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The first boundary value problem and the first eigenvalue problem for the elliptic equations degenerate on the boundary

By

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Introduction

The degenerate elliptic equations have been studied by many authors. Mikhlin [4] discussed those degenerate on a part of the boundary. However, he treated only the weak solutions of the problem. Il'in [2] and Oleinik [6] discussed the degenerate elliptic equation

$$\sum_{i,j=1}^{m} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{m} b_i \frac{\partial u}{\partial x_i}+cu=f.$$

They proved the uniqueness and existence theorems for the genuine solutions of the boundary value problem. They imposed some conditions on the equation, especially they required essentially that the equation is reduced to the case when $c \leqq c_0 < 0$ in the domain.

We treat the elliptic equation in the domain

$$\sum_{i,j=1}^{m} \frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial u}{\partial x_i}\right)-qu=f$$

which may be degenerate on the entire boundary. We prove, by the variational method, the uniqueness and existence theorems for the genuine solutions of the first boundary value problem and the existence theorem for the genuine solutions of the first eigenvalue problem. We do not assume that $q \geqq q_0 > 0$ in the domain. However, in order that Ladyzhenskaya-Ural'tseva's estimation for the solutions be applicable, we have to impose certain restrictions on the "order of de-

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generacy".

In §1 some notations and terminologies are introduced. We develop the main results in §2–§4; especially the boundary problems and the eigenvalue problems are stated in §3 and in §4 respectively. Section 5 is devoted to the lemmas used in §2–§4. In §6 we give some sufficient conditions for the validity of the assumptions made in the main theorems. Finally we arrange, in §7, some examples to which our theorems are applicable.

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§1. Notations and terminologies

Let Ω be a bounded domain in the m-dimensional Euclidean space, and $\partial\Omega$ its boundary.

By $C_{l,\alpha}(\overline{\Omega})$ $(0<\alpha<1)$ we denote the space of functions u, the l-th order derivatives of which are Hölder continuous on Ω and for which

$$\|u\|_{C_{l,\alpha}(\overline{\Omega})}\equiv\sum_{k=0}^{l}\sum_{|k|=k}\max_{\overline{\Omega}}|D^k u|+H_{l,\alpha,\Omega}<\infty,$$

where

$$D^k u\equiv\frac{\partial^{k_1+\cdots+k_m}u}{\partial x_1^{k_1}\cdots\partial x_m^{k_m}},\quad |k|\equiv k_1+\cdots+k_m$$

and

$$H_{l,\alpha,\Omega}\equiv\max_{|k|=l}\sup_{x,x'\in\Omega}|D^k u(x)-D^k u(x')|/|x-x'|^{\alpha}.$$

Also $C_{l,0}(\overline{\Omega})$ will denote the space of functions, the l-th order derivatives of which are continuous on $\overline{\Omega}$.

By $C_{l,\alpha}(\Omega)$ we denote the space of functions which belongs to $C_{l,\alpha}(\overline{\Omega'})$ for any domain Ω' strictly contained in Ω.

If for any $x_0\in\partial\Omega$, there exist a neighbourhood $U_{x_0}=\{x\,|\,|x-x_0|<\sigma_{x_0}\}$ and the local coordinates $y_1(x),\cdots,y_m(x)\in C_{l,\alpha}(\overline{U}_{x_0})$ $(0\leqq\alpha<1)$ such that

$$U_{x_0}\cap\overline{\Omega}=\{x\,|\,y_m(x)\geqq y_m(x_0)=0\}\cap U_{x_0}$$

and

$$\left|\frac{D(y_1,\cdots,y_m)}{D(x_1,\cdots,x_m)}\right| \neq 0 \text{ in } \overline{U}_{x_0},$$

then we say that $\partial\Omega$ belongs to $C_{l,\alpha}$.

Let $\Omega_1,\cdots,\Omega_n,\cdots$ be the sequence of domains such that each member Ω_n is strictly contained in the next Ω_{n+1}, and the boundary $\partial\Omega_n$ belongs to $C_{l,\alpha}$. If $\Omega=\bigcup_{n=1}^{\infty}\Omega_n$, then we say that $\partial\Omega$ belongs to $\overline{C}_{l,\alpha}$ and we call $\{\Omega_n\}$ an approximating sequence of domains for $\Omega(\partial\Omega\in\overline{C}_{l,\alpha})$. Hereafter we assume $\partial\Omega$ belongs to $\overline{C}_{l,\alpha}$.

By $L_p(\Omega)$ we denote the space of functions u which are measurable in Ω and for which

$$\|u\|_{L_p(\Omega)}\equiv\left[\int_\Omega |u|^p d V\right]^{1/p}<\infty.$$

Let $\dot{C}_\infty(\Omega)$ be the set of functions which are infinitely differentiable in Ω and with supports strictly contained in Ω.

When we have

$$\int_\Omega u\frac{\partial\zeta}{\partial x_i}dV=-\int_\Omega w_i\zeta dV$$

for the function u defined in Ω and for every function $\zeta\in\dot{C}_\infty(\Omega)$, we call w_i the generalized derivative of u with respect to x_i and denote it by $\frac{\partial u}{\partial x_i}$.

By $W_p^{(1)}(\Omega)$ we denote the space of functions u measurable in Ω and having the first order generalized derivatives also measurable in Ω, and for which

$$\|u\|\, W_p^{(1)}(\Omega)\equiv\left[\int_\Omega \{\sum_{i=1}^{m}\left(\frac{\partial u}{\partial x_i}\right)^p+u^p d\}\, V\right]^{1/p}<\infty.$$

By $\overset{\circ}{W}{}_p^{(1)}(\Omega)$ we denote the closure of $\dot{C}_\infty(\Omega)$ in $W_p^{(1)}(\Omega)$. For $\varphi\in C_{1,0}(\overline{\Omega})$ we shall denote by $\overset{\varphi}{W}{}_p^{(1)}(\Omega)$ the subset of $W_p^{(1)}(\Omega)$ defined by

$$\overset{\varphi}{W}{}_p^{(1)}(\Omega)\equiv\{u\,|\,u-\varphi\in\overset{\circ}{W}{}_p^{(1)}(\Omega)\}.$$

For $u\in\overset{\varphi}{W}{}_p^{(1)}(\Omega)$ and a real number K, we define

$$u^{(K)}\equiv\max\{u-K,0\}$$

and

$$\Omega^{(K)} \equiv \{x \in \Omega \mid u(x) > K\}.$$

For $p_{ij}(i, j=1, \cdots, m)$, q, $\rho \in C_{0,0}(\overline{\Omega})$ and $u, v \in W_2^{(1)}(\Omega)$ we set

$$G_\Omega[u,v] \equiv \int_\Omega \{\sum_{i,j=1}^m p_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + q uv\} dV,$$

$$D_\Omega[u,v] \equiv \int_\Omega \sum_{i,j=1}^m p_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dV,$$

$$H_{\Omega,\rho}[u, v] \equiv \int_\Omega \rho\, uv\ dV,$$

$$G_\Omega[u] \equiv G_\Omega[u, u],\ D_\Omega[u] \equiv D_\Omega[u, u]$$

and

$$H_{\Omega,\rho}[u] \equiv H_{\Omega,\rho}[u, u].$$

Now we shall consider the boundary value problem

(1) $\quad L[u] \equiv \sum_{i,j=1}^m \frac{\partial}{\partial x_j}(p_{ij} \frac{\partial u}{\partial x_i}) - qu = f \qquad$ in Ω,

(2) $\quad u = \varphi \qquad$ on $\partial\Omega$,

where $p_{ij} = p_{ji}$, q, $f \in C_{0,0}(\overline{\Omega})$ and $\varphi \in C_{1,0}(\overline{\Omega})$. We say that u is a weak solution of (1)-(2) if

(i) $\quad G_\Omega[u, \zeta] + H_{\Omega,1}[f, \zeta] = 0 \qquad$ for every $\zeta \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$,

and

(ii) $\quad D_\Omega[u] < \infty$, and there exists $\{u_h \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)\}$ such that

$$D_\Omega[u - u_h] \to 0 \text{ and } \|u - u_h\|_{L_2(\Omega)} \to 0 \text{ as } h \to \infty.$$

If $u \in C_{2,0}(\Omega) \cap C_{0,0}(\overline{\Omega})$ satisfies (1)-(2) and $D_\Omega[u] < \infty$, then we call u a solution of (1)-(2) (in this case $p_{ij} \in C_{1,0}(\Omega) \cap C_{0,0}(\overline{\Omega})$).

For the operator L and a boundary point $x_0 \in \partial\Omega$, we define a strong barrier function $v(x)$ as such that, for some $\sigma > 0$,

(i) $\quad v(x) \in C_{0,0}(\overline{\omega(x_0, \sigma)})$

(ii) $\quad v(x) = 0$ at x_0

(iii) $\quad v(x) > 0$ in $\overline{\omega(x_0, \sigma)} - \{x_0\}$

(iv) $L[v]<0$ in $\omega(x_0, \sigma)$,

where $\omega(x_0, \sigma)=\{x\in\Omega||x-x_0|<\sigma\}$.

To abbreviate the main theorems, we define Properties (A) and (B) of the domain and the operator L, as follows:

Property (A). For any boundary point x_0 there exists some $\sigma>0$, such that $\omega(x_0, \sigma)\equiv\{x\in\Omega||x-x_0|<\sigma\}$ satisfies the following condition.

If $v\in C_{2,0}(\Omega)\cap C_{0,0}(\overline{\Omega})$ satisfies

$$L[v]\leqq \min_{\overline{\omega(x_0,\sigma)}} f \qquad \text{in } \omega(x_0, \sigma)$$

$$v\geqq \max_{\partial\omega(x_0,\rho)} \varphi \qquad \text{on } \partial\omega(x_0 ,\sigma)$$

resp.

$$L[v]\geqq \max_{\overline{\omega(x_0,\sigma)}} f \qquad \text{in } \omega(x_0, \sigma)$$

$$v\leqq \min_{\partial\omega(x_0,\sigma)} \varphi \qquad \text{on } \partial\omega(x_0, \sigma),$$

then for any solution u of (1)-(2), we have

$$u\leqq v \qquad \text{in } \overline{\omega(x_0, \sigma)}$$

resp.

$$u\geqq v \qquad \text{in } \overline{\omega(x_0, \sigma)}.$$

Property (B). For any boundary point there is a strong barrier function.

§2. Weak solutions

In this section we consider the boundary value problem for the degenerate elliptic equation (1). In the first place we show that any solution of (1)-(2) is a weak solution of (1)-(2) (Theorem 1), and in the next place we give an a priori estimation for the weak solutions of Ladyzhenskaya-Ural'tseva's type (Theorem 2).

Theorem 1. *If u is a solution of*

$$(1) \quad L[u]\equiv\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial u}{\partial x_i}\right)-qu=f \qquad \text{in } \Omega,$$

$$(2) \quad u=\varphi \qquad \text{on } \partial\Omega,$$

then u is a weak solution of (1)−(2), *where*

$$p_{ij}=p_{ji}\in C_{1,0}(\Omega)\cap C_{0,0}(\overline{\Omega});\ q, f\in C_{0,0}(\overline{\Omega});$$
$$\varphi\in C_{1,0}(\overline{\Omega});\ \partial\Omega\in\overline{C}_{1,0}$$

and

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j\geqq 0\ \ ((\xi_1, \cdots, \xi_m)\ \textit{is any real vector}).$$

Proof. For any $\xi\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ we take a sequence $\{\zeta_h\in\dot{C}_\infty(\Omega)\}$ such that $\zeta_h\to\zeta$ in $W_2^{(1)}(\Omega)$ as $h\to\infty$.

Since $u\in C_{2,0}(\Omega)$ satisfies (1), we have

$$G_\Omega[u, \zeta_h]+H_{\Omega,1}[f, \zeta_h]=0.$$

Therefore, by Lemma 1 and in a similar way we have

$$G_\Omega[u, \zeta]+H_{\Omega,1}[f, \zeta]$$
$$=\lim_{h\to\infty}(G_\Omega[u, \zeta_h]+H_{\Omega,1}[f, \zeta_h])=0.$$

Moreover we can show that there exists a sequence $\{u_h\in W_2^{(1)}(\Omega)\}$ such that $D_\Omega[u-u_h]\to 0$ and $\|u-u_h\|_{L_2(\Omega)}\to 0$, as follows.

Since $D_\Omega[u]$ is finite, we get

$$D_\Omega[u-\varphi]\leqq(\sqrt{D_\Omega[u]}+\sqrt{D_\Omega[\varphi]})^2<\infty.$$

Choose an approximating sequence $\{\Omega_h\}$ for $\Omega(\partial\Omega\in\overline{C}_{1,0})$, and set

$$u_h\equiv\begin{cases} u-K_h & \text{in } \Omega_h^+\equiv\{x\in\Omega_h\,|\,u(x)-\varphi(x)>K_h\}\\ u+K_h & \text{in } \Omega_h^-\equiv\{x\in\Omega_h\,|\,u(x)-\varphi(x)<-K_h\}\\ \varphi & \text{in } \Omega_h^0\ \ \{x\in\Omega_h\,|\,|u(x)-\varphi(x)|\leqq K_h\}\\ \varphi & \text{in } \Omega-\Omega_h\end{cases}$$

where $K_h\equiv\max_{\partial\Omega_h}|u-\varphi|\to 0$ as $h\to\infty$.

Then we have

$$\|u-u_h\|_{L_2(\Omega)}=\|u-u_h\|_{L_2(\Omega-\Omega_h)}+\|K_h\|_{L_2(\Omega_h^+)}$$
$$+\|-K_h\|_{L_2(\Omega_h^-)}+\|u-\varphi\|_{L_2(\Omega_h^0)}$$
$$\to\lim_{h\to\infty}\|u-\varphi\|_{L(\Omega_h^0)}$$

and

$$D_\Omega[u-u_h]$$
$$=D_{\Omega-\Omega_h}[u-\varphi]+D_{\Omega_h^+}[K_h]+D_{\Omega_h^-}[-K_h]+D_{\Omega_h^o}[u-\varphi]$$
$$\to\lim_{h\to\infty}D_{\Omega_h^o}[u-\varphi].$$

On the other hand

$$\text{mes}(\Omega_h^0-\{x\in\Omega\,|\,u(x)=\varphi(x)\})\to 0,$$

and therefore

$$\lim_{h\to\infty}\|u-\varphi\|_{L_2(\Omega_h^o)}=0\,,$$
$$\lim_{h\to\infty}D_{\Omega_h^o}[u-\varphi]=0\,.$$

Thus

$$\|u-u_h\|_{L_2(\Omega)}\to 0\quad(h\to\infty)$$

and

$$D_\Omega[u-u_h]\to 0\quad(h\to\infty)\,.$$

Finally we show that $u_h\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)$. Note that $u-\varphi\in C_{1,0}(\Omega_h)$, which implies that $u-\varphi\in \overset{u-\varphi}{W}{}_2^{(1)}(\Omega_h)$. Therefore, if we set

$$v_h^+\equiv\begin{cases}\max\{u-\varphi-K_h,0\} & \text{in } \Omega_h\\ 0 & \text{in } \Omega-\Omega_h\end{cases}$$

$$v_h^-\equiv\begin{cases}\max\{u-\varphi+K_h,0\} & \text{in } \Omega_h\\ 0 & \text{in } \Omega-\Omega_h,\end{cases}$$

then $v_h^+,v_h^-\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ (cf. Lemma 2). Thus

$$u_h-\varphi=v_h^++v_h^-\in \overset{\circ}{W}{}_2^{(1)}(\Omega),$$

i. e.,

$$u_h\in \overset{\varphi}{W}{}_2^{(1)}(\Omega).$$

Theorem 2. *Assume that* $p_{ij},=p_{ji}$ q, $f\in C_{0,0}(\overline{\Omega})$ *and* $\varphi\in C_{1,0}(\overline{\Omega})$, *and that* p_{ij} *satisfies*

$$\sum_{i,j=1}^m p_{ij}\xi_i\xi_j\geqq p_0(x)\sum_{i=1}^m \xi_i^2\geqq 0$$

$$((\xi_1, \cdots, \xi_m) \textit{ is any real vector})$$

where

$$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega) \ \left(\frac{m}{m+2} < k < 1 \textit{ when } m \geqq 2 \textit{ or } \frac{1}{2} < k < 1\right.$$
when $m=1$).

If u *is a weak solution of* (1)–(2) *which satisfies* $D_\Omega[u] < d$, *then*

$$\operatorname*{vrai\,max}_{\Omega} |u| < C,$$

when C depends on m, k, d and C' (C' is any positive constant such that $\left\|\frac{1}{p_0}\right\|_{L_{k/(1-k)}(\Omega)}$, $\max_{\bar\Omega}|q|$, $\max_{\bar\Omega}|f|$, $\max_{\bar\Omega}|\varphi|$, $D_\Omega[\varphi]$, mes $\Omega < C'$).

Proof. We begin with the case $m \geqq 2$. Since u is a weak solution of (1)-(2), and $u^{(K)} \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ $(K \geqq \max_{\partial\Omega} \varphi, K \geqq 1)$, we have

$$\int_{\Omega^{(K)}} \left[\sum_{i,j=1}^m p_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u^{(K)}}{\partial x_j} + quu^{(K)} + fu^{(K)}\right] dV = 0.$$

Therefore, we have

$$\begin{aligned}
&\int_{\Omega^{(K)}} \sum_{i,j=1}^m p_{ij} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} dV \\
&\leqq \max_{\bar\Omega} |q| \int_{\Omega^{(K)}} u(u-K) \ dV + \int_{\Omega^{(K)}} |f| (u-K) dV \\
&\leqq \max_{\bar\Omega} |q| \left[\frac{3}{2} \int_{\Omega^{(K)}} (u-K)^2 dV + \frac{K^2}{2} \text{mes } \Omega^{(K)}\right] \\
&\quad + \frac{1}{2} \int_{\Omega^{(K)}} (u-K)^2 dV + \frac{1}{2} \max_{\bar\Omega} |f|^2 \text{mes } \Omega^{(K)} \\
&\leqq 2 \max\{\max_{\bar\Omega} |q|, \max_{\bar\Omega} |f|^2, 1\} \left[\int_{\Omega^{(K)}} (u-K)^2 dV + K^2 \text{mes} \Omega^{(K)}\right].
\end{aligned}$$

Now by Corollary 1 to Lemma 4 and Corollary 1 to Lemma 8, we get

$$\begin{aligned}
&\|\nabla u\|^2_{L_{2k}(\Omega^{(K)})} \leqq 2 \left\|\frac{1}{p_0}\right\|_{L_{k/(1-k)}(\Omega)} \\
&\times \max\{\max_{\bar\Omega} |q|, \max_{\bar\Omega} |f|^2, 1\} \left[B_{mk}'^2 \|\nabla u\|^2_{L_{2k}(\Omega)^{(K)}} + K^2 \text{mes } \Omega^{(K)}\right].
\end{aligned}$$

Whereas

$$(K^{2km/(m-2k)}\text{ mes }\Omega^{(K)})^{\{k(m+2)-m\}/km}$$

$$\leqq [\int_{\Omega^{(K)}} u^{2km/(m-2k)} dV]^{\{k(m+2)-m\}/km} \leqq \|u\|_{L_{2km/(m-2k)}(\Omega)}^{2\{k(m+2)-m\}/(m-2k)}$$

$$\leqq [\|u-\varphi\|_{L_{2km/(m-2k)}(\Omega)} + \|\varphi\|_{L_{2km/(m-2k)}(\Omega)}]^{2\{k(m+2)-m\}/(m-2k)},$$

and $u-\varphi \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$, so that from Lemma 3 and Lemma 8 follows

$$(K^{2km/(m-2k)}\text{mes }\Omega^{(K)})^{\{k(m+2)-m\}/km}$$

$$\leqq \{\frac{k(m-1)}{m-2k}(\|\frac{1}{p_0}\|_{L_{k/(1-k)}(\Omega)} D_\Omega[u-\varphi])^{1/2}$$

$$+\|\varphi\|_{L_{2km/(m-2k)}(\Omega)}\}^{2\{k(m+2)-m\}/(m-2k)}$$

$$\leqq \{\frac{k(m-1)}{m-2k}(\|\frac{1}{p_0}\|_{L_{k/(1-k)}(\Omega)})^{1/2}(\sqrt{D_\Omega[u]}+\sqrt{D_\Omega[\varphi]})$$

$$+\|\varphi\|_{L_{2km/(m-2k)}(\Omega)}\}^{2\{(m+2)-m\}/(m-2k)},$$

which implies that mes $\Omega^{(K)} \to 0$ as $K \to \infty$.
Thus if we choose $K_0 (\geqq \max_{\partial\Omega} \varphi, 1)$ so large that

$$2\|\frac{1}{p_0}\|_{L_{k/(1-k)}(\Omega)} \max\{\max_{\bar{\Omega}}|q|, \max_{\bar{\Omega}}|f|^2, 1\} B_{mk'}$$

$$\leqq 2\|\frac{1}{p_0}\|_{L_{k/(1-k)}(\Omega)} \max\{\max_{\bar{\Omega}}|q|, \max_{\bar{\Omega}}|f|^2, 1\}$$

$$\times (\frac{k(m-1)}{m-2k})^2 (\text{mes }\Omega^{(K_0)})^{\{(m+2)k-m\}/mk} \leqq \frac{1}{2},$$

then for any $K \geqq K_0$ we have

$$\|\nabla u\|^2_{L_{2k}(\Omega^{(K)})} \leqq 4\|\frac{1}{p_0}\|_{L_{k/(1-k)}(\Omega)} \times \max\{\max_{\bar{\Omega}}|q|, \max_{\bar{\Omega}}|f|^2, 1\} K^2 \text{mes }\Omega^{(K)}.$$

Hence the assumption of Lemma 7 is satisfied.

Next we show that $\|u\|_{L_1(\Omega^{(K_0)})}$ is bounded by a constant which does not depend on u.

Since

$$\|u\|_{L_1(\Omega^{(K_0)})} \leqq (\text{mes }\Omega)^{1/2} \|u\|_{L_2(\Omega^{(K_0)})} \leqq (\text{mes }\Omega)^{1/2} \|u^{(K_0)} + K_0\|_{L_2(\Omega^{(K_0)})}$$

$$\leqq (\text{mes }\Omega)^{1/2} (\|u^{(K_0)}\|_{L_2(\Omega^{(K_0)})} + K_0 (\text{mes }\Omega)^{1/2}),$$

from Corollary 2 to Lemma 8 follows

$$\|u\|_{L_1(\Omega(K_0))} \leqq (\text{mes }\Omega)^{1/2}$$

$$\times \{ \frac{k(m-1)}{m-2k} (\text{mes }\Omega)^{\{(m+2)k-m\}/2mk} \sqrt{\| \frac{1}{p_0} \|_{Lk/(1-k)(\Omega)} D_{\Omega(K_0)}[u]}$$

$$+ K_0 (\text{mes }\Omega)^{1/2} \}.$$

Therefore by Lemma 7 $\operatorname{vrai}\max_{\Omega} u$ is bounded by a constant C which depends on m, k, d and C'.

Since $-u$ is a weak solution of (1)-(2) in which f and φ are replaced by $-f$ and $-\varphi$ respectively, $\operatorname{vrai}_{\Omega} \max(-u)$ is also bounded by C.

Thus we have

$$\operatorname{vrai}\max_{\Omega} |u| < C,$$

which completes the proof in the case $m \geqq 2$.

When $m=1$, by Lemmas 2, 8, and (5) in the proof of Lemma 4, we set

$$|u-\varphi| \leqq \frac{1}{2} (\text{mes }\Omega)^{(2k-1)/2k} \sqrt{\| \frac{1}{p_0} \|_{Lk/(1-k)(\Omega)} D_{\Omega}[u]}.$$

Therefore

$$|u| \leqq |u-\varphi| + |\varphi| < C.$$

§3. Boundary value problems

We treat the boundary value problem for the degenerate elliptic equation

$$(1) \qquad \sum_{i,j=1}^{m} \frac{\partial}{\partial x_j} (p_{ij} \frac{\partial u}{\partial x_i}) - qu = f \qquad \text{in } \Omega$$

with the boundary condition

$$(2) \qquad u = \varphi \qquad \text{on } \partial\Omega.$$

We give the uniqueness theorem for weak solutions (Theorem 3) and the existence theorem for genuine solutions (Theorem 4).

Theorem 3. *Assume that* $p_{ij}=p_{ji}, q$, $f \in C_{0,0}(\overline{\Omega})$; $\varphi \in C_{1,0}(\overline{\Omega})$ *and* $\partial\Omega \in \overline{C}_{1,0}$ *and also that* p_{ij} *and* q *satisfy*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x) \sum_{i=1}^{m} \xi_i^2 \geqq 0$$

($(\xi_1, \cdots, \xi_m)$ is any real vector)

$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega)$ *(for some k with $\frac{m}{m+2} < k < 1$ when $m \geqq 2$, or with $\frac{1}{2} < k < 1$ when $m = 1$)*

and

$$q \geqq 0$$

in Ω.

Then the weak solution of (1)-(2) is unique.

Proof. Let u_1 and u_2 be two weak solutions of (1)-(2). Then we have

$$G_\Omega[u_1, \zeta] + H_{\Omega,1}[f, \zeta] = 0$$

and

$$G_\Omega[u_2, \zeta] + H_{\Omega,1}[f, \zeta] = 0$$

for every $\zeta \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$. Therefore we get

$$G_\Omega[u_1 - u_2, \zeta] = 0.$$

By the definition of a weak solution, there exist two sequences $\{u_{1h} \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)\}$ and $\{u_{2h} \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)\}$ such that

$$D_\Omega[u_1 - u_{1h}] \to 0, \ \|u_1 - u_{1h}\|_{L_2(\Omega)} \to 0$$

and

$$D_\Omega[u_2 - u_{2h}] \to 0, \ \|u_2 - u_{2h}\|_{L_2(\Omega)} \to 0 \quad \text{as } h \to \infty.$$

Now set $\zeta = u_{1h} - u_{2h}$. Then we have

$$G_\Omega[u_1 - u_2, u_{1h} - u_{2h}] = 0.$$

From Lemma 1 follows

$$G_\Omega[u_1 - u_2] = 0$$

and therefore, because of $q \geqq 0$,

$$D_\Omega[u_1 - u_2] = 0.$$

Thus by Corollary 2 to Lemma 8 we have

$$\|u_1 - u_2\|_{L_2(\Omega)} = \lim_{h \to \infty} \|u_{1h} - u_{2h}\|_{L_2(\Omega)}$$

$$\leqq \lim_{h\to\infty} B'_{mk} \sqrt{\| \frac{1}{p_0} \|_{L_{k/(1-k)}(\Omega)} D_\Omega [u_{1h} - u_{2h}]} = 0,$$

i. e., $u_1 = u_2$ almost everywhere in Ω.

Corollary. *Assume that* $p_{ij} = p_{ji} \in C_{1,0}(\Omega) \cap C_{0,0}(\overline{\Omega})$; $q, f \in C_{0,0}(\overline{\Omega})$; $\varphi \in C_{1,0}(\overline{\Omega})$ *and* $\partial\Omega \in \overline{C}_{1,0}$ *and also that* p_{ij} *and* q *satisfy*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x) \sum_{i=1}^{m} \xi_i^2 \geqq 0$$

$((\xi_1, \cdots, \xi_m)$ *is any real vector*$)$

$$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega)$$

(for some k *with* $\frac{m}{m+2} < k < 1$ *when* $m \geqq 2$*, or with* $\frac{1}{2} < k < 1$ *when* $m = 1$*)*

and

$$q \geqq 0$$

in Ω.

Then the solution of (1)-(2) is unique.

This corollary is clear by Theorem 1 and Theorem 3.

Theorem 4. *Assume that the bounded domain* Ω *in m-dimensional Euclidean space and the coefficients of the problem (1)-(2) satisfy the following conditions*:

(i) $p_{ij} = p_{ji}$, $q, f \in C_{0,\alpha}(\Omega) \cap C_{0,0}(\overline{\Omega})$,

$\frac{\partial p_{ij}}{\partial x_j} \in C_{0,\alpha}(\Omega)$ $(i, j = 1, \cdots, m)$, $\varphi \in C_{2,\alpha}(\Omega) \cap C_{1,0}(\overline{\Omega})$,

$\partial\Omega \in \overline{C}_{2,\alpha}$,

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x) \sum_{i=1}^{m} \xi_i^2 \geqq 0 \quad ((\xi_1, \cdots, \xi_m) \text{ is any real vector}),$$

$$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega)$$

(for some k *with* $\frac{m}{m+2} < k < 1$ *when* $m \geqq 2$*, or with*

$\frac{1}{2}<k<1$ *when* $m=1$),

$$p_0(x)>0 \quad \text{in } \Omega,$$

$$q\geqq 0 \qquad \text{in } \overline{\Omega},$$

(ii) *the domain and the operator have properties* (A) *and* (B). *Then there exists a unique solution* u_0 *of* $(1)-(2)$. *Moreover we have*

$$G_\Omega[u_0]+2H_{\Omega,1}[f,u_0]$$
$$=\inf_{u\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)}(G_\Omega[u]+2H_{\Omega,1}[f,u]).$$

Proof. By Lemma 9, there exist d and d_n such that

$$\inf_{u\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)}(G_\Omega[u]+2H_{\Omega,1}[f,u])=d$$

and

$$\inf_{u\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)}(G_{\Omega_n}[u]+2H_{\Omega_n,1}[f,u])=d_n,$$

where $\{\Omega_n\}$ is an approximating sequence for $\Omega(\partial\Omega\in\overline{C}_{2,\alpha})$. Clearly

$$d_1\geqq\cdots\geqq d_n\geqq\cdots\geqq d.$$

Moreover we can show that $\lim_{n\to\infty} d_n=d$.

In fact, there exists $\{u_h\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)\}$ such that

$$\lim_{h\to\infty}(G_\Omega[u_h]+2H_{\Omega,1}[f,u_h])=d$$

and there exists $\{u'_h\,|\,u'_h-\varphi\in\dot{C}_\infty(\Omega)\}$ such that

$$|G_\Omega[u_h]-G_\Omega[u'_h]|<\frac{1}{h}$$

and

$$|H_{\Omega,1}[f,u_h]-H_{\Omega,1}[f,u'_h]|<\frac{1}{h}.$$

By choosing n so large that $u'_h\in \overset{\varphi}{W}{}_2^{(1)}(\Omega_n)$, we have

$$d=\lim_{h\to\infty}(G_\Omega[u_h]+2H_{\Omega,1}[f,u_h])$$
$$=\lim_{h\to\infty}(G_\Omega[u'_h]+2H_{\Omega,1}[f,u'_h])$$

$$\geqq \lim_{h\to\infty} \inf_{u\in \overset{\varphi(1)}{W2}(\Omega_n)} (G_{\Omega_n}[u]+2H_{\Omega_n,1}[f,u])$$

$$=\lim_{h\to\infty} d_{n(h)}.$$

Hence

$$\lim_{u\to\infty} d_n = d.$$

Whereas it is certain that for each n there exists a solution $u_n \in C_{2,\alpha}(\Omega_n)$ of (1)-(2) for Ω_n, which satisfies

$$G_{\Omega_n}[u_n]+2H_{\Omega_n,1}[f,u_n] = \inf_{u\in \overset{\varphi(1)}{W2}(\Omega_n)} (G_{\Omega_n}[u]+2H_{\Omega_n,1}[f,u]).^{1)}$$

Therefore we have

$$d_n = G_{\Omega_n}[u_n]+2H_{\Omega_n,1}[f,u_n]$$

which are bounded above by d_1. By Lemma 10 $D_{\Omega_n}[u_n]$ is bounded and therefore, if we define $u_n=\varphi$ in $\Omega-\Omega_n$,

$$D_\Omega[u_n]=D_{\Omega_n}[u_n]+D_{\Omega-\Omega_n}[\varphi]$$

is also bounded. Since by Theorem 1 u_n is a weak solution of (1)-(2) for Ω_n, from Theorem 2 follows

$$\max_{\bar{\Omega}_n}|u_n|<C,$$

where C depends only on m, k, d_1, $\|\frac{1}{p_0}\|_{L^{k/(1-k)}(\Omega)}$, $\max_{\bar{\Omega}}|q|$, $\max_{\bar{\Omega}}|f|$, $\max_{\bar{\Omega}}|\varphi|$, $D_\Omega[\varphi]$ and mes Ω. Hence

$$\max_{\bar{\Omega}}|u_n|<C',$$

where C' depends on the same quantities as C.

By Lemma 12 $u_{n'}$ $(n'>n)$ satisfies

$$\|u_{n'}\|_{C_{2,\alpha}(\bar{\Omega}_n)} \leqq C''[\|Lu_{n'}\|C_{0,\alpha}(\bar{\Omega}_{n+1})+\max_{\bar{\Omega}_{n+1}}|u_{n'}|]$$

$$<C''[\|f\|_{C_{0,\alpha}(\bar{\Omega})}+C']<C''',$$

where C''' depends on α, $\|\frac{\partial p_{ij}}{\partial x_j}\|_{C_{0,\alpha}(\bar{\Omega}_{n+1})}$, $\|q\|_{C_{0,\alpha}(\bar{\Omega}_{n+1})}$, $\min_{\bar{\Omega}_{n+1}} p_0$, the diame-

1) This is shown in the same way as in Suzuki [9], where the solution of the first eigenvalue problem are derived by the variational method.

ter of Ω_{n+1}, the distance between Ω_n and $\partial\Omega_{n+1}$, $\|f\|_{C_{0,\alpha}(\bar{\Omega})}$ and C'.

Therefore we can choose the subsequence of $\{u_n\}$ such that

$$u_{11}, u_{12}, \cdots, u_{1n}, \cdots \to u_0 \quad \text{in } C_{2,0}(\overline{\Omega_1})$$
$$u_{21}, u_{22}, \cdots, u_{2n}, \cdots \to u_0 \quad \text{in } C_{2,0}(\overline{\Omega_2})$$
$$\cdots$$
$$u_{n1}, u_{n2}, \cdots, u_{nn}, \cdots \to u_0 \quad \text{in } C_{2,0}(\overline{\Omega_n})$$
$$\cdots,$$

where the upper subsequence contains the lower subsequence and $u_0 \in C_{2,\alpha}(\Omega)$,[2)] the absolute value of which is bounded by C'''. Obviously u_0 satisfies (1) in Ω, and $D_\Omega[u_0] = \lim_{n\to\infty} D_{\Omega_n}[u_0] = \lim_{n\to\infty} \lim_{n'\to\infty} D_{\Omega_n}[u_{n'n'}]$ is finite.

Let us show that u_0 satisfies (2) at the boundary.

By Property (B), there exists a strong barrier function $v(x)$ for any boundary point x_0. Set

$$\psi_1(x) = \varphi(x_0) + \varepsilon + k_1 v(x)$$

and

$$\psi_2(x) = \varphi(x_0) - \varepsilon - k_2 v(x).$$

Then we have

$$\psi_1(x) \geqq \psi_2(x).$$

In the first place we choose $\omega(x_0, \sigma)$, which appears in Properties (A) and (B), so small that

$$\psi_1(x) \geqq \max_{\partial\Omega \cap \partial\omega(x_0,\sigma)} u_n$$

and

$$\psi_2(x) \leqq \min_{\partial\Omega \cap \partial\omega(x_0,\sigma)} u_n$$

on $\partial\Omega \cap \partial\omega(x_0, \sigma)$. In the next place we choose k_1, k_2 so large that

$$L[\psi_1(x)] \leqq \min_{\overline{\omega(x_0,\sigma)}} f,$$

2) If $\|u_n\|_{C_{2,\alpha}(\overline{\Omega})}$ are bounded and $u_n \to u_0$ in $C_{2,0}(\overline{\Omega})$, then $u_0 \in C_{2,\alpha}(\overline{\Omega})$ (cf. Suzuki [9] p. 68, Theorem 9).

$$L[\psi_2(x)] \geqq \max_{\omega(x_0,\sigma)} f$$

in $\omega(x_0, \sigma)$, and

$$\psi_1(x) \geqq \max_{\partial\omega(x_0,\sigma)} u_n,$$

$$\psi_2(x) \leqq \min_{\partial\omega(x_0,\sigma)} u_n$$

on $\partial\omega(x_0, \sigma)$. Then by Property (A), we have

$$\psi_2(x) \leqq u_n(x) \leqq \psi_1(x) \text{ in } \omega(x_0, \sigma).$$

Therefore, letting $n \to \infty$, we have

$$\psi_2(x) \leqq u_0(x) \leqq \psi_1(x) \text{ in } \omega(x_0, \sigma).$$

Hence, letting $x \to x_0$ and letting $\varepsilon \to 0$, we obtain

$$\varphi(x_0) = \varliminf_{x \to x_0} u_0(x) \leqq \varlimsup_{x \to x_0} u_0(x) = \varphi(x_0).$$

Therefore

$$\lim_{x \to x_0} u_0(x) = \varphi(x_0).$$

Thus we have shown the existence of a solution of the problem (1)–(2). Because of Theorem 3, this solution is unique.

Finally we shall show that we have

$$G_\Omega[u_0] + 2H_{\Omega,1}[f, u_0] = d.$$

Since u_0 is a solution of the problem (1)-(2), by Theorem 1 u_0 is also a weak solution of the problem (1)-(2), i. e., there exists a sequence $\{u_h \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)\}$ such that $D_\Omega[u_0 - u_h] \to 0$, $\|u_0 - u_h\|_{L_2(\Omega)} \to 0$ as $h \to \infty$, and for any $\zeta \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ we have

$$G_\Omega[u_0, \zeta] + H_{\Omega,1}[f, \zeta] = 0.$$

Therefore, from Lemma 1 it follows that

$$\begin{aligned} G_\Omega[u_0] &+ 2H_{\Omega,1}[f, u_0] \\ &= \lim_{h \to \infty} (G_\Omega[u_h] + 2H_{\Omega,1}[f, u_h]) \\ &\geqq \lim_{h \to \infty} d = d. \end{aligned}$$

For any $u \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)$, put

$$u=u_0+\zeta,\ u=u_h+\zeta_h.$$

Clearly $\xi_h\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ and

$$D_\Omega[\zeta-\zeta_h]\to 0,\ \|\zeta-\zeta_h\|_{L_2(\Omega)}\to 0 \text{ as } h\to\infty.$$

Therefore

$$\begin{aligned}(G_\Omega[u]&+2H_{\Omega,1}[f,u])-(G_\Omega[u_h]+2H_{\Omega,1}[f,u_h])\\&=G_\Omega[\zeta_h]+(G_\Omega[u_h,\zeta_h]+H_{\Omega,1}[f,\zeta_h])\\&\geqq G_\Omega[u_h,\zeta_h]+H_{\Omega,1}[f,\zeta_h].\end{aligned}$$

By Lemma 1, letting $h\to\infty$, we have

$$\begin{aligned}(G_\Omega[u]&+2H_{\Omega,1}[f,u])-(G_\Omega[u_0]+2H_{\Omega,1}[f,u_0])\\&\geqq G_\Omega[u_0,\zeta]+H_{\Omega,1}]f,\zeta]\\&=\lim_{h\to\infty}(G_\Omega[u_0,\zeta_h]+H_{\Omega,1}[f,\zeta_h])=0.\end{aligned}$$

Thus we have

$$G_\Omega[u_0]+2H_{\Omega,1}[f,u_0]\leqq d.$$

Hence we have

$$G_\Omega[u_0]+2H_{\Omega,1}[f,u_0]=d.$$

§4. Eigenvalue problems

We consider the eigenvalue problem for the degenerate elliptic equation

$$\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial u}{\partial x_i}\right)-qu+\lambda\rho u=0 \quad \text{in } \Omega, \tag{3}$$

with the boundary condition

$$u=0 \quad \text{on } \partial\Omega. \tag{4}$$

We arrange the fundamental properties of the eigenfunctions and eigenvalues (Proposition 1–2). Next we prove the discreteness of the spectrum (Theorem 5), and finally we give the solution of the problem for (3)–(4) which shows that the number of eigenvalues is indeed countable (Theorem 6).

Proposition 1. *If u is one of the weak eigenfunctions corresponding to a weak eigenvalue λ, i.e., a non-trivial weak solution* *of (3)-(4), where*

$$p_{ij}=p_{ji},\ q,\ \rho\in C_{0,0}(\bar{\Omega})\,;\partial\Omega\in\bar{C}_{1,0}$$

and

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j\geqq 0\quad ((\xi_1,\cdots,\xi_m)\ \textit{is any real vector}),$$

then we have

$$G_\Omega[u]-\lambda H_{\Omega,\rho}[u]=0.$$

Proof. By the definition of a weak solution, $D_\Omega[u]<\infty$, and there exists a sequence $\{u_h\in \overset{\circ}{W}{}_2^{(1)}(\Omega)\}$ such that

$$D_\Omega[u-u_h]\to 0,\ \|u-u_h\|_{L_2(\Omega)}\to 0,$$

and for any $\zeta\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ we have

$$G_\Omega[u,\zeta]-\lambda H_{\Omega,\rho}[u,\zeta]=0.$$

Putting $\zeta=u_h$, we get

$$G_\Omega[u,u_h]-\lambda H_{\Omega,\rho}[u,u_h]=0.$$

Therefore from Lemma 1 follows

$$G_\Omega[u]-\lambda H_{\Omega,\rho}[u]=0.$$

Proposition 2. *Assume that $p_{ij}=p_{ji}$, q, $\rho\in C_{0,0}(\bar{\Omega})$ and that p_{ij} satisfy*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j\geqq 0((\xi_1,\cdots,\xi_m)\ \textit{is any real vector}).$$

Then for two weak eigenfunctions u_n, $u_{n'}$ corresponding to weak eigenvalues $\lambda_n\neq\lambda_{n'}$, we have

$$H_{\Omega,\rho}[u_n,u_{n'}]=0.$$

Proof. By the definition of a weak solution, $D_\Omega[u_n]$, $D_\Omega[u_{n'}]<\infty$, and there exist two sequences $\{u_{nh}\in \overset{\circ}{W}{}_2^{(1)}(\Omega)\}$, $\{u_{n'h}\in \overset{\circ}{W}{}_2^{(1)}(\Omega)\}$ such that

$$D_\Omega[u_n-u_{nh}]\to 0,\ \|u_n-u_{nh}\|_{L_2(\Omega)}\to 0$$

$$D_\Omega[u_{n'}-u_{n'h}]\to 0,\ \|u_{n'}-u_{n'h}\|_{L_2(\Omega)}\to 0$$

and for any $\zeta \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ we have

$$G_\Omega[u_n, \zeta] - \lambda_n H_{\Omega,\rho}[u_n, \zeta] = 0,$$
$$G_\Omega[u_n, \zeta] - \lambda_{n'} H_{\Omega,\rho}[u_{n'}, \zeta] = 0.$$

Without loss of generality we can assume $\lambda_n \neq 0$. Therefore from Lemma 1 follows

$$H_{\Omega,\rho}[u_n, u_{n'}] = \lim_{h\to\infty} H_{\Omega,\rho}[u_n, u_{n'h}]$$
$$= \lambda_n^{-1} \lim_{h\to\infty} G_\Omega[u_n, u_{n'h}] = \lambda_n^{-1} G_\Omega[u_n, u_{n'}]$$
$$= \lambda_n^{-1} \lim_{h\to\infty} G_\Omega[u_{nh}, u_{n'}]$$
$$= \lambda_n^{-1}\lambda_{n'} \lim_{h\to\infty} H_{\Omega,\rho}[u_{nh}, u_{n'}].$$
$$= \lambda_n^{-1}\lambda_{n'} H_{\Omega,\rho}[u_n, u_{n'}].$$

Since $\lambda_n^{-1}\lambda_{n'} \neq 1$, we have

$$H_{\Omega,\rho}[u_n, u_{n'}] = 0.$$

Theorem 5. *Assume that* $p_{ij} = p_{ji}$, q, $\rho \in C_{0,0}(\overline{\Omega})$; $\partial\Omega \in \overline{C}_{1,0}$ *and also that* p_{ij} *and* ρ *satisfy*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x) \sum_{i=1}^{m} \xi_i^2 \geqq 0$$

$((\xi_1, \cdots, \xi_m)$ *is any real vector.*),

$$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega)$$

(*for some* k *with* $\frac{m}{m+2} < k < 1$ *when* $m \geqq 2$, *or with* $\frac{1}{2} < k < 1$ *when* $m = 1$),

$$\rho \geqq \rho_0 > 0.$$

Then the weak spectrum of the eigenvalue problem for (3)-(4) *are discrete, i.e., the totality of weak eigenvalues taken with respective multiplicity does not have any finite limiting point.*

Proof. Suppose on the contrary that we had a sequence of eigenvalues $\{\lambda_n\}$ such that $\lim_{n\to\infty} \lambda_n = \lambda_0 \neq \infty$. By Proposition 2 we can construct a system of weak eigenfunctions $\{u_n\}$ corresponding to $\{\lambda_n\}$, normalized and orthogonal with respect to $H_{\Omega,\rho}[\ ,\ ]$. From Proposition 1 follows $G_\Omega[u_n]=\lambda_n$.

Let ε be any number such that $0<\varepsilon<\frac{1}{2}$. Then there exists an $n_0(\varepsilon)$ such that

$$\lambda_0-\varepsilon<G_\Omega[u_n]<\lambda_0+\varepsilon$$

for every $n>n_0(\varepsilon)$. Since u_n is a weak solution of (3)-(4) for $\lambda=\lambda_n$, there exists $v_n\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ such that

$$D_\Omega[u_n-v_n]<\varepsilon,\ H_{\Omega,\rho}[u_n-v_n]<\varepsilon\ ,$$

and therefore we have

$$G_\Omega[v_n]<\lambda_0+\eta(\varepsilon)\ ,$$
$$H_{\Omega,\rho}[v_n[<(1+\sqrt{\varepsilon})^2\ \ (n>n_0, (\varepsilon))\ ,$$

where $\eta(\varepsilon)$ is some positive number.

Because

$$D_\Omega[v_n]\leqq G_\Omega[v_n]+\frac{\max_{\bar\Omega}|q|}{\min_{\bar\Omega}\rho}H_{\Omega,\rho}[v_n]$$
$$\leqq\lambda_0+\eta(\varepsilon)+\frac{\max_{\bar\Omega}|q|}{\min_{\bar\Omega}\rho}(1+\sqrt{\varepsilon})^2\ \ (n>n_0(\varepsilon)),$$

from the corollary to Lemma 11 follows the relative compactness of $\{v_n\}$. Now take a Cauchy sequence in $L_2(\Omega)$ from $\{v_n\}$ and denote it again by $\{v_n\}$. Obviously

$$H_{\Omega,\rho}[v_n-v_{n'}]\to 0\ \ (n, n'\to\infty).$$

Whereas by the orthogonality of $\{u_n\}$ we have

$$\sqrt{H_{\Omega,\rho}[v_n-v_{n'}]}$$
$$\geqq\sqrt{H_{\Omega,\rho}[u_n-u_{n'}]}-\sqrt{H_{\Omega,\rho}[u_n-v_n]}-\sqrt{H_{\Omega,\rho}[u_{n'}-v_{n'}]}$$
$$\geqq\sqrt{H_{\Omega,\rho}[u_n]-2H_{\Omega,\rho}[u_n, u_{n'}]+H_{\Omega,\rho}[u_{n'}]}-2\sqrt{\varepsilon}$$
$$=\sqrt{2}\,(1-\sqrt{2\varepsilon})\ \ (n>n_0(\varepsilon)).$$

Therefore we have

$$H_{\Omega,\rho}[v_n - v_{n'}] \not\to 0 \quad (n, n' \to \infty).$$

This is a contradiction.

Theorem 6. *Assume that the bounded domain Ω in m-dimensional Euclidean space and the coefficients in* (3)-(4) *satisfy the following conditions*:

(i) $p_{ij} = p_{ji},\ q,\ \rho \in C_{0,\alpha}(\Omega) \cap C_{0,0}(\overline{\Omega})$,

$\frac{\partial p_{ij}}{\partial x_j} \in C_{0,\alpha}(\Omega)\ (i, j = 1, \cdots, m)$,

$\partial\Omega \in \overline{C}_{2,\alpha}$,

$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x) \sum_{i=1}^{m} \xi_i^2 \geqq 0$

$((\xi_1, \cdots, \xi_m)$ *is any real vector.*),

$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega)$

(*for some k with* $\frac{m}{m+2} < k < 1$ *when* $m \geqq 2$, $\frac{1}{2} < k < 1$ *when* $m = 1$),

$p_0 > 0$ in Ω,

$\rho \geqq \rho_0 > 0$ in $\overline{\Omega}$,

(ii) *the domain and the coefficients have Properties* (*A*) *and* (*B*). *Then there exist eigenfunctions* $u_1, u_2, \cdots, u_n, \cdots (\in C_{2,\alpha}(\Omega) \cap C_{0,0}(\overline{\Omega}))$ *corresponding to the eigenvalues* $\lambda_1 \leqq \lambda_2 \leqq \cdots \leqq \lambda_n \leqq \cdots$ *such that*

$$G_\Omega[u_n] = \inf_{u \in \mathfrak{W}_n} G_\Omega[u] = \lambda_n,$$

where

$$\mathfrak{W}_n = \{u \in \overset{\circ}{W}{}_2^{(1)}(\Omega) \mid H_{\Omega,\rho}[u] = 1,\ H_{\Omega,\rho}[u, u_k] = 0 \quad (k = 1, \cdots, n-1)\}$$

Proof. Since this theorem is proved in the same way as Theorem 4, suffice it to say that we give a rough proof.

Let $\{\Omega_k\}$ be an approximating sequence for $\Omega(\partial\Omega \in \overline{C}_{2,\alpha})$. Consider the problem for Ω_k, denote by $\mathfrak{W}_n(\Omega_k)$ the set of functions for Ω_k corresponding to $\mathfrak{W}_n$ for Ω and set

$$\inf_{u\in\mathfrak{W}_n} G_\Omega[u]=\lambda_n\,,$$

$$\inf_{u\in\mathfrak{W}_n(\Omega_k)} G_{\Omega_k}[u]=\lambda_{nk}\,.$$

It follows that $\lim_{k\to\infty}\lambda_{nk}=\lambda_n$. Now for any k there exists an eigenfunction $u_{nk}\in C_{2,\alpha}(\Omega_k)\cap \overset{\circ}{W}{}_2^{(1)}(\Omega)$ such that

$$G_{\Omega_k}[u_{nk}]=\inf_{u\in\mathfrak{W}_n(\Omega_k)} G_{\Omega_k}[u]=\lambda_{nk}\,,$$

and therefore $G_{\Omega_k}[u_{nk}]$ is bounded. From this fact results the boundedness $D_\Omega[u_{nk}]$, $\underset{\Omega}{vrai\ max}\ |u_{nk}|$, and thereafter $\|u_{nk'}\|_{C_{2,\alpha}(\Omega_k)}$ $(k'>k)$.

By a diagonal process, we get a subsequence of $\{u_{nk}\}$ (hereafter we denote this subsequence again by $\{u_{nk}\}$) which converges in $C_{2,0}(\Omega)$ to a solution u_n of (3) corresponding to λ_n. Using a strong barrier function we conclude that u_n also satisfies (4). Obviously $D_\Omega[u_n]<\infty$.

Finally we confirm that u_n satisfies

$$G_\Omega[u_n]=\inf_{u\in\mathfrak{W}_n} G_\Omega[u]=\lambda_n\,.$$

Since u_n is an eigenfunction for (3)-(4) corresponding to λ_n, it follows from Proposition 1 that

$$G_\Omega[u_n]-\lambda_n H_{\Omega,\rho}[u_n]=0\,.$$

By the corollary to Lemma 11 for some subsequence $\{u_{nk'}\}$ of $\{u_{nk}\}$ we have

$$\lim_{k'\to\infty} H_{\Omega,\rho}[u_{nk'}]=H_{\Omega,\rho}[u_n]\,.$$

Therefore

$$H_{\Omega,\rho}[u_n]=\lim_{k'\to\infty} H_{\Omega,\rho}[u_{nk'}]$$

$$=\lim_{k'\to\infty} H_{\Omega_{k'},\rho}[u_{nk'}]=\lim_{k'\to\infty}1=1.$$

Thus

$$G_\Omega[u_n]=\lambda_n\,.$$

§5. Lemmas

In this section we shall state several lemmas, some of which were used in the proofs of the theorems in the preceding sections, while others are of more preliminary nature. Throughout this section we assume that $p_{ij}=p_{ji}$, q, ρ, $f\in C_{0,0}(\Omega)$ and $\partial\Omega\in\overline{C}_{1,0}$.

Lemma 1. *Let* u, $v\in L_2(\Omega)$, $\{u_h\in W_2^{(1)}(\Omega)\}$ *and* $\{v_k\in W_2^{(1)}(\Omega)\}$ *be such that*

$$D_\Omega[u]<\infty,\ D_\Omega[u-u_h]\to 0,\ \|u-u_h\|_{L_2(\Omega)}\to 0,$$
$$D_\Omega[v]<\infty,\ D_\Omega[v-v_k]\to 0,\ \|v-v_k\|_{L_2(\Omega)}\to 0,$$

where p_{ij} *satisfies*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j\geqq 0 \quad ((\xi_1,\cdots,\xi_m) \text{ is any real vector}).$$

Then we have

$$D_\Omega[u,v]=\lim_{h\to\infty} D_\Omega[u_h,v]=\lim_{h,k\to\infty} D_\Omega[u_h,v_k],$$
$$(u,v)_\Omega=\lim_{h\to\infty}(u_h,v)_\Omega=\lim_{h,k\to\infty}(u_h,v_k)_\Omega,$$

where $(u,v)_\Omega=\int_\Omega uv\,dV$.

Proof. From the assumption of the lemma follows

$$D_\Omega[u_h],\ D_\Omega[v_k]<C,$$

where C does not depend on h, k. Obviously $D_\Omega[u,v]<\infty$ and

$$D_\Omega[u,v]=D_\Omega[u-u_h,v-v_k]+D_\Omega[u-u_h,v_k]$$
$$+D_\Omega[u_h,v-v_k]+D_\Omega[u_h,v_k].$$

Letting $h,k\to\infty$, we get

$$D_\Omega[u,v]=\lim_{h,k\to\infty} D_\Omega[u_h,v_k].$$

In a similar way we have

$$D_\Omega[u,v]=\lim_{h\to\infty} D_\Omega[u_h,v].$$

The equalities for $(\ ,\ )_\Omega$ are obvious.

Lemma 2. *If* $u\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)$ *and* $K\geqq\max_{\partial\Omega}\varphi$, *then* $u^{(K)}\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ (Cf. Ladyzhenskaya-Ural'tseva [3] p. 75, Lemma 3.3).

Lemma 3. *If* $u\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$, *then*

$$\|u\|_{L_{2km/(m-2k)}(\Omega)}\leqq\frac{k(m-1)}{m-2k}\|\nabla u\|_{L_{2k}(\Omega)}\ \left(\frac{1}{2}<k<\frac{m}{2}\right).$$

Proof. When $u\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$, we have

$$\|u\|_{L_{m/(m-1)}(\Omega)}\leqq\frac{1}{2}\|\nabla u\|_{L_1(\Omega)}\quad(m\geqq 2)$$

(cf. Nirenberg [5] p. 14). If we put $u=u^{2k(m-1)/m-2k)}$, then we get the inequality of the lemma (more precisely we must show in the first place that we have this inequality for $u\in\dot{C}_\infty(\Omega)$, and in the next place that we have it for $u\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$).

Lemma 4. *For any* $u\in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ *we have*

$$\|u\|_{L_2(\Omega)}\leqq B_{mk}\|\nabla u\|_{L_{2k}(\Omega)}\,,$$

where

$$B_{mk}=\begin{cases}\dfrac{k(m-1)}{m-2k}(\text{mes }\Omega)^{\{(m-2)k-m\}/2km} & (m\geqq 2)\\[2mm] \dfrac{1}{2}(\text{mes }\Omega)^{(3k-1)/2k} & (m=1)\end{cases}$$

and

$$\frac{m}{m+2}<k<\frac{m}{2}\qquad(m\geqq 2)$$

$$k>\frac{1}{2}\qquad(m=1).$$

Proof. When $m\geqq 2$, by Hölder's inequality we have

$$\|u\|_{L_2(\Omega)}\leqq\|u\|_{L_{2km/(m-2k)}(\Omega)}(\text{mes }\Omega)^{\{(m+2)k-m\}/2mk}.$$

Therefore from Lemma 3 follows the inequality of the lemma.

When $m=1$, by the inequality

$$|u(x)|\leqq\frac{1}{2}\int_{-\infty}^{\infty}|\nabla u|\,dV$$

and Hölder's inequality we have

$$|u|\leqq\frac{1}{2}(\text{mes }\Omega)^{(2k-1)/2k}\|\nabla u\|_{L_{2k}(\Omega)}. \tag{5}$$

Therefore we get the inequality of the lemma.

Corollary 1. *If* $u\in \overset{\varphi}{W}{}_2^{(1)}(\Omega)$, *then for any* $K\geqq\max_{\partial\Omega}\varphi$ *we have*

$$\|u^{(K)}\|_{L_2(\Omega^{(K)})} \leqq B'_{mk}\|\nabla u\|_{L_{2k}(\Omega^{(K)})},$$

where

$$B'_{mk} = \begin{cases} \dfrac{k(m-1)}{m-2k}(\text{mes } \Omega^{(K)})^{\{(m+2)k-m\}/2km} & (m \geqq 2) \\ \dfrac{1}{2}(\text{mes } \Omega^{(K)})^{(3k-1)/2k} & (\text{m}=1) \end{cases}$$

and

$$\frac{m}{m+2} < k < \frac{m}{2} \quad (m \geqq 2)$$

$$k > \frac{1}{2} \quad (m=1).$$

This inequality follows from Lemma 2 and Lemma 4.

Corollary 2. For any $u \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ we have

$$\|u\|_{L_{2k}(\Omega)} \leqq B''_{mk}\|\nabla u\|_{L_{2k}(\Omega)},$$

where

$$B''_{mk} = \begin{cases} \dfrac{k(m-1)}{m-2k}(\text{mes } \Omega)^{1/m} & (m \geqq 2), \\ \dfrac{1}{2}\text{mes } \Omega & (m=1) \end{cases}$$

and

$$\frac{1}{m+2} < k < \frac{m}{2} \quad (m \geqq 2),$$

$$k > \frac{1}{2} \quad (m=1).$$

This inequality follows from Hölder's inequality and Lemma 4.

Lemma 5. *For any* $u \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$ *we have*

$$\|u\|^2_{L_2(\Omega)} \leqq \left(\frac{k(m-1)}{m-2k}\right)^{m(1-k)/k}\|u\|^{\{(m+2)k-m\}/k}_{L_{2k}(\Omega)}\|\nabla u\|^{m(1-k)/k}_{L_{2k}(\Omega)}$$

$$\left(m \geqq 2, \frac{m}{m+2} < k < 1\right)$$

or we have

$$\|u\|^2_{L_2(\Omega)} \leqq \{\frac{1}{2}(\text{mes } \Omega)^{(2k-1)/2k}\}^{2(1-k)}\|u\|^{2k}_{L_{2k}(\Omega)}\|\nabla u\|^{2(1-k)}_{L_{2k}(\Omega)}$$

$$(m=1, \frac{1}{2}<k<1).$$

Proof. We begin with the case $m \geqq 2$. From Hölder's inequality follows

$$\int_\Omega u^2 dV = \int_\Omega u^\alpha u^{2-\alpha} dV$$

$$\leqq [\int_\Omega u^{2k} dV]^{\alpha/2k} [\int_\sigma u^{(2-\alpha)\cdot 2k/2k-\alpha} dV]^{(2k-\alpha)/2k}$$

for every $\alpha (0<\alpha<2k)$. Putting $\alpha = \{(m+2)k-m\}/k$, we have

$$\int_\Omega u^2 dV \leqq \|u\|^{\{(m+2)k-m\}/k}_{L_{2k}(\Omega)} \|u\|^{m(1-k)/k}_{L_{2mk/(m-2k)}(\Omega)}.$$

Therefore from Lemma 3 follows the inequality of the lemma.

When $m=1$, by the inequality (5) in the proof of Lemma 4 we get

$$\int_\Omega u^2 dV = \int_\Omega u^{2k} u^{2-2k} dV$$

$$\leqq \int_\Omega u^{2k} dV \{\frac{1}{2}(\text{mes } \Omega)^{(2k-1)/2k} \|\nabla u\|_{L_{2k}(\Omega)}\}^{2(1-k)}$$

$$\leqq \{\frac{1}{2}(\text{mes } \Omega)^{(2k-1)/2k}\}^{2(1-k)} \|u\|^{2k}_{L_{2k}(\Omega)} \|\nabla u\|^{2(1-k)}_{L_{2k}(\Omega)}.$$

Lemma 6. *If* $u \in \overset{\varphi}{W}{}^{(1)}_2(\Omega)$ *satisfies*

$$\int_{\Omega^{(K)}} (u-K) dV \leqq CK(\text{mes } \Omega^{(K)})^{1+\varepsilon}$$

for some positive ε *and for every* $K \geqq K_0 > 0$, *then*

$$\operatorname*{vrai\,max}_\Omega u \leqq C',$$

where

$$C' = [K_0^{1/(1+\varepsilon)} + C^{1/(1+\varepsilon)}\{K_0 \text{ mes } \Omega^{(K_0)} + \|u\|_{L_1(\Omega^{(K_0)})}\}^{\varepsilon/(1+\varepsilon)}]^{(1+\varepsilon)/\varepsilon}$$

(Cf. Ladyzhenskaya-Ural'tseva [3] p. 92, Lemma 5. 1)

Lemma 7. *If* $u \in \overset{\varphi}{W}{}^{(1)}_2(\Omega)$ *satisfies*

$$\|\nabla u\|_{L_{2k}(\Omega^{(K)})} \leqq CK(\text{mes }\Omega^{(K)})^{1/2}$$
$$\left(\frac{m}{m+2} < k < \frac{m}{2},\ m \geqq 2\right)$$

for every $K \geqq K_0 > 0$, *then*

$$\text{vari}\max_{\Omega} u < C',$$

where C' corresponds to C' in Lemma 6, in which we replace ε by $\dfrac{(m+2)k-m}{2mk}$, C by $\dfrac{k(m-1)}{m-2k}C$, and K_0 by $\max\{K_0, \max_{\partial\Omega}\varphi\}$.

Proof. By Schwartz's inequality we have

$$\int_{\Omega^{(K)}} (u-K)dV \leqq \|u^{(K)}\|_{L_2(\Omega^{(K)}, \rho)}(\text{mes }\Omega^{(K)})^{1/2}.$$

Therefore from Corollary 1 to Lemma 4 and the assumption of the present lemma follows

$$\int_{\Omega^{(K)}} (u-K)dV \leqq \frac{k(m-1)}{m-2k} CK\ (\text{mes }\Omega^{(K)})^{1+\{(m+2)k-m\}/2mk}$$

for every $K \geqq \max\{K_0, \max_{\partial\Omega}\varphi\}$. Hence the assumption of Lemma 6 is satisfied.

Lemma 8. *Assume*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x)\sum_{i=1}^{m} \xi_i^2 \geqq 0$$
$$((\xi_1, \cdots, \xi_m) \text{ is any real vector})$$

and

$$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega) \quad (0<k<1).$$

Then for any $u \in \overset{\circ}{W}{}_2^{(1)}(\Omega)$, *we have*

$$\|\nabla u\|^2_{L_{2k}(\Omega)} \leqq \left\|\frac{1}{p_0}\right\|_{L_{k/(1-k)}(\Omega)} D_\Omega[u].$$

Proof. From Hölder's inequality and from the assumption of the lemma follows

$$\|\nabla u\|^2_{L_{2k}(\Omega)} = \left[\int_\Omega \left(\frac{1}{p_0}\right)^k \left\{\sum_{i=1}^m p_0\left(\frac{\partial u}{\partial x_i}\right)^2\right\}^k dV\right]^{1/k}$$

$$\leqq \|1/p_0\|_{L_{k/(1-k)}(\Omega)} \int_\Omega \sum_{i,j=1}^{m} p_{ij}\left(\frac{\partial u}{\partial x_i}\right)\left(\frac{\partial u}{\partial x_j}\right) dV.$$

Corollary 1. *Under the same assumption as in the lemma, if* $u \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)$, *we have*

$$\|\nabla u\|^2_{L_{2k}(G^{(K)})} \leqq \left\| \frac{1}{p_0} \right\|_{L_{k/(1-k)}(\Omega)} D_{\Omega^{(K)}}[u]$$

for every $K \geqq \max_{\partial\Omega} \varphi$.

This inequality follows from Lemma 2 and Lemma 8.

Corollary 2. *Under the same assumption as in the lemma, if* $u \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)$, *we have*

$$\|u^{(K)}\|_{L_2(G^{(K)})} \leqq B'_{mk} \sqrt{\left\| \frac{1}{p_0} \right\|_{L_{k/(1-k)}(\Omega)} D_{\Omega^{(K)}}[u]}$$

for every $K \geqq \max_{\partial\Omega} \varphi$, *where* B'_{mk} *and* k *are the same constants as in Corollary* 1 *to Lemma* 4.

In fact, by Corollary 1 to Lemma 4 and Corollary 1 to Lemma 8 we get the desired inequality.

Lemma 9. *Assume*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq p_0(x) \sum_{i=1}^{m} \xi_i^2 \geqq 0$$

$((\xi_1, \cdots, \xi_m)$ is any real vector),

$\frac{1}{p_0} \in L_{k/(1-k)}(\Omega)$ *(for some k with $\frac{m}{m+1} < k < 1$ when $m \geqq 2$ or with $\frac{1}{2} < k < 1$ when $m=1$)*

$$q \geqq 0,$$

and

$$f \in C_{0,0}(\overline{\Omega}).$$

Then we have

$$\inf_{u \in \overset{\varphi}{W}{}_2^{(1)}(\Omega)} (G_\Omega[u] + 2H_{\Omega,1}[f, u]) > -\infty.$$

Proof. Let K be a constant larger than $\max_{\partial\Omega} \varphi$. Then we have

$$G_\Omega[u]+2H_{\Omega,1}[f,u]$$
$$\geqq\int_\Omega\sum_{i,j=1}^m p_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dV+\int_\Omega qu^2dV-2\int_\Omega(u^{(K)}+K)|f|dV$$
$$\geqq\int_{\Omega^{(K)}}\sum_{i,j=1}^m p_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dV-\varepsilon\int_{\Omega^{(K)}}\{u^{(K)}\}^2dV$$
$$-\frac{1}{\varepsilon}\int_\Omega f^2dV-2K\max_\Omega|f|\text{ mes }\Omega\quad(\varepsilon>0).$$

By Corollary 1 to Lemma 4 and Corollary 1 to Lemma 8, we get

$$G_\Omega[u]+2H_{\Omega,1}[f,u]$$
$$\geqq\left\{\frac{1}{\left\|\frac{1}{p_0}\right\|_{L_{k/(1-k)}(\Omega)}}-\varepsilon B'^2_{mk}\right\}\|\nabla u\|^2_{L_{2k}(\Omega^{(K)})}$$
$$-\left(\frac{1}{\varepsilon}\max_\Omega|f|+2K\right)\max_\Omega|f|\text{ mes }\Omega.$$

Therefore, by choosing ε so small that the coefficient of $\|\nabla u\|^2_{L_{2k}(\Omega^{(K)})}$ is positive, we have

$$G_\Omega[u]+2H_{\Omega,1}[f,u]$$
$$\geqq-\left(\frac{1}{\varepsilon}\max_{\bar\Omega}|f|+2K\right)\max_\Omega|f|\text{ mes }\Omega.$$

Lemma 10. *Under the same assumption as in Lemma* 9, *if* $u\in\overset{\varphi}{W}{}_2^{(1)}(\Omega)$ *satisfies*

$$G_\Omega[u]+2H_{\Omega,1}[f,u]\leqq C,$$

then we have

$$D_\Omega[u]\leqq G_\Omega[u]\leqq C',$$

where C' *depends on* m, k, C, $\|1/p_0\|_{L_{k/(1-k)}(\Omega)}$, $\max_{\bar\Omega}|f|$, $\max_{\partial\Omega}\varphi$ *and* mes Ω.

Proof. Let K be a constant larger than $\max_{\partial\Omega}\varphi$. Then we have

$$G_\Omega[u]\leqq C+2\int_\Omega|f|(u^{(K)}+K)dV$$
$$\leqq C+2\int_{\Omega^{(K)}}|f|u^{(K)}dV+2K\max_{\bar\Omega}|f|\text{ mes }\Omega$$
$$\leqq C+\left(\frac{1}{\varepsilon}\max_\Omega|f|+2K\right)\max_{\bar\Omega}|f|\text{ mes }\Omega+\varepsilon\int_{\Omega^{(K)}}\{u^{(K)}\}^2dV.$$

From Corollary 2 to Lemma 8 follows

$$G_\Omega[u] \leqq C + \left(\frac{1}{\varepsilon}\max_{\bar{\Omega}}|f| + 2K\right)\max_{\bar{\Omega}}|f|\,\mathrm{mes}\,\Omega$$
$$+ \varepsilon B'^2_{mk}\left\|\frac{1}{p_0}\right\|_{L_{k/(1-k)}(\Omega)} D_{\Omega(K)}[u]$$
$$\leqq C + \left(\frac{1}{\varepsilon}\max_{\bar{\Omega}}|f| + 2K\right)\max_{\bar{\Omega}}|f|\,\mathrm{mes}\,\Omega$$
$$+ \varepsilon B'^2_{mk}\left\|\frac{1}{p_0}\right\|_{L_{k/(1-k)}(\Omega)} G_\Omega[u].$$

Therefore, by choosing ε so small that the coefficient of $G_\Omega[u]$ is less than $\frac{1}{2}$, we have

$$G_\Omega[u] \leqq 2C + 2\left(\frac{1}{\varepsilon}\max_{\bar{\Omega}}|f| + 2K\right)\max_{\bar{\Omega}}|f|\,\mathrm{mes}\,\Omega.$$

Lemma 9′. *Assume*

$$\sum_{i,j=1}^{m} p_{ij}\xi_i\xi_j \geqq 0 \quad ((\xi_1, \cdots, \xi_m) \textit{ is any real vector})$$

and

$$\rho \geqq \rho_0 > 0.$$

Then

$$\inf_{u\in\mathfrak{W}_1} G_\Omega[u] > -\infty,$$

where $\mathfrak{W}_1 = \{u \in \overset{\circ}{W}{}^{(1)}_2(\Omega) \mid H_{\Omega,\rho}[u] = 1\}$.

Proof.

$$G_\Omega[u] \geqq \int_\Omega \sum_{i,j=1}^{m} p_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}dV + \int_\Omega qu^2 dV$$
$$\geqq -\max_{\bar{\Omega}}\frac{|q|}{\rho}H_{\Omega,\rho}[u] = -\max_{\bar{\Omega}}\frac{|q|}{\rho}.$$

Lemma 10′. *Under the same assumption as in Lemma* 9′, *if* $u \in \mathfrak{W}_1$, *then we have*

$$D_\Omega[u] \leqq G_\Omega[u] + \max_{\bar{\Omega}}\frac{|q|}{\rho}.$$

Proof.

$$D_\Omega[u] \leqq G_\Omega[u] + |H_{\Omega,q}[u]|$$

$$\leqq G_{\Omega}[u]+\max_{\bar{\Omega}}\frac{|q|}{\rho}H_{\Omega,\rho}[u].$$

Lemma 11. *Any bounded set in* $\overset{\circ}{W}{}_p^{(1)}(\Omega)$ *is relatively compact in* $L_p(\Omega)$.

(Cf. Smirnov [8] p. 351.[3])

Corollary. *Assume*

$$\sum_{i,j=1}^{m}p_{ij}\xi_i\xi_j\geqq p_0(x)\sum_{i=1}^{m}\xi_i^2\geqq 0$$

$$((\xi_1,\cdots,\xi_m)\ \textit{is any real vector}),$$

$$\frac{1}{p_0}\in L_{k/(1-k)}(\Omega)\qquad (\textit{for some } k \textit{ with } \frac{m}{m+2}<k<1$$

$$\textit{when } m\geqq 2, \textit{ or with } \frac{1}{2}<k<1 \textit{ when } m=1).$$

Then

$$\mathfrak{A}=\{u\in \overset{\circ}{W}{}_2^{(1)}(\Omega)\,|\,D_{\Omega}[u]<C\}$$

is relatively compact in $L_2(\Omega)$.

Proof. From the assumption of the corollary and from Lemma 8 follows $\|\nabla u\|_{L_{2k}(\Omega)}<C'$, and therefore from Corollary 2 to Lemma 4 follows $\|u\|_{L_{2k}(\Omega)}<C''$. Thus by Lemma 11 $\mathfrak{A}$ is relatively compact in $L_{2k}(\Omega)$.

New Let $\{u_n\}$ be a Cauchy sequence in $L_{2k}(\Omega)$. Since

$$\|\nabla(u_n-u_{n'})\|_{L_{2k}(\Omega)}$$
$$\leqq\|\nabla u_n\|_{L_{2k}(\Omega)}+\|\nabla u_{n'}\|_{L_{2k}(\Omega)}<2C'$$

and

$$\|u_n-u_{n'}\|_{L_{2k}(\Omega)}\to 0\ \ (n,n'\to\infty),$$

from Lemma 5 follows

$$\|u_n-u_{n'}\|_{L_2(\Omega)}\to 0\ \ (n,n'\to\infty),$$

so that $\{u_n\}$ is also a Cauchy sequence in $L_2(\Omega)$. Hence $\mathfrak{A}$ is rela-

3) In [8] only the relative compactness in $L_p(\Omega')$ (Ω' is strictly contained in Ω) are stated. When $u\in \overset{\circ}{W}{}_p^{(1)}(\Omega)$, a similar discussion leads to the relative compactness in $L_p(\Omega)$ (cf. Suzuki [9], p. 44).

tively compact in $L_2(\Omega)$.

Lemma 12. *Let the elliptic operator*

$$Lu\equiv\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial u}{\partial x_i}\right)-qu$$

be such that

$$\sum_{i,j=1}^{m}p_{ij}\xi_i\xi_j\geqq p_0\sum_{i=1}^{m}\xi_i^2$$

$$(p_0>0,(\xi_1,\cdots,\xi_m)\ \textit{is any real vector})$$

$$\frac{\partial p_{ij}}{\partial x_i},\ q\in C_{0,\alpha}(\overline{\Omega}).$$

If $u\in C_{2,\alpha}(\overline{\Omega})$, *then we have*

$$\|u\|_{C_{2,\alpha}(\overline{\Omega}')}\leqq C\,[\|Lu\|_{C_{0,\alpha}(\overline{\Omega})}+\max_{\overline{\Omega}}|u|],$$

where Ω' *is any subdomain of* Ω *which is strictly contained in* Ω, *and* C *depends on* α, $\|\frac{\partial p_{ij}}{\partial x_i}\|_{C_{0\alpha},(\overline{\Omega})}$, $\|q\|_{C_{0,\alpha}(\overline{\Omega})}$, p_0, *the diameter of* Ω, *and the distance between* Ω' *and* $\partial\Omega$ (Cf. Schauder [7]).

§6. Property (A) and Property (B)

In the first place we shall discuss a sufficient condition for the domain and the coefficients in the problem (1)-(2) to have Property (A).

Assume

$$p_{ij}\in C_{1,0}(\Omega)\cap C_{0,0}(\overline{\Omega});q,f\in C_{0,0}(\overline{\Omega});\varphi\in C_{0,0}(\partial\Omega)$$

and

$$\sum_{i,j=1}^{m}p_{ij}\xi_i\xi_j\geqq 0\ \ ((\xi_1,\cdots,\xi_m)\ \textit{is any real vector}).$$

If $q>0$ in Ω and $v\in C_{2,0}(\Omega)\cap C_{0,0}(\overline{\Omega})$ satisfies

$$L[v]\leqq\min_{\Omega}f\quad\text{in }\Omega$$

$$v\geqq\max_{\partial\Omega}\varphi\quad\text{on }\partial\Omega$$

resp.

$$L[v]\geqq\max_{\overline{\Omega}} f \qquad \text{in } \Omega$$

$$v\leqq\min_{\partial\Omega} \varphi \qquad \text{on } \partial\Omega,$$

then by the well known maximum principle, we have

$$u\leqq v \qquad \text{in } \overline{\Omega}$$

resp.

$$u\geqq v \qquad \text{in } \overline{\Omega}$$

for any solution u of the problem (1)-(2) (cf. Courant-Hilbert [1], p. 329).

Suppose on the other hand $q \not> 0$ at $x'\in\overline{\Omega}$. If there exists i_0 such that $p_{i_0 i_0}(x')>0$ and $\frac{\partial p_{i_0 j}}{\partial x}\in C_{0,0}(\overline{\Omega})$ $(j=1,\cdots,m)$, then there exists a subdomain Ω' of Ω such that $\overline{\Omega'}\ni x'$ and

$$a-e^{-b(x_{i_0}-x'_{i_0})}>0,$$

$$be^{-b(x_{i_0}-x'_{i_0})}\left((p_{i_0 i_0}b-\sum_{j=1}^{m}\frac{\partial p_{i_0 j}}{\partial x_j}\right)+q(a-e^{-b(x_{i_0}-x'_{i_0})})>0$$

in Ω', for some positive constants a, b. Hence by the transformation

$$u=(a-e^{-b(x_{i_0}-x'_{i_0})})w,$$

the problem is reduced to the case where $q>0$ in Ω'.

Therefore, if $\partial\Omega=\Gamma_0\cup\Gamma_1\cup\cdots\cup\Gamma_m$, then domain Ω and the coefficients in the problem have Property (A), where

$$\Gamma_0=\{x\in\partial\Omega\,|\,q(x)>0\},$$

$$\Gamma_i=\{x\in\partial\Omega\,|\,p_{ii}(x)>0, \text{ and } \frac{\partial p_{ij}}{\partial x_j}\ (j=1,\cdots,m)$$

are continuous in a neighbourhood of $x\}$.

We now turn to the discussion of Property (B). Let x_0 be a boundary point, and $x_0\in S$, where (i) S belongs to $C_{2,0}$ ($\partial\Omega$ need not belong to $C_{2,0}$), i.e., there exist a neighbourhood $U_{x_0}=\{x\,|\,|x-x_0|<\sigma\}$ and the local coordinates $y_1(x),\cdots,y_m(x)\in C_{2,0}(U_{x_0})$ such that

$$U_{x_0}\cap S=\{x\in U_{x_0}\,|\,y_m(x)=y_m(x_0)=0\}$$

and

$$\left|\frac{D(y_1,\cdots,y_m)}{D(x_1,\cdots,x_m)}\right| \neq 0 \text{ in } U_{x_0},$$

and (ii) S satisfies

$$U_{x_0}\cap\Omega\subset\{x\in U_{x_0}\,|\,y_m(x)>y_m(x_0)=0\}.$$

Assume, rewriting $U_{x_0}\cap\Omega$ as $\omega(x_0,\sigma)$, that

$$p_{ij}\in C_{1,0}(\omega(x_0,\sigma))\cap C_{0,0}(\overline{\omega(x_0,\sigma)}),$$
$$q\in C_{0,0}(\overline{\omega(x_0,\sigma)}),$$
$$\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial y_k}{\partial x_j}\right)(y_k-y_k(x_0))\ ((k=1,\cdots,m-1) \text{ is bounded in } \omega(x_0,\sigma))$$

and that there exist constants $a, b, c(0<a<1, b>0, c>0)$ such that

$$(6)\qquad (a-1)\sum_{i,j=1}^{m}p_{ij}\frac{\partial y_m}{\partial x_i}\frac{\partial y_m}{\partial x_j}y_m^{a-2}+\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial y_m}{\partial x_i}\right)y_m^{a-1} < -by_m^{-c} \text{ in } \omega(x_0,\sigma).$$

Then, by setting

$$v=\sum_{i=1}^{m-1}(y_i-y_i(x_0))^2+y_m^a,$$

we can construct a strong barrier function.

In fact, taking σ sufficiently small, we have

$$L[v]=\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{vv}{x_i}\right)-qv$$
$$=\sum_{i,j=1}^{m}\sum_{k,l=1}^{m}p_{ij}\frac{\partial y_k}{\partial x_i}\frac{\partial y_l}{\partial x_j}\frac{\partial^2 v}{\partial y_k\partial y_l}+\sum_{i,j=1}^{m}\sum_{k=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial y_k}{\partial x_i}\right)\frac{\partial v}{\partial y_k}-qv$$
$$=2\sum_{k=1}^{m-1}\sum_{i,j=1}^{m}p_{ij}\frac{\partial y_k}{\partial x_i}\frac{\partial y_k}{\partial x_j}+2\sum_{k=1}^{m-1}\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial y_k}{\partial x_j}\right)(y_k-y_k(x_0))-qv$$
$$+a(a-1)\sum_{i,j=1}^{m}p_{ij}\frac{\partial y_m}{\partial x_i}\frac{\partial y_m}{\partial x_j}y_m^{a-2}+a\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial y_m}{\partial x_i}\right)y_m^{a-1}$$
$$<0 \text{ in } \omega(x_0,\sigma).$$

Obviously v satisfies all other conditions for a strong barrier function.

§7. Examples

1) Let

$$\Omega=\{x\,|\sum_{k=1}^{m}x_k^2<1(m\geqq 2)\}.$$

Consider the problem

$$\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\left(1-\sqrt{\sum_{k=1}^{m}x_k^2}\right)^{2/m-\varepsilon_i}\frac{\partial u}{\partial x_i}\right\}-qu=f\,(0<\varepsilon_i\leqq 2/m)\quad \text{in } \Omega,$$

$$u=\varphi \quad \text{on } \partial\Omega,$$

where

$$q, f\in C_{0,\alpha}(\Omega)\cap C_{0,0}(\overline{\Omega}),$$

$$\varphi\in C_{2,\alpha}(\Omega)\cap C_{0,0}(\overline{\Omega}),$$

and

$$q\geqq 0 \text{ in } \overline{\Omega}\ (q\neq 0 \ on\ \partial\Omega).$$

This problem has a unique solution.

Proof. Here suffice it to say that we prove that Property (B) is satisfied. Let $x_0\in\partial\Omega$. In $\omega(x_0,\sigma)$ take $y_l=1-\sqrt{\sum_{k=1}^{l}x_k^2}$ $(l=1,\cdots,m)$ as local coordinates. If we choose a number a such that $0<a<1-\frac{2}{m}+\min\varepsilon_i$, then we have

$$\begin{aligned}
&(a-1)\sum_{i,j=1}^{m}p_{ij}\frac{\partial y_m}{\partial x_i}\frac{\partial y_m}{\partial x_j}y_m^{a-2}+\sum_{i,j=1}^{m}\frac{\partial}{\partial x_j}\left(p_{ij}\frac{\partial y_m}{\partial x_i}\right)y_m^{a-1}\\
&=(a-1)\sum_{k=1}^{m}\left(1-\sqrt{\sum_{k=1}^{m}x_k^2}\right)^{2/m-\varepsilon_i}\left(\frac{\partial y_m}{\partial x_i}\right)^2y_m^{a-2}\\
&\qquad+\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left\{\left(1-\sqrt{\sum_{k=1}^{m}x_k^2}\right)^{2/m-\varepsilon_i}\frac{\partial y_m}{\partial x_i}\right\}y_m^{a-1}\\
&\approx(a-1)\sum_{i=1}^{m}\left(1-\sqrt{\sum_{k=1}^{m}x_k^2}\right)^{2/m-\varepsilon_i}\left(\frac{x_i}{\sqrt{\sum_{k=1}^{m}x_k^2}}\right)^2y_m^{a-2}\\
&\qquad+\sum_{i=1}^{m}\left(\frac{2}{m}-\varepsilon_i\right)\left(1-\sqrt{\sum_{k=1}^{m}x_k^2}\right)^{2/m-\varepsilon_i-1}\left(\frac{x_i}{\sqrt{\sum_{k=1}^{m}x_k^2}}\right)^2y_m^{a-1}\\
&=\sum_{i=1}^{m}\{(a-1)+(\frac{2}{m}-\varepsilon_i)\}\left(\frac{x_i}{\sqrt{\sum_{k=1}^{m}x_k^2}}\right)^2y_m^{2/m-\varepsilon_i+a-1}\\
&\leqq\sum_{i=1}^{m}\{(a-1)+(\frac{2}{m}-\min\varepsilon_i)\}\left(\frac{x_i}{\sqrt{\sum_{k=1}^{m}x_k^2}}\right)^2y_m^{a-1+2/m-\max\varepsilon_i}.
\end{aligned}$$

$$=\{(a-1)+(\frac{2}{m}-\min \varepsilon_i)\}\, y_m^{a-1+2/m-\max\varepsilon_i}.$$

Thus the inequality (6) is satisfied.

2) Let

$$\Omega=\{x\,|\,0<x_i<1(i=1,\cdots,m)(m\geqq 2)\}.$$

Consider the problem

$$\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(x_i^{2/m-\varepsilon_i}\frac{\partial u}{\partial x_i}\right)-qu=f\left(0<\varepsilon_i\leqq\frac{2}{m}\right)\text{ in }\Omega,$$

$$u=\varphi \quad \text{on } \partial\Omega,$$

where

$$q,f\in C_{0,\alpha}(\Omega)\cap C_{0,0}(\bar{\Omega}),$$

$$\varphi\in C_{2,\alpha}(\Omega)\cap C_{0,0}(\bar{\Omega})$$

and

$$q>0 \text{ in } \bar{\Omega}(q\neq 0 \text{ at } x=0).$$

Again this problem has a unique solution.

3) Let

$$\Omega=\{x\,|\,1<\sum_{k=1}^{m}x_k^2<4(m\geqq 2)\}.$$

Consider the eigenvalue problem for

$$\sum_{i=1}^{m'}\frac{\partial}{\partial x_i}\left\{\left(\sqrt{\sum_{k=1}^{m}x_k^2}-1\right)^{2/m-\varepsilon_i}\frac{\partial u}{\partial x_i}\right\}+\sum_{i=m'+1}^{m}\frac{\partial}{\partial x_i}\left\{\left(2-\sqrt{\sum_{k=1}^{m}x_k^2}\right)^{2/m-\varepsilon_i}\frac{\partial u}{\partial x_i}\right\}-qu+\lambda\rho u=0$$

$$\left(0<\varepsilon_i\leqq\frac{2}{m},\ 1\leqq m'<m\right)\text{ in }\Omega,$$

$$u=0 \quad \text{on } \partial\Omega,$$

where

$$q,\rho,f\in C_{0,\alpha}(\Omega)\cap C_{0,0}(\bar{\Omega}),$$

and

$$\rho\geqq\rho_0>0 \quad \text{in } \bar{\Omega}.$$

This problem has countable and discrete eigenvalues.

4) Let

$$\Omega=\{x\,|\,0<x_i<1\,(i=1,\cdots,2m)\,(m\geqq 1)\}.$$

Consider the eigenvalue problem for

$$\sum_{i=1}^{m}\frac{\partial}{\partial x_i}\left(x_i^{1/m-\varepsilon_i}\frac{\partial u}{\partial x_i}\right)+\sum_{i=1}^{m}\frac{\partial}{\partial x_{m+i}}\left\{(1-x_i)^{1/m-\varepsilon_i}\frac{\partial u}{\partial x_{m+i}}\right\}$$

$$-qu+\lambda\rho u=0\left(0<\varepsilon_i\leqq\frac{1}{m}\right)\quad\text{in }\Omega,$$

$$u=0\qquad\text{on }\partial\Omega,$$

where

$$q,\rho,f\in C_{0'\alpha}(\Omega)\cap C_{0,0}(\overline{\Omega})$$

and

$$\rho\geqq\rho_0>0\ \text{ in }\overline{\Omega}.$$

Again this problem has countable and discrete eigenvalues.

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