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Derivations of uniformly hyperfinite C*-algebras

By

Shôichirô SAKAI*

1. Introduction

Recently, the author [7] proved that every derivation of W*-algebras is ineer. On the other hand, it has been known that there are many examples of C*-algebras having outer derivations ([4], [5]). However, those examples were constructed in the frame of general C*-algebras.

In the present paper, we shall study derivations of uniformly hyperfinite C*-algebras which were introduced by Glimm [1] and are appearing in the quantum field theory.

The main result of this paper is as follows: every derivation of uniformly hyperfinite C*-algebras is inner. Also, we shall show that such algebras have many outer *-automorphisms.

2. Derivations of uniformly hyperfinite C*-algebras

Let $\mathfrak{A}$ be a C*-algebra with the unit 1. $\mathfrak{A}$ is called a uniformly hyperfinite C*-algebra, if it has a sequence of type I_{n_i}-subfactors $\{\mathfrak{M}_i\}$ $(n_i < +\infty)$ as follows: (i) $1 \in \mathfrak{M}_i$ for all i; (ii) $\mathfrak{M}_i \subset \mathfrak{M}_{i+1}$; (iii) $n_i \to \infty (i \to \infty)$; (iv) $\mathfrak{A}$ is the uniform closure of $\bigcup_{i \leq 1} \mathfrak{M}_i$.

Let $\mathfrak{A}$ be a uniformly hyperfinite C*-algebra, then $\mathfrak{A}$ has the unique trace τ with $\tau(1)=1$, where 1 is the identity of $\mathfrak{A}$.

Let $\{\pi_\tau, \mathfrak{H}_\tau\}$ be the *-representation on a Hilbert space $\mathfrak{H}_\tau$ constructed via τ and let M be the weak closure of $\pi_\tau(\mathfrak{A})$, then M is a hy-

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* Department of Mathematics, University of Pennsylvania and Research Institute for Mathematical Sciences, Kyoto University.

perfinite II_1-factor and the mapping $a\to\pi_\tau(a)$ $(a\in\mathfrak{A})$ is one-to-one, because $\mathfrak{A}$ is simple ([1]). We shall identify $\mathfrak{A}$ with $\pi_\tau(\mathfrak{A})$.

In the following discussions, we shall show that $\mathfrak{A}$ has only inner derivations. Let D be a derivation of $\mathfrak{A}$, then by Theorem 2 in [7], there exists an element a in M such that $D(x)=[a, x]$ for $x\in\mathfrak{A}$. Put $D^*(x)=[a^*, x]$ for $x\in\mathfrak{A}$, then D^* is also a derivation of $\mathfrak{A}$ (cf. [7]); hence it is enough to assume that a is self-adjoint.

By considering $\|a\|1+a$, we can take a positive element a of M such that $[a, x]=D(x)$ for $x\in\mathfrak{A}$. Let $\mathfrak{C}=\{a\,|\,a\geqq 0,\ a\in M \text{ and } [a, x]=D(x) \text{ for all } x\in\mathfrak{A}\}$, then there exists an element b in $\mathfrak{C}$ such that $[b, x]=D(x)$ for all $x\in\mathfrak{A}$ and $\hat{\tau}(b)=\inf_{a\in\mathfrak{C}}\hat{\tau}(a)$, where $\hat{\tau}$ is the trace on M with $\hat{\tau}(1)=1$.

Lemma. Let $\mathfrak{K}$ be an uniformly closed convex subset of the self-adjoint portion M^s of M generated by $\{u^*bu\,|$ all unitary $u\in\mathfrak{A}\}$. Then $d(\mathfrak{A}, \mathfrak{K})=0$, where $d(\mathfrak{A}, \mathfrak{K})=\inf_{\substack{a\in\mathfrak{C}\\ k\in\mathfrak{K}}}\|a-k\|$.

Proof. Suppose that $d(\mathfrak{A}, \mathfrak{K})>0$, then there exists a bounded self-adjoint linear functional f on M such that $f(\mathfrak{A})=0$ and $f(\mathfrak{K})>\varepsilon$ for some positive $\varepsilon(>0)$.

Let $\mathfrak{M}_i^u$ be the compact group of all unitary elements of $\mathfrak{M}_i$ under the uniform topology, and let $d\mu_i$ be the Haar measure on $\mathfrak{M}_i^u$ such that $\mu_i(\mathfrak{M}_i^u)=1$.

Let M^u be the group of all unitary elements of M. The mapping $u\to u^*yu$ of M^u into M is uniformly continuous for each $y\in M$. For each $g\in M^*$, define $g^u(y)=g(u^*yu)$ for $y\in M$ and $u\in M^u$, where M^* is the dual space of M. Then, the uniform continuity of the mapping $u\to u^*yu$ implies that the mapping $u\to g^u$ of M^u with the uniform topology into M^* with the topology $\sigma(M^*, M)$ is continuous.

Put $f_i(y)=\int_{\mathfrak{M}_i^u} f^u(y)d\mu_i(u)$, then $f_i(v^*yv)=f_i(y)$ for all $v\in\mathfrak{M}_i^u$ and $y\in M$. The sequence $\{f_i\}$ is bounded and the unit sphere of M^* is $\sigma(M^*, M)$-compact; hence $\{f_i\}$ is relatively $\sigma(M^*, M)$-compact.

Let f_0 be an accumulate point of $\{f_i\}$, then $f_0(v^*yv)=f_0(y)$ for all $v\in\mathfrak{M}_i^u$ and $y\in M$. Therefore $f_0(v^*y)=f_0(v^*yv^*v)=f_0(yv^*)$ for

$v\in\mathfrak{M}_i^u$ and $y\in M$; hence $f_0(xy)=f_0(yx)$ for $x\in\bigcup_{i=1}\mathfrak{M}_i$ and $y\in M$, because elements of $\mathfrak{M}_i$ are finite linear combinations of unitary elements in $\mathfrak{M}_i$ for all i; therefore $f_0(xy)=f_0(yx)$ for $x\in\mathfrak{A}$ and $y\in M$.

Moreover, $f(\mathfrak{K})>\varepsilon$ and so $f^u(b)>\varepsilon$; hence $f_0(b)\geqq\varepsilon$ and so f_0 is a non-zero bounded self-adjoint linear functional on M.

Let $f_0=f_0^+-f_0^-$ be the unique decomposition of f_0 such that f_0^+, $f_0^-\geqq 0$, $\|f_0\|=\|f_0^+\|+\|f_0^-\|$ (cf. [2], [8]). Then $f_0^u=f_0$ for all $u\in\mathfrak{A}^u$, where $\mathfrak{A}^u$ is the set of all unitary elements in $\mathfrak{A}$; by the unicity, $(f_0^+)^u=f_0^+$ and $(f_0^-)^u=f_0^-$ for all $u\in\mathfrak{A}^u$; hence $f_0^+(xy)=f_0^+(yx)$ and $f_0^-(xy)=f_0^-(yx)$ for $x\in\mathfrak{A}$ and $y\in M$. Therefore we proved that there exist two different states φ_1,φ_2 on M such that $\varphi_1(xy)=\varphi_1(yx)$ and $\varphi_2(xy)=\varphi_2(yx)$ for $x\in\mathfrak{A}$ and $y\in M$, and moreover $\varphi_1(b)\neq\varphi_2(b)$.

Now let $\mathfrak{B}$ be a C^*-subalgebra of M generated by $\mathfrak{A}$ and b. Let Ω be the set of all states φ on $\mathfrak{B}$ such that $\varphi(xy)=\varphi(yx)$ for $x\in\mathfrak{A}$ and $y\in\mathfrak{B}$, then by the preceding discussions, Ω contains at least two points; moreover Ω is a $\sigma(\mathfrak{B}^*,\mathfrak{B})$-compact convex set, where $\mathfrak{B}^*$ is the dual of $\mathfrak{B}$.

Let ψ be an extreme point of Ω, and let $\{\pi_\psi,\mathfrak{H}_\psi\}$ be the $*$-reperesentation of $\mathfrak{B}$ on a Hilbert space $\mathfrak{H}_\psi$ constructed via ψ.

Let $\mathfrak{N}$ (resp. $\mathfrak{O}$) be the weak closure of $\pi_\psi(\mathfrak{B})$ (resp. $\pi_\psi(\mathfrak{A})$) on $\mathfrak{H}_\psi$. Then by Theorem 2 in [7], there exists a self-adjoint element $c\in\mathfrak{O}$ such that $\pi_\psi(D(x))=[c,\pi_\psi(x)]$ for $x\in\mathfrak{A}$; hence $\pi_\psi(D(x))=\pi_\psi([b,x])=[\pi_\psi(b),\pi_\psi(x)]=[c,\pi_\psi(x)]$ for $x\in\mathfrak{A}$; hence $\pi_\psi(b)-c\in\pi_\psi(\mathfrak{A})'$, where $\pi_\psi(\mathfrak{A})'$ is the commutant of $\pi_\psi(\mathfrak{A})$. Therefore $\pi_\psi(b)$ belongs to the W^*-algebra generated by $\mathfrak{O}$ and $\pi_\psi(b)-c$; hence $\mathfrak{N}$ is a W^*-algebra generated by $\mathfrak{O}$ and $\pi_\psi(b)-c$, and so $\pi_\psi(b)-c$ belongs to the center Z of $\mathfrak{N}$. Suppose that Z contains a non-trivial central projection z. Put $\psi'(y)=\frac{1}{\psi(z)}\langle\pi_\psi(y)zl_\psi,l_\psi\rangle$ and $\psi''(y)=\frac{1}{1-\psi(z)}\langle\pi_\psi(y)(1-z)l_\psi,l_\psi\rangle$, where l_ψ is the image of the identity 1 in $\mathfrak{H}_\psi$, and $<\ ,\ >$ is the scalar product of $\mathfrak{H}_\psi$.

Let h be a positive element of $\mathfrak{A}$, then $\psi(yh)=\psi(yh^{1/2}h^{1/2})=\psi(h^{1/2}yh^{1/2})\geqq 0$ for $y(\geqq 0)\in\mathfrak{B}$; hence by Proposition 1 in [6], $\psi(yh)$

$\leqq \|h\| \psi(y)$ for $y(\geqq 0) \ni \mathfrak{B}$; therefore there exists a positive element d' in $\pi(\mathfrak{B})'$ such that $\psi(yh) = \langle \pi_\psi(y) d' l_\psi, l_\psi \rangle = \langle \pi_\psi(y) \pi_\psi(h) l_\psi, l_\psi \rangle$ for $y \in \mathfrak{B}$, where $\pi_\psi(\mathfrak{B})'$ is the commutant of $\pi_\psi(\mathfrak{B})$ on $\mathfrak{H}_\psi$.

Hence $d' l_\psi = \pi_\psi(h) l_\psi$, because $\pi_\psi(\mathfrak{B}) l_\psi$ is dense in $\mathfrak{H}_\psi$. Now we shall consider

$$\begin{aligned}
\psi'(yh) &= \frac{1}{\psi(z)} \langle \pi_\psi(yh) z l_\psi, l_\psi \rangle = \frac{1}{\psi(z)} \langle \pi_\psi(y) \pi_\psi(h) z l_\psi, l_\psi \rangle \\
&= \frac{1}{\psi(z)} \langle \pi_\psi(y) z \pi_\psi(h) l_\psi, l_\psi \rangle = \frac{1}{\psi(z)} \langle \pi_\psi(y) z d' l_\psi, l_\psi \rangle \\
&= \frac{1}{\psi(z)} \langle \pi_\psi(y) d' z l_\psi, l_\psi \rangle = \frac{1}{\psi(z)} \langle d' \pi_\psi(y) z l_\psi, l_\psi \rangle \\
&= \frac{1}{\psi(z)} \langle \pi_\psi(y) z l_\psi, d' l_\psi \rangle = \frac{1}{\psi(z)} (\pi_\psi(y) z l_\psi, \pi_\psi(h) 1_\psi \rangle \\
&= \frac{1}{\psi(z)} \langle \pi_\psi(h) \pi_\psi(y) z l_\psi, l_\psi \rangle = \psi'(hy) \text{ for } y \in \mathfrak{B}.
\end{aligned}$$

Hence $\psi' \in \Omega$ and analogously $\psi'' \in \Omega$; hence $\psi = \psi' = \psi''$ and so $z = 1 - z = 1$, a contradiction. Therefore $Z = (\lambda 1)$, where λ are complex numbers.

Hence $\mathfrak{N} = \mathfrak{O}$; therefore, we can define an *-isomorphism ρ of M onto $\mathfrak{N}$ such that $\rho(x) = \pi_\psi(x)$ for $x \in \mathfrak{A}$, because $\psi = \tau$ on $\mathfrak{A}$.

Now, let $\|y\|_2 = \hat{\tau}(y^* y)^{1/2}$ for $y \in M$. For $y \in M$ and $\varepsilon > 0$, there exists an i such that $\|y - d_j\|_2 < \varepsilon$ for some $d_j \in \mathfrak{M}_j (j \geqq i)$; hence $\|\int_{\mathfrak{M}_j''} u^* y u d\mu_j(u) - \int_{\mathfrak{M}_j'} u^* d_j u d\mu_j(u)\|_2 = \|\int_{\mathfrak{M}_j''} u^* y u d\mu_j(u) - \tau(d_j) 1\|_2 \leqq \varepsilon$. Therefore

$$\begin{aligned}
&\|\hat{\tau}(y) 1 - \int_{\mathfrak{M}_j''} u^* y u d\mu_j(u)\|_2 \leqq \|\hat{\tau}(y) 1 - \tau(d_j) 1\|_2 \\
&+ \|\int_{\mathfrak{M}_j''} u^* y u d\mu_j(u) - \tau(d_j) 1\|_2 \leqq \|y - d_j\|_2 + \varepsilon < 2\varepsilon \text{ for } j \geqq i.
\end{aligned}$$

Put $y_j = \int_{\mathfrak{M}_j''} u^* y u d\mu_j(u)$, then the sequence $\{y_j\}$ converges strongly to $\hat{\tau}(y) 1$ in M.

Let $a = \rho^{-1}(\pi_\psi(b))$ and suppose that $\hat{\tau}(b) \neq \psi(b) = \langle \pi_\psi(b) l_\psi, l_\psi \rangle$, then $\hat{\tau}(b) \neq \hat{\tau}(a)$, because $y \to \langle \rho(y) l_\psi, l_\psi \rangle$ $(y \in M)$ is the trace on M and so by the unicity, $\hat{\tau}(y) = \langle \rho(y) l_\psi, l_\psi \rangle$.

$$\begin{aligned}[a, x] &= [\rho^{-1}(\pi_\psi(b)), x] = [\rho^{-1}(\pi_\psi(b)), \rho^{-1}(\pi_\psi(x))] \\ &= \rho^{-1}([\pi_\psi(b), \pi_\psi(x)]) = \rho^{-1}(\pi_\psi([b, x])) \\ &= [b, x] \text{ for } x \in \mathfrak{A};\end{aligned}$$

hence $a \in \mathfrak{C}$; therefore $\hat{\tau}(a) \geqq \hat{\tau}(b)$. Now suppose that $\hat{\tau}(a) > \hat{\tau}(b)$, and consider the *-representation π: $b \to b \oplus \pi_\psi(b)$ of $\mathfrak{B}$ on the Hilbert space $\mathfrak{H}_\tau \oplus \mathfrak{H}_\psi$, then the weak closure $\overline{\pi(\mathfrak{A})}$ of $\mathfrak{A}$ on $\mathfrak{H}_\tau \oplus \mathfrak{H}_\psi$ consists of all elements $\{y \oplus \rho(y) \mid y \in M\}$.

$$[b, x] \oplus [\pi_\psi(b), \pi_\psi(x)] = D(x) \oplus \pi_\psi(D(x)) \in \pi(\mathfrak{A})$$

and so $(bu - ub) \oplus \pi_\psi(bu - ub) \in \pi(\mathfrak{A})$ for $u \in \mathfrak{A}^u$; hence $(u^*bu - b) \oplus \pi_\psi(u^*bu - b) \in \pi(\mathfrak{A})$ for $u \in \mathfrak{A}^u$. Therefore,

$$\begin{aligned}&\{u^*bu - b - \hat{\tau}(b)1\} \oplus \{\pi_\psi(u^*bu - b) - \hat{\tau}(b)1\} \\ &= \{u^*bu - b - \hat{\tau}(b)1\} \oplus \{\rho(u^*)\rho(a)\rho(u) - \pi_\psi(b) - \hat{\tau}(b)1\} \\ &= \{u^*bu - b - \hat{\tau}(b)1\} \oplus \{\rho(u^*au) - \pi_\psi(b) - \hat{\tau}(b)1\} \in \pi(\mathfrak{A}).\end{aligned}$$

Hence

$$\{b_j - b - \hat{\tau}(b)1\} \oplus \{\rho(a_j) - \pi_\psi(b) - \hat{\tau}(b)1\} \in \pi(\mathfrak{A})$$

and so

$$\{\hat{\tau}(b)1 - b - \hat{\tau}(b)1\} \oplus \{\hat{\tau}(a)1 - \pi_\psi(b) - \hat{\tau}(b)1\} \in \overline{\pi(\mathfrak{A})}.$$

Hence

$$-b \oplus [\{\hat{\tau}(a) - \hat{\tau}(b)\}1 - \pi_\psi(b)] \in \overline{\pi(\mathfrak{A})}.$$

On the other hand, $b \oplus \rho(b) \in \overline{\pi(\mathfrak{A})}$; hence $0 \oplus [\{\hat{\tau}(a) - \hat{\tau}(b)\}1 - \pi_\psi(b) + \rho(b)] \in \overline{\pi(\mathfrak{A})}$ and so $\{\hat{\tau}(a) - \hat{\tau}(b)\}1 + \rho(b) = \pi_\psi(b)$; therefore $\|\pi_\psi(b)\| > \|\rho(b)\| = \|b\|$, a contradiction. Hence, we have $\hat{\tau}(a) = \psi(b) = \hat{\tau}(b)$.

Therefore $\psi(b) = \hat{\tau}(b)$ for all extreme elements ψ of Ω; hence $\varphi(b) = \hat{\tau}(b)$ for all $\varphi \in \Omega$. This contradicts that $\varphi_1(b) \neq \varphi_2(b)$, and completes the proof.

Now we shall show the following theorem.

Theorem. Every derivation D of $\mathfrak{A}$ is inner.

Proof. For $u \in \mathfrak{A}^u$, $[b, u] = bu - ub \in \mathfrak{A}$; hence $u^*bu - b \in \mathfrak{A}$. For arbitrary $\varepsilon>0$, by the above lemma there exist an element $a\in\mathfrak{A}$, finite families of unitary elements $\{u_n|n=1,2,\cdots,m\}$ in $\mathfrak{A}$ and positive numbers $\{\lambda_n|n=1,2,\cdots,m\}$ such that $\sum_{n=1}^{m}\lambda_n=1$ and $\|\sum_{n=1}^{m}\lambda_n u_n^* bu_n-a\|<\varepsilon$. Therefore,

$$\|b-\{a+(b-\sum_{n=1}^{m}\lambda_n u_n^* bu_n)\}\|$$
$$=\|\sum_{n=1}^{m}\lambda_n u_n^* bu_n-a\|<\varepsilon.$$

On the other hand, $b-\sum_{n=1}^{m}\lambda_n u_n^* bu_n=\sum_{n=1}^{m}\lambda_n(b-u_n^* bu_n)\in\mathfrak{A}$; hence $a+(b-\sum_{n=1}^{m}\lambda_n u_n^* bu_n)\in\mathfrak{A}$ and so b belongs to $\mathfrak{A}$.

This completes the proof.

3. Concluding remarks

We can extend the definition of uniformly hyperfinite C*-algebras to the non-separable case as follows: $\mathfrak{A}$ is called uniformly hyperfinite if it has a directed family of type I_{n_α} subfactors $\{\mathfrak{M}_\alpha\}$ ($\alpha\in\Pi$, $n_\alpha<+\infty$) such that (i) $1\in\mathfrak{M}_\alpha$ for all $\alpha\in\Pi$; (ii) $\mathfrak{M}_\alpha\subset\mathfrak{M}_\beta$ if $\alpha\leqq\beta$; (iii) $\mathfrak{A}$ is infinite-dimensional; (iv) $\mathfrak{A}$ is the uniform closure of $\bigcup_{\alpha\in\Pi}\mathfrak{M}_\alpha$.

Then, we can prove that every derivation of such algebras is inner, because our proof is available for these algebras.

Finally we shall remark that uniformly hyperfinite C*-algebras have outer *-automorphisms.

By induction, we shall define a sequence of unitary elements $\{u_i\}$ of $\mathfrak{A}$ such that $u_i\in\mathfrak{M}_i$.

Take an one-dimensional projection e_1 in $\mathfrak{M}_1$ and put $u_1=e_1-(1-e_1)$. Now suppose that $u_i (i\leqq j)$ are defined. Put $\mathfrak{M}_{j+1}=\mathfrak{M}_j\otimes(\mathfrak{M}_j'\cap\mathfrak{M}_{j+1})$, where $\mathfrak{M}_j'$ is the commutant of $\mathfrak{M}_j$ in $\mathfrak{A}$.

Take an one-dimensional projection e_{j+1} from $\mathfrak{M}_j'\cap\mathfrak{M}_{j+1}$ and put $v_{j+1}=e_{j+1}-(1-e_{j+1})$.

Then, define $u_{j+1}=u_j v_{j+1}$.

Next, by induction, we shall define a sequence of pure states $\{\psi_i\}$ on $\mathfrak{M}_i$ as follows:

Take a pure state ψ_1 on $\mathfrak{M}_1$ such that $\psi_1(e_1)=1$. Such state is unique, because $\mathfrak{M}_1$ is a type $I_{n_1}(n_1<+\infty)$ factor. Now suppose that

$\varphi_i(i\leqq j)$ are defined. Write $\mathfrak{M}_{j+1}=\mathfrak{M}_j\otimes(\mathfrak{M}_j'\cap\mathfrak{M}_{j+1})$. Take the unique pure state ξ_{j+1} on $\mathfrak{M}_j'\cap\mathfrak{M}_{j+1}$ such that $\xi_{j+1}(e_{j+1})=1$ and put $\psi_{j+1}=\psi_j\otimes\xi_{j+1}$, then ψ_{j+1} is pure on $\mathfrak{M}_{j+1}$, and clearly $\psi_{j+1}=\psi_j$ on $\mathfrak{M}_j$. Therefore we have the unique state φ on $\mathfrak{A}$ such that $\varphi=\psi_j$ on $\mathfrak{M}_j$ for all j. Clearly φ is pure on $\mathfrak{A}$.

Let $\{\pi_\varphi,\mathfrak{H}_\varphi\}$ be the *-representation of $\mathfrak{A}$ on a Hilbert space $\mathfrak{H}_\varphi$ constructed via φ, then π_φ is faithful, because $\mathfrak{A}$ is simple. For $a\in\mathfrak{M}_i$ and $m<n$,

$$\begin{aligned}
&\varphi(a^*(u_m-u_n)^*(u_m-u_n)a)=\varphi(a^*(u_m-u_n)^2a)\\
&=\varphi(a^*(1-v_{m+1}v_{m+2}\cdots v_n)^2a)\\
&=\varphi(a^*a(1-v_{m+1}v_{m+2}\cdots v_n)^2)\quad (m>i)\\
&\leqq\varphi(a^*a(1-v_{m+1}v_{m+2}\cdots v_n)^2a^*a)^{1/2}\varphi((1-v_{m+1}\cdots v_m)^2)^{1/2}\\
&\leqq 4\varphi((a^*a)^2)^{1/2}\varphi(2-2v_{m+1}v_{m+2}\cdots v_n)^{1/2}\\
&=4\varphi((a^*a)^2)^{1/2}\{2-2\psi_{m+1}(v_{m+1})\cdots\psi_n(v_n)\}^{1/2}\\
&=0.
\end{aligned}$$

Hence $\{\pi_\varphi(u_n)\}$ is a Cauchy sequence in the strong operator topology.

Let u be the strong limit of $\{\pi_\varphi(u_n)\}$, then u is a unitary operator on $\mathfrak{H}_\varphi$, because $\pi_\varphi(u_n)$ is self-adjoint.

Moreover, for $d\in\pi_\varphi(\mathfrak{M}_i)$

$$\begin{aligned}
u\pi_\varphi(d)u&=\text{strong}-\lim_n\pi_\varphi(u_n)\pi_\varphi(d)\pi_\varphi(u_n)\\
&=\pi_\varphi(u_1v_2\cdots v_i)\pi_\varphi(d)\pi_\varphi(u_1v_2\cdots v_i)\in\pi_\varphi(\mathfrak{M}_i).
\end{aligned}$$

Hence $u\pi_\varphi(\mathfrak{A})u\subset\mathfrak{A}$; therefore the mapping $\rho:x\to\pi_\varphi^{-1}(u\pi_\varphi(x)u)$ $(x\in\mathfrak{A})$ is an *-automorphism of $\mathfrak{A}$ such that $\rho(\mathfrak{M}_i)\subset\mathfrak{M}_i$ for all i.

Now suppose that there exists a unitary element $v\in\mathfrak{A}$ such that $\rho(x)=v^*xv$ for all $x\in\mathfrak{A}$. Then $\pi_\varphi(v)u\pi_\varphi(x)(\pi_\varphi(v)u)^*=\pi_\varphi(x)$ for $x\in\mathfrak{A}$; hence $\pi_\varphi(v)u=\lambda I$, where $|\lambda|=1$, and I is the identity; therefore $u\in\pi_\varphi(\mathfrak{A})$.

Let $\pi_\varphi(w)=u(w\in\mathfrak{A})$, then w is a self-adjoint unitary element in $\mathfrak{A}$. For arbitrary $\varepsilon>0$, there exist an i such that some self-adjoint $w_j\in\mathfrak{M}_j$

$$\|w-w_j\|<\varepsilon \text{ for } j\geqq i.$$

On the other hand, for $x_j\in\mathfrak{M}_j$

$$\pi_\varphi(w)\pi_\varphi(x_j)\pi_\varphi(w)=u\pi_\varphi(x_j)u$$
$$=\text{strong}-\lim_n\pi_\varphi(u_j)\pi_\varphi(v_{j+1})\cdots\pi_\varphi(v_n)\pi_\varphi(x_j)\pi_\varphi(u_j)\pi_\varphi(v_{j+1})\cdots\pi_\varphi(v_n)$$
$$=\pi_\varphi(u_j)\pi_\varphi(x_j)\pi_\varphi(u_j)=\pi_\varphi(u_jx_ju_j).$$

Hence $(u_jw)x_j(u_jw)=x_j$ for all $x_j\in\mathfrak{M}_j$ and so $u_jw\in\mathfrak{M}_j'$, where $\mathfrak{M}_j'$ is the commutant of $\mathfrak{M}_j$ in $\mathfrak{A}$; hence $w=u_ju_j'$, where $u_j'\in\mathfrak{M}_j'$.
Therefore

$$\|w-w_j\|=\|u_ju_j'-w_j\|=\|u_j'-u_jw_j\|$$
$$<\varepsilon \text{ for } j\geqq i.$$

Hence

$$\left\|\int_{\mathfrak{M}_j''}g^*u_j'g\,d\mu_j(g)-\int_{\mathfrak{M}_j''}g^*u_jw_jg\,d\mu_j(g)\right\|$$
$$=\|u_j'-\tau(u_jw_j)1\|\leqq\int_{\mathfrak{M}_j''}\|g^*(w-w_j)g\|d\mu_j(g)$$
$$\leqq\varepsilon \text{ for } j\geqq i.$$

Since $\pi_\varphi(u_j')=\pi_\varphi(u_j)\pi_\varphi(w)=\pi_\varphi(u_j)u$
$=\text{strong-}\lim_n v_{j+1}v_{j+2}\cdots v_n$, u_j' is

self-adjoint. Suppose that $\pi_\varphi(u_j')=1$ for $j\geqq i$, then $\pi_\varphi(w)=u=u_j$ for $j\geqq i$. On the other hand $u_j\neq u_k$, if $j\neq k$, a contradiction; hence $u_j'\neq 1$; therefore there exists a non-trivial projection p in $\mathfrak{A}$ such that $u_j'=p-(1-p)$. Then

$$\|p-(1-p)-\tau(u_jw_j)1\|$$
$$=\max\{|1-\tau(u_jw_j)|,\ |1+\tau(u_jw_j)|\}\leqq\varepsilon.$$

Hence $-\varepsilon\leqq 1-\tau(u_jw_j)\leqq\varepsilon$ and $-\varepsilon\leqq 1+\tau(u_jw_j)\leqq\varepsilon$; therefore $2\leqq 2\varepsilon$, a contradiction.

This implies that $u\notin\pi_\varphi(\mathfrak{A})$——namely ρ is an outor *-automorphism of $\mathfrak{A}$.

REFERENCES

[1] Glimm, J., On a certain class of operator algebras, Trans. Amer. Math. Soc. **95** (1960), 318-340.
[2] Grothendieck, A., Un resultat sur le dual d'une C*-algèbre, J. Math. Pures Appl. **36** (1957), 97-108.
[3] Kadison, R., Derivations of operator algebras, Ann. of Math. **83** (1966), 280-293.
[4] Kadison, R., E. Lance and J. Ringrose, Derivations and automorphisms of opera-

tor algebras, II, to appear.
[5] Miles, P., Derivations on B*-algebras, Pacific J. Math. **14** (1964), 1359–1366.
[6] Sakai, S., A Radon-Nikodym theorem in W*-algebras, Bull. Amer. Math. Soc. **71** (1965), 149–151.
[7] ———, Derivations of W*-algebras, Ann. of Math. **83** (1966), 273–279.
[8] ———, The theory of W*-algebras, Lecture notes, Yale University, 1962.