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An algebra of pseudo difference schemes and its application

By

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1. Introduction

In this paper we discuss an algebra of one-parameter families of bounded operators mapping the space L^2 of square integrable vector valued functions into itself. As we know, the algebra of pseudo differential operators by Caldéron-Zygmund and its extensions are very useful devices to obtain the energy inequality for the Cauchy problem of non-symmetric hyperbolic systems of differential equations [1] [2] [4] [6] [7]. Analogously we introduce the algebra of one-parameter families of bounded operators for the purpose of getting some local energy inequality to assure the stability of a finite difference scheme for regularly non-symmetric hyperbolic systems [3]. The authors are greatly indebted to the advice of Prof. S. Matsuura particularly for the formalism in section 3. We wish to thank him for this advice.

2. An example

In the theory of pseudo differential operators, we have a special operator denoted by Λ which is the Fourier transform of the multiplication operator $|\xi|$ in L^2_ξ where ξ means a real vector $(\xi_1, \cdots, \xi_n)$ in R^n_ξ. This operator plays an important role in the theory of differential equations with variable coefficients. In that case the commutator $a(x)\Lambda-\Lambda a(x)$, where $a(x)$ is a smooth bounded function, is a bounded operator in the L^2 sense. While in the L^2 theory of finite difference

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schemes with variable coefficients, a commutator $a(x)T-Ta(x)$, where T is a translation operator defined by $Tu(x)=u(x+he)$ (h is a positive number, e is some unit n-vector), have the property that it is a one-parameter family of bounded operators with norm $O(h)$. This property corresponds to the boundedness of operators in the theory of pseudo differential operators. Now we ask what is a family corresponding to the operator Λ. Maybe there are several families corresponding to this operator. Here we consider one example which is defined in the following way:

$$\Lambda_h=\mathcal{F}^{-1}|\sin h\xi|\mathcal{F} \tag{1}$$

where $\mathcal{F}$ means usual Fourier transformation and $\sin h\xi$ means a vector $(\sin h\xi_1, \sin h\xi_2, \cdots, \sin h\xi_n)$. $|\sin h\xi|$ means, of course, the absolute value of this vector.

We shall show first this family of operators has a similar property to that of T. We assume that $a(x)$ is smooth and is equal to a constant for large $|x|$. Then we can prove that $a(x)\Lambda_h-\Lambda_h a(x)$ is a family of bounded operators with norm $O(h)$. Because, for every square integrable function u putting $a_1(x)=a(x)-a(\infty))$, we get

$$\begin{aligned}
&\mathcal{F}[a(x)\Lambda_h-\Lambda_h a(x)]u\\
&\quad=\mathcal{F}[a_1(x)\Lambda_h-\Lambda_h a_1(x)]u\\
&\quad=\int\hat{a}_1(\xi-\eta)|\sin h\eta|\hat{u}(\eta)d\eta-\int\hat{a}_1(\xi-\eta)|\sin h\xi|\hat{u}(\eta)d\eta\\
&\quad=\int\hat{a}_1(\xi-\eta)[|\sin h\eta|-|\sin h\xi|]\hat{u}(\eta)d\eta.
\end{aligned}$$

$$\begin{aligned}
\|[a_1(x)\Lambda_h-\Lambda_h a_1(x)]u\|&\leqq\left\|\int|\hat{a}_1(\xi-\eta)||\sin h\eta-\sin h\xi||\hat{u}(\eta)|d\eta\right\|\\
&=\left\|\int|\hat{a}_1(\xi-\eta)|\left|2\sin h\frac{(\xi-\eta)}{2}\right|\left|\cos\frac{h(\xi+\eta)}{2}\right||\hat{u}(\eta)|d\eta\right\|\\
&\leqq h\left\|\int|\hat{a}_1(\xi-\eta)||\xi-\eta||\hat{u}(\eta)|d\eta\right\|\\
&\leqq h\||\hat{a}_1(\xi)|\cdot|\xi|\|_{L_1}\|u\|.
\end{aligned}$$

3. An algebra of one-parameter families of operators

Definition. *A one parameter family H_h of bounded operators* *mapping L^2 into itself is called a "null scheme" if it satisfies the following inequality. (And we denote the set of all null schemes by $\mathcal{N}_0$).*

$$\|H_h\| = O(h). \tag{2}$$

Next we consider the set $\mathcal{K}$ of $p \times p$ matrix valued functions k defined for x in R^n_x and ξ in R^n_ξ with the following properties

i) k is homogeneous of degree zero in ξ.

ii) each k is independent of x for $|x| > R$; R is a fixed positive constant.

iii) $k(x, \xi)$ belongs to $C^\infty(R^n_x \times (R^n_\xi - \{0\}))$.

The set $\mathcal{K}$ forms an algebra of matrix valued functions by pointwise addition and multiplication.

Now we associate a one-parameter family of operators K_h with each function $k \in \mathcal{K}$ by the following formula:

$$K_h u = \text{l.i.m.} \int e^{i\langle x, \xi \rangle} k(x, \lambda(h\xi)) \hat{u}(\xi) d\xi \quad u \in L^2_x \tag{3}$$

where $\lambda(\xi) = (\lambda_1(\xi), \lambda_2(\xi), \cdots, \lambda_n(\xi))$ is a real valued vector function which satisfies the following two conditions:

$$(\alpha) \qquad |\lambda(\xi) - \lambda(\eta)| \leqq C|\xi - \eta|, \ |\lambda(\xi)| \leqq M \quad \text{for } \xi, \eta \in R^n_\xi$$

$$(\beta) \qquad \lim_{h \to \infty} \frac{\lambda(h\xi)}{h} = \xi \quad \text{for } \xi \in R^n_\xi.$$

The existence of the limit in the mean in (3) is based on the following estimate (4) and the expansion in lemma 2 below (see (10)).

$$\|K_h\| \leqq \sup_{\substack{x \in R^n \\ \xi \in \omega \\ |\beta| \leq m}} |D^\beta_\xi k(x, \xi)|. \tag{4}$$

with ω a fixed compact in $R^n_\xi - \{0\}$, m an integer depending only on the dimension n of the space, C a constant.

Now we consider the algebra of one-parameter families of bounded operators mapping L^2 into itself generated by K_h defined above. We denote this algebra by $\mathcal{P}$ and we call it the algebra of "pseudo difference schemes."

Next we need to define an operator Λ_h (of which one example was stated in section 2) by the following:

(5) $$\Lambda_h = \mathcal{F}^{-1} |\lambda(h\xi)| \mathcal{F}.$$

Definition. *A family K_h which belongs to $\mathcal{P}$ is called "negligible" if $\Lambda_h K_h$ and $K_h \Lambda_h$ belong to $\mathcal{N}_0$. The set of all negligible schemes is denoted by $\mathcal{N}$.*

Lemma 1. *If $a(x)$ belongs to $\mathcal{K}$, then $[a(x), \Lambda_h] = a(x)\Lambda_h - \Lambda_h a(x)$ is a null scheme.*

Proof is completely similar to the proof for the case $\lambda_h(x) = \sin h\xi$ which is stated in section 2. We only used the condition (α) of $\lambda_h(\xi)$. In fact, for all $u \in L^2_x$, there exists a constant C

$$\| [a(x), \Lambda_h] u \| \leqq C \| \hat{a}_1(\xi) |\xi| \|_{L^1} \cdot \|u\|_{L^2} \cdot h, \quad a_1(x) = a(x) - a(\infty).$$

Lemma 2. (*P. D. Lax*)

Every k in $\mathcal{K}$ can be expanded in a series

(6) $$k(x, \xi) = \sum_{\alpha} a_\alpha \exp\left(i \left\langle \alpha, \frac{\xi}{|\xi|} \right\rangle \right)$$

α varying over all multiindices so that the series, as well as the differentiated series with respect to x or ξ, converge uniformly. For the proof see [4]. Therefore we see that the following finite sums of special kernels are dense in $\mathcal{K}$ with respect to the topology of $C^1(R^n_x \times (R^n_\xi - \{0\}))$:

(7) $$\sum_{\alpha}^{finite} a_\alpha(x) k_\alpha(\xi), \qquad k_\alpha(\xi), a_\alpha(x) \in \mathcal{K}.$$

Here we can assume that $k_\alpha(\xi)$ are scalar ones according to the lemma 2.

We try to compute the corresponding families A_α and $K_{h\alpha}$ in $\mathcal{P}$ for the kernels $a_\alpha(x)$ and $k_\alpha(\xi)$, then we get

(8) $$A_\alpha u = a_\alpha(x) u, \quad u \in L^2_x$$

(9) $$K_{h\alpha} u = \mathcal{F}^{-1} k_\alpha(\lambda(h\xi)) \mathcal{F} u, \quad u \in L^2_x.$$

Consequently we can express the family K_h associated with general element $k(x, \xi)$ in $\mathcal{K}$ in the following manner:

$$K_h=\sum_{\alpha} A_\alpha K_{h\alpha} \tag{10}$$

because of the continuity of mapping $\mathcal{M}: \mathcal{K}\to\mathcal{P}$ with respect to C^m topology (see (4)).

Now we state a lemma:

Lemma 3. *If K_h is a family in $\mathcal{P}$ assocaited with k in $\mathcal{K}$, then the commutator $[K_h, \Lambda_h]$ is a null scheme.*

Proof. First we remark that if $\|[K_h^{(n)}, \Lambda_h]\|\leqq Ch$ with C independent of n, and if $K_h^{(n)}\to K_h$ $(n\to+\infty)$ with respect to the operator norm, then $[K_h, \Lambda_h]$ belongs to $\mathcal{N}_0$. Therefore it suffices to prove this lemma for the case of $K_h^{(n)}$ associated with the finite sum:

$$k^{(n)}(x, \xi)=\sum_{\alpha}^{finite} a_\alpha(x)k_\alpha(\xi).$$

We have already

$$\|a_\alpha(x)\Lambda_h-\Lambda_h a_\alpha(x)\|\leqq h\|\hat{a}_{1\alpha}(\xi)|\xi|\|_{L_1}$$

but we also have

$$\|\hat{a}_{1\alpha}(\xi)|\xi|\|_{L_1}\leqq\frac{M}{|\alpha|^k} \quad \textit{for some constant } M$$

because using the fact that the support of $a_{1\alpha}(x)$ are contained in a fixed compact

$$\begin{aligned}\|\hat{a}_{1\alpha}(\xi)|\xi|\|_{L_1}&\leqq\left(\int\frac{d\xi}{(1+|\xi|)^{2n}}\cdot\int(1+|\xi|)^{2(n+1)}|\hat{a}_{1\alpha}(\xi)|^2d\xi\right)^{1/2}\\ &\leqq M\sup_{\substack{x\\ |\beta|\leqq n+1}}|D_x^\beta a_{1\alpha}(x)|\\ &\leqq\frac{M}{(1+|\alpha|)^k}.\end{aligned}$$

Here we can take k large enough since $a_{1\alpha}(x)\in C_0^\infty$. Then we get

$$\|[K_h^{(n)}, \Lambda_h]\|\leqq Ch$$

with C independent of n.

Now we state another important lemma

Lemma 4. *If $a(x)$ and $k(\xi)$ are in $\mathcal{K}$, then the commutator $[A, K_h]$ of A and K_h is negligible.*

Proof. For every $u \in L^2_x$, we have

$$\|\Lambda_h[A, K_h]u\| = \|(\Lambda_h A K_h - \Lambda_h K_h A)u\|$$
$$= \left\| \int \hat{a}(\xi-\eta) |\lambda(h\xi)| [k(\lambda(h\eta)) - k(\lambda(h\xi))] \hat{u}(\eta) d\eta \right\|$$
$$\leqq \left\| \int \hat{a}(\xi-\eta) [|\lambda(h\xi)| - |\lambda(h\eta)|] k(\lambda(h\eta)) \hat{u}(\eta) d\eta \right\|$$
$$+ \left\| \int \hat{a}(\xi-\eta) [|\lambda(h\eta)| k(\lambda(h\eta)) - |\lambda(h\xi)| k(\lambda(h\xi))] \hat{u}(\eta) d\eta \right\|$$
$$\leqq \left\| \int |\hat{a}(\xi-\eta)| |\lambda(h\xi) - \lambda(h\eta)| |k(\lambda(h\eta))| |\hat{u}(\eta)| d\eta \right\|$$
$$+ \left\| \int |\hat{a}(\xi-\eta)| |\lambda(h\xi) - \lambda(h\eta)| \times \right.$$
$$\left. \times \left| \frac{|\lambda(h\eta)| k(\lambda(h\eta)) - |\lambda(h\xi)| k(\lambda(h\xi))}{|\lambda(h\xi) - \lambda(h\eta)|} \right| |\hat{u}(\eta)| d\eta \right\|$$
$$\leqq hC \| \dot{a}_1(\xi) |\xi| \|_{L_1} \|u\| + h \| \dot{a}_1(\xi) |\xi| \|_{L_1} S \|u\|$$

$S =$ *Lipshitz constant of* $|\xi| k(\xi)$ *for* $|\xi| \leqq 1$.

The proof is same for $[A, K_h]\Lambda_h$.

Thus we get the following:

Theorem 1. *The set* $\mathcal{N}$ *is a two sided ideal in* $\mathcal{P}$.

Proof. For every element N_h in $\mathcal{N}$, we will show that any product $H_h N_h$ and $N_h H_h$ belong to $\mathcal{N}$ taking H_h in $\mathcal{P}$. Because H_h is formed by addition and multiplication from fininite number of families K_h associated with $k(x, \xi)$ in $\mathcal{K}$. Then it is sufficient to show that any product $K_h N_h$ and $N_h K_h$ belong to $\mathcal{N}$. On the other hand by the lemma 3, $[K_h, \Lambda_h] \in \mathcal{N}_0$, therefore we get

$$K_h N_h \Lambda_h = K_h(N_h \Lambda_h) \in \mathcal{N}_0$$
$$N_h K_h \Lambda_h = (N_h \Lambda_h) K_h + N_h [K_h, \Lambda_h] \in \mathcal{N}_0 .$$

This means $K_h N_h$ and $N_h K_h$ are negligible. c.q.f.d.

Now we state the

Theorem 2. *The mapping* $\mathcal{M}$ *from* $\mathcal{K}$ *onto* $\mathcal{P}$ *is a homomorphism with respect to addition and multiplication modulo* $\mathcal{N}$.

Proof. The homomorphism with respect to addition is evident. For multiplication we only need the following:

Lemma 5. *If k_1 and k_2 are in $\mathcal{K}$, then $k_1 \cdot k_2 = k_3$ also belongs to $\mathcal{K}$. If we denote K_{1h}, K_{2h}, $K_{1h} \circ K_{2h}$ the associated family with k_1, k_2, k_3 respectively, then we have*

$$K_{1h}K_{2h} - K_{1h} \circ K_{2h} \in \mathfrak{N}.$$

Proof of lemma 5. We can assume as usual that k_1 and k_2 are some finite sums of type (7), because of lemma 2. We put

$$k_1 = \sum_{\alpha} a_{\alpha}(x) k_{h\alpha}^{(1)}(\xi), \quad k_2 = \sum_{\beta} b_{\beta}(x) k_{h\beta}^{(2)}(\xi).$$

Then we get

$$k_3 = \sum_{\alpha,\beta} a_{\alpha}(x) b_{\beta}(x) k_{h\alpha}^{(1)}(\xi) k_{h\beta}^{(2)}(\xi)$$

and

$$K_{1h} = \sum_{\alpha} A_{\alpha} K_{h\alpha}^{(1)}, \quad K_{2h} = \sum_{\beta} B_{\beta} K_{h\beta}^{(2)}$$

$$K_{1h} \circ K_{2h} = \sum_{\alpha,\beta} A_{\alpha} B_{\beta} K_{h\alpha}^{(1)} K_{h\beta}^{(2)}$$

$$K_{1h} K_{2h} = \sum_{\alpha,\beta} A_{\alpha} K_{h\alpha}^{(1)} B_{\beta} K_{h\beta}^{(2)}.$$

Therefore we have

$$K_{1h}K_{2h} - K_{1h} \circ K_{2h} = \sum_{\alpha,\beta} A_{\alpha} (B_{\beta} K_{h\alpha}^{(1)} - K_{h\alpha}^{(1)} B_{\beta}) K_{\beta h}^{(2)}.$$

If we prove the commutator $[B_{\beta}, K_{h\alpha}^{(1)}]$ is negligible then the proof of lemma 5 will finish. But this is the simple consequence of lemma 4. Of course, by the proof of lemma 4, we get

$$\| [B_{\beta}, K_{h\alpha}^{(1)}] \Lambda_h \| \leq Ch$$

with C determined only by the Lipshitz constant of $\lambda(\xi)$ and $|\xi| k_i(x, \xi)$ and the C^1 norm of $k_i(x, \xi)$ in x $(i=1, 2)$.

Corollary. *If K_{1h} and K_{2h} are the same families in lemma 5, and M_h is $K_{1h} \circ K_{2h}$, then $K_h \Lambda_h^2 - K_{1h} \Lambda_h K_{2h} \Lambda_h$ is a null scheme.*

Proof.

$$M_h \Lambda_h^2 - K_{1h} \Lambda_h K_{2h} \Lambda_h = (M_h \Lambda_h - K_{1h} \Lambda_h K_{2h}) \Lambda_h$$

$$= (M_h - K_{1h} K_{2h}) \Lambda_h^2 + K_{1h} [\Lambda_h, K_{2h}] \Lambda_h \in \mathfrak{N}_0.$$

Proposition. *The homomorphism in theorem 2 is a *-homomorphism.*

Proof. If k is in $\mathcal{K}$, then k^* is also in $\mathcal{K}$, we denote K_h and $K_h^\#$ associated family with k and k^* respectively. We can prove K_h^* adjoint of K_h is equal to $K_h^\#$ modulo the negligible schemes $\mathcal{N}$. The proof of this fact can be proved by the same kind of reasoning as in lemma 5.

Now we prove the

Theorem 2'. *The homomorphism $\mathcal{M}$ from $\mathcal{K}$ onto $\mathcal{P}$ is an isomorphism modulo $\mathcal{N}$.*

Proof. We are making use of the trick by P. D. Lax [4]. We suppose that there exists a non zero element k in $\mathcal{K}$ whose associated family in $\mathcal{P}$ is negligible. Then for k^*k, and also for the rotation in ξ of $k^*(x,\xi)k(x,\xi)$, the same is true. By the integration with respect to the invariant measure over the whole rotation group. We have a non negative matrix valued function $s(x)$ whose associate family S_h is negligible. Therefore $\Lambda_h s(x)$, $s(x)\Lambda_h$ are null schemes.

On the other hand, because $s(x)$ is a non zero positive semidefinite matrix, we can assume that there exists a component $s_{ii}(x) \geqq \delta > 0$ in some open set Ω. If we take a scalar function $u(x)$ which belongs to $\mathcal{D}(\Omega)$, then we get

$$\|\Lambda_h s_{ii}(x)u(x)\| = O(h).$$

But we can put $s_{ii}(x)u(x) = v(x)$ for any function of $\mathcal{D}(\Omega)$, then we have

$$\|\Lambda_h v(x)\| = O(h)$$

which means for some constant $B > 0$

$$\left\|\frac{\Lambda_h v}{h}\right\| \leqq B\|v\|, \qquad \forall v \in \mathcal{D}(\Omega).$$

Consequently by the Lebesgue theorem, we have

$$\|\Lambda v\| \leqq B\|v\|, \qquad \forall v \in \mathcal{D}(\Omega)$$

that means

$$\left\|\frac{\partial}{\partial x_i}v\right\| \leqq B\|v\|, \quad \forall v \in \mathcal{D}(\Omega)$$

which is a contradiction.

Now we state a theorem which is important for the later application.

Theorem 3. *If $p(x, \xi)$ is positive definite and belongs to $\mathcal{K}$, then the associated family P_h satisfies the following inequality for $u \in L^2_x$.*

$$\mathcal{R}e\langle P_h \Lambda_h^2 u, u\rangle \geqq -O(h)\|u\|^2. \tag{11}$$

Proof. We can find a non singular matrix function $r(x, \xi)$ in $\mathcal{K}$ such that

$$p(x, \xi) = r^*(x, \xi) r(x, \xi).$$

Then using corollary of lemma 5, lemma 2, and lemma 3, we get

$$\begin{aligned}\langle P_h \Lambda_h^2 u, u\rangle &= \langle \Lambda_h R_h^* R_h \Lambda_h u, u\rangle + O(h)\|u\|^2 \\ &= \langle R_h \Lambda_h u, R_h \Lambda_h u\rangle + O(h)\|u\|^2 \\ &\geqq -O(h)\|u\|^2.\end{aligned}$$

c.q.f.d.

Now we specialize the vector function $\lambda(\xi)$ for the later use. Let us consider a constant coefficient finite difference scheme D_{jh} which approximates the partial derivative $\frac{\partial}{\partial x_j}$ with accuracy 1. Let us write this scheme $D_{jh} = \sum_l C_l T_j^l$, $T_j u(x) = u(x_1, \cdots, x_j + h, \cdots x_n)$. Then we can use $\frac{1}{i}\sum_l C_l e^{i\xi_j l}$, as a $\lambda_j(\xi)$ if $\sum_l C_l e^{i\xi_j l}$ is pure imaginary for real ξ. In fact by the consistency we can prove

$$\frac{\lambda_j(h\xi)}{h} \to \xi_j \qquad \text{for } h \downarrow 0$$

and $\lambda(\xi) = \frac{1}{i}\sum_l C_l e^{\xi_j l}$ is analytic in ξ. Since all derivatives with respect to ξ are bounded, we have

$$|\lambda_j(\xi) - \lambda_j(\eta)| \leqq C_j |\xi - \eta|.$$

Consequently, we get

$$|\lambda(\xi) - \lambda(\eta)| \leqq C|\xi - \eta|$$

where C is a constant independent of ξ, η

4. Calculus of difference schemes

Let S_h be a family of usual scalar difference operators of the following form:

$$S_h=\sum_j b_j(x)\,T^j \qquad (\textit{finite sum}) \tag{12}$$

where $b_j(x)$ is C^∞ and constant for $|x|>R$ and $T^j=T_1^{j_1}\cdot T_2^{j_2}\cdot\cdots\cdot T_n^{j_n}$, $T_p^{j_p}u(x)=u(x+j_p he_p)$ $(e_p=(0\cdots01_p0\cdots0))$. As we have shown in section 2, $T_p^{j_p}$ commute with $a_j(x)$ modulo $\mathfrak{N}_0$ and of course Λ_h commutes with $T_p^{j_p}$ in usual sense. Thus we have the following.

Lemma 6. *Let $K_h\Lambda_h$ be the family corresponding to $k\in K$, then $K_h\Lambda_h S_h-S_hK_h\Lambda_h$ is a null scheme.*

Proof. We use Lemma 1 and Lemma 3 and Lemma 4.

$$K_h\Lambda_h S_h=\sum_j\sum_e a_j(x)K_{hj}\Lambda_h b_e(x)\,T^e$$

$$S_hK_h\Lambda_h=\sum_j\sum_e b_j(x)\,T^j a_e(x)K_{he}\Lambda_h$$

$$\begin{aligned}
&a_j(x)K_{hj}\Lambda_h b_e(x)\,T^e-b_e(x)\,T^e a_j(x)K_{hj}\Lambda_h\\
&=a_j(x)K_{hj}(\Lambda_h b_e(x)-b_e\Lambda_h)\,T^e\\
&\quad+a_j(x)\,(K_{hj}b_e(x)-b_e(x)K_{hj}\Lambda_h T^e\\
&\quad+b_e(x)\,(a_j(x)\,T^e-T^e a_j(x))K_{hj}\Lambda_h\in\mathfrak{N}_0\,.
\end{aligned}$$

We can write:

$$\begin{aligned}
a_j(x)K_{hj}\Lambda_h b_e(x)\,T^e&\equiv a_j(x)K_{hj}b_e(x)\Lambda_h T^e\\
&\equiv a_j(x)b_e(x)K_{hj}\Lambda_h T^e\equiv a_j(x)b_e(x)\,T^eK_{hj}\Lambda_h\\
&\equiv b_e(x)\,T^e a_j(x)K_{hj}\Lambda_h \qquad (\textit{mod } \mathfrak{N}_0)
\end{aligned}$$

5. Application

The stability of Friedrichs' scheme for regularly hyperbolic systems.

Let us consider the system of equations with variable coefficients

$$\frac{\partial u}{\partial t}=\sum_{j=1}^n A_j(x)\frac{\partial u}{\partial x_j} \tag{13}$$

where $A_j(x)$ are given smooth bounded $N\times N$ matrices and equal to constant for large x while u is an unknown N-vector. We assume that (13) is a regularly hyperbolic system, that is to say, that the matrix $\sum_j A_j(x)\xi_j = A(x, \xi)$ have only real distinct eigenvalues for all real $\xi = (\xi_1, \cdots, \xi_n)$ and every pair of eigenvalues $\mu_p(x, \xi)$ and $\mu_q(x, \xi)$ satisfy the following condition: there exists a positive constant d such that

$$|\mu_p(x, \xi) - \mu_q(x, \xi)| \geqq d \qquad \text{for all } x \in R^n, \ \xi \in S^{n-1} \quad (p \neq q) \ p, q = 1, \cdots, N. \tag{14}$$

Then we know that there is a nonsingular smooth matrix $N(x, \xi)$ whose determinant is bounded away from zero for all $x \in R^n$, $\xi \in S^{n-1}$, and that it is a diagonalizer of $A(x, \xi)$, i.e.

$$D(X, \xi) = N(x, \xi)A(x, \xi)N^{-1}(x, \xi) = \begin{pmatrix} \mu_1(x, \xi) & & \\ & \ddots & \\ & & \mu_N(x, \xi) \end{pmatrix}. \tag{15}$$

Now we consider the numerical solution of the Cauchy problem for this system. Among useful finite difference schemes, we consider here exclusively one which is called Friedrichs' scheme. Replacing $\frac{\partial u}{\partial t}$ by $u(x, t+k) - \overline{u(x, t)}/k$ and $\frac{\partial u}{\partial x_j}$ by $\frac{u(x+\delta_j, t) - u(x-\delta_j, t)}{2h}$ where h and k are space and time mesh length respectively and we denoted $\overline{u(x, t)} = \frac{\sum_{j=1}^{n} \{u(x+\delta_j, t) + u(x-\delta_j, t)\}}{2n}$ ($\delta_j = he_j$, e_j is a unit n-vectors $j = 1, \cdots, n$), we can write down the Friedrichs' scheme for the system (13) as follows: for every $u \in L_x^2$

$$u(x, t+k) = \sum_{j=1}^{n} \left[\frac{u(x+\delta_j, t) + u(x-\delta_j, t)}{2n} + kA_j(x) \frac{u(x+\delta_j, t) - u(x-\delta_j, t)}{2h} \right]. \tag{16}$$

Or denoting $\lambda = \frac{k}{h}$, we have

(17) $\qquad u(x, t+k) = S_h u(x, t)$, *where* S_h *is defined by*

$$S_h u=\sum_{j=1}^{n}\left\{\frac{u(x+\delta_j,t)+u(x-\delta_j,t)}{2n}+\lambda A_j(x)\frac{u(x+\delta_j,t)-u(x-\delta_j,t)}{2}\right\}. \tag{18}$$

Now we state the stability theorem.

Theorem 3. *If* $|\lambda|<\frac{1}{\sqrt{n}\,\mu_0}$, *then the scheme (17), (18) is stable in the sense of Lax-Richtmyer. Here* μ_0 *is* $\max_{\substack{p=1,\dots,N\\|\xi|\le 1\\ x\in R^n}}|\mu_p(x,\xi)|$.

Proof of the theorem. First we remark that the stability means for any m positive integer, $\|S_h^m u(x,0)\|\leqq C\|u(x,0)\|$, where C is a constant independent of m and h and $mh\leqq T$.

To show this inequality it is convenient to introduce a new norm which is equivalent to the usual L^2 norm. First we explain its construction and equivalence.

i) Construction of a new norm.

Let H_h be a bounded operator corresponding to $h(x,\xi)=N^*(x,\xi)N(x,\xi)$ which is a strictly positive definite function for $x\in R^n$, $\xi\in S^{n-1}$ and belongs to $\mathcal{K}$. Then we can show for any u with small fixed support, $\langle \mathcal{R}e\, H_h u, u\rangle$ is positive definite. Because

$$\langle H_h u, u\rangle=\langle H_{0h}u, u\rangle+\sum_{j=1}^{n}\langle (x^j-x_0^j)H_{1jh}u, u\rangle \tag{19}$$

H_{0h}: *operator corresponding to* $h(x_0,\xi)$

H_{1jh}: *operator corresponding to* $h_{1j}(x,\xi)$ *such that*

$$h(x,\xi)=h(x_0,\xi)+\sum_{j=1}^{n}(x^j-x_0^j)h_{1j}(x,\xi).$$

Naturally $\mathcal{R}e\langle H_{0h}u, u\rangle\geqq d_1\|u\|^2$, then if we take the diameter of that fixed support very small, we can show (although H_{1jh} doesn't belong to the operator family discussed in §3.)

$$|\langle (x^j-x_0^j)H_{1jh}u, u\rangle|\leqq\varepsilon\|u\|^2.$$

We get

$$\mathcal{R}e\langle H_h u, u\rangle\geqq d_2\|u\|^2. \tag{20}$$

We fix now a partition of unity $\{\varphi_p\}$ such that $\sum_p \varphi_p^2 = 1$. Because of the assumption about $A_j(x)$, we consider only a finite partition of unity. If we take the maximum of diameter of the support of φ_p sufficiently small, then

$$\|u\|_H^2 = \sum_j \mathcal{R}e\langle H_h \varphi_j u, \varphi_j u\rangle \tag{21}$$

is equivalent to the L^2 norm. For the inequality $\|u\|_H \leqq c\|u\|$ is evident by the boundedness of H_h. The inverse inequality results from (20) by summing up the inequality (20) for $\varphi_j u$.

Now we can say that it suffices to prove

$$\|S_h u\|_H \leqq (1+O(h))\|u\|_H \tag{22}$$

for the stability of (17). But to establish (22), we only have to show

$$\mathcal{R}e\langle H_h S_h \varphi_j u, S_h \varphi_j u\rangle \leqq (1+O(h))\mathcal{R}e\langle H_h \varphi_j u, \varphi_j u\rangle, \tag{23}$$

because the left hand side of (22) is

$$\sum_j \mathcal{R}e\langle \varphi_j H_h \varphi_j S_h u, S_h u\rangle$$

and

$$\begin{aligned} &|\langle H_h \varphi_j S_h u, \varphi_j S_h u\rangle - \langle H_h S_h \varphi_j u, S_h \varphi_j u\rangle| \\ &= \langle H_h(\varphi_j S_h - S_h \varphi_j)u, \varphi_j S_h u\rangle + \langle H_h S_h \varphi_j u, (\varphi_j S_h - S_h \varphi_j)u\rangle| \\ &\leqq O(h)\|u\|^2. \end{aligned}$$

ii) Proof of (23). Putting $v = \varphi_j u$, we'll show

$$\mathcal{R}e\langle H_h S_h v, S_h v\rangle \leqq (1+O(h))\mathcal{R}e\langle H_h v, v\rangle \tag{24}$$

that means

$$\mathcal{R}e\langle (H_h - S_h^* H_h S_h)v, v\rangle \geqq -O(h)\|v\|^2. \tag{25}$$

Now we can write S_h and S_h^* in the following form (here we take $\lambda(\xi) = \sin\xi$ as in the section 2)

$$\begin{aligned} S_h &= E_h + i\lambda Q_h \Lambda_h \\ S_h^* &= E_h^* - i\lambda \Lambda_h Q_h^* . \end{aligned} \tag{26}$$

Where $E_h = \mathcal{F}^{-1}\left[I \cdot \sum_{j=1}^n \frac{\cos h\xi_j}{n}\right]\mathcal{F}$ and Q_h is a family corresponding to

$\sum^n A_j(X)\frac{\xi_j}{|\xi|}\in\mathcal{K}$. Now we put $P_h=H_h-S_h^*H_hS_h$ and we get, using (26),

$$P_h=P_h^{(0)}+i\lambda P_h^{(1)}+\lambda^2P_h^{(2)}$$

where

$$P_h^{(0)}=H_h-E_h^*H_hE_h$$

$$P_h^{(1)}=A_hQ_h^*H_hE_h-E_h^*H_hQ_hA_h$$

$$P_h^{(2)}=A_hQ_h^*H_hQ_hA_h\,.$$

Then we have $P_h^{(1)}=0 \pmod{\mathcal{N}_0}$. In fact using Lemmas 3, 4, 5 and the fact that $Q_h^\sharp\circ H_h=H_h\circ Q_h$, we get

$$\begin{aligned}A_hQ_h^*H_hE_h&\equiv A_hQ_h^\sharp H_hE_h\equiv Q_h^\sharp H_hA_hE_h\\&\equiv Q_h\circ H_hA_hE_h\equiv H_h\circ Q_hA_hE_h\\&\equiv H_hQ_hA_hE_h\equiv H_hQ_hE_hA_h\\&\equiv H_hE_hQ_hA_h\equiv E_hH_hQ_hA_h\,.\end{aligned}$$

Furthermore by Lemmas 3, 4 and Theorem 1,

$$\begin{aligned}P_h^{(2)}&\equiv Q_h^\sharp A_hH_hQ_hA_h\equiv Q_h^\sharp H_hA_hQ_hA_h\\&\equiv Q_h\circ H_hA_hQ_hA_h\equiv Q_h\circ H_hQ_hA_h^2\\&\equiv Q_h\circ H_h\circ Q_hA_h^2\,.\end{aligned}$$

And we have also

$$P_h^{(0)}\equiv H_h(I-E_h^*E_h).$$

So that

$$P_h\equiv H_h(I-E_h^*E_h)-Q_h^\sharp\circ H_h\circ Q_hA_h^2.$$

The Fourier transform of $I-E_h^*E_h$ is expressed by

$$I\cdot\left(1-\left(\frac{\sum\cos h\xi_j}{n}\right)^2\right)$$

and

$$\begin{aligned}&1-\left(\frac{\sum\cos h\xi_j}{n}\right)^2\\&=1-\frac{\sum\cos^2h\xi_j}{n}+\frac{\sum_{j>k}(\cos h\xi_j-\cos h\xi_k)^2}{n^2}\\&=\frac{1}{n}|\sin h\xi|^2+\frac{1}{n^2}\sum_{j>k}(\cos h\xi_j-\cos h\xi_k)^2.\end{aligned}$$

Hence we can write using a notation $T_{jk}=T_j+T_j^{-1}-T_k-T_k^{-1}$ and lemma 6:

$$P_h \equiv \frac{1}{n^2}\sum_{j>k} H_h T_{jk}^2 + \left(\frac{1}{n} H_h - \lambda^2 Q_h^\sharp \circ H_{h} H_h \circ Q_h\right) \Lambda_h^2$$

$$\equiv \frac{1}{n^2}\sum_{j>k} T_{jk} H_h T_{jk} + \left(\frac{1}{n} H_h - \lambda^2 Q_h^\sharp \circ H_h \circ Q_h\right) \Lambda_h^2,$$

because of the fact

$$\frac{1}{n} H - Q_h^\sharp \circ H_h \circ Q_h = N_h^\sharp \circ \left(\frac{1}{n} - \lambda^2 \mathscr{D}_h^\sharp \circ \mathscr{D}_h\right) \circ N_h$$

where $\mathscr{D}_h$ is the pseudo difference scheme corresponding to $D(x,\xi)$. Applying the Theorem 3 for the operator $N_h^\sharp \circ \left(\frac{1}{n} - \lambda^2 \mathscr{D}_h \circ \mathscr{D}_h\right) \circ N_h$ whose symbol satisfies the condition of this theorem, we get using the fact $T_{jk}^* = T_{jk}$

$$\mathscr{R}e(P_h v, v) = \mathscr{R}e \frac{1}{n^2} \sum_{j>k} (H_h T_{jk} v,\ T_{jk} v)$$

$$+ \mathscr{R}e\left(N_h \circ \left(\frac{1}{n} - \lambda^2 \mathscr{D}_h^\sharp \circ \mathscr{D}_h\right) \circ N_h v,\ v\right)$$

$$\geqq \frac{1}{n^2} d_2 \| T_{jk} v\|^2 - O(h)\|v\|^2$$

$$\geqq -O(h)\|v\|^2$$

$$\geqq -O(h)\,\mathscr{R}e(H_h v, v).$$

That means (25), and consequently (22) that was to be proved.

Remark. This method works as well in proving the stability of another scheme. For example, the modified Lax-Wendroff scheme of accuracy 2 which proposed recently by Richtmyer [7], can also be proved to be stable under the same assumption of the coefficients as in (18). This scheme can be written as

$$S_h u = \left\{ I + i\lambda Q_h \Lambda_h \left(E_h + i\frac{\lambda}{2} Q_h \Lambda_h \right) \right\} u. \tag{27}$$

Essential feature of these schemes is that they are both polynomials of the same matrix Q_h that comes from the original system (18). It might be possible to give a general theory for this kind of schemes.

BIBLIOGRAPHY

[1] Caldéron, A. P. and A. Zygmund, Singular integral operators and differential equations, Amer. J. Math. **79** (1957), 901-921.

[2] ——————, Uniqueness in Cauchy problem, Amer. J. Math. **80** (1958), 16-36.

[3] Lax, P. D., Differential equations, difference equations and matrix theory, Comm. Pure Appl. Math. **11** (1958), 175-194.

[4] ——————, The L_2 operator calculus of Mikhlin, Caldéron and Zygmund, Mimeographed lecture note, New York University.

[5] Yamaguti, M., Sur l'inégalité d'énergie pour le système hyperbolique, Proc. Japan Acad. **35** (1959), 37-41.

[6] ——————, Le problème de Cauchy et les opérateurs d'intégrale singulière, Mem. Coll. Sci. Univ. Kyoto Ser. A Math. **32** (1959), 121-151.

[7] Wendroff, B., Well posed problems and stable difference operators, Manuscript.