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On the principle of limiting amplitude

By

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§1. Introduction and theorem

We study the behavior for large time of solutions of wave equations with a harmonic forcing term in the three dimensional euclidean space and we prove the so-called limiting amplitude principle. The principle states that every solution $u(x, t)$ for the initial value problem,

$$\left\{\frac{\partial^2}{\partial t^2}+b(x)\frac{\partial}{\partial t}-\Delta+c(x)\right\}u(x, t)=f(x)e^{i\omega t} \tag{1.1}$$

$$u(x, t)|_{t=0}=\frac{\partial}{\partial t}u(x, t)|_{t=0}=0 \tag{1.2}$$

tends to the steady state solution, $e^{i\omega t}v(x, i\omega)$ uniformly on bounded sets at $t\to\infty$. Here $v(x, i\omega)$ satisfies the elliptic equation,

$$\{-\Delta+c(x)+i\omega b(x)-\omega^2\}\, v(x, i\omega)=f(x); \tag{1.3}$$

and the Sommerfeld radiation conditions at infinity. Δ denotes the Laplacian in E^3 and ω is a real number. In the case when $b(x)\equiv 0$ and the real valued function $c(x)$ is once continuously differentiable and its support is compact, this principle has been proved by O. A. Ladyzenskaja [1]. Here the rate of approach to steady state is like $e^{-\varepsilon t}$, $\varepsilon>0$ as $t\to\infty$. When $b(x)$ and $c(x)$ satisfy that $b(x)\geq 0$, $b(x)=0\left(\frac{1}{|x|^{3+\varepsilon}}\right)$, $c(x)=0\left(\frac{1}{|x|^{2+\varepsilon}}\right)$ as $|x|\to\infty$, and others, S. Mizohata and K. Mochizuki [2] had shown the principle, but they did not give the rate of approach. In this paper we shall obtain the rate $e^{-\varepsilon t}$ under the assumption that the real valued functions $b(x)\geq 0$, $c(x)\geq 0$ are bounded and their supports are compact.

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Theorem 1. *Let $f(x)$, $b(x)$ and $c(x)$ be functions which satisfy the following conditions,*

i) *$f(x)$, $b(x)$ and $c(x)$ vanish outside a bounded set*

ii) $\sum_{|\alpha|\le 2}|D^{\alpha}f|\in L^2(E^3)$

iii) *$b(x)\geq 0$, $c(x)\geq 0$, and they are bounded functions.*

And let $u(x,t)$ be a solution for initial value problem (1. 1), (1. 2). *Then there exists a steady state $e^{i\omega t}v(x)$, such that*

$$\max_{x\in K}|u(x,t)-v(x)e^{i\omega t}|\leq C\cdot e^{-\varepsilon t},\quad \exists\varepsilon>0,\ \text{as } t\to\infty, \tag{1.4}$$

and $v(x)$ is a solution of (1. 3) *satisfying the Sommerfeld radiation conditions at infinity, that is,*

$$\int_{|x|=R}|u|^2ds=0(1),\ \int_{|x|=R}\left|\frac{d}{d|x|}u+i\omega u\right|^2ds=o(1)\quad as\ R\to\infty,$$

where K is a bounded set of E^3.

We can regard a solution $u(x,t)$ as a twice continuously differentiable function $u(t)$ from $[0,\infty)$ to $L^2(E^3)$ and as a continuous function to $\mathcal{D}^2_{L^2}(E^3)$. In this sence there exists the unique solution of (1. 1), (1. 2) if $f(x)\in\mathcal{D}^1_{L^2}(E^3)$. Let $\tilde{u}(\lambda)$ be the Laplace image of $u(t)$ with respect to t,

$$\tilde{u}(\lambda)=\int_0^\infty e^{-\lambda t}u(t)dt\quad \text{in } L^2. \tag{1.5}$$

Then

$$\tilde{u}(\lambda)=v(\lambda)/\lambda-iw$$

and

$$u(t)=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma-i\tau}^{\sigma+i\tau}\frac{v(\lambda)}{\lambda-i\omega}e^{\lambda t}d\lambda\quad \text{in } L^2 \tag{1.6}$$

for large $\sigma>0$. Where

$$\{-\Delta+c(x)+\lambda b(x)+\lambda^2\}v(\lambda)=f(x),\ v(\lambda)\in L^2,\ \operatorname{Re}\lambda>0. \tag{1.7}$$

(1. 7) has the unique solution belonging to $L^2(E^3)$ if $f(x)$ belongs to $L^2(E^3)$ and $\operatorname{Re}\lambda$ is sufficiently large positive. Therefore we study the analyticity of $v(\lambda)$ with respect to λ and the order of $\|v(\lambda)\|_{L^2(K)}$ as $|\operatorname{Im}\lambda|\to\infty$.

§2. Some lemmas

1) In the case when $b(x)\equiv c(x)\equiv 0$

$$(-\Delta+\lambda^2)v(x,\lambda)=f(x),\quad f(x)\in L^2$$

has the unique solution $v(x,\lambda)$ in $\mathcal{D}^2_{L^2}$ at $\mathrm{Re}\,\lambda>0$ and $v(x,\lambda)$ is an analytic function of λ to L^2. $v(x,\lambda)\equiv R(\lambda)f$ is represented by a fundamental solutions $E(\lambda)$ as following

$$R(\lambda)f=E(\lambda)*f,\ \text{where}\ E(\lambda)=\overline{\mathcal{F}}\left\{\frac{1}{(4\pi^2|\xi|^2+\lambda^2)}\right\}=\frac{e^{-\lambda|x|}}{4\pi|x|}.$$

Let $Q(\delta)$ denote a Hilbert space consisting of all functions f such that $e^{\delta|x|}f\in L^2(E^3)$ with the inner product $(f,g)_\delta=(e^{\delta|x|}f, e^{\delta|x|}g)_{L^2(E^3)}$, $(-\infty<\delta<+\infty)$. Now it is clear that $Q(\delta)\subset Q(\delta')$ if $\delta\geq\delta'$. Using these space,

Lemma 1. *Let*

$$R(\lambda)f=\frac{1}{4\pi}\int_{E^3}\frac{e^{-\lambda|x-y|}}{|x-y|}f(y)dy.$$

Then $R(\lambda)$, which values a bounded operator from $Q(2\delta)$ to $Q(-2\delta)$, is an analytic function of λ and satisfies the following estimates at $\mathrm{Re}\,\lambda\geq-\delta$ $(\delta>0)$.

i) $|R(\lambda)f|_{-2\delta}\leq\{C(\delta)/(1+|\lambda|)(1+|R_e|)\}\cdot|f|_{2\delta}$

ii) $|DR(\lambda)f|_{-2\delta}\leq\{C(\delta)/(1+|R_e\lambda|)\}\,|f|_{2\delta}$

iii) $|D^2R(\lambda)f|_{-2\delta}\leq\{C(\delta)(1+|\lambda|)/(1+|R_e\lambda|)\}\,|f|_{2\delta}$

iv) $|R(\lambda)-R(\lambda+h)\}f|_{-2\delta}\leq\{C(\delta)|h|/(1+|\lambda|)(1+|R_e\lambda|)\}\,|f|_{2\delta}$

where $|\ |_\delta$ *denote the norm of* $Q(\delta)$, i.e. $|f|^2_\delta=\int_{E^3}|e^{\delta|x|}f|^2dx$ *and* $C(\delta)$ *are constants.*

2) The case when $b(x)\equiv 0$, $c(x)\not\equiv 0$.

Lemma 2. *Let*

$$L_1(\lambda)u=(-\Delta+\lambda^2+c(x)u),\quad u\in\mathcal{D}^2_{L^2}$$

and $G_1(\lambda)$ be the Green operators of $L_1(\lambda)$, that is,

$$G_1(\lambda)\cdot L_1(\lambda)\subset L_1(\lambda)\cdot G_1(\lambda)=I\colon L^2\to L^2.$$

Then we can consider $G_1(\lambda)$ *as bounded operators from* $Q(\delta)$ *to* $Q(-\delta)$. *In this sense we can analytically continue* $G_1(\lambda)$ *to an analytic function of* λ *in* $\mathrm{Re}\lambda \geq -\delta' < 0$, *which satisfies the following estimates (we denote the extension also by* $G_1(\lambda)$*)*,

$$\text{i)} \quad |G_1(\lambda)f|_{-\delta} \leq \frac{C}{(1+|\lambda|)(1+|\mathrm{Re}\lambda|)}|f|_\delta$$

$$\text{ii)} \quad |\{G_1(\lambda)-G_1(\lambda+h)\}f|_{-\delta} \leq \frac{C|h|}{(1+|\lambda|)(1+|\mathrm{Re}\lambda|)}|f|_\delta$$

and $G_1(\lambda)$ *are compact operators from* $Q(\delta)$ *to* $Q(-\delta)$ *(which map any bounded set to a precompact set), where* $c(x) \geq 0$ *is a bounded function with compact support.*

3) The case when $b(x) \not\equiv 0$.

Lemma 3. *Let*

$$L_2(\lambda)u = (-\Delta + \lambda^2 + c(x) + \lambda b(x))u, \quad u \in \mathscr{D}^2_{L^2},$$

and $G_2(\lambda)$ *be the Green operators of* $L_2(\lambda)$, *that is,*

$$G_2(\lambda) \cdot L_2(\lambda) \subset L_2(\lambda) \cdot G_2(\lambda) = I \colon L^2 \to L^2.$$

Then we can consider $G_2(\lambda)$ *as bounded operators from* $Q(\delta)$ *to* $Q(-\delta)$. *In this sense we can analytically continue* $G_2(\lambda)$ *to analytic function of* λ *in* $\mathrm{Re}\lambda \leq -\delta'' < 0$, *which satisfies the following estimate (we denote the extension also by* $G_2(\lambda)$*)*,

$$|G_2(\lambda)f|_{-\delta} \leq \frac{C}{(1+|\lambda|)(1+|\mathrm{Re}\lambda|)}|f|_\delta .$$

Where $b(x) \geq 0$ *and* $c(x) \geq 0$ *are bounded functions with compact supports.*

§3. Proof of lemmas

1) **Proof of Lemma 1** (the case when $c(x) \equiv b(x) \equiv 0$).

We first prove that $R(\lambda)$ is an analytic function of λ which values the bounded operators from $Q(2\delta)$ to $Q(-2\delta)$. In order to do so it is sufficient to show that $f(x) \to \varphi_i(x)$; $\varphi_i(x) = \int_{E^3} |x-y|^i e^{-\lambda|x-y|} f(y)\,dy$ $(i = -1, 0, 1, 2 \cdots)$ are bounded operators from $Q(2\delta)$ to $Q(-2\delta)$.

Since $|e^{-\lambda|x-y|}|=e^{-\mathrm{Re}\,\lambda|x-y|}\leq e^{+\delta|x-y|}\leq e^{+\delta|x|}e^{+\delta|y|}$, $\mathrm{Re}\,\lambda\geq-\delta<0$,

$$\int|e^{-2\delta|x|}\varphi_i(x)|^2dx\leq\int\left\{e^{-2\delta|x|}\int|x-y|^i e^{-\lambda|x-y|}f(x)dy\right\}^2dx$$

$$\leq\iint\{e^{-\delta|x|}|x-y|^i e^{-\delta|y|}\}^2dxdy\cdot\int|e^{+2\delta|y|}f(y)|^2dy$$

that is,

$$|\varphi_i|_{-2\delta}\leq C_{i,\delta}|f|_{2\delta}\,.$$

Hence $f(x)\to\varphi_i(x)$ are bounded operators from $Q(+2\delta)$ to $Q(-2\delta)$. Next we estimate $R(\lambda)$. Since $Q(\delta)$ densely contains $\mathscr{D}$ (compact support, C^∞-function), it is sufficient to show the inequality in the case $f\in\mathscr{D}$. When $\mathrm{Re}\,\lambda>0$, we can write

$$R(\lambda)f=\overline{\mathscr{F}}\left[\frac{1}{(2\pi|\xi|)^2+\lambda^2}\mathscr{F}(f)\right]$$

$$=\int\frac{e^{2\pi ix\cdot\xi}}{4\pi^2|\xi|^2+\lambda^2}\hat{f}(\xi)d\xi,\quad \hat{f}(\xi)=\mathscr{F}(f) \tag{3.1}$$

$$=\sum_{(i,j,k)}\int_{\Gamma(i,j,k)}\frac{e^{2\pi ix\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi, \tag{3.2}$$

where $\mathscr{F}$ and $\overline{\mathscr{F}}$ are Fourier transform and Fourier inverse transform, respectively, i, j, k take a sign of $+$ or $-$, and

$$\Gamma(+,+,+)=[0,\infty)\times[0,\infty)\times[0,\infty)$$
$$\Gamma(+,+,-)=[0,\infty)\times[0,\infty)\times(-\infty,0]$$
$$\vdots$$

Now we prove the estimate i). To do so we divide λ into four cases which are $\{\lambda;\mathrm{Re}\,\lambda\geq N>0\}$, $\{\lambda;-\delta\leq\mathrm{Re}\,\lambda\leq N,\mathrm{Im}\,\lambda\geq N\}$, $\{\lambda;-\delta\leq\mathrm{Re}\,\lambda\leq N,\mathrm{Im}\,\lambda\leq-N\}$ and $\{\lambda;-\delta\leq\mathrm{Re}\,\lambda\leq N,|\mathrm{Im}\,\lambda|\leq N\}$. When λ is in $\{\mathrm{Re}\,\lambda\geq N>0\}$, it follows that

$$\sup_{\xi\in R^3}\left|\frac{1}{(2\pi|\xi|)^2+\lambda^2}\right|\leq\frac{C}{(1+|\lambda|)(1+|\mathrm{Re}\,\lambda|)}.$$

Hence, we have from (3.1)

$$\|R(\lambda)f\|_{L^2}^2\leq\frac{C}{(1+|\lambda|)^2(1+|\mathrm{Re}\,\lambda|)^2}\|f\|_{L^2}^2\,.$$

Since $Q(2\delta)\subset L^2\subset Q(-2\delta)$, $(\delta>0)$, we conclude that

$$|R(\lambda)f|_{-2\delta}\leq\frac{C}{(1+|\lambda|)(1+|\mathrm{Re}\lambda|)}|f|_{2\delta}.$$

When λ is in $\{-\delta<\mathrm{Re}\lambda\leq N, \mathrm{Im}\lambda\geq N\}$, we consider that ξ is of three-dimension complex space: C^3. Then $\hat{f}(\xi)$ is an analytic function in C^3 and satisfies that

$$\sup_{|\mathrm{Im}\,\xi|\leq A}(1+|\xi|)^M|\hat{f}(\xi)|\leq C_{M,A}, \tag{3.3}$$

where A and M are real numbers and $C_{M,A}$ is a constant depending on A, M and f. And $1/4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2$ is analytic in ξ unless the points are such that $\xi_1^2+\xi_2^2+\xi_3^2=-\lambda^2/4\pi^2$. When $\lambda=\alpha+i\beta$, $\alpha>0$, $\beta\geq N>0$, we set

$$R_{(i,j,k)}(\lambda)f\equiv\int_{\Gamma(i,j,k)}\frac{e^{2\pi ix\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)\cdot d\xi. \tag{3.4}$$

We move lines of integration of the right hand side of (3.4) as following,

$$[0,\infty)\to[0,+\delta i/2\pi]+[+\delta i/2\pi,+\infty+\delta i/2\pi)\equiv I_++J_+$$
$$(-\infty,0]\to[0,-\delta i/2\pi]+[-\delta i/2\pi,-\infty-\delta i/2\pi)\equiv I_-+J_-.$$

Then

$$R_{(i,j,k)}(\lambda)f=\int_{(I_i+J_i)\times(I_j+J_j)\times(I_k+J_k)}\frac{e^{2\pi ix\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi. \tag{3.5}$$

In fact, $e^{2\pi ix\cdot\xi}\{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2\}^{-1}$ is analytic and bounded in ξ on $D_i\times D_j\times D_k$, and (3.3) holds there, where

$$D_+=\{\mathrm{Re}\,\eta\geq0,\ \delta/2\pi\geq\mathrm{Im}\,\eta\geq0\},\ D_-=\{\mathrm{Re}\,\eta\leq0,\ -\delta/2\pi\leq\mathrm{Im}\,\eta\geq0\}.$$

(3.5) shows that $R_{(i,j,k)}(\lambda)f$ is analytic in $\{\mathrm{Re}\lambda>-\delta, \mathrm{Im}\lambda\geq N\}$ as a $Q(-\delta)$-valued function of λ. Let

$$S(K_i,K_j,K_k)(\lambda)f\equiv\int_{K_i\times K_j\times K_k}\frac{e^{2\pi ix\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2}\hat{f}(\xi)d\xi \tag{3.6}$$

where K_i is I_i or J_i.

Lemma 4. *Let* $p(\xi,\lambda)=1/|4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2|$ *and* $\lambda\in\{-\delta<\mathrm{Re}\lambda\leq N, \mathrm{Im}\lambda\geq N\}$. *Then*

i) $\sup_{\xi \in I_i \times I_j \times I_k} p(\xi, \lambda) \leq \frac{C}{|\lambda|^2}$

ii) $\sup_{\xi \in K_i \times K_j \times K_k} p(\xi, \lambda) \leq \frac{C}{(1+|\lambda|)(\delta + \mathrm{Re}\,\lambda)}$

where K_h is I_h or J_h and one of K_i, K_j and K_k is J_+ or J_-.

Proof. We may assume $N \geq \delta$. Then $\mathrm{Re}\,\lambda^2 \leq 0$.

i) Since $4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2) = z \in [-3\delta^2, 0]$ if $\xi \in I_i \times I_j \times I_k$, we have from $\mathrm{Re}\,\lambda^2 \leq 0$

$$p(\xi, \lambda) \leq \sup_{z \in [-3\delta^2, 0]} \left(\frac{1}{|\lambda^2 + z|} \right) \leq \frac{C}{|\lambda|^2}.$$

ii) Since $\xi \in K_i \times K_j \times K_k$ and one of K_i, K_j and K_k is J_+ or J_-, we can write

$$4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2) = 4\pi^2 z^2 t(\xi), \quad z \in J_+, \ t(\xi) \geq 1.$$

Hence

$$\begin{aligned}\sup_{\xi \in K_i \times K_j \times K_k} p(\xi, \lambda) &\leq \sup_{Z \in J_+} \frac{1}{|\lambda^2 + 4\pi^2 z^2|} \\ &\leq \frac{C}{(|\lambda|+1)(\mathrm{Re}\,\lambda + \delta)}.\end{aligned}$$

This proves Lemma 4.

1° $$\begin{aligned} S(I_+, I_+, I_+)(\lambda) f &\equiv \int_{I_+ \times I_+ \times I_+} \frac{e^{2\pi i x \cdot \xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2} \hat{f}(\xi) d\xi \\ &= -i \int_0^{\delta/2\pi} \int_0^{\delta/2\pi} \int_0^{\delta/2\pi} \frac{e^{-2\pi(s_1x_1+s_2x_2+s_3x_2)}}{\lambda^2 - 4\pi^2(s_1^2+s_2^2+s_3^2)} \hat{f}(+is) ds_1 ds_2 ds_3, \ s=(s_1, s_2, s_3).\end{aligned}$$

Since

$$\begin{aligned}|f(is)|^2 &\leq \left\{ \int |e^{2\pi(s_1x_1+s_2x_2+s_3x_3)}|f|(x)|dx \right\}^2 \\ &\leq \int e^{-2\varepsilon|x|} ds \cdot \int |e^{(\sqrt{3}\delta+\varepsilon)|x|} f(x)|^2 dx, \quad \left(|s_1|, |s_2|, |s_3| \leq \frac{\delta}{2\pi} \right),\end{aligned}$$

$$\begin{aligned}&\int |e^{-(\sqrt{3}\delta+\varepsilon)|x|} S(I_+, I_+, I_+)(\lambda) f|^2 dx \\ &\leq \left\{ \int e^{-2\varepsilon|x|} dx \right\}^2 \left\{ \iiint_0^{2\pi} ds_1 ds_2 ds_3 \right\}^2 \cdot \int |e^{(\sqrt{3}\delta+\varepsilon)|x|} f(x)|^2 dx \\ &\times \sup_{0 \leq s_1, s_2, s_3 \leq \delta/2\pi} \left| \frac{1}{\lambda^2 - 4\pi^2(s_1^2+s_2^2+s_3^2)} \right|.\end{aligned}$$

From Lemma 4, i)

$$\leq \frac{(4\pi)^2}{(2\varepsilon)^6}\left(\frac{\delta}{2\pi}\right)^6 \int |e^{(\sqrt{3}\delta+\varepsilon)x'} f(x)|^2 dx \cdot \frac{C}{|\lambda|^4}.$$

2° $$S(I_+, J_+, J_+)(\lambda)f \equiv \int_{I_+ \times J_+ \times J_+} \frac{e^{2\pi i x\cdot\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2} \hat{f}(\xi)d\xi$$

$$= i\int_0^{2\pi} ds_1 \int_0^\infty ds_2 \int_0^\infty \frac{e^{-2\pi s_1 x_1} e^{2\pi i s_2 x_2 - \delta x_2} e^{2\pi i s_3 x_3 - \delta x_3}}{4\pi^2\left\{(is_1)^2 + \left(s_2 + i\frac{\delta}{2\pi}\right)^2 + \left(s_3 + i\frac{\delta}{2\pi}\right)^2\right\} + \lambda^2} \hat{f}(\sigma) ds_3,$$

where σ denotes $\left(is_1, s_2 + i\frac{\delta}{2\pi}, s_3 + i\frac{\delta}{2\pi}\right) = ie^{-\delta x_2 - \delta x_3}$

$$\times \int_0^\infty \int_0^\infty e^{2\pi i(s_2 x_2 + s_3 x_3)} \left\{\int_0^{\delta/2\pi} \frac{e^{-2\pi s_1 x_1} g_1(s_1, s_2, s_3)}{q_1(s_1, s_2, s_3, \lambda)} ds_1\right\} ds_2 ds_3$$

where
$$\begin{cases} q_1(s_1, s_2, s_3, \lambda) = 4\pi^2\left\{\left(s_2 + i\frac{\delta}{2\pi}\right)^2 + \left(s_3 + i\frac{\delta}{2\pi}\right)^2 - s_1^2\right\} + \lambda^2 \\ g_1(s_1, s_2, s_3) = \hat{f}\left(is_1, s_2 + i\frac{\delta}{2\pi}, s_3 + i\frac{\delta}{2\pi}\right). \end{cases}$$

Since we can regard the above equality as Fourier transform from (s_2, s_3) to (x_2, x_3), using Plancherel's theorem, we have the following inequality.

$$\iint |e^{\delta x_2 + \delta x_3} S(I_+, J_+, J_+)(\lambda) f|^2 ds_2 ds_3$$

$$= \iint \left|\int_0^{\delta/2\pi} \frac{e^{-\pi s_1 x_1} g_1(s_1, s_2, s_3)}{q_1(s_1, s_2, s_3)} ds_1\right|^2 ds_2 ds_3$$

$$\leq \frac{e^{2\delta|x_1|}}{\inf\limits_{(s_1, s_2, s_3)} |q_1(s_1, s_2, s_3)|} \left(\frac{\delta}{2\pi}\right) \iint_0^{-\infty} \left\{\int_0^{-2/2\pi} |g_1(s_1, s_2, s_3)|^2 ds_1\right\} ds_2 ds_3$$

and

$$\iint_0^\infty |g_1(s_1, s_2, s_3)|^2 ds_2 ds_3 \leq \iint_{-\infty}^{+\infty} \left|\hat{f}\left(is_1, s_2 + i\frac{\delta}{2\pi}, s_3 + i\frac{\delta}{2\pi}\right)\right|^2 ds_2 ds_3$$

also using Plancherel's theorem on $(y_2, y_3) \rightarrow (s_2, s_3)$,

$$= \iint_{-\infty}^{+\infty} \left|\int_{-\infty}^{+\infty} e^{\delta y_2 + \delta y_3} e^{2\pi s_1 y_1} f(y) dy_1\right|^2 dy_2 dy_3$$

$$\leq \int_{-\infty}^{+\infty} e^{-2\varepsilon|y_1|} dy_1 \cdot \iiint |e^{\sqrt{3}\delta|y| + \varepsilon|y_1|} f(y)|^2 dy_1 dy_2 dy_3.$$

In fact, $|\delta y_1 + \delta y_2 + \delta y_3| \leq \sqrt{3}\delta|y|$.

Since

$$\inf_{\substack{(0\le s_1\le\delta/2\pi)\\(0\le s_2,s_3<\infty)}} |q_1(s_1, s_2, s_3, \lambda)|^2 = \{\sup_{\xi\in I_+\times J_+\times J_+} p(\xi, \lambda)^2\}^{-1}$$
$$\geq C(1+|\lambda|)^2(\delta+\mathrm{Re}\,\lambda)^2$$

from Lemma 4, ii), it follows that

$$\int |e^{-(\sqrt{3}\delta+\varepsilon)|x|} S(I_+, J_+, J_+)(\lambda) f|^2 dx$$
$$\leq \int_{-\infty}^{\infty} e^{-2(\delta+\varepsilon)x_1|} \left\{ \iint_{-\infty}^{+\infty} |e^{\delta x_2+\delta x_3} \cdot S(I_+, J_+, J_+)(\lambda) f|^2 dx_2 dx_3 \right\} dx_1$$
$$\leq C(1+|\lambda|)^{-2}(\delta+\mathrm{Re}\,\lambda)^{-2} \cdot \left\{ \int_{-\infty}^{+\infty} e^{-2\varepsilon|x_1|} dx_1 \right\}^2 \left(\frac{\delta}{2\pi}\right)^2$$
$$\times \int_{R^3} |e^{(\sqrt{3}\delta+\varepsilon)|x|} f(x)|^2 dx.$$

3° $$S(J_+, J_+, J_+)(\lambda) f \equiv \int_{J_+\times J_+\times J_+} \frac{e^{2\pi i x\xi}}{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2} \hat{f}(\xi) d\xi$$
$$= e^{-\delta(x_1+x_2+x_3)} \int_0^\infty \int_0^\infty \int_0^\infty e^{2\pi i(x_1 s_1+x_2 s_2+x_3 s_3)} \frac{g_2(s_1, s_2, s_3)}{q_2(s_1, s_2, s_3, \lambda)} ds_1 ds_2 ds_3$$

where

$$\begin{cases} g_2(s_1, s_2, s_3) = \hat{f}\left(s_1 + i\frac{\delta}{2\pi}, s_2 + i\frac{\delta}{2\pi}, s_3 + i\frac{\delta}{2\pi}\right) \\ q_2(s_1, s_2, s_3, \lambda) = 4\pi^2 \left\{ \left(s_1 + i\frac{\delta}{2\pi}\right)^2 + \left(s_2 + i\frac{\delta}{2\pi}\right)^2 + \left(s_3 + i\frac{\delta}{2\pi}\right)^2 \right\} + \lambda^2. \end{cases}$$

In the same manner with 2°, using Plancherel's theorem and Lemma 4, ii), we have

$$\int_{R^3} |e^{-(\sqrt{3}3\delta+\varepsilon)|x|} S(J_+, J_+, J_+)(\lambda) f|^2 dx$$
$$\leq \int_{R^3} e^{\delta(x_1+x_2+x_3)} S(J_+, J_+, J_+)(\lambda) f|^2 dx$$
$$\leq C(1+|\lambda|)^{-2}(\delta+\mathrm{Re}\,\lambda)^{-2} \int_{R^3} |e^{-\delta(x_1+x_2+x_3)} f(x)|^2 dx$$
$$\leq C(1+|\lambda|)^{-2}(\delta+\mathrm{Re}\,\lambda)^{-2} \int_{R^3} |e^{(\sqrt{3}\delta+\varepsilon)|x|} f(x)|^2 dx.$$

In the same manner we have the estimates of other $S(\cdots)(\lambda)f$, that is,

(3.7) $$\int |e^{-(\sqrt{3}+\varepsilon)|x|} S(\cdots)(\lambda) f|^2 dx$$
$$\leq C(1+|\lambda|)^{-2}(\delta+\mathrm{Re}|\lambda|)^{-2} \int |e^{(\sqrt{3}\delta+\varepsilon)|x|} f|^2 dx.$$

Thus, from (3.2), (3.4), (3.6) and (3.7) we can estimate $R(\lambda)f$ as following.

$$R(\lambda)f=\sum_{(i,j,k)}R_{(i,j,k)}(\lambda)f$$
$$=\sum_{(i,j,k)}\{\sum_{(K_i,K_j,K_k)}S(K_i,K_j,K_k)(\lambda)f\}.$$

Hence

$$\int_{R^3}|e^{-\gamma|x|}R(\lambda)f|^2dx$$
$$\leq\sum_{(i,j,k)}\{\sum_{(K_i,K_j,K_k)}\int_{R^3}|e^{-\gamma|x|}S(K_i,K_j,K_k)(\lambda)f|^2dx\}$$
$$\leq C(1+|\lambda|)^{-2}(\delta+\mathrm{Re}\,\lambda)^{-2}\int_{R^3}|e^{\gamma|x|}f(x)|^2dx.$$

Where $\lambda=\alpha+i\beta$, $-\delta<\alpha\leq N$, $\beta\geq N$ and $\gamma=\sqrt{3}\,\delta+\varepsilon>0$. If we take ε such that $\gamma\leq 2\delta$, we have the estimate of Lemma 1.

When $\mathrm{Im}\,\lambda\leq -N<0$, we can have the estimates in the same manner with the above case if we move lines of integration of the right hand side of (3.4) as following

$$[0,\infty)\to\left[0,\,-i\frac{\delta}{2\pi}\right]+\left[-i\frac{\delta}{2\pi},\,+\infty-i\frac{\delta}{2\pi}\right)$$
$$(-\infty,0]\to\left[0,\,+i\frac{\delta}{2\pi}\right]+\left[+i\frac{\delta}{2\pi},\,-\infty+i\frac{\delta}{2\pi}\right).$$

Thus we obtain the estimate i) of Lemma 1. We can prove the estimates ii), iii) and iv) of Lemma 1, using the fact that $p_1(\lambda)$, $p_2(\lambda)$ and $p_3(\lambda)$ satisfy the following inequalities,

$$\sup_{\xi\in K_i\times K_j\times K_k}|p_1(\lambda)|^2\leq C(\delta+\mathrm{Re}\,\lambda)^{-2}$$
$$\sup_{\xi\in K_i\times K_j\times K_k}|p_2(\lambda)|^2\leq C(1+|\lambda|)^2(\delta+\mathrm{Re}\,\lambda)^{-2}$$
$$\sup_{\xi\in K_i\times K_j\times K_k}|p_3(\lambda)|^2\leq C\frac{|h|^2|2\lambda+h|^2}{(1+|\lambda|)^4(\delta+\mathrm{Re}\,\lambda)^2(\delta+\mathrm{Re}\,\lambda+h)^2}$$

where

$$p_0(\lambda)=\{4\pi^2(\xi_1^2+\xi_2^2+\xi_3^2)+\lambda^2\}^{-1}$$
$$p_1(\lambda)=2\pi\xi_\nu p_0(\lambda),\quad p_2(\lambda)=4\pi^2\xi_\nu\cdot\xi_\mu p_0(\lambda)$$
$$p_3(\lambda)=p_0(\lambda+h)-p_0(\lambda)=-p_0(\lambda+h)p_0(\lambda)h(2\lambda+h).$$

The proof of Lemma 1 is complete.

2) **Proof of Lemma 2** (the case when $b(x)\equiv 0$, $c(x)\not\equiv 0$).

Lemma 5. *Let $c(x)$ be a bounded function with compact support and $c(x)\geq 0$. Let $u(x)$ satisfy that*

$$u(x)+\frac{1}{4\pi}\int_{E^3}\frac{e^{-\lambda|x-y|}}{|x-y|}c(y)u(y)dy=0, \quad u\in L^2_{loc}. \tag{3.8}$$

Then $u(x)\equiv 0$ in E^3 if $\mathrm{Re}\,\lambda\geq 0$.

Proof. From (3.8) and the assumption that $c(x)$ is a bounded function with compact support, we have the condition of $u(x)$ such that

$$\{-\Delta+\lambda^2+c(x)\}u(x)=0 \tag{3.9}$$

$$u(x)=0(|x|^{-1})e^{-\mathrm{Re}\lambda|x|}, \quad \frac{d}{dx}u(x)+\lambda u(x)=0(|x|^{-1})e^{-Re\lambda|x|}, \tag{3.10}$$

as $|x|\to\infty$. Let $\lambda=\alpha+i\beta$. When $\beta=0$, we have from (3.9) that

$$\begin{aligned} 0&=\int_{|x|\leq R}\{-\Delta+c(x)+\alpha^2\}u(x)\cdot\overline{u(x)}dx \\ &=\int_{|x|\leq R}|\mathrm{grad}\,u(x)|^2dx+\int_{|x|\leq R}c(x)|u(x)|^2dx \\ &\quad+\alpha^2\int_{|x|\leq R}|u(x)|^2dx-\int_{|x|=R}\left(\frac{d}{d|x|}u\right)\bar{u}ds. \end{aligned} \tag{3.11}$$

Since the last term of (3.11) vaishes when $R\to\infty$ and $c(x)\geq 0$, we have that

$$\int_{|x|\leq R}|\mathrm{grad}\,u(x)|^2dx=0, \text{ that is, } |\mathrm{grad}\,u(x)|=0.$$

This implies that $u(x)$ is constant. Thus $u(x)\equiv 0$ in E^3 from (3.10) if $\alpha\geq 0$. When $\alpha\beta\neq 0$, we have $u(x)\equiv 0$ since $2\alpha\beta\int|u(x)|^2dx=0$ which is the imaginary part of $\int\{-\Delta+\lambda^2+c(x)\}u(x)\cdot\overline{u(x)}dx$. The case when $\alpha=0$ and $\beta\neq 0$ is left. In this case it is sufficient to prove the two fact such that a) $u(x)$, which satisfies (3.8), is a function of L^2; b) if $(\Delta+\beta^2)u(x)=0$ at $|x|\geq R$ and $u(x)\in L^2(|x|\geq R)$, then $u(x)\equiv 0$ in $|x|\geq R$. In fact, if $u(x)\equiv 0$ in $|x|\geq R$, $u(x)\equiv 0$ in whole space E^3 from the unique continuation theorem of solutions of elliptic equation of second order (Refer to Eidus [3] or Povzner [4] for details of Lemma 5). q. e. d.

We now prove Lemma 2. Let $\mathcal{A}(\delta)$ be the Banach space of bounded operators from $Q(-\delta)$ to $Q(-\delta)$ and let $\mathcal{B}(\delta)$ be that from $Q(\delta)$ to $Q(-\delta)$. We denote the norm of each Banach space by

$$\|A\|_{\delta}\equiv\sup_{|f|_{-\delta}\leq 1}|Af|_{-\delta}, \qquad A\in\mathcal{A}(\delta)$$

$$\|B\|_{\delta}\equiv\sup_{|f|_{\delta}\leq 1}|Bf|_{-\delta}, \qquad B\in\mathcal{B}(\delta).$$

Let $u(x,\lambda)$ be a solution in L^2 of the equation $L_1(\lambda)u=f$ where $f\in L^2$ and $\mathrm{Re}\,\lambda>0$. Then $u(x,\lambda)$ is a solution of the integral equation

$$u(x,\lambda)+R(\lambda)\cdot c(x)\cdot u(x,\lambda)=R(\lambda)\cdot f(x)$$

which is obtained by operating $R(\lambda)$ to the both sides of $L_1(\lambda)u=f$, where $R(\lambda)$ is defined in Lemma 1 and $c(x)\cdot$ is an operator which multiplies $c(x)$. That is,

$$(3.12) \qquad \{I+R(\lambda)\cdot c(x)\cdot\}G_1(\lambda)=R(\lambda), \quad \text{at } \mathrm{Re}\,\lambda>0,$$

where $I+R(\lambda)\cdot c(x)$, $G_1(\lambda)$ and $R(\lambda)$ are bounded operators on L^2. To obtain the analytic continuation of $G_1(\lambda)$ we shall show that the equation of (3.12) can be solved at $\mathrm{Re}\,\lambda\leq 0$ if we consider $G_1(\lambda)$ as an element of $\mathcal{B}(\delta)$.

(3.13): *$R(\lambda)\in\mathcal{B}(\delta)$ and $R(\lambda)\cdot c(x)\cdot\in\mathcal{A}(\delta)$ are compact operators, that is, they map a bounded set to a pre-compact set, and they are analytic functions of λ, which value in $\mathcal{B}(\delta)$ and $\mathcal{A}(\delta)$, respectively, at* $\mathrm{Re}\,\lambda\geq-\delta'$ *for some* $\delta'>0$.

In fact an integral operator having a kernel $e^{-\delta|x|}\frac{e^{-\lambda|x-y|}}{|x-y|}e^{-\delta|y|}$, which is the Hilbert-Schmidt type, is a compact operator on L^2, $e^{+\delta|x|}$ is a bounded operator from $Q(\gamma)$ to $Q(\gamma-\delta)$, and $c(x)\cdot$ is a bounded operator from $Q(-\delta)$ to $Q(\delta)$ from the assumption that $c(x)$ is a bounded function with compact support. Hence we have that $R(\lambda)$ and $R(\lambda)\cdot c(x)\cdot$ are compact operators. The analyticity follows from Lemma 1 and the assumption for $c(x)\cdot$

(3.14): *$\{I+R(\lambda)\cdot c(x)\cdot\}$ has an inverse $\{I+R(\lambda)\cdot c(x)\cdot\}^{-1}\in\mathcal{A}(\delta)$ at* $\mathrm{Re}\,\lambda\geq 0$.

In fact, since $R(\lambda)\cdot c(x)\cdot$ is a compact operator on $Q(-\delta)$ from (3.13) it is sufficient to show that, if $\{I+R(\lambda)\cdot c(x)\cdot\}u=0$ and $u\in Q(-\delta)$, then $u=0$. This follows from Lemma 5. Let us put

$$R_1(\lambda)=\{I+R(\lambda)\cdot c(x)\cdot\}^{-1}.$$

We now assume that there exists the inverse $R_1(\lambda)$ at $\lambda=\lambda_0$. If we operate $R_1(\lambda_0)$ to the both sides of $\{I+R(\lambda)\cdot c(x)\cdot\}u=f$; u and $f\in Q(-\delta)$, we have

$$\{I+S(\lambda)\}u=R_1(\lambda_0)f, \text{ where } S(\lambda)=R_1(\lambda_0)\{R(\lambda)-R(\lambda_0)\}C(x):$$

Since $R(\lambda)\cdot c(x)\cdot$ is analytic at $\lambda=\lambda_0$, we have $\|S(\lambda)\|_\delta\leq 1/2$ in some neighborhood of $\lambda=\lambda_0$. Hence the Neumann series

$$\{I+S(\lambda)\}^{-1}=I+S(\lambda)+S(\lambda)^2+\cdots$$

uniformly converge in the neighborhood of $\lambda=\lambda_0$. This implies that $\{I+S(\lambda)\}^{-1}$ is analytic in the neighborhood of $\lambda=\lambda_0$ for $S(\lambda)$ is analytic there. Thus, there exists $R_1(\lambda)\equiv\{I+S(\lambda)\}^{-1}R_1(\lambda_0)$ which is analytic in the neighborhood of $\lambda=\lambda_0$. Considering (3.14) and above, we have that

(3.15): $R_1(\lambda)=\{I+R(\lambda)\cdot c(x)\cdot\}^{-1}$ *is an analytic function of* λ *at* $\mathrm{Re}\lambda\geq 0$ *which values in* $\mathcal{A}(\delta)$.

Let us estimate $R_1(\lambda)$. From Lemma 1, i) we have, at $\mathrm{Re}\geq-\delta/2$,

$$|R(\lambda)\cdot c(x)\cdot f(x)|_{-\delta}\leq C(1+|\lambda|)^{-1}(1+|\mathrm{Re}\lambda|)^{-1}\cdot|c(x)\cdot f(x)|_\delta$$
$$\leq C(1+|\lambda|)^{-1}(1+|\mathrm{Re}\lambda|)^{-1}\sup_x|c(x)\cdot e^{2\delta|x|}|\cdot|f(x)|_{-\delta},$$

that is,

$$\|R(\lambda)\cdot c(x)\cdot\|_\delta\leq C(1+|\lambda|)^{-1}(1+|\mathrm{Re}\lambda|)^{-1}.$$

Hence, since $\|R(\lambda)\cdot c(x)\cdot\|_\delta\leq 1/2$ when $|\lambda|$ is large enough and $\mathrm{Re}\lambda\geq-\delta/2$, there exists the inverse of $\{I+R(\lambda)\cdot c(x)\cdot\}$ by Neumann's series;

$$R_1(\lambda)=I+R(\lambda)\cdot c(x)\cdot+(R(\lambda)\cdot c(x)\cdot)^2+\cdots,$$

which implies that $R_1(\lambda)$ is analytic in λ and $\|R_1(\lambda)\|_\delta\leq$Const. in $\mathrm{Re}\lambda\geq-\delta/2$ and $|\lambda|\geq M$. Since (3.15) holds where $|\lambda|$ is bounded, we have that

(3.16): *$R_1(\lambda)$ can be continued to an analytic function in* $\mathrm{Re}\,\lambda \geq -\delta'$, $\delta' > 0$, *which value* $\mathcal{A}(\delta)$. *If we denote the extension also by* $R_1(\lambda)$, *we can estimate* $R_1(\lambda)$ *as following,*

$$\|R_1(\lambda)\|_\delta \leq \text{Const.} < \infty; \quad \mathrm{Re}\,\lambda \geq -\delta'.$$

Using (3.16) and Lemma 1, we conclude that

(3.17): *$G_1(\lambda)$ can be continued to an analytic function in* $\mathrm{Re}\,\lambda \geq -\delta'$, $\delta' > 0$, *which value* $\mathcal{B}(\delta)$ *and has the estimates as*

1° $|G_1(\lambda)|_\delta \leq \text{Const.}(1+|\lambda|)^{-1}(1+|\mathrm{Re}\,\lambda|)^{-1}$

2° $|G_1(\lambda) - G_1(\lambda+h)|_\delta \leq \text{Const.}|h|(1+|\lambda|)^{-1}(1+|\mathrm{Re}\,\lambda|)^{-1}$, $0 \leq \mathrm{Re}\,h < 1$, $|h| \leq 1$.

In fact, from (3.12) we have $G_1(\lambda) = R_1(\lambda) \cdot R(\lambda)$ at $\mathrm{Re}\,\lambda > 0$, the right hand side of which is analytic in $\mathrm{Re}\,\lambda \geq -\delta'$ from (3.16). This defines the continuation. Since $|G_1(\lambda)|_\delta \leq \|R_1(\lambda)\|_\delta \cdot |R(\lambda)|_\delta$, the estimate 1° of (3.17) follows from the estimates of (3.16) for $R_1(\lambda)$ and of Lemma 1, i) for $R(\lambda)$ which is $|R(\lambda)|_\delta \leq C(1+|\lambda|)^{-1}(1+|\mathrm{Re}\lambda|)^{-1}$.

Using (3.12), we have

$$\{I + R(\lambda) \cdot c(x) \cdot\} \cdot \{G_1(\lambda) - G_1(\lambda+h)\}$$
$$= \{R(\lambda+h) - R(\lambda)\} \cdot c(x) \cdot G_1(\lambda+h) - \{R(\lambda+h) - R(\lambda)\}.$$

Operating $R_1(\lambda) \equiv \{I + R(\lambda) \cdot c(x) \cdot\}^{-1}$ to the both sides of above equality,

$$|G_1(\lambda) - G_1(\lambda+h)|_\delta$$
$$\leq |R_1(\lambda)[\{R(\lambda+h) - R(\lambda)\} \cdot c(x) \cdot G_1(\lambda+h) - \{R(\lambda+h) - R(\lambda)\}]|_\delta$$
$$\leq \|R_1(\lambda)\|_\delta \cdot [\|\{R(\lambda+h) - R(\lambda)\} \cdot c(x)\|_\delta \cdot |G_1(\lambda+h)|_\delta$$
$$+ |\{R(\lambda+h) - R(\lambda)\}|_\delta].$$

Using the estimates of iv) of Lemma 1, of (3.16) and of 1° of (3.17) and the inequality;

$$\|\{R(\lambda+h) - R(\lambda)\} \cdot c(x)\|_\delta \leq \text{Const.}|R(\lambda+h) - R(\lambda)|_\delta,$$

we have the estimate 2° of (3.17), that is,

$$|G_1(\lambda) - G_1(\lambda+h)|_\delta$$
$$\leq |\{R(\lambda+h) - R(\lambda)\}|_\delta \{\text{Const.}\|R_1(\lambda)\|_\delta \cdot |G_1(\lambda+h)|_\delta + 1\}$$
$$\leq \text{Const.}|h|(1+|\lambda|)^{-1}(1+|\mathrm{Re}\,\lambda|)^{-1}; \quad 0 \leq \mathrm{Re}\,h, \ |h| \leq 1.$$

The proposition of (3.17) is nothing but Lemma 2. Since $R(\lambda)$ is a compact operator and $R_1(\lambda)$ is a bounded operator, $G_1(\lambda)=R_1(\lambda)\cdot R(\lambda)$ is also a compact operator.

The proof of Lemma 2 is complete.

3) **Proof of Lemma 3** (the case when $b(x)\not\equiv 0$).

From the assumption of Lemma 3, that is, $b(x)\geq 0$, $c(x)\geq 0$ are bounded functions with compact supports, we may put $b(x)=a^2(x)$, where $a(x)$ is a bounded real valued function with compact support, and there exists the Green operator $G_1(\lambda)$ of $L_1(\lambda)$ which satisfies Lemma 2.

Operating $G_1(\lambda)$ to the both sides of the following equality

$$L_2(\lambda)u\equiv\{-\Delta+\lambda^2+c(x)+\lambda b(x)\}u=f,$$
$$u\in D(L_2(\lambda)),\quad f\in L^2,\quad \mathrm{Re}\,\lambda>0$$

we have

$$\{I+\lambda G_1(\lambda)\cdot b(x)\cdot\}u(x)=G_1(\lambda)\cdot f(x)\equiv g(x). \tag{3.18}$$

From Lemma 2 $G_1(\lambda)\cdot b(x)\cdot$ is a compact operator on $Q(-\delta)$ at $\mathrm{Re}\,\lambda\geq-\delta'<0$ and $g(x)\in Q(-\delta)$ if $f(x)\in Q(\delta)$. In the same manner as in the proof of Lemma 2 we first prove that

(3.19): *There exists the inverse* $R_2(\lambda)\equiv\{I+\lambda G_1(\lambda)\cdot b(x)\cdot\}^{-1}\in\mathcal{A}(\delta)$ *in* $\mathrm{Re}\,\lambda\geq-\delta''$ *for some positive* δ'' *which is analytic there and satisfies that*

$$|||R_2(\lambda)|||_\delta\leq\text{Const.},$$

where $\mathcal{A}(\delta)$ *is the Banach space defined in the proof of Lemma 2 and* $|||\ |||_\delta$ *is the norm of* $\mathcal{A}(\delta)$.

It is clear from the analyticity of $\lambda G_1(\lambda)\cdot b(x)\cdot$ that $R_2(\lambda)$ is analytic where $R_2(\lambda)$ exists. To prove (3.19) we note that the following two problems are equivalent, which are to solve the equation;

$$\{I+\lambda G_1(\lambda)\cdot b(x)\cdot\}u(x)=g(x)\quad\text{in } Q(-\delta)$$

and to solve in L^2 the equation;

$$\{I+\lambda a(x)\cdot G_1(\lambda)\cdot a(x)\cdot\}v(x)=a(x)\cdot g(x)\equiv h(x).$$

Here the relations of $u(x)$ and $v(x)$ are given by

$$(3.20)\qquad \begin{cases} v(x)=a(x)\cdot u(x) \\ u(x)=-\lambda G_1(\lambda)\cdot a(x)\cdot v(x)+g(x). \end{cases}$$

We put

$$T_\lambda \equiv \lambda a(x)\cdot G_1(\lambda)\cdot a(x).$$

Since T_λ is a compact operator on L^2, there exists the inverse of $\{I+T_\lambda\}$ if we can show the estimate $\|v\|_{L^2}\leq C\|\{I+T_\lambda\}v\|_{L^2}$ for all $v\in L^2$. We show it when $\alpha>0$; $\lambda=\alpha+i\beta$. The proof of Lemma 5 gives that $L_1=-\Delta+c(x)$, $D(L_1)=\mathcal{D}^2_{L^2}$ is a positive definite selfadjoint operator, that is, there exists the resolution of the identity $E_\mu(0\leq\mu<\infty)$ such that

$$L_1 f=\int_0^\infty \mu dE_\mu f, \quad \text{for all } f\in D(L_1)=\mathcal{D}^2_{L^2}.$$

Hence we can write

$$G_1(\lambda)f=\int_0^\infty \frac{1}{\mu+\lambda^2}dE_\mu f.$$

Therefore

$$T_\lambda v=\lambda a(x)\cdot\int_0^\infty \frac{1}{\mu+\lambda^2}dE_\mu a(x)\cdot v.$$

If we denote the inner product of L^2 by $(\ ,\ )$,

$$\begin{aligned}\mathrm{Re}(T_\lambda v, v)&=\mathrm{Re}\Big(a(x)\cdot\int_0^\infty\frac{\lambda}{\mu+\lambda^2}dE_\mu a(x)\cdot v, v\Big)\\ &=\mathrm{Re}\int_0^\infty\frac{\lambda}{\mu+\lambda^2}d(E_\mu a(x)\cdot v, a(x)\cdot v)\\ &=\int_0^\infty\frac{(\mu+\alpha^2+\beta^2)\alpha}{(\mu+\alpha^2-\beta^2)^2+(2\alpha\beta)^2}d(E_\mu a(x)\cdot v, a(x)\cdot v);\\ &\quad \lambda=\alpha+i\beta,\ \alpha>0.\end{aligned}$$

Since $(E_\mu a(x)\cdot v, a(x)\cdot v)$ is a monotone increasing function and the integrand is a positive function from $\alpha>0$ and $\mu\geq 0$, it follows that

$$(3.21)\qquad \mathrm{Re}(T_\mu v, v)\geq 0,\quad \mathrm{Re}\,\lambda>0,\quad \text{for all } v\in L^2.$$

Applying it to the equality;

$$\|v^2_{L^2}\|+\mathrm{Re}(T_\lambda v, v)=\mathrm{Re}(\{I+T_\lambda\}v, v),$$

$$\|v\|_{L^2} \leq \| \{I + T_\lambda\} v\|_{L^2} \quad (\mathrm{Re}\,\lambda > 0). \tag{3.22}$$

The inequalities (3.21) and (3.22) hold at $\mathrm{Re}\,\lambda \geq 0$ since T_λ is continuous in λ (The method of the proof of (3.22) is the same one as in Lemma 3, 4 of Mizohata and Mochizuki [2]). We now study the case $\mathrm{Re}\,\lambda \leq 0$.

$$\begin{aligned} &\{T_\lambda - T_{\lambda-h}\} v \\ &= \{\lambda a(x) \cdot G_1(\lambda) \cdot a(x) \cdot - (\lambda - h) a(x) \cdot G_1(\lambda - h) \cdot a(x) \cdot \} v \\ &= [h \cdot a(x) \cdot G_1(\lambda) \cdot a(x) \cdot + (\lambda - h) \cdot a(x) \{G_1(\lambda) - G_1(\lambda - h)\} a(x)] v. \end{aligned} \tag{3.23}$$

Applying i) and ii) of Lemma 2 to (3.23) for sufficiently small $h \geq 0$ and $\mathrm{Re}\,\lambda - h \geq -\delta' < 0$, we have

$$\begin{aligned} &\| \{T_\lambda - T_{\lambda-h}\} v\|_{L^2} \\ &\leq |h| (\sup_x |a(x) e^{+\delta'|x|}|) \cdot |G_1(\lambda) a(x) v|_{-\delta} \\ &\quad + |\lambda - h| (\sup_x |a(x) e^{+\delta|x|}) | \{G_1(\lambda) - G_1(\lambda - h)\} a(x) v|_{-\delta} \\ &\leq \mathrm{Const.} |h| (1 + |\lambda|)^{-1} |a(x) \cdot v|_\delta \\ &\quad + \mathrm{Const.} |\lambda - h| \cdot (1 + |\lambda|)^{-1} \cdot |h| \cdot |a(x) \cdot v|_\delta \\ &\leq \mathrm{Const.} |h| \|v\|_{L^2}, \qquad \text{for } \mathrm{Re}\,\lambda \geq 0. \end{aligned} \tag{3.24}$$

On the other hand,

$$\|v\|_{L^2}^2 + \mathrm{Re}(T_\lambda v, v) = \mathrm{Re}(\{I + T_{\lambda-h}\} v, v) + \mathrm{Re}(\{T_\lambda - T_{\lambda-h}\} v, v).$$

Since $\mathrm{Re}(T_\lambda v, v) \geq 0$ from (3.21) when $\mathrm{Re}\,\lambda = 0$,

$$\|v\|_{L^2}^2 \leq \| \{I + T_{\lambda-h}\} v\|_{L^2} \|v\|_{L^2} + \| \{T_\lambda - T_{\lambda-h}\} v\|_{L^2} \|v\|_{L^2}.$$

Since $\| \{T_\lambda - T_{\lambda-h}\} v\|_{L^2} \leq (1/2) \|v\|_{L^2}$ for $|h| \leq \delta''$ from (3.24) if we choose sufficiently small $\delta'' > 0$, replacing $\lambda - h$ with λ, we have the estimate

$$\|v\|_{L^2} \leq 2 \| \{I + T_\lambda\} v\|_{L^2} \quad \text{at } \mathrm{Re}\,\lambda \geq -\delta'' < 0.$$

Thus there exists the inverse $(I + T_\lambda)^{-1}$ at $\mathrm{Re}\,\lambda \geq -\delta'' < 0$, which satisfies

$$\|(I + T_\lambda)^{-1} h(x)\|_{L^2} \leq 2 \|h(x)\|_{L^2}. \tag{3.25}$$

This implies the existence of the inverse of $\{I + \lambda G_1(\lambda) \cdot b(x) \cdot\}$ in $\mathcal{A}(\delta)$ at $\mathrm{Re}\,\lambda \geq -\delta'' < 0$ by (3.20). In fact

$$\{I+\lambda G_1(\lambda)\cdot b(x)\cdot\}^{-1}=-\lambda G_1(\lambda)\cdot a(x)\cdot(I+T_\lambda)^{-1}\cdot a(x)\cdot+I\in\mathcal{A}(\delta).$$

From (3. 25) we obtain the estimate of norm of this inverse, that is,

$$\begin{aligned}&|||\{I+\lambda G_1(\lambda)\cdot b(x)\cdot\}^{-1}|||_\delta\\&\leq|\lambda|\|G_1(\lambda)\|_\delta\cdot\sup_x|e^{\delta|x|}a(x)|\cdot\|\{I+T_\lambda\}^{-1}\|_{L^2}\cdot\sup_x|e^{\delta|x|}a(x)|+1\\&\leq\text{Const.}<\infty.\end{aligned}$$

This proves (3. 19).

When $\mathrm{Re}\lambda$ is sufficiently large, $G_2(\lambda)=\{I+\lambda G_1(\lambda)\cdot b(x)\}^{-1}\cdot G_1(\lambda)$ from (3. 18). The right hand side of this equality can be continued to an analytic function in $\mathrm{Re}\lambda\geq-\delta''<0$ from (3. 19) and Lemma 2. Applying the estimates of i) of Lemma 2 for $G_1(\lambda)$ and of (3. 19) the estimate of Lemma 3 for $G_2(\lambda)$ follows.

The proof of Lemma 3 is complete.

§4. Proof of Theorem 1

Let $u(x,t)$ be a solution for initial value problem (1. 1), (1. 2). And let $\tilde{u}(x,\lambda)$ be the Laplace image of $u(x,t)$ with respect to t in the sence of L^2. Then by the inversion formula of Laplace transform for a positive constant σ_0,

$$u(x,t)=\frac{1}{2\pi i}\lim_{\tau\to\infty}\int_{\sigma_0-i\tau}^{\sigma_0+i\tau}e^{\lambda t}\tilde{u}(x,\lambda)d\lambda,\quad\text{in } L^2.$$

And $\tilde{u}(x,\lambda)$ is a solution in L^2 for the equation that

$$\{-\Delta+c(x)+\lambda b(x)+\lambda^2\}\tilde{u}(x,\lambda)=\frac{f(x)}{\lambda-i\omega},$$

that is,

$$\tilde{u}(x,\lambda)=\frac{1}{\lambda-i\omega}G_2(\lambda)\cdot f(x).$$

If we regard $\tilde{u}(x,\lambda)$ as a function in $Q(-\delta)$, we can apply Lemma 3 to $\tilde{u}(x,\lambda)$. When $f(x)$ belongs to $Q(\delta)$, $\tilde{u}(x,\lambda)$ is analytic in $\mathrm{Re}\lambda\geq-\delta''<0$ except for one point $\lambda=i\omega$ which is a simple pole with the residue $G_2(i\omega)\cdot f(x)$ and, where $|\lambda|$ is large, $\tilde{u}(x,\lambda)$ satisfies the estimate

$$\|e^{-\delta|x|}\tilde{u}(x,\lambda)\|_{L^2}\leq\text{Const.}(1+|\lambda|^2)^{-1}\|e^{\delta|x|}f(x)\|_{L^2}.$$

This implies by means of Cauchy integral formula

$$(4.1)\qquad u(x,t)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}e^{\lambda t}\tilde{u}(x,\lambda)d\lambda,\quad \text{in } Q(-\delta)$$

$$=\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty}e^{\lambda t}\tilde{u}(x,\lambda)d\lambda+G_2(i\omega)\cdot f(x)e^{i\omega t},\quad \text{in } Q(-\delta),$$

where $\delta''\geq\varepsilon>0$.

In order to prove the rate of approach of Theorem 1 we require the stronger estimate of $G_2(\lambda)\cdot f(x)$ for λ under the assumption to $f(x)$ in Theorem 1.

Since $\Delta f\in Q(-\delta)$, we have

$$G_2(\lambda)\cdot f(x)=\frac{f(x)-g(x,\lambda)}{\lambda^2},$$

$$g(x,\lambda)=G_2(\lambda)\cdot[\{\lambda b(x)+c(x)\}f(x)+\Delta f(x)].$$

Applying Lemma 3, for sufficiently large $|\lambda|$,

$$(4.2)\qquad \|e^{-\delta|x|}G_2(\lambda)\cdot f(x)\|_{L^2}\leq\text{Const.}(|\lambda|+1)^{-2},\quad \text{at } \mathrm{Re}\lambda\geq-\delta''<0.$$

On the other hand, $V(x,\lambda)=G_2(\lambda)\cdot f(x)$ satisfies the equation

$$(4.3)\qquad V(x,\lambda)+\frac{1}{4\pi}\int\frac{e^{-\lambda|x-y|}}{|x-y|}c(y)V(y,\lambda)dy$$

$$+\frac{\lambda}{4\pi}\int\frac{e^{-\lambda|x-y|}}{|x-y|}b(y)V(y,\lambda)dy=\frac{1}{4\pi}\int\frac{e^{-\lambda|x-y|}}{|x-y|}f(y)dy.$$

This implies that

$$(4.4)\qquad \sup_{x\in K}|V(x,y)|$$

$$\leq\text{Const.}[\|e^{-\delta|x|}V(x,\lambda)\|_{L^2}+|\lambda|\,\|e^{-\delta|x|}V(x,y)\|_{L^2}$$

$$+(|\lambda|+1)^{-2}\|e^{\delta|x|}\{f(x)+\Delta f(x)\}\|_{L^2}],$$

where K is an arbitrary bounded set in E^3. In fact,

$$\sup_{x\in K}\left|\frac{1}{4\pi}\int\frac{e^{-\lambda|x-y|}}{|x-y|}p(y)dy\right|$$

$$\leq\left[\sup_{x\in K}\frac{1}{4\pi}\int\left|\frac{e^{-\mathrm{Re}\lambda|x-y|}e^{-\delta'|y|}}{|x-y|}\right|^2dy\right]\cdot\int|e^{\delta|y|}p(y)|^2dy$$

and

$$\frac{1}{4\pi}\int\frac{e^{-\lambda|x-y|}}{|x-y|}f(y)dy=\frac{1}{\lambda^2}\left\{f(x)-\frac{1}{4\pi}\int\frac{e^{-\lambda|x-y|}}{|x-y|}\Delta f(y)dy\right\}.$$

Applying (4. 2) to (4. 4), we have

$$\sup_{x\in K}|V(x,\lambda)|\leq \text{Const.}(|\lambda|+1)^{-1}, \quad \text{at } \operatorname{Re}\lambda\geq -\delta''<0.$$

Since $\tilde{u}(x,\lambda)=(\lambda-i\omega)^{-1}V(x,\lambda)$ and from (4. 1)

$$\begin{aligned}
&\sup_{x\in K}|u(x,t)-G_2(i\omega)\cdot f(x)e^{i\omega t}|\\
&\leq \sup_{x\in K}\left|\frac{1}{2\pi i}\int_{-\varepsilon-i\infty}^{-\varepsilon+i\infty} e^{\lambda t}\frac{V(x,\lambda)}{\lambda-i\omega}d\lambda\right|\\
&\leq \frac{1}{2\pi i}e^{-\varepsilon t}\int_{-i\infty}^{+i\infty}\{\sup_{x\in K}|V(x,\lambda-\varepsilon)|\}\left|\frac{1}{\lambda-\varepsilon-i\omega}\right|d\lambda\\
&\leq \text{Const.}e^{-\varepsilon t}\int_{-i\infty}^{+i\infty}(|\lambda|+1)^{-2}d\lambda, \quad \text{where } \varepsilon>0\\
&=0(e^{-\varepsilon t}), \quad (t\to\infty).
\end{aligned}$$

This proves the rate of approach of Theorem 1. Since $G_2(i\omega)\cdot f(x)=V(x)$ satisfies (4. 3) at $\lambda=i\omega$, it follows that $V(x)$ is a solution of (1. 3) and Sommerfeld's radiation conditions.

The proof of Theorem 1 is complete.

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