Publ. RIMS, Kyoto Univ. Ser. A
Vol. 3 (1968), pp. 289—297

A remark on complex analytic families of complex tori

By

Takashi OKANO*

1. It is well known that the field of meromorphic functions on an irreducible compact complex space is an algebraic function field whose transcendence degree over the complex number field $\boldsymbol{C}$ is not greater than the dimension of the space (for example, see [4] or [6]).

Let X and Y be complex spaces and π a proper holomorphic mapping of X onto Y with irreducible fibers. For a point t of Y we put K_t the meromorphic function field of the fiber $\pi^{-1}(t)$. We ask how many functions of K_t can be extended to meromorphic functions on neighborhoods of the fiber.

In this paper, we solve this problem only in a special case, the case of complex analytic families of complex tori (Corollary 1 of Theorem 2).

Let Y be an irreducible complex space and $\Omega=(\omega_{ij})$ be a $(n, 2n)$-matrix, where ω_{ij} $(i=1,2,\cdots,n;\ j=1,2,\cdots,2n)$ are holomorphic functions on Y. We suppose that the $(2n, 2n)$-matrix $\begin{pmatrix}\Omega\\ \overline{\Omega}\end{pmatrix}$ (where $\overline{\Omega}$ means the complex conjugate of the matrix Ω) is non-singular for each point of Y.

Let $\boldsymbol{C}^n$ be the space of n complex variables $z=\begin{pmatrix}z_1\\ \vdots\\ z_n\end{pmatrix}$ and G be the discontinuous abelian group of analytic automorphisms of $\boldsymbol{C}^n\times Y$ generated by

$$g_j\colon\ (z,t)\rightarrow(z+\omega_j(t),t),\ \ j=1,\cdots,2n,$$

Received October 12, 1967.
Communicated by S. Nakano.
* Faculty of Liberal Arts and Education, Nagoya City University.

where $\omega_j(t)=\begin{pmatrix}\omega_{1j}(t)\\ \vdots \\ \omega_{nj}(t)\end{pmatrix}$. Then the factor space $X=(C^n\times Y)/G$ is a complex analytic family of complex tori over the space Y. We denote the natural projection of X to Y by π. The fiber $X_t=\pi^{-1}(t)$ is the complex torus with periods $\omega_j(t)$, $j=1,\dots,2n$.

2. From now on, we denote by $td(K_t)$ the transcendence degree of the field K_t over the complex number field.

We put $Y_k=\{t\in Y\,|\,td(K_t)=k\}$, $k\geqq 0$. Let t_0 be a point of Y_k, where we assume $k>0$. Then there is a linear transformation of the variables z,

$$\begin{pmatrix}w_1\\ \vdots \\ w_n\end{pmatrix}=Q\begin{pmatrix}z_1\\ \vdots \\ z_n\end{pmatrix},\ \text{where } Q=\begin{pmatrix}a_{11}, & a_{12}, & \dots, & a_{1n}\\ \vdots & \vdots & & \vdots \\ a_{t1} & a_{t2}, & \dots, & a_{kn}\\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2}, & \dots, & a_{nn}\end{pmatrix}$$ is a non-singular (n, n)-matrix, such that any function of K_{t_0} is independent of the $n-k$ variables $w_{k+1},\cdots,w_n$. We put $P=\begin{pmatrix}a_{11}, & \dots a_{1n}\\ \vdots & \vdots \\ a_{k1}, & \dots a_{kn}\end{pmatrix}$.

Then clearly,

(i) $rank\ P=k$,

and 2n vectors $P\omega_j(t_0)$, $j=1,\dots,2n$, of the space C^k (of the variables $w=\begin{pmatrix}w_1\\ \vdots \\ w_k\end{pmatrix}$) form a lattice G^k in the space C^k (see [5], p. 103).

Hence we obtain a k-dimensional complex torus C^k/G^k and a natural holomorphic mapping of X_{t_0} onto the torus C^k/G^k which is induced from the linear mapping $w=Pz$. Further, the field K_{t_0} is naturally isomorphic to the field of meromorphic functions on C^k/G^k, and hence the torus C^k/G^k is an abelian variety.

Let $\tilde{\omega}_1, \tilde{\omega}_2, \dots, \tilde{\omega}_{2k}$ be a free base of the group G^k, where $\tilde{\omega}_i=\begin{pmatrix}\tilde{\omega}_{i1}\\ \\ \tilde{\omega}_{ik}.\end{pmatrix}$. Then there are $(2n, 2k)$-matrix H_1 and a $(2k, 2n)$-matrix H_2 with integral elements such that;

$$P\Omega(t_0)H_1=(\tilde{\omega}_1, \dots, \tilde{\omega}_{2k}),\ \text{where } \Omega(t_0)=(\omega_{ij}(t_0)),$$

and,

(ii) $P\Omega(t_0)H_1H_2 = P\Omega(t_0)$.

Further, since C^k/G^k is abelian, there is a non-singular skew-symmetric $(2k, 2k)$-matrix A with integral elements such that,

(iii) $P\Omega(t_0)H_1A(P\Omega(t_0)H_1)' = 0$,

(iv) $\sqrt{-1}\overline{P\Omega(t_0)H_1}A(P\Omega(t_0)H_1)' < 0$, where $'$ means the transposition.

3. We assume that the set Y_n is of second category.

With each point t of Y_n, we associate a non-singular skew-symmetric $(2n,2n)$-matrix A with integral elements such that

$$\Omega(t_0)A\Omega(t_0)' = 0 \quad \text{and,}$$
$$\sqrt{-1}\overline{\Omega(t_0)}A\Omega(t_0)' < 0.$$

Let $\mathfrak{A}$ be the set of those matrices A. For each A of $\mathfrak{A}$ we consider the analytic set $Y(A) = \{t \in Y \mid \Omega(t)A\Omega(t)' = 0\}$ in Y. Then there is a matrix A of $\mathfrak{A}$, such that $Y(A) = Y$. By the definition of A, there is a point t_0 of Y_n such that $\sqrt{-1}\,\overline{\Omega(t_0)}A\Omega(t_0)' < 0$. Hence $\sqrt{-1}\,\overline{\Omega(t)}A\Omega(t)' < 0$ for each point t of Y, as Y is connected. Therefore $Y = Y(A) = Y_n$.

In this case, using the theory of ϑ functions, we see, for each $t \in Y$, that any function of K_t can be extended to a meromorphic function on a neighborhood of X_t.

4. Lemma 1. *Let F_1 be a (l, p)-matrix and F_2 be a (p, l)-matrix, where $l \geqq p$. Then the rank of the (l, l)-matrix $F_1F_2 - E^{(l)}$ is not smaller than $l-p$, where $E^{(l)}$ is the identity matrix of rank l.*

Proof. Let q be the number of multiplicity of the root 1 of the characteristic polynomial of the matrix $F_1 F_2$. Then there is a non-

$$JF_1\,F_2J^{-1} = \begin{pmatrix} 1 & & & & & & \\ & 1 & & & & * & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & \alpha_{q+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \alpha_l \end{pmatrix}, \qquad \text{where } \alpha_i \neq 1,\ i = q+1, \dots, l.$$

singular (l, l)-matrix J such that

Hence the *rank* $(F_1F_2-E^{(l)})\geqq l-q$.

Let $\boldsymbol{u}_i$ be the i-th unit column vector $\begin{pmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{pmatrix}\cdots i,\ i=1,\dots,q$. Then q vectors $JF_1F_2J^{-1}\boldsymbol{u}_i$ are independent. Hence $q\leqq p$ and therefore *rank* $(F_1F_2-E^{(l)})\geqq l-p$.

Lemma 2. *Let T be a real $(2n, 2n)$-matrix and S be a $(n, 2n)$-matrix.*

Then 2 *rank* $ST\geqq$ *rank* $\begin{pmatrix}S\\ \overline{S}\end{pmatrix}T$.

Proof. The rank of a matrix is the dimension of the image of the linear mapping defined by the matrix.

We consider the matrices ST and $\overline{S}T$ as the linear mappings from the vector space $\boldsymbol{C}^{2n}$ to the vector space $\boldsymbol{C}^n$ respectively. Since T is real, the rank of ST and the rank of $\overline{S}T$ are equal. On the other hand the image of $\begin{pmatrix}S\\ \overline{S}\end{pmatrix}T$: $\boldsymbol{C}^{2n}\to\boldsymbol{C}^n\oplus\boldsymbol{C}^n$ is contained in the (direct) sum of the images of ST and $\overline{S}T$. Hence we have 2 *rank* $ST\geqq$ *rank* $\begin{pmatrix}S\\ \overline{S}\end{pmatrix}T$.

Now t_0 let be a point of Y_k, where $k>0$. Then, as mentioned in §2, there is a system of matrices H_1, H_2 and A (with integral elements) and P of types described in § 2 such that,

(i) *rank* $P=k$,

(ii) $P\Omega(t_0)H_1H_2=P\Omega(t_0)$,

(iii) $P\Omega(t_0)H_1A(P\Omega(t_0)H_1)'=0$,

(iv) $\sqrt{-1}\,\overline{P\Omega(t_0)}H_1A(P\Omega(t_0)H_1)'<0$.

Conversely, let t be a point of Y and assume that there are integral matrices H_1, H_2 and P with properties (i) *rank* $P=k$ and (ii) $P\Omega(t)H_1H_2=P\Omega(t)$, then the $2n$ column vectors of $P\Omega(t)$ generate a lattice in $\boldsymbol{C}^k$ and the $2k$ column vectors of $P\Omega(t)H_1$ give a system of a free base of this lattice.

We fix a system of matrices H_1, H_2 and A with integral elements and consider the set $Y(H_1, H_2)=\{t\in Y|$ there is a (k,n)-matrix P such that the above conditions (i) and (ii) are satisfied at the point $t\}$ and the set $Y(H_1,H_2,A)=\{t\in Y|$ there is a (k,n)-matrix P such that the above conditions (i), (ii), (iii) and (iv) are satisfied at $t\}$.

Let Π be the vector space of all (k,n)-matrices, and $\mathfrak{P}$ the analytic subspace of $\Pi\times Y$ defined by the equation $P\Omega(t)(H_1H_2-E^{(2n)})=0$ $(P\in\Pi, t\in Y)$ with the natural projection p to Y. Then $p^{-1}(t)$ is linear subspace of Π for each point t of Y. Further, the space $p^{-1}(t)$ contains a matrix of rank k if and only if the rank of the matrix $\Omega(t)(H_1H_2-E^{(2n)})$ is not greater than $n-k$, because the equation $P\Omega(t)(H_1H_2-E^{(2n)})=0$ is a system of k independent equations $\boldsymbol{a}_i\Omega(t)(H_1H_2-E^{(2n)})=0$, $i=1,\dots,k$, where $\boldsymbol{a}_i$ is the i-th row vector of the matrix P.

Proposition 1. *The set $Y(H_1, H_2)$ is an analytic subset of Y defined by the equation $\mathrm{rank}\,\Omega(t)(H_1H_2-E^{(2n)})=n-k$, and $\mathfrak{P}|Y(H_1, H_2)$ is a complex analytic vector bundle of dimension k^2.*

Further, let t_1 be a point of $Y(H_1, H_2)$ and P_0 be a matrix of $p^{-1}(t_1)$ of rank k. Then, for each matrix P_1 of $p^{-1}(t_1)$, there is a (k,k)-matrix L with $LP_0=P_1$.

Proof. Since the matrix $\begin{pmatrix}\Omega(t)\\ \overline{\Omega(t)}\end{pmatrix}$ is non-singular for each point t of Y the rank of $\Omega(t)(H_1H_2-E^{(2n)})$ is not smaller than $n-k$ by Lemma 1 and Lemma 2. Thus the set $Y(H_1, H_2)$ is defined by the equation *rank* $\Omega(t)(H_1H_2-E^{(2n)})=n-k$.

Let $\mathfrak{L}$ be the space of all (k,k)-matrices and b the linear map of $\mathfrak{L}$ to the vector space $p^{-1}(t_1)$ defined by $L\to LP_0$, then it is trivial that the map b is surjective.

Proposition 2. *The set $Y(H_1, H_2, A)$ is an analytic subset of $Y(H_1, H_2)$.*

Proof. Let t_1 be a point of $Y(H_1, H_2)$. Then there is an open neighborhood U of t_1 in the space $Y(H_1, H_2)$ and a holomorphic section $P(t)$ of the vector bundle $\mathfrak{P}|Y(H_1, H_2)$ over U such that the rank of $P(t)$ is k for any t of U.

We consider the equation $P(t)\Omega(t)H_1A(P(t)\Omega(t)H_1)'=0$ on U.

Then the solution of this equation is an analytic subset of U and it is independent of the choise of the section $P(t)$ by the last assertion of Proposition 1. Therefore the set $\{t\in Y(H_1, H_2) \mid$ there is a (k, n)-matrix P of $p^{-1}(t)$ such that the condition (i) and (iii) are satisfied$\}$ is an analytic subset N of $Y(H_1, H_2)$.

Let $N_1, N_2, \dots$ be the connected components of N. Then the signature of the Hermitian matrix $\sqrt{-1}\,\overline{P(t)\Omega(t)}H_1A(P(t)\Omega(t)H_1)'$ is constant on $N_i, i=1, 2, \dots$, because the determinant of the matrix can not be zero at any point of N. Therefore the set $Y(H_1, H_2, A)$ is an analytic subset of $Y(H_1, H_2)$.

Theorem 1. *We suppose that $Y_j\neq\phi$ for some $j, n>j\geqq 0$. Then, for $q>j$, the set $Y(q)=Y_q\cup Y_{q+1}\cup\cdots\cup Y_n$ is a countable union of thin analytic sets.*

Proof. Let t_0 be a point of $Y(q)$. Then t_0 is a point of Y_k, where $k\geqq q$, and there are matrices H_1, H_2 and A as before such that $t_0\in Y(H_1, H_2, A)$. Thus the set $Y(q)$ is equal to the union of such thin analytic sets of Y.

5. We put $p=\inf_{t\in Y}\{td(K_t)\}$. Then the set Y_p is of second category by Theorem 1.

Let t_0 be a point of Y_p. We assume $p>0$. Then we obtain, as mentioned in § 4, three integral matrices H_1, H_2 and A of type $(2n, 2p)$, $(2p, 2n)$ and $(2p, 2p)$ respectively such that $t_0\in Y(H_1, H_2, A)$. The analyticity of $Y(H_1, H_2, A)$ and the fact that Y_p is of second category imply the existence of matrices H_1, H_2 and A, of type described above, such that $Y=Y(H_1, H_2, A)$.

We fix such a system of matrices $\{H_1 H_2, A\}$. Then we obtain a complex analytic vector bundle $\mathfrak{P}$ on Y of dimension p^2 which is embedded in the space $\Pi\times Y$ (where Π is the vector space of all (p, n)-matrices).

Let t_0 be a point of Y. Then there are an open neighborhood U of t_0 and a holomorphic section $P(t)$ of $\mathfrak{P}$ on U such that *rank* $P(t)=p$ for each t of U. Using this section $P(t)$, we can construct, as mentioned in § 1, a complex analytic family X'_U of abelian varieties over U whose period matrices are $P(t)\Omega(t)H_1$, and we obtain naturally a holomorphic mapping σ_U of $X|U$ onto X'_U, which is induced by the linear mapping $\begin{pmatrix} w_1 \\ \vdots \\ w_p \end{pmatrix} = P(t) \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$.

Another such section of $\mathfrak{P}$ over U defines the same family X'_U and the mapping σ_U but alters only the holomorphic coordinates w. Hence we have;

Theorem 2. *Let* $p = \inf_{t\in Y} \{td(K_t)\}$, $p>0$. *Then we have a complex analytic vector bundle* $\mathfrak{B}\to Y$ *of dimension* p *and a holomorphic mapping* $\bar{\sigma}$ *of* $C^n \times Y$ *onto* $\mathfrak{B}$ *such that* $\bar{\sigma}$ *is locally defined by* $(z, t)\to(P(t)z, t)$, *where* $P(t)$ *is a* (p, n)-*matrix of holomorphic functions on an open set* U *of* Y, *and* $P(t)\Omega(t)$ *gives a discontinuous abelian group* G' *of analytic automorphisms of* $\mathfrak{B}|U$ *and the factor space* $X'=\mathfrak{B}/G'$ *is a family of abelian varieties of dimension* p *over the space* Y.

The map $\bar{\sigma}$ *induces naturally a holomorphic mapping* σ *of* X *onto* X' *such that*
$$\begin{array}{ccc} X & \xrightarrow{\sigma} & X' \\ & \pi\searrow \quad \swarrow\pi' & \\ & Y & \end{array}$$
is commutative.

Corollary 1. *We denote by* K'_t *the subfield of* K_t *consisting of all elements of* K_t *which can be extended to meromorphic functions on some neighborhoods of* X_t.

Then, for each point t *of* X;

(a) *the transcendence degree of the field* K'_t *is equal to* $\inf_{t\in Y} \{td(K_t)\}$, *and*

(b) *the field* K'_t *is algebraically closed in* K_t.

Proof. Let $f_1, \dots, f_s$ be meromophic functions on a neighborhood of X_t such that the analytic restriction of $f_1, \dots, f_s$ to the fiber X_t are independent. Then, for t' sufficiently near to t, the restriction of $f_1, \dots. f_s$ to X'_t are independent. Hence we see that the *tr. degree* of $K'_t \leqq p$ by Theorem 1.

Now let K''_t be the subfield of K_t obtained from the field of meromorphic functions on $X'_t = \pi'^{-1}(t)$ by $\sigma_t : X_t \to X'_t$. Then $K''_t \subset K'_t$, because the family $X' \xrightarrow{\pi'} Y$ is a family of abelian varieties (see § 2). Hence (a) is proved.

If a meromorqhic function on the torus X_t is dependent on meromorphic functions on X_t which are independent of the variables $w_{p+1}, \dots, w_n$, then it is also independent of $w_{p+1}, \dots, w_n$. The assertion (b) follows from this.

Remark. In [3], we considered the property (b) of the corollary in the case of general complex analytic fiber spaces. There we obtained;

Let X and Y be normal and connected complex spaces and $X \xrightarrow{\pi} Y$ a proper holomorphic mapping of X onto Y with irreducible fibers. Then the set $\{t \in Y \mid$ the field K_t' is not algebraically closed in $K_t\}$ is nowhere dense in Y.

Corollary 2. *We assume that $Y_0 = \phi$ and $Y_n \neq Y$. Then every abelian variety of a member of the family $X \xrightarrow{\pi} Y$ is always 'singular' (in the sense of [1]).*

Let M be a connected complex manifold and $\mathfrak{B} \xrightarrow{\pi} M$ be a complex analytic family, in the sense of Kodaira-Spencer [2], of complex tori. Then, by Theorem 18.6 in [2], the family $\mathfrak{B} \xrightarrow{\pi} M$ is locally the same as our family of complex tori constructed in § 1.

Hence we get:

Theorem 3. *Let $\mathfrak{B} \xrightarrow{\pi} M$ be a complex analytic family, in the sense of Kodaira-Spencer, of complex tori. We put $p = \inf_{t \in M} \{td(K_t)\}$ and assume $p > 0$. Then there exists a complex analytic family $\mathfrak{B}' \xrightarrow{\pi'} M$ of abelian varieties of dimension p over M and a holomorphic mapping σ of $\mathfrak{B}$ onto $\mathfrak{B}'$ with $\pi = \pi' \circ \sigma$, such that σ is locally the same as mentioned in Theorem 2.*

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