t, x)$ is a real function defined in $I\times R^m$, and is measurable in t for each fixed $x\in R^m$, and continuous in x for each fixed $t\in I$, and is bounded from below.

If $u(t)$ is a measurable function in R^r, $F(t, x, u(t))$ is measurable in t for each fixed $x\in R^m$, and is continuous in x for each fixed $t\in I$. Therefore for each measurable function $u(t)$ the system of equations

$$\begin{cases} dx(t)/dt\in F(t, x(t), u(t)) & \text{for almost all } t\in I, \\ x(t_0)=x_0 \end{cases}$$

has an absolutely continuous solution $x(t)$ for every $x_0\in R^m$, if $F(t,x, u)$ satisfies the assumptions stated above.

We say that $x(t)$ is the trajectory corresponding to a control $u(t)$ (measurable in t and $\in Q(t, x(t))$ on I) if $x(t)$ is an m-dimensional, absolutely continuous function satisfying the above system of equations.

We say that a control $u(t)$, defined for $t_0\leq t\leq \bar{t}$, $\bar{t}\in I$, transfers K to $K(t)$ if one of the trajectories $x(t)$ corresponding to $u(t)$ satisfies the relations $x(t_0)\in K$ and $x(\bar{t})\in K(\bar{t})$.

We shall consider the problem of finding a control function $u(t)$ which transfers K to $K(t)$ and which minimizes the cost functional

$$J(x)=\int_{t_0}^{\bar{t}} f(t, x(t))dt,$$

where $x(t)$ is one of the solutions corresponding to $u(t)$, and $\bar{t}$ represents a value of t such that $x(t)\in K(t)$.

Theorem 4. *Suppose that the conditions stated above are satisfied. Also suppose that there exists at least one control $u(t)$ which transfers K to $K(t)$ on I.*

Then there exists an optimal control, i.e., a measurable func-

tion $u^*(t)$ for which one of the corresponding solutions, $x^*(t)$, with initial condition $x^*(t_0)$ K, attains $K(t^*)$ for some t^* in I, and

$$\inf J(x)=J(x^*)=\int_{t_0}^{t^*} f(t, x^*(t))dt,$$

where, in addition, $u^*(t)\in Q(t, x^*(t))$.

Proof. Now consider the set of all the $x(t)$ satisfying

$$dx(t)/dt\in F(t, x(t), u(t))$$

almost everywhere on I, $x(t_0)\in K$ and $x(\bar{t})\in K(\bar{t})$ for some $\bar{t}\in I$, where, in addition, $u(t)\in Q(t, x(t))$ for some control $u(t)$. Since one such solution exists by hypothesis, this set is not empty. Consequently we can select a sequence of trajectories $\{x_n(t)\}$ on I, with

$$J(x_n)=\int_{t_0}^{t_n} f(t, x_n(t))dt$$

decreasing monotonically to inf $J(x)$, where t_n represents a value of t such that $x_n(t)\in K(t)$. $x_n(t)$ satisfy the following relations

$$dx_n(t)/dt\in R(t, x_n(t))$$

almost everywhere on I and $x_n(t_0)\in K$. By the compactness of solutions of the contingent equation, we conclude that

$$dx^*(t)/dt\in R(t, x^*(t)),\ x^*(t_0)\in K,$$

where $x^*(t)$ is a limit function of a subsequence of $\{x_n(t)\}$. Also we can select a further subsequence (without changing the notation) such that $\{t_n\}$ converges to some t^* in I since I is a compact interval. Further, making use of the equi-continuity of $\{x_n(t)\}$ and the upper semi-continuity of $K(t)$, we conclude that $x^*(t_0)\in K$ and $x^*(t^*)\in K(t^*)$. From Proposition 8 we can select a measurable function $u^*(t)$ such that

$$dx^*(t)/dt\in F(t, x^*(t), u^*(t))$$

almost everywhere on I and $u^*(t)\in Q(t, x^*(t))$ on I. Finally

$$J(x_n)=\int_{t_0}^{t_n} f(t, x_n(t))dt$$

approaches

$\int_{t_0}^{t^*} f(t, x^*(t))dt$ as $n \to \infty$ and hence $\inf J(x) = J(x^*)$. Thus $x^*(t)$ on $t_0 \leq t \leq t^*$ is optimal.

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