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On the theory of linear equations and Fredholm determinants

By

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Introduction

In the present paper we concern ourselves with the study of linear equations of types

$$u-T(z)u=f \tag{1}$$

and

$$T(z)u=f, \tag{2}$$

where z is a complex parameter. As is well-known, integral equations of type (1) have been studied succesfully by making use of the (classical) Fredholm determinants. One of the important results is the fact that the so called resolvent kernel is a meromorphic function of z.

To treat linear equations of type (1) in a Hilbert space, it is necessary to give another definition of the Fredholm determinants, for the original Fredholm's definition of determinants can not be used any more in an abstract theory. An extension of Fredholm's determinant theory has been carried out by several authors. Among recent works we mention Dunford-Schwartz [3], Gohberg-Krein [6] and Kuroda [9]. Especially the treatment in [6] is a very elegant one. The generalized Fredholm determinant $\det_p(1-zT)$ is defined for any compact operator T belonging to the class $\boldsymbol{C}_p$ of von Neumann and Schatten. This fact enables us to obtain an abstract version of the classical Fredholm theory.

The study of linear equations of type (2) requires somewhat

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different approach. Our method is essentially along the lines of Gohberg-Krein [5].

In §1 we give preliminaries to the whole study. The contents of this section is essentially included, e.g., in [3] except for Lemma 1. 1. In §2 we define the generalized Fredholm determinant $\det_\nu(1-zT)$ and derive some of its fundamental properties. Our treatment resembles in many points to that of [6]. However some results (for example, Lemma 2. 7 (ii) and Corollary (i) to Theorem 2. 5) are stated in more general forms than those which have appeared in the literature. In §3 we study linear equations of type (2). One of main results can be described as follows: let $\mathscr{B}_0(\mathfrak{H})$ be the set of all bounded Fredholm operators with index 0 in a Hilbert space $\mathfrak{H}$ and $T(z)$ be a $\mathscr{B}_0(\mathfrak{H})$-valued meromorphic function with finite rank singular parts.[1] Then $T(z)^{-1}$ (if exists) has the same property.

In the forthcoming papers, we shall study some related problems to this subject.

§1. Preliminaries

1. NOTATIONS. Let $\mathfrak{H}, \mathfrak{H}_1, \cdots$ be Hilbert spaces and the set of all closed linear operators T with domain $\mathscr{D}(T)\subset\mathfrak{H}$ and range $\mathscr{R}(T)\subset\mathfrak{H}_1$ be denoted by $\mathscr{L}(\mathfrak{H}, \mathfrak{H}_1)$. $\mathscr{L}_0(\mathfrak{H}, \mathfrak{H}_1)$ and $\mathscr{B}(\mathfrak{H}, \mathfrak{H}_1)$ denote the subset of $\mathscr{L}(\mathfrak{H}, \mathfrak{H}_1)$ consisting of all $T\in\mathscr{L}(\mathfrak{H}, \mathfrak{H}_1)$ with $\mathscr{D}(T)$ dense in $\mathfrak{H}$ and $\mathscr{D}(T)=\mathfrak{H}$, respectively. All $T\in\mathscr{B}(\mathfrak{H}, \mathfrak{H}_1)$ are bounded and the bound (norm) of T is denoted by $\|T\|$. $\mathscr{B}(\mathfrak{H}, \mathfrak{H}_1)$ is a Banach space with norm $\| \ \|$. For $T\in\mathscr{L}(\mathfrak{H}, \mathfrak{H}_1)$, $\mathfrak{N}(T)$ denotes the null space of T, $\mathfrak{N}(T)=\{u\in\mathscr{D}(T)\,;\ Tu=0\}$. $\mathfrak{N}(T)$ is a closed subspace of $\mathfrak{H}$. If $T\in\mathscr{L}_0(\mathfrak{H}, \mathfrak{H}_1)$, the adjoint operator T^* exists and belongs to $\mathscr{L}_0(\mathfrak{H}_1, \mathfrak{H})$, and $T^{**}=T$.

We call $T\in\mathscr{L}_0(\mathfrak{H}, \mathfrak{H}_1)$ a Fredholm operator from $\mathfrak{H}$ to $\mathfrak{H}_1$, if $\alpha(T)=\dim\mathfrak{N}(T)<\infty$, $\mathscr{R}(T)$ is closed, and $\beta(T)=\dim\mathscr{R}(T)^{\perp}<\infty$.[2] $\alpha(T)$ and $\beta(T)$ are called nullity and deficiency of T, respectively.

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1) See p. 442.

2) $\mathscr{R}(T)^{\perp}$ denotes the orthogonal complement of $\mathscr{R}(T)$ in $\mathfrak{H}_1$.

We define the index $\kappa(T)$ of T by $\kappa(T)=\alpha(T)-\beta(T)$. $\Phi(\mathfrak{H},\mathfrak{H}_1)$ denotes the set of all Fredholm operators from $\mathfrak{H}$ to $\mathfrak{H}_1$, and we put $\Phi_j(\mathfrak{H},\mathfrak{H}_1)=\{T\in\Phi(\mathfrak{H},\mathfrak{H}_1);\kappa(T)=j\}$, $j=0,\pm1,\cdots$. If $T\in\Phi_j(\mathfrak{H},\mathfrak{H}_1)$, then $T^*\in\Phi_{-j}(\mathfrak{H}_1,\mathfrak{H})$ with $\alpha(T)=\beta(T^*)$ and $\beta(T)=\alpha(T^*)$. We write $\mathcal{L}(\mathfrak{H})$ for $\mathcal{L}(\mathfrak{H},\mathfrak{H})$ and define similarly $\mathcal{L}_0(\mathfrak{H})$, $\mathcal{B}(\mathfrak{H})$, $\Phi(\mathfrak{H})$ and $\Phi_j(\mathfrak{H})$.

For $T\in\mathcal{L}_0(\mathfrak{H})$, we put $\Phi_j(T)=\{\lambda\in\boldsymbol{C};\lambda-T\in\Phi_j(\mathfrak{H})\}$ and $\Phi(T)=\{\lambda\in\boldsymbol{C};\lambda-T\in\Phi(\mathfrak{H})\}=\bigcup\Phi_j(T)$. The resolvent set, the spectrum, the set of all eigenvalues and the set of all isorated eigenvalues with finite multiplicities of T are dencted by $\rho(T)$, $\sigma(T)$, $\sigma_p(T)$ and $\sigma_d(T)$, respectively. We put $\bar{\rho}(T)=\rho(T)\cup\sigma_d(T)$ and $\sigma_e(T)=\Phi(T)^c$. $\sigma_e(T)$ is called the essential spectrum of T. Clearly $\bar{\rho}(T)\subset\Phi_0(T)$ and $\rho(T)=\{\lambda\in\Phi_0(T);\alpha(\lambda-T)=0\}$. $\rho(T)$, $\bar{\rho}(T)$, $\Phi_j(T)$ and $\Phi(T)$ are open. We denote by $n(\lambda;T)(\leq\infty)$ the algebraic multiplicity of $\lambda\in\rho_p(T)$, and by $\Sigma_p(T)$ (resp.$\Sigma_d(T)$) the set of $\lambda\in\sigma_p(T)$ (resp.$\sigma_d(T)$) each repeated $n(\lambda;T)$-times. We put $n(\lambda;T)=0$ for $\lambda\in\rho(T)$.

For a set M whose elements are complex numbers, M^0 denotes the set of all non-zero elements of M, and $M^*=\{\bar{\lambda};\lambda\in M\}$. For $T\in\mathcal{L}_0$ we have that $\sigma_e(T)^*=\sigma_e(T^*)$, $\Phi_j(T)^*=\Phi_j(T^*)$ $(j=0,\pm1,\cdots)$, $\rho(T)^*=\rho(T^*)$, $\sigma_d(T)^*=\sigma_d(T^*)$ and $\Sigma_d(T)^*=\Sigma_d(T^*)$. The following lemma will be sometimes usefull.

Lemma 1.1. *Let $T\in\mathcal{B}(\mathfrak{H},\mathfrak{H}_1)$ and $S\in\mathcal{B}(\mathfrak{H}_1,\mathfrak{H})$. Then, $\sigma_e(TS)^0=\sigma_e(ST)^0$ and $\Sigma_p(TS)^0=\Sigma_p(ST)^0$. For $\lambda\in\Phi(TS)^0=\Phi(ST)^0$, $\alpha(\lambda-TS)=\alpha(\lambda-ST)$ and $\beta(\lambda-TS)=\beta(\lambda-ST)$. Thus, $\Phi_j(TS)^0=\Phi_j(ST)^0$ $(j=0,\pm1,\cdots)$, $\rho(TS)^0=\rho(ST)^0$ and $\Sigma_d(TS)^0=\Sigma_d(ST)^0$.*

For the proof, see Shizuta [14].

If X is a Banach space and U is an open set in $\boldsymbol{C}^m$, we denote by $\mathcal{A}(U;X)$ and $\mathcal{M}(U;X)$ the set of all X-valued analytic functions defined in U and the set of all X-valued meromorphic functions in U, respectively. If U is an open set in R^m, we denote by $C^k(U;X)$ the set of all X-valued functions defined in U which are k-times continuously differentiable, $k=0,1,\cdots$. If X is the complex plane $\boldsymbol{C}$, we simply write $\mathcal{A}(U)$ for $\mathcal{A}(U;\boldsymbol{C})$, and define similarly $\mathcal{M}(U)$ and $C^k(U)$. If σ is an arbitrary set in $\boldsymbol{C}^m$, $\mathcal{A}(\sigma)$ denotes the set of all functions $f(\xi)$ which are analytic in some neighbourhood of σ. Let U be an open set in $\boldsymbol{C}$. For $f\in\mathcal{M}(U)$, the order of f at a point $a\in U$ is denoted by $n(a;f)$. In other words, if $f(z)=\sum_{j=n}^{\infty}a_j(z-a)^j$, $a_n\neq 0$, $-\infty<n<\infty$, is Laurent's expansion of $f(z)$ at $z=a$, then $n(a;f)=n$.

We denote by $C(a,r)$ a circle in the complex plane with center a and radius r.

Let $\mathfrak{H}$ be a finite dimensional Hilbert space and $S\in\mathcal{B}(\mathfrak{H})$. If $\{\varphi_j;\ j=1,\cdots,l\}$, $l=\dim\mathfrak{H}$, is a base of $\mathfrak{H}$, $S\varphi_j$ has the expansion: $S\varphi_j=\sum_{k=1}^{l}s_{kj}\varphi_k$, $j=1,\cdots,l$. We define $\det S$ by $\det S=\det(s_{jk})$. $\det S$ does not depened on the employed base and is determined only by S. It is called the (usual) determinant of S. If $\Sigma_p(S)=\{\lambda_j;\ j=1,\cdots,l\}$, then $\det S=\Pi\lambda_j$.

2. Classes of compact operators. We denote by $\boldsymbol{C}_\infty(\mathfrak{H},\mathfrak{H}_1)$ the set of all compact linear operators from $\mathfrak{H}$ to $\mathfrak{H}_1$. If $T\in\boldsymbol{C}_\infty(\mathfrak{H},\mathfrak{H}_1)$, $|T|=(T^*T)^{1/2}$ is a compact and non-negative self-adjoint operator in $\mathfrak{H}$. Let $\{\mu_j(T)\}=\Sigma_d(|T|)^0$ be the set of non-zero eigenvalues of $|T|$ numbered in the decreasing order. We say $T\in\boldsymbol{C}_p(\mathfrak{H},\mathfrak{H}_1)$, $0<p<\infty$, if $\|T\|_p=\{\Sigma\mu_j(T)^p\}^{1/p}<\infty$. Note that

$$\mu_j(T)=\min_{\{u_1\cdots,u_{j-1}\}}\ \max_{(u,u_1)=\cdots=(u_{j-1})=0}\|Tu\|, \tag{1.1}$$

where $\{u_1,\cdots,u_{j-1}\}$ runs over the set of all orthonormal systems consisting of $(j-1)$ vectors of $\mathfrak{H}$. We call $\mu_j(T)$ the j-th characteristic value of T. If we put $\|T\|_\infty=\sup\mu_j(T)=\mu_1(T)$, then we have

$$\|T\|_\infty=\|T\|\leq\|T\|_p,\qquad 0<p<\infty.$$

For $p<q$, $\boldsymbol{C}_p(\mathfrak{H},\mathfrak{H}_1)\subset\boldsymbol{C}_q(\mathfrak{H},\mathfrak{H}_1)$ with $\|T\|_q\leq\|T\|^{1-p/q}\|T\|_p^{p/q}$. If $T\in\boldsymbol{C}_p(\mathfrak{H},\mathfrak{H}_1)$ and $S\in\boldsymbol{C}_q(\mathfrak{H}_1,\mathfrak{H}_2)$, then $ST\in\boldsymbol{C}_r(\mathfrak{H},\mathfrak{H}_2)$ with $\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$ and

$$\|ST\|_r\leq 2^{1/r}\|S\|_q\|T\|_p,\qquad 0<r<\infty, \tag{1.2}$$

$$\|ST\|_r\leq\|S\|_q\|T\|_p,\qquad 1\leq r<\infty.\text{[3]} \tag{1.3}$$

For $1\leq p\leq\infty$, $\boldsymbol{C}_p(\mathfrak{H},\mathfrak{H}_1)$ is a Banach space with norm $\|\ \|_p$. Moreover,

3) This follows from (1.11) and Riesz convexity theorem.

if we denote by $\boldsymbol{C}_0(\mathfrak{H}, \mathfrak{H}_1)$ the set of all $T \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}_1)$ with finite dimensional $\mathscr{R}(T)$ ($\dim \mathscr{R}(T)$ is sometimes called the rank of T), $\boldsymbol{C}_p(\mathfrak{H}, \mathfrak{H}_1)$ is the completion of $\boldsymbol{C}_0(\mathfrak{H}, \mathfrak{H}_1)$ with norm $\| \ \|_p$. In fact, suppose that $T \in \boldsymbol{C}_p(\mathfrak{H}, \mathfrak{H}_1)$, $1 \leq p < \infty$. Then

$$|T| = \sum_{j=1}^{\infty} \mu_j(T) P_j ,$$

where $P_j = (\ , \varphi_j)\varphi_j$ and $\{\varphi_j\}$ is an orthonormal system (in $\mathfrak{H}$) consisting of eigenvectors of $|T|$. We put

$$E_n = \sum_{j=1}^{n} P_j , \qquad E_n' = 1 - E_n ,$$

$$T_n = T E_n . \tag{1.4}$$

Then we have

$$\begin{aligned}(T - T_n)^*(T - T_n) &= E_n' T^* T E_n' = E_n' |T|^2 E_n' \\ &= \sum_{j \geq n+1} \mu_j(T)^2 P_j .\end{aligned}$$

Hence

$$\| T - T_n \|_p = \{ \sum_{j \geq n+1} \mu_j(T)^p \}^{1/p} \to 0 \quad (n \to \infty).$$

The case $p = \infty$ is treated similarly. We call $\{T_n\}$ defined by (1.4) the standard approximation of $T \in \boldsymbol{C}_p(\mathfrak{H}, \mathfrak{H}_1)$. Note that $\Sigma_d(|T_n|)^0 = \{\mu_j(T_n)\} = \{\mu_j(T);\ j = 1, \cdots, n\}$.

Let $T \in \mathscr{B}(\mathfrak{H}, \mathfrak{H}_1)$. Then Lemma 1.1 shows that $\Sigma_p(T^*T)^0 = \Sigma_p(TT^*)^0$, i.e. $\Sigma_p(|T|^2)^0 = \Sigma_p(|T^*|^2)^0$. Hence $T \in \boldsymbol{C}_p(\mathfrak{H}, \mathfrak{H}_1)$ if and only if $T^* \in \boldsymbol{C}_p(\mathfrak{H}_1, \mathfrak{H})$ $(0 \leq p \leq \infty)$, and $\|T\|_p = \|T^*\|_p$ $(0 < p \leq \infty)$.

If $A \in \mathscr{B}(\mathfrak{H}_0, \mathfrak{H}_1)$, $T \in \boldsymbol{C}_p(\mathfrak{H}, \mathfrak{H}_1)$ and $B \in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H}_2)$, then (1.1) implies that $\mu_j(BT) \leq \|B\| \mu_j(T)$ and $\mu_j(TA) = \mu_j(A^* T^*) \leq \|A^*\| \mu(T^*) = \|A\| \mu_j(T)$. Hence $TA \in \boldsymbol{C}_p(\mathfrak{H}_0, \mathfrak{H}_1)$ and $BT \in \boldsymbol{C}_p(\mathfrak{H}_1, \mathfrak{H}_2)$ with

$$\begin{aligned}&\| TA \|_p \leq \| T \|_p \| A \| , \\ &\| BT \|_p \leq \| B \| \| T \|_p .\end{aligned} \tag{1.5}$$

Now we give the definition of the trace of $T \in \boldsymbol{C}_1(\mathfrak{H}) = \boldsymbol{C}_1(\mathfrak{H}, \mathfrak{H})$. Let $F = \{f_\alpha;\ \alpha \in \mathfrak{A}\}$ is a complete orthonormal system in $\mathfrak{H}$. We put

$$F(T) = \sum_{\alpha} (T f_\alpha, f_\alpha).$$

Since $\mathfrak{R}(T)^{a}$ [4] is a separable subspace of $\mathfrak{H}$, (Tf_α, f_α) vanishes except for at most countable numbers of α, and that $T\in \boldsymbol{C}_1(\mathfrak{H})$ ensures the absolute convergence of the above series:

$$\sum_{\alpha} |(Tf_\alpha, f_\alpha)| \leq \Sigma \mu_j(T) = \|T\|_1 .$$

This shows that $F(T)$ is a continuous linear form on the Banach space $\boldsymbol{C}_1(\mathfrak{H})$. Let $S\in \boldsymbol{C}_1(\mathfrak{H}, \mathfrak{H}_1)$, $A\in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H})$ and G be a complete orthonormal system in $\mathfrak{H}_1$. Then $F(AS)=G(SA)$. Hence, if $U\in \mathscr{B}(\mathfrak{H})$ is unitary, we have $F(U^*TU)=F(UU^*T)=F(T)$. This shows that $F(T)$ depends only on T and nct on F employed. Thus we define $\mathrm{tr}(T)$ by

$$\mathrm{tr}(T)=F(T).$$

Obviously we have

$$|\mathrm{tr}(T)| \leq \|T\|_1, \quad \mathrm{tr}(AS)=\mathrm{tr}(SA), \tag{1.6}$$

for $T\in \boldsymbol{C}_1(\mathfrak{H})$, $S\in \boldsymbol{C}_1(\mathfrak{H}, \mathfrak{H}_1)$ and $A\in \mathscr{B}(\mathfrak{H}_1, \mathfrak{H})$.

Let $T\in \boldsymbol{C}_1(\mathfrak{H})$ and P be a (not necessarily orthogonal) projection such that $\mathfrak{R}(P)\supset \mathfrak{R}(T)$. Then

$$\mathrm{tr}(T)=\mathrm{tr}(PT)=\mathrm{tr}(PTP). \tag{1.7}$$

Let $T\in \boldsymbol{C}_0(\mathfrak{H})$ and P be a projection such that $\mathfrak{R}(P)\supset \mathfrak{R}(T)$. Then

$$\Sigma_d(T)^0=\Sigma_d(PTP)^0, \tag{1.8}$$

$$\mathrm{tr}(T)=\sum_{\lambda\in\Sigma_d(PTP)^0} \lambda = \sum_{\lambda\in\Sigma_d(T)^0} \lambda. \tag{1.9}$$

Hence, if $P\in \boldsymbol{C}_0(\mathfrak{H})$ is a projection and $N\in \boldsymbol{C}^0(\mathfrak{H})$ is nilpotent, i.e., $N^l=0$ for an integer l, we have

$$\mathrm{tr}(P)=\dim \mathfrak{R}(P), \quad \mathrm{tr}(N)=0. \tag{1.10}$$

One of important properties of trace is the following duality: if $T\in \boldsymbol{C}_p(\mathfrak{H}, \mathfrak{H}_1)$, $1\leq p<\infty$, then

4) $\mathfrak{R}(T)^a$ denotes the closure of $\mathfrak{R}(T)$.

$$\|T\|_p = \sup_{\substack{S\in \boldsymbol{C}_q(\mathfrak{H}_1,\mathfrak{H}) \\ \|S\|_q=1}} |\mathrm{tr}(ST)|, \tag{1.11}$$

where $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $q=\infty$ is permitted.

3. Operator calculus. Let $T\in\mathcal{B}(\mathfrak{H})$. A subset σ of $\sigma(T)$ is called a spectral set, if σ is both open and closed in $\sigma(T)$. Let $f\in\mathcal{A}(\sigma)$, and let D be an open set in $\boldsymbol{C}$ whose boundary γ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense customary in the theory of complex variables. We call such an open set D J-type. Suppose that $D\supset\sigma$ and $D^a\cap\{\sigma(T)-\sigma\}=\phi$, and that D^a is contained in the domain of analyticity of f. Then we can define the operator $f(T)_\sigma$[5] $\in\mathcal{B}(\mathfrak{H})$ by the following formula:

$$f(T)_\sigma=\frac{1}{2\pi i}\int_\gamma f(\xi)(\xi-T)^{-1}d\xi. \tag{1.12}$$

Since $(\xi-T)^{-1}\in\mathcal{A}(\rho(T);\mathcal{B}(\mathfrak{H}))$, it follows from Cauchy integral theorem that $f(T)_\sigma$ depends only on σ and f, and not on D (or γ). If σ_1 and σ_2 are two spectral sets of $\sigma(T)$, $f\in\mathcal{A}(\sigma_1)$ and $g\in\mathcal{A}(\sigma_2)$, then $\sigma_1\cap\sigma_2$ is also a spectral set of $\sigma(T)$, and

$$f(T)_{\sigma_1}g(T)_{\sigma_2}=(f\cdot g)(T)_{\sigma_1\cap\sigma_2}. \tag{1.13}$$

For $f(\xi)\equiv 1$ and $g(\xi)\equiv\xi$, we put $f(T)_\sigma=P(\sigma;T)=P_\sigma$ and $g(T)_\sigma=T_\sigma$. Clearly P_σ is a projection, and $P_\sigma T=TP_\sigma=T_\sigma$. If we put $\sigma'=\sigma(T)-\sigma$, then $P_{\sigma'}=1-P_\sigma$. Hence, putting $\mathfrak{H}_\sigma=\mathcal{R}(P_\sigma)$ and $\mathfrak{H}_{\sigma'}=\mathcal{R}(P_{\sigma'})$, we have the following decomposition of $\mathfrak{H}$ and T:

$$\mathfrak{H}=\mathfrak{H}_\sigma\dot{+}\mathfrak{H}_{\sigma'} \quad (\textit{direct sum}),$$
$$T=T_\sigma\dot{+}T_{\sigma'}.$$

In the above decomposition we denote by T_σ and $T_{\sigma'}$ the restriction of T in $\mathfrak{H}_\sigma$ and $\mathfrak{H}_{\sigma'}$, respectively. This convention will cause no confusion. For $\xi\in\rho(T)$ we have

$$\begin{aligned}(\xi-T)^{-1}&=(\xi-T)^{-1}P_\sigma+(\xi-T)^{-1}P_{\sigma'}\\&=(\xi-T_\sigma)^{-1}\dot{+}(\xi-T_{\sigma'})^{-1}.\end{aligned}$$

5) We define $f(T)_\phi=0$ for empty set ϕ for the sake of convenience.

Considering analytic continuation of the resolvents of T_σ and $T_{\sigma'}$, we see that $\sigma(T_\sigma)=\sigma$ and $\sigma(T_{\sigma'})=\sigma'$. If T_σ is considered as an operator in $\mathfrak{H}$, we have $\sigma(T_\sigma)=\sigma\cup\{0\}$ (if $\sigma'\neq\phi$) and

$$(1.14)\qquad (\xi-T_\sigma)^{-1}=(\xi-T)^{-1}P_\sigma+\xi^{-1}P_{\sigma'}.$$

We write $f(T)$ for $f(T)_{\sigma(T)}$. Clearly $T_{\sigma(T)}=T$ and $P(\sigma(T);T)=1$. From (1.13) and (1.14), we obtain

$$(1.15)\qquad \begin{aligned} &f(T)P_\sigma=P_\sigma f(T)=f(T)_\sigma,\\ &f(T_\sigma)=f(T)_\sigma+f(0)P_{\sigma'} \end{aligned}$$

for any spectral set σ of $\sigma(T)$.

If $\lambda\in\sigma_d(T)$, we write $P(\lambda;T)$ for $P(\{\lambda\};T)$. Clearly $n(\lambda;T)=\dim\mathcal{R}(P(\lambda;T))=\mathrm{tr}(P(\lambda;T))$, $\lambda\in\sigma_d(T)$. Note that Laurent's expansion of $(\xi-T)^{-1}$ at $\xi=\lambda\in\sigma_d(T)$ has the form

$$(1.16)\qquad \begin{aligned}(\xi-T)^{-1}&=(\xi-T)^{-1}P(\lambda;T)+(\xi-T)^{-1}P(\{\lambda\}';T)\\ &=\frac{N^{l-1}}{(\xi-\lambda)^l}+\cdots+\frac{N}{(\xi-\lambda)^2}+\frac{P}{\xi-\lambda}+(\xi-T_{\{\lambda\}'})^{-1}P(\{\lambda\}';T),\end{aligned}$$

where $P=P(\lambda;T)$, and $N=(T-\lambda)P(\lambda;T)$ with $\mathcal{R}(N)\subset\mathcal{R}(P)$ and $N^l=0$, $0<l\leq n(\lambda;T)$.

If $T\in\mathcal{B}(\mathfrak{H})$ and $f\in\mathcal{A}(\sigma(T))$, we have

$$(1.17)\qquad \sigma(f(T))=f(\sigma(T)).$$

This is so called spectral mapping theorem. Let $f\in\mathcal{A}(\sigma(T))$, and $g\in\mathcal{A}(\sigma(f(T)))$ and $(g\circ f)(\xi)=g(f(\xi))$. Then $g\circ f\in\mathcal{A}(\sigma(T))$ and

$$(1.18)\qquad g(f(T))=g\circ f(T).$$

More generally, if σ is a spectral set of $\sigma(f(T))$, then $f^{-1}(\sigma)\cap\sigma(T)$ is a spectral set of $\sigma(T)$, and

$$(1.18')\qquad g(f(T))_\sigma=(g\circ f)(T)_{f^{-1}(\sigma)\cap\sigma(T)}.$$

Let $T\in\boldsymbol{C}_p(\mathfrak{H})$, $0\leq p\leq\infty$, and $f\in\mathcal{A}(\sigma(T))$. Suppose that $f(0)=0$. Then $g(\xi)=f(\xi)/\xi\in\mathcal{A}(\sigma(T))$ and $f(T)=g(T)\in\boldsymbol{C}_p(\mathfrak{H})$. Under this situation, we have the following spectral mapping property of another type:

$$\Sigma_d(f(T))=f(\Sigma_d(T)) \qquad (\text{if } \dim\mathfrak{H}<\infty),$$
(1.19)
$$\Sigma_d(f(T))=f(\Sigma_d(T))^0 \qquad (\text{if } \dim\mathfrak{H}=\infty).$$

In fact, if $\lambda\in\sigma_d(f(T))$, $\sigma=f^{-1}(\{\lambda\})\cap\sigma(T)$ is a spectral set of $\sigma(T)$ and a finite set of points in $\sigma_d(T)$: $\sigma=\{\xi_1,\cdots,\xi_n\}$. By (1.18′) we have $P(\lambda; f(T))=P(\sigma; T)=\Sigma P(\xi_j; T)$. Hence

$$n(\lambda; f(T))=\mathrm{tr}(P(\lambda; f(T)))=\Sigma\,\mathrm{tr}(P(\xi_j; T))=\Sigma n(\xi_j; T).$$

Lemma 1.2. *Let D be an open set of J-type and $f\in\mathcal{A}(D)$.*

(i) *Let U be an open set in R^m and $T(x)\in C^k(U; \mathcal{B}(\mathfrak{H}))$. Suppose that $\sigma(T(x))$ is divided into two spectral sets $\sigma(x)$ and $\sigma(x)'$ so that $\sigma(x)\subset D$ and $\sigma(x)'\cap D^a=\phi$. Then $f(T(x))_{\sigma(x)}\in C^k(U; \mathcal{B}(\mathfrak{H}))$.*

(ii) *Let U be an open set in $\boldsymbol{C}^m$ and $T(z)\in\mathcal{A}(U; \mathcal{B}(\mathfrak{H}))$. Suppose that $\sigma(T(z))$ is divided into two spectral sets $\sigma(z)$ and $\sigma(z)'$ so that $\sigma(z)\subset D$ and $\sigma(z)'\cap D^a=\phi$. Then $f(T(z))_{\sigma(z)}\in\mathcal{A}(U; \mathcal{B}(\mathfrak{H}))$.*

(iii) *Let U and $T(z)$ be as in* (ii). *Suppose that $T_j(z)\in\mathcal{A}(U; \mathcal{B}(\mathfrak{H}))$, $j=1,2,\cdots$, and $T_j(z)\to T(z)$ in $\mathcal{B}(\mathfrak{H})$ uniformly in U. Then each $\sigma(T_j(z))$ is divided into two spectral sets $\sigma_j(z)$ and $\sigma_j(z)'$ so that $\sigma_j(z)\subset D$, $\sigma_j(z)'\cap D^a=\phi$ and $\sigma_j(z)\to\sigma(z)$ as $j\to\infty$,[6] and $f(T_j(z))_{\sigma_j(z)}\to f(T(z))_{\sigma(z)}$ in $\mathcal{B}(\mathfrak{H})$ uniformly in U.*

4. Some lemmas. We define a subclass $\mathcal{A}_0(U; \boldsymbol{C}_0(\mathfrak{H}))$ of $\mathcal{A}(U; \mathcal{B}(\mathfrak{H}))$ and the corresponding subclasses $\mathcal{M}_0(U; \boldsymbol{C}_0(\mathfrak{H}))$ and $C_0^k(U; \boldsymbol{C}_0(\mathfrak{H}))$, through $\boldsymbol{C}_0(\mathfrak{H})$ is not a Banach space. By $T(z)\in\mathcal{A}_0(U; \boldsymbol{C}_0(\mathfrak{H}))$ we mean that i) $T(z)\in\mathcal{A}(U; \mathcal{B}(\mathfrak{H}))$, ii) for each $a\in U$ there exists a neighbourhood $U(a)$ of a so that in $U(a)$ $T(z)$ has the expansion of the form

$$T(z)=\sum_{j=1}^{k}(\ ,\chi_j(z))\varphi_j(z)=\sum_{j=1}^{k}\varphi_j(z)\chi_j(z)^*, \tag{1.20}$$

where $\varphi_j(z)\in\mathcal{A}(U(a); \mathfrak{H})$ and $\chi_j(z)^*=(\ ,\chi_j(z))\in\mathcal{A}(U(a); \mathfrak{H}^*)$, $\mathfrak{H}^*=\mathcal{B}(\mathfrak{H}, \boldsymbol{C})$, and iii) $\{\varphi_j(z)\}$ are linearly independent for $z\in U(a)$. $\mathcal{M}_0(U; \boldsymbol{C}_0(\mathfrak{H}))$ and $C_0^k(U; \boldsymbol{C}_0(\mathfrak{H}))$ are defined similarly. However, for

6) This means that $\sup\limits_{\lambda\in\sigma(z),\lambda'\in\sigma_j(z)}|\lambda-\lambda'|\to 0$ as $j\to\infty$.

$T(z)\in\mathcal{M}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$ we always assume that in the expansion (1.20) $\varphi_j(z)\in\mathcal{A}(U(a);\ \mathfrak{H})$ and $\chi_j(z)^*\in\mathcal{M}(U(a);\ \mathfrak{H}^*)$.

Lemma 1.3. *$\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))\subset\mathcal{A}(U;\ \boldsymbol{C}_1(\mathfrak{H}))$, and $\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))\times\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))\subset\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$. The same inclusion relations hold if $\mathcal{A}$ is replaced by $\mathcal{M}$ or C^k. If $T(z)\in\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))$ and $T(z)^{-1}\in\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))$ exists, then $T(z)\times\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))\subset\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$ and $T(z)\times\mathcal{M}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))\subset\mathcal{M}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$.*

Lemma 1.4. (i) *Suppose $T(z)\in\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))$. Then $T(z)\in\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$ if and only if for each $a\in U$ a neighbourhood $U(a)$ and $P(z)\in\mathcal{A}(U(a);\ \mathcal{B}(\mathfrak{H}))$ are attached so that $P(z)\in\boldsymbol{C}_0(\mathfrak{H})$, $P(z)$ is a (not necessarily orthogonal) projection (i.e. $P(z)^2=P(z)$) and $\mathcal{R}(T(z))\subset\mathcal{R}(P(z))$ for each $z\in U(a)$.*

(ii) *Let U be a domain in $\boldsymbol{C}^m$. Suppose $P(z)\in\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))$ and $P(z)$ is a projection for each $z\in U$. Then $P(z)\in\mathcal{A}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$ if and only if $T(z)\in\boldsymbol{C}_0(\mathfrak{H})$ for some (or equivalently any) $z\in U$.*

(iii) *If we replace $\mathcal{A}$ by $\mathcal{M}$ or C^k, the assertions of* (i) *and* (ii) *hold in the corresponding sense.*

Proof. First we prove the if part of (ii). We note that if P_0 and P_1 are projections and $\|P_0-P_1\|<1$, then $\dim\mathcal{R}(P_0)=\dim\mathcal{R}(P_1)$. Therefore we have $\dim\mathcal{R}(P(z))=\text{const}=k<\infty$ for $z\in U$. Take $a\in U$ and a neighbourhood $U(a)$ of a so that $\|P(z)-P(a)\|<1$ for $z\in U(a)$. Take a base $\{\varphi_i;\ i=1,\cdots,k\}$ of $\mathcal{R}(P(a))$. Since $\|P(z)-P(a)\|<1$, $\{P(z)\varphi_i\}$ are linearly independent and form a base of $\mathcal{R}(P(z))$, $z\in U(a)$. We put

$$\varphi_i(z)=P(z)\varphi_i\,,\qquad i=1,\cdots,k. \tag{1.21}$$

We take a biorthogonal base $\{\psi_j;\ j=1,\cdots,k\}$ for $\{\varphi_i(a)=\varphi_i\}$: $(\varphi_i,\psi_j)=\delta_{ij}$, and put

$$P_{ij}(z)=(\varphi_i(z),\psi_j),\qquad i,j=1,\cdots,k. \tag{1.22}$$

Clearly $p_{ij}(z)\in\mathcal{A}(U(a))$ and $p_{ij}(a)=\delta_{ij}$. Hence if we take a sufficiently small neighbourhood $V(a)$ of a, the inverse matrix $(q_{ij}(z))=(p_{ij}(z))^{-1}$ of $(p_{ij}(z))$ exists, and each $q_{ij}(z)\in\mathcal{A}(V(a))$. We put

$$\psi_j(z)=\sum_{i=1}^{k}\overline{q_{ij}(z)\psi_i}, \qquad j=1,\cdots,k. \tag{1.23}$$

Then by (1.21), (1.22) and (1.23), $\varphi_j(z)\in\mathcal{A}(V(a);\mathfrak{H})$, $\psi_j(z)^*\in\mathcal{A}(V(a);\mathfrak{H}^*)$ and $(\varphi_i(z),\psi_j(z))=\delta_{ij}$ for $z\in V(a)$. Thus we have

$$P(z)=\sum_{j=1}^{k}(\ ,\psi_j(z))\varphi_j(z)=\Sigma\varphi_j(z)\psi_j(z)^*, \quad z\in V(a), \tag{1.24}$$

and hence $P(z)\in\mathcal{A}_0(U;\boldsymbol{C}_0(\mathfrak{H}))$.

The if part of (i) follows easily from (ii) and Lemma 1.3, since $P(z)T(z)=T(z)$. To prove the only if part of (i), we have only to construct a biorthogonal base $\{\psi_j(z)\}$ of $\{\varphi_j(z)\}$ of (1.20) in a similar way to (1.22)—(1.23) and define $P(z)$ by (1.24).

Note that if we put

$$t_{ij}(z)=(T(z)\varphi_j(z),\quad \psi_i(z))=(\varphi_j(z),\chi_i(z)),$$

then we have

$$T(z)P(z)=\Sigma\varphi_i(z)t_{ij}(z)\psi_j(z)^*.$$

The proof of (iii) is carried over similarly.

Lemma 1.5. *Let $P_n(z)\in\mathcal{A}(U;\mathcal{B}(\mathfrak{H}))$, $n=0,1,\cdots$, and each $P_n(z)$ is a projection for $z\in U$. Suppose that $P_n(z)\in\boldsymbol{C}_0(\mathfrak{H})$ and $P_n(z)\to P_0(z)$ $(n\to\infty)$ in $\mathcal{B}(\mathfrak{H})$ uniformly on any compact set of U. Then $P_n(z)\in\mathcal{A}_0(U;\boldsymbol{C}_0(\mathfrak{H}))$ and $P_n(z)\to P_0(z)$ $(n\to\infty)$ in $\boldsymbol{C}_1(\mathfrak{H})$ uniformly on any compact set of U.*

The proof is quite similar to that of Lemma 1.4 (ii).

§2. Generalized Fredholm determinants

1. Definition of the generalized Fredholm determinant of order ν. Let $T\in\boldsymbol{C}_p(\mathfrak{H})$, $0<p<\infty$ and let ν be an integer not smaller than p. Set

$$\begin{aligned}\sigma_\nu(\lambda;\,T)&=-\operatorname{tr}\frac{1}{\lambda}\{(1-\lambda)^{-1}-(1+\lambda T+\cdots+\lambda^{\nu-1}T^{\nu-1})\}\\&=-\operatorname{tr}\{\lambda^{\nu-1}T^\nu(1-\lambda T)^{-1}\}.\end{aligned}$$

Since $T^\nu\in\boldsymbol{C}_{p/\nu}(\mathfrak{H})\subset\boldsymbol{C}_1(\mathfrak{H})$, $\sigma_\nu(\lambda;\,T)$ is a well-defined function on $\boldsymbol{C}$.

We note that $\sigma_\nu(\lambda;\ T)\in\mathcal{M}(\boldsymbol{C})$. The only singularities of $\sigma_\nu(\lambda;\ T)$ are simple poles at $\lambda_i^{-1}(\{\lambda_i\}=\sigma_d(T))$ and residues are positive integers $n(\lambda_i;\ T)$ by (1.16) and (1.10). We define the function $\delta_\nu(\lambda;\ T)$ on $\boldsymbol{C}$ by

$$\frac{\delta_\nu'(\lambda;\ T)}{\delta_\nu(\lambda;\ T)}=\sigma_\nu(\lambda;\ T), \tag{2.1}$$

$$\delta_\nu(0;\ T)=1. \tag{2.2}$$

Thus defined entire function $\delta_\nu(\lambda;\ T)$ is called the generalized Fredholm determinant of order ν associated with T. When there is no possibility of confusion we write simply $\delta_\nu(\lambda)$ instead of $\delta_\nu(\lambda;\ T)$.

Theorem 2.1. *Let $T\in\boldsymbol{C}_p(\mathfrak{H})$, $0<p<\infty$ and let ν be an integer not smaller than p. Then $\delta_\nu(\lambda;\ T)$ defined as above is an entire function of λ. We have $\delta_\nu(\lambda;\ T)\neq 0$ if and only if $1\in\rho(\lambda T)$. Hence $\delta_\nu(\lambda;\ T)=0$ if and only if $\lambda^{-1}\in\sigma_d(T)$. Moreover we have $n(\lambda^{-1};\ T)=n(\lambda;\ \delta_\nu(\cdot\,;\ T))$, $\lambda\in\boldsymbol{C}$, and hence $\delta_\nu(\lambda;\ T)(1-\lambda T)^{-1}\in\mathcal{A}(\boldsymbol{C};\ \mathcal{B}(\mathfrak{H}))$.*

2. Explicit form of the generalized Fredholm determinant. Since our definition of $\delta_\nu(\lambda;\ T)$ is implicit, it is desirable to write it in a concrete form.

Let $T\in\boldsymbol{C}_p(\mathfrak{H})$, $0<p<\infty$. For the moment we assume that $1\in\rho(T)$. Let D be an open set of J-type containing $\sigma(T)$. We further assume that $0\in D$ and $1\notin D^a$. Define the single-valued analytic function $\mathrm{Log}(1-\xi)$ on D as follows. On the component of D containing 0, $\mathrm{Log}(1-\xi)$ is taken to be on the branch of $\log(1-\xi)$ such that $\log 1=0$. Denoting the components of D not containing 0 by D_k $(k=1, 2, \cdots, n)$, $\mathrm{Log}(1-\xi)$ on D_k is taken to be on an arbitrary branch of $\log(1-\xi)$ depending on k. Put

$$f_\nu(\xi)=\mathrm{Log}(1-\xi)+\left(\xi+\cdots+\frac{1}{\nu-1}\xi^{\nu-1}\right) \tag{2.3}$$

and

$$g_\nu(\xi)=\frac{1}{\xi^\nu}f_\nu(\xi). \tag{2.4}$$

Noting that $\mathrm{Log}(1-\xi)$, $f_\nu(\xi)$ and $g_\nu(\xi)\in\mathcal{A}(\sigma(T))$, we have

(2.5) $$f_\nu(T)=\mathrm{Log}(1-T)+\left(T+\cdots+\frac{1}{\nu-1}T^{\nu-1}\right)$$

and

(2.6) $$f_\nu(T)=T^\nu g_\nu(T)=g_\nu(T)T^\nu.$$

If $\nu\geq p$, then $f_\nu(T)\in \boldsymbol{C}_1(\mathfrak{H})$, and it is easily seen that $\exp[\mathrm{tr}\{f_\nu(T)\}]$ does not depend on the arbitrariness in defining $\mathrm{Log}(1-\xi)$. In fact, let $\widetilde{\mathrm{Log}}(1-\xi)$ and $\tilde{f}_\nu(\xi)$ be corresponding to another choice of the branches of $\log(1-\xi)$. Then we have

$$\mathrm{Log}(1-T)-\widetilde{\mathrm{Log}}(1-T)=\Sigma' 2\pi i m_j P(\lambda_j;\ T),$$

where Σ' is a finite sum and m_j's are integers. Hence

$$\mathrm{tr}\{f_\nu(T)\}-\mathrm{tr}\{\tilde{f}_\nu(T)\}=\Sigma' 2\pi i m_j n(\lambda_j;\ T).$$

This proves the above assertion. It should be noted that the choice of D is also arbitrary.

Let now $T\in \boldsymbol{C}_p(\mathfrak{H})$ be arbitrary and ν be an integer $\geq p$. Then $\det_\nu(1-T)$ will be defined by the formula

(2.7) $$\det_\nu(1-T)=\exp[\mathrm{tr}\{f_\nu(T)\}],\quad \textit{if } 1\in\rho(T),$$
$$=0,\quad \textit{if } 1\in\sigma(T).$$

We shall state two lemmas on some properties of $\det_\nu(1-T)$ thus defined.

Lemma 2.1. *Let $T\in \boldsymbol{C}_p(\mathfrak{H})$ and let ν be an integer not smaller than p. Let γ_0 be a closed contour such that* i) *γ_0 is contained in $\rho(T)$,* ii) *γ_0 is entirely inside $C(0,1)$ and* iii) *0 is inside γ_0. Denote the set of all points in $\sigma(T)$ which are inside γ_0 by σ_0 and the set of all points in $\sigma(T)$ which are outside γ_0 by σ_1. Put $P_j=P(\sigma_j;\ T)$ and $T_j=TP_j$, $j=0,1$. Then we have*

(2.8) $$\det_\nu(1-T)=\det_\nu(1-T_1)\det_\nu(1-T_0),$$

where

$$\det_\nu(1-T_1)=\det_1(1-T_1)\exp\left[\mathrm{tr}(T_1)+\cdots+\frac{1}{\nu-1}\mathrm{tr}(T_1^{\nu-1})\right]$$

and

$$\det_\nu(1-T_0)=\exp[\mathrm{tr}\{f_\nu(T_0)\}].$$

Proof. Let $1\in\rho(T)$. Then by (1.15) we have

$$\begin{aligned} f_\nu(T) &= f_\nu(T)P_1+f_\nu(T)P_0 \\ &= f_\nu(T_1)+f_\nu(T_0). \end{aligned}$$

Hence it follows that

$$\begin{aligned} \det{}_\nu(1-T) &= \exp[\mathrm{tr}\{f_\nu(T)\}] \\ &= \det{}_\nu(1-T_1)\det{}_\nu(1-T_0). \end{aligned}$$

Since T_1, $\mathrm{Log}(1-T_1)\in \boldsymbol{C}_0(\mathfrak{H})$, we obtain

$$\begin{aligned} \det{}_\nu(1-T_1) &= \exp\left[\mathrm{tr}\{\mathrm{Log}(1-T_1)\}+\mathrm{tr}(T_1)+\cdots+\frac{1}{\nu-1}\mathrm{tr}(T_1^{\nu-1})\right] \\ &= \det{}_1(1-T_1)\exp\left[\mathrm{tr}(T_1)+\cdots+\frac{1}{\nu-1}\mathrm{tr}(T_1^{\nu-1})\right]. \end{aligned}$$

If $1\in\sigma(T)$, then both sides of (2.8) will vanish. Thus we have the asserted equality.

Next we shall examine $\det_1(1-T)$ for $T\in \boldsymbol{C}_0(\mathfrak{H})$.

Lemma 2.2. (i) *Let $S\in \boldsymbol{C}_0(\mathfrak{H})$ and let $\{\lambda_j;\ j=1,\cdots,n\}$ be the set of all non-zero eigenvalues of S repeated according to multiplicities. Let $P\in \boldsymbol{C}_0(\mathfrak{H})$ be a (not necessarily orthogonal) projection such that $\mathfrak{R}(P)\supset\mathfrak{R}(S)$. Then we have*

$$\det{}_1(1-S)=\det{}_1(1-PSP)=\det(1_P-S_P)=\prod_{j=1}^{n}(1-\lambda_j), \tag{2.9}$$

where 1_P is the identity in $\mathfrak{R}(P)$ and S_P is the restriction of $SP=PSP$ to $\mathfrak{R}(P)$.

(ii) *Let S and $\{\lambda_j\}$ be as in* (i). *Let, furthermore, $\{\varphi_i;\ i=1,\cdots,l\}$ be a base of $\mathfrak{R}(P)$ and suppose that $\{\psi_j;\ j=1,\cdots,l\}$ is biorthogonal to $\{\varphi_i\}$, i.e., $(\varphi_i,\psi_j)=\delta_{ij}$. Put $s_{ij}=(S\varphi_j,\psi_i)=(PSP\varphi_j,\psi_i)$. Then we have*

$$\det{}_1(1-S)=\det(\delta_{ij}-s_{ij}). \tag{2.10}$$

Proof. Suppose that $1\in\rho(S)$. Then we have by definition

$$\det{}_1(1-S)=\exp[\mathrm{tr}\{\mathrm{Log}(1-S)\}].$$

Since $S\in \boldsymbol{C}_0(\mathfrak{H})$, it is seen that $\mathrm{Log}(1-S)=f_1(S)=Sg_1(S)\in \boldsymbol{C}_0(\mathfrak{H})$. By (1.9) and (1.19) (the spectral mapping theorem), we obtain

$$\mathrm{tr}\{\mathrm{Log}(1-S)\}=\sum_{j=1}^{n}\mathrm{Log}(1-\lambda_j).$$

Hence

$$\det{}_1(1-S)=\prod_{j=1}^{n}(1-\lambda_j).$$

We have also

$$\det{}_1(1-PSP)=\prod_{j=1}^{n}(1-\lambda_j)$$

and

$$\det(1_P-S_P)=\prod_{j=1}^{n}(1-\lambda_j).$$

If $1\in\sigma(S)$, each member of (2.9) is zero. Thus (i) is established. (ii) follows at once from (i). The proof of Lemma 2.2 is completed.

Let $T\in\boldsymbol{C}_p(\mathfrak{H})$ and ν be an integer not smaller than p. We shall show that $\det_\nu(1-\lambda T)\equiv\delta_\nu(\lambda;\ T)$, $\lambda\in\boldsymbol{C}$. Put

$$\delta(\lambda)=\det{}_\nu(1-\lambda T).$$

Clearly thus defined $\delta(\lambda)$ is single-valued on $\boldsymbol{C}$. Moreover $\delta(\lambda)$ is an entire function of λ by virtue of Theorem 2.3, since in the proof of Theorem 2.3 we need only the expression (2.8) of $\det_\nu(1-\lambda T)$. Fix $\lambda_0\in\boldsymbol{C}$ so that $1\in\rho(\lambda_0 T)$. Let D be an open set of J-type such that $\sigma(\lambda_0 T)\subset D$, $0\in D$ and $1\notin D^a$. For any λ belonging to a sufficiently small neighbourhood $U(\lambda_0)$ of λ_0, we have also $\sigma(\lambda T)\subset D$. Let us fix $\mathrm{Log}(1-\xi)$ on D. Then $\mathrm{Log}(1-\xi)$, $f_\nu(\xi)$, $g_\nu(\xi)\in\mathcal{A}(\sigma(\lambda T))$ for all $\lambda\in U(\lambda_0)$. Note that D is independent of $\lambda\in U(\lambda_0)$. Thus we obtain

$$f_\nu(\lambda T)=\mathrm{Log}(1-\lambda T)+\left\{\lambda T+\cdots+\frac{1}{\nu-1}(\lambda T)^{\nu-1}\right\}$$

and

$$f_\nu(\lambda T)=(\lambda T)^\nu g_\nu(\lambda T)=g_\nu(\lambda T)(\lambda T)^\nu.$$

It is obvious that $g_\nu(\lambda T)\in\mathcal{A}(U(\lambda_0);\mathcal{B}(\mathfrak{H}))$ (Lemma 1.2 (ii)) and $(\lambda T)^\nu\in\mathcal{A}(U(\lambda_0);\boldsymbol{C}_1(\mathfrak{H}))$. Hence we see that $f_\nu(\lambda T)\in\mathcal{A}(U(\lambda_0);\boldsymbol{C}_1(\mathfrak{H}))$. Clearly we have

$$\begin{aligned}\frac{d}{d\lambda}f_\nu(\lambda T)&=-(1-\lambda T)^{-1}T+\{T+\cdots+\lambda^{\nu-2}T^{\nu-1}\}\\&=-\frac{1}{\lambda}\{(1-\lambda T)^{-1}-(1+\cdots+\lambda^{\nu-1}T^{\nu-1})\},\end{aligned}$$

$$\frac{d}{d\lambda}\operatorname{tr}\{f_\nu(\lambda T)\}=\operatorname{tr}\left\{\frac{d}{d\lambda}f_\nu(\lambda T)\right\}=\sigma_\nu(\lambda;\ T).$$

Thus $\delta(\lambda)$ is an entire function satisfying the following equalities

$$\frac{\delta'(\lambda)}{\delta(\lambda)}=\sigma_\nu(\lambda;\ T),$$
$$\delta(0)=1.$$

Hence $\delta(\lambda)=\det_\nu(1-\lambda T)$ coincides with the generalized Fredholm determinant $\delta_\nu(\lambda;\ T)$. These facts may be summed up as follows.

Theorem 2.2. *Let $T\in\mathfrak{C}_p(\mathfrak{H})$ and let ν be an integer not smaller than p. Suppose that $\lambda^{-1}\notin\sigma(T)$. Then we have*

$$\delta_\nu(\lambda;\ T)=\exp\left[\operatorname{tr}\left\{\operatorname{Log}(1-\lambda T)+\lambda T+\cdots+\frac{1}{\nu-1}(\lambda T)^{\nu-1}\right\}\right]. \tag{2.11}$$

3. Dependence of the generalized Fredholm determinant on T.

Theorem 2.3. *Let $1\leq p<\infty$ and let ν be an integer not smaller than p. Then, if U is an open set in $\mathbf{C}^m$ and $T(z)\in\mathcal{A}(U;\ \mathfrak{C}_p(\mathfrak{H}))$, we have* $\det_\nu(1-T(z))\in\mathcal{A}(U)$. *Similarly, if U is an open set in R^m and $T(x)\in C^k(U;\ \mathfrak{C}_p(\mathfrak{H}))$ $(0\leq k\leq\infty)$, we have* $\det_\nu(1-T(x))\in C^k(U)$.

Proof. Let $T(z)\in\mathcal{A}(U;\ \mathfrak{C}_p(\mathfrak{H}))$. Let us fix $a\in U$, and choose r, $0<r<1$, such that $C(0,r)\subset\rho(T(a))$. If $U(a)$ is a sufficiently small neighbourhood of a, we have also $C(0,r)\subset\rho(T(z))$ for any $z\in U(a)$. Let $\sigma_1(z)=\{\lambda\in\sigma(T(z));\ |\lambda|>r\}$ and let $\sigma_0(z)=\{\lambda\in\sigma(T(z));\ |\lambda|<r\}$. We put $P_j(z)=P(\sigma_j(z);\ T(z))$ and $T_j(z)=T(z)P_j(z)=P_j(z)T(z)$, $j=0,1$. Clearly $P_1(z)\in\mathfrak{C}_0(\mathfrak{H})$. In view of Lemma 1.2, we have $P_1(z)\in\mathcal{A}(U(a);\ \mathcal{B}(\mathfrak{H}))$. Hence it follows from Lemma 1.4 that $P_1(z)$, $T_1(z)\in\mathcal{A}_0(U(a);\ \mathfrak{C}_0(\mathfrak{H}))$. It is easily seen that $P_0(z)\in\mathcal{A}(U(a);\ \mathcal{B}(\mathfrak{H}))$ and $T_0(z)\in\mathcal{A}(U(a);\ \mathfrak{C}_p(\mathfrak{H}))$.

In virtue of Lemma 2.2, $\det_\nu(1-T(z))$ is expressible as

$$\det_\nu(1-T(z))=\det_1(1-T_1(z))\exp\left[\operatorname{tr}(T_1(z))+\cdots+\frac{1}{\nu-1}\operatorname{tr}(T_1(z))^{\nu-1}\right]\exp[\operatorname{tr}\{f_\nu(T_0(z))\}].$$

Since $f_\nu(T_0(z)) = g_\nu(T_0(z))T_0(z)^\nu \in \mathcal{A}(U(a); \boldsymbol{C}_1(\mathfrak{H}))$ (Lemma 1.2), we see that $\exp[\mathrm{tr}\{f_\nu(T_0(z))\}] \in \mathcal{A}(U(a))$. It is also seen that $\exp\left[\mathrm{tr}(T_1(z)) + \cdots + \frac{1}{\nu-1}\mathrm{tr}(T_1(z)^{\nu-1})\right] \in \mathcal{A}(U(a))$. Now it remains to show that $\det_1(1-T_1(z))$ depends analytically on z in some neighbourhood $V(a)$ of a. Using the expression (1.24) of $P_1(z)$ and putting $t_{ij}(z) = (T(z)\varphi_j(z), \psi_i(z))$, we have by Lemma 2.1 $\det_1(1-T_1(z)) = \det(\delta_{ij} - t_{ij}(z))$. Since $t_{ij}(z) \in \mathcal{A}(V(a))$, it follows at once that $\det_1(1-T_1(z)) \in \mathcal{A}(V(a))$. Thus $\det_\nu(1-T(z))$ is analytic in $V(a)$ and consequently in U. The other part of the theorem can be proved in a similar way.

Theorem 2.4. *Let $1 \leq p < \infty$ and let ν be an integer not smaller than p. Suppose that $T(z)$, $T_n(z)$ $(n=1, 2, \cdots) \in \mathcal{A}(U; \boldsymbol{C}_p(\mathfrak{H}))$. If $T_n(z) \to T(z)$ $(n \to \infty)$ in $\boldsymbol{C}_p(\mathfrak{H})$ uniformly on any compact set in U, then*

$$\det_\nu(1-T_n(z)) \to \det_\nu(1-T(z))$$

uniformly on any compact set in U. In particular, if $T_n \to T$ in $\boldsymbol{C}_p(\mathfrak{H})$, then

$$\delta_\nu(\lambda;\ T_n) \to \delta_\nu(\lambda;\ T)$$

uniformly on any compact set in $\boldsymbol{C}$.

Proof. The proof of this theorem can be done in a similar way as that of Theorem 2.3.

As a consequence of the above theorem, we obtain the following

Corollary. ([6], [9]) *Let T_1, $T_2 \in \boldsymbol{C}_1(\mathfrak{H})$. Then*

$$\det_1(1-T_1^*) = \overline{\det(1-T_1)}, \tag{2.12}$$

$$\det_1((1-T_1)(1-T_2)) = \det_1(1-T_1)\det_1(1-T_2). \tag{2.13}$$

4. Power series expansion. Since $\delta_\nu(\lambda;\ T)$ is an entire function of λ, we can expand it in an everywhere convergent power series in λ. We write

$$\delta_\nu(\lambda;\ T) = \sum_{n=0}^{\infty} \delta_n^{(\nu)} \lambda^n \tag{2.14}$$

and want to obtain explicit formulas for $\delta_n^{(\nu)}$.

Theorem 2.5. *(Plemelj's formula,* [6], [11], [15]) *Let* $T \in \boldsymbol{C}_p(\mathfrak{H})$, $1 \leq p < \infty$ *and let* ν *be an integer not smaller than* p. *Then*

(2.15)
$$\delta_0^{(\nu)} = 1, \quad \delta_1^{(\nu)} = \cdots = \delta_{n-1}^{(\nu)} = 0,$$
$$\delta_n^{(\nu)} = \frac{(-1)^n}{n!} \begin{vmatrix} 0 & n-1 & & & & \\ \vdots & \ddots & n-2 & 0 & & \\ 0 & \ddots & \ddots & \ddots & \ddots & \\ \sigma_\nu & & \ddots & \ddots & \ddots & 1 \\ \vdots & \ddots & & \ddots & \ddots & \ddots \\ \sigma_n & \cdots\cdots & \sigma_\nu & & 0 & \cdots\cdots\, 0 \end{vmatrix} \quad (n = \nu, \nu+1, \cdots)$$

where
$$\sigma_k = \operatorname{tr}(T^k) \quad (k = \nu, \nu+1, \cdots).$$

Proof. First we assume that $|\lambda|$ is sufficiently small. Then we have a convergent series in $\mathscr{B}(\mathfrak{H})$,
$$(1 - \lambda T)^{-1} = 1 + \lambda T + (\lambda T)^2 + \cdots.$$
It follows at once that
$$\lambda^{\nu-1} T^\nu (1 - \lambda T)^{-1} = \lambda^{\nu-1} T^\nu + \lambda^\nu T^{\nu+1} + \cdots,$$
the series being convergent in $\boldsymbol{C}_1(\mathfrak{H})$. Hence we get
$$\begin{aligned} \sigma_\nu(\lambda; T) &= -\operatorname{tr}(\lambda^{\nu-1} T^\nu + \lambda^\nu T^{\nu+1} + \cdots) \\ &= -(\sigma_\nu \lambda^{\nu-1} + \sigma_{\nu+1} \lambda^\nu + \cdots), \end{aligned}$$
so that
$$\frac{\delta_\nu'(\lambda; T)}{\delta_\nu(\lambda; T)} = -\sum_{n=\nu}^{\infty} \sigma_n \lambda^{n-1}.$$
Next let us put
$$d(\lambda) = \exp\left\{-\sum_{n=1}^{\nu-1} \frac{1}{n} \sigma_n \lambda^n\right\} \delta_\nu(\lambda; T),$$
where $\{\sigma_1, \cdots, \sigma_{\nu-1}\}$ are arbitrary but fixed complex numbers. Then it is obvious that
$$\frac{d'(\lambda)}{d(\lambda)} = -\sum_{n=1}^{\infty} \sigma_n \lambda^{n-1},$$
when $|\lambda|$ is sufficiently small. We also have
$$d(\lambda) = \sum_{n=0}^{\infty} d_n \lambda^n.$$

Combining the last two equations, we obtain

$$\sum_{n=0}^{\infty}(n+1)d_{n+1}\lambda^n = -\sum_{k=0}^{\infty} d_k\lambda^k \sum_{l=0}^{\infty}\sigma_{l+1}\lambda^l. \tag{2.16}$$

Equate the coefficients of λ^{n-1} on the two sides of (2.16). This gives

$$d_0 = 1,$$

$$nd_n = -\sum_{k=0}^{n-1} d_k\sigma_{n-k}, \quad n=1, 2, \cdots. \tag{2.17}$$

Solving the first n of these equations, we have

$$d_n = \frac{(-1)^n}{n!}\begin{vmatrix} \sigma_1 & n-1 & & & \\ \sigma_2 & \sigma_1 & n-2 & & 0 \\ \vdots & & \ddots & \ddots & \\ \vdots & & & \ddots & 1 \\ \vdots & & & & \ddots \\ \sigma_n & \cdots & \cdots & \cdots & \sigma_1 \end{vmatrix}, \quad n=1, 2, \cdots.$$

Since $\{\sigma_1, \cdots, \sigma_{\nu-1}\}$ are arbitrary complex numbers, we may choose $\sigma_1=\sigma_2=\cdots=\sigma_{\nu-1}=0$, so that $d(\lambda)=\delta_\nu(\lambda; T)$. Thus the generalized Plemelj's formula (2.14) is obtained.

Corollary. ([9]) Let $T\in \boldsymbol{C}_1(\mathfrak{H})$, $1\leq p<\infty$ and let ν be an integer not smaller than p. Suppose that P, $Q\in\mathscr{B}(\mathfrak{H})$ are (not necessrily orthogonal) projections such that $PT=T$ and $TQ=T$, respectively. Then we have

$$\det_\nu(1-T) = \det_\nu(1-PTP) = \det_\nu(1-QTQ).$$

5. Infinite product representation.

Theorem 2.6. ([2], [6], [10]) *Let $T\in\boldsymbol{C}_p(\mathfrak{H})$, $1\leq p<\infty$, and let ν be an integer not smaller than p. Let $\{\lambda_j\}$ be an enumeration of non-zero eigenvalues of T with repetition according to multiplicities. Then we have*

$$\delta_\nu(\lambda; T) = \prod_{j=1}^{\infty}(1-\lambda\lambda_j)\exp\left\{\lambda\lambda_j + \cdots + \frac{1}{\nu-1}(\lambda\lambda_j)^{\nu-1}\right\}. \tag{2.18}$$

To prove the above theorem, we need to prepare a series of lemmas. The first lemma is due to Weyl.

Lemma 2.3. ([16]) *Let $\mathfrak{H}$ be a Hilbert space of finite dimension l. Let $\{\lambda_i(S)\}$ and $\{\mu_i(S)\}$ be eigenvalues and characteristic values of S, respectively. Then we have*

$$\prod_{j=1}^{l} |\lambda_j(S)| \leq \prod_{j=1}^{l} \mu_j(S), \tag{2.19}$$

$$\sum_{j=1}^{l} |\lambda_j(S)|^p \leq \sum_{j=1}^{l} \mu_j(T)^p. \tag{2.20}$$

Proof. See Weyl [16].

By means of the inequalities in the above lemma we obtain the following estimation of $\det_1(1-T)$, which plays a crucial role in the proof of Theorem 2.6.

Lemma 2.4. ([3], [6]) *Let $T \in \boldsymbol{C}_1(\mathfrak{H})$ and let $\{\mu_j(T)\}$ be characteristic values of T. Then we have*

$$|\det_1(1-T)| \leq \prod_{j=1}^{\infty} (1+\mu_j(T)). \tag{2.21}$$

Proof. Let $T_n = TE_n$ be the standard approximation of T defined by (1.4). Applying (2.19) to $(1-E_nT_n)$ restricted on $\mathfrak{R}(E_n)$ and noting Lemma 2.2, we have

$$|\det_1(1-T_n)| \leq \prod_{j=1}^{n} \mu_j((1-E_nT_n)\,|\,\mathfrak{R}(E_n)).$$

It follows by the min-max principle (1.1) that

$$\mu_j((1-E_nT_n)\,|\,\mathfrak{R}(E_n)) \leq 1+\mu_j(T_n), \quad j=1, 2, \cdots, n.$$

Since $\mu_j(T_n)=\mu_j(T)$, $j=1, \cdots, n$, we have

$$|\det_1(1-T_n)| \leq \prod_{j=1}^{n} (1+\mu_j(T)).$$

Letting $n \to \infty$, we arrive at the asserted inequality.

Lemma 2.5. ([3], [6], [13]) *Let $T \in \boldsymbol{C}_p(\mathfrak{H})$, $0<p<\infty$, and let $\{\lambda_j(T)\}$ be an enumeration of non-zero eigenvalues of T repeated according to multiplicities. Then we have*

$$\Sigma |\lambda_j(T)|^p \leq \|T\|_p^p. \tag{2.22}$$

Proof. Since non-zero eigenvalues of T_n and E_nTE_n coincide by (1.8), it is shown by (2.20) that

$$\sum_{j=1}^{n}|\lambda_j(T_n)|^p\leq\sum_{j=1}^{n}\mu_j(E_nTE_n|\mathfrak{R}(E_n))^p.$$

If follows from the definition of T_n and the min-max principle (1.1) that

$$\mu_j(E_nTE_n|\mathfrak{R}(E_n))=\mu_j(E_nTE_n)$$
$$\leq\mu_j(TE_n)=\mu_j(T_n)=\mu_j(T),\quad j=1,\cdots,n.$$

Hence

$$\sum_{j=1}^{n}|\lambda_j(T_n)|^p\leq\sum_{j=1}^{n}\mu_j(T)^p.$$

Since there exist enumerations of eigenvalues $\lambda_j(T_n)$ and $\lambda_j(T)$ such that $\lambda_j(T_n)\to\lambda_j(T)$ $(n\to\infty)$ for every $j=1,2,\cdots$, we have at once

$$\sum_{j=1}^{k}|\lambda_j(T)|^p\leq\|T\|_p^p.$$

Going to the limit $k\to\infty$, we obtain the required result (2.22).

Now we turn to the proof of a special case of Theorem 2.6. We state it as the following

Lemma 2.6. *Let $T\in\mathfrak{C}_1(\mathfrak{H})$ and let $\{\lambda_j\}$ be an enumeration of eigenvalues of T repeated according to multiplicities. Then we have*

$$\delta_1(\lambda;T)=\prod_{j=1}^{\infty}(1-\lambda\lambda_j) \tag{2.23}$$

Proof. The following argument is essentially due to Carlemann [1]. By Lemma 2.4 we have

$$|\delta_1(\lambda;\ T)|\leq\prod_{j=1}^{N}(1+|\lambda|\mu_j(T))\exp\{|\lambda|\sum_{j=N+1}^{\infty}\mu_j(T)\}.$$

Let us fix $\varepsilon>0$ arbitrarily. Choosing N sufficiently large, we get

$$|\delta_1(\lambda;\ T)|\leq\exp(\varepsilon|\lambda|) \tag{2.24}$$

for sufficiently large $|\lambda|$. Noting that $\sum_{j=1}^{\infty}|\lambda_j|\leq\|T\|_1<\infty$, we put

$$\delta(\lambda)=\prod_{j=1}^{\infty}(1-\lambda\lambda_j).$$

Then we have also

(2. 25) $$|\delta(\lambda)|\leq\exp(\varepsilon|\lambda|)$$

for sufficiently large $|\lambda|$.

Now define $\gamma(z)$ by

$$\gamma(z)=\delta(\lambda)\delta(\omega\lambda)\delta(\omega^2\lambda),$$

where $z=\lambda^3$ and $\omega^3=1$, $\omega\neq1$. Consequently,

$$\gamma(z)=\prod_{j=1}^{\infty}(1-z\lambda_j)^3.$$

Since $\Sigma|\lambda_j^3|^{1/3}<\infty$, the order of the entire function $\gamma(z)$ does not exceed 1/2. Then, by Wimann's theorem there exists a sequence of circles centered at 0 and with radius r_k tending to infinity, so that

(2. 26) $$|\gamma(z)|=|\delta(\lambda)||\delta(\omega\lambda)||\delta(\omega^2\lambda)|>1$$

for $|z|=|\lambda|^3=r_k$, $k=1, 2, \cdots$. Using the estimate (2. 25) and (2. 26), we get for $|\lambda|=r_k^{1/3}$

$$|\delta(\lambda)|^{-1}\leq\exp(2\varepsilon|\lambda|),$$

when $|\lambda|$ is sufficiently large. Hence it follows that

$$\left|\frac{\delta_1(\lambda;\ T)}{\delta(\lambda)}\right|\leq\exp(3\varepsilon|\lambda|)$$

for $|\lambda|=r_k^{1/3}$, $|\lambda|$ being sufficiently large. Since Lemma 2. 4 implies that the entire function $\delta_1(\lambda;\ T)$ is of order not exceeding 1, we have by making use of Hadamard's theorem

$$\delta_1(\lambda;\ T)=e^{a+b\lambda}\delta(\lambda).$$

Thus we obtain for $|\lambda|=r_k^{1/3}$

$$|e^{a+b\lambda}|\leq\exp(3\varepsilon|\lambda|),$$

$|\lambda|$ being sufficiently large. This gives $|b|\leq3\varepsilon$. Since ε is arbitrary, we have $b=0$. In view of $\delta(0;\ T)=\delta(0)=1$, we see $a=0$.

Lemma 2. 7. (i) *Let $T\in\mathfrak{C}_1(\mathfrak{H})$ and let $\{\lambda_j(T)\}$ be an enumeration of non-zero eigenvalues of T with repetition according to multiplicities. Then we have*

$$(2.27) \qquad \operatorname{tr}(T)=\sum_{j=1}^{\infty}\lambda_j(T).$$

(ii) *Let* $T\in\mathscr{B}(\mathfrak{H},\mathfrak{H}_1)$ *and* $S\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H})$. *Assume that* $ST\in \boldsymbol{C}_1(\mathfrak{H})$ *and* $TS\in\boldsymbol{C}_1(\mathfrak{H}_1)$. *Then we have*

$$(2.28) \qquad \operatorname{tr}(ST)=\operatorname{tr}(TS).$$

Proof. By Theorem 2.5, we have $\delta_1^{(1)}=\operatorname{tr}(T)$. On the other hand Lemma 2.6 shows that $\delta_1^{(1)}=\Sigma\lambda_i(T)$. Hence (i) holds. (ii) is an easy consequence of (i) and Lemma 1.1.

Proof of Therem 2.6. Let $1\in\rho(\lambda T)$. Then by Theorem 2.2 the entire function $\delta^{(\nu)}(\lambda;\ T)$ is expressible as

$$\delta^{(\nu)}(\lambda;\ T)=\exp[\operatorname{tr}\{f_\nu(\lambda T)\}].$$

By virtue of the spectral mapping theorem (1.19) and Lemma 2.7 (i), it follows that

$$\operatorname{tr}\{f_\nu(\lambda T)\}=\sum_{j=1}^{\infty}f_\nu(\lambda\lambda_j)$$
$$=\sum_{j=1}^{\infty}\left\{\operatorname{Log}(1-\lambda\lambda_j)+(\lambda\lambda_j)+\cdots+\frac{1}{\nu-1}(\lambda\lambda_j)^{\nu-1}\right\}.$$

Hence (2.18) holds. When $1\in\sigma(\lambda T)$, both sides of (2.18) will be zero. This completes the proof of Theorem 2.5.

As a direct consequence of Theorem 2.5 and Lemma 1.1, we have the following

Corollary. (i) *Let* $T\in\mathscr{B}(\mathfrak{H},\mathfrak{H}_1)$ *and let* $S\in\mathscr{B}(\mathfrak{H}_1,\mathfrak{H})$. *Assume that* $ST\in\boldsymbol{C}_p(\mathfrak{H})$ *and* $TS\in\boldsymbol{C}_p(\mathfrak{H}_1)$, $0<p<\infty$. *Then we have*

$$\det_\nu(1-ST)=\det_\nu(1-TS),$$

where ν *is an integer not smaller than* p.

(ii) *Let* $T\in\boldsymbol{C}_p(\mathfrak{H})$, $0<p<\infty$, *and let* ν *be an integer not smaller than* p. *Then we have*

$$\det_\nu(1-T^*)=\overline{\det_\nu(1-T)}.$$

6. Representation of $(1-\lambda T)^{-1}$. Using a $\mathscr{B}(\mathfrak{H})$-valued entire function $\Delta_\nu(\lambda;\ T)=\delta_\nu(\lambda;\ T)(1-\lambda T)^{-1}$, $(1-\lambda T)^{-1}$ can be expressible as

$$(1-\lambda T)^{-1}=\frac{\Delta_\nu(\lambda;\ T)}{\delta_\nu(\lambda;\ T)}\,.$$

We write

$$\Delta_\nu(\lambda;\ T)=\sum_{n=0}^{\infty}\Delta_n^{(\nu)}\lambda^n,$$

the series being convergent in $\mathscr{B}(\mathfrak{H})$ for all $\lambda\in\boldsymbol{C}$. We wish to give explicit formulas for $\Delta_n^{(\nu)}$.

Theorem 2.8. (*Plemelj's formula*, [6], [11], [15]) *Let* $T\in\boldsymbol{C}_p(\mathfrak{H})$ *and let* ν *be an integer not smaller than* p. *Then*

$$\Delta_n^{(\nu)}=\frac{(-1)^n}{n!}\begin{vmatrix} 1 & n & 0 & \cdots & 0 \\ T & 0 & n-1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \sigma_\nu & & \ddots & 1 \\ T^n & \sigma_n & \cdots\ \sigma_\nu\ \cdots & & 0 \end{vmatrix}\quad (n=0, 1, \cdots). \tag{2.29}$$

Proof. It is obvious that

$$(1-\lambda T)\cdot\Delta_\nu(\lambda;\ T)=\delta_\nu(\lambda;\ T).$$

In this equation we replace $\delta_\nu(\lambda;\ T)$ and $\Delta_\nu(\lambda;\ T)$ by the power series expansions. It follows that

$$\sum_{n=0}^{\infty}\Delta_n^{(\nu)}\lambda^n-\sum_{n=0}^{\infty}T\Delta_n^{(\nu)}\lambda^{n+1}=\sum_{n=0}^{\infty}\delta_n^{(\nu)}\lambda^n. \tag{2.30}$$

Equating the coefficients of two sides of (2.30), we obtain

$$\Delta_0^{(\nu)}=1,$$
$$\Delta_n^{(n)}=\delta_n^{(\nu)}+T\Delta_{n-1}^{(\nu)}\,.$$

Let the expression of the right hand side of (2.29) be denoted by $\widetilde{\Delta}_n^{(\nu)}$. Then we have $\widetilde{\Delta}_n^{(\nu)}=1$. By expanding the determinant in terms of its first row, we get

$$\widetilde{\Delta}_n^{(\nu)}=\delta_n^{(\nu)}-n\left(-\frac{1}{n}\right)T\widetilde{\Delta}_{n-1}^{(\nu)}=\delta_n^{(\nu)}+T\widetilde{\Delta}_{n-1}^{(\nu)}\,.$$

Thus we see that both $\Delta_n^{(\nu)}$ and $\widetilde{\Delta}_n^{(\nu)}$ satisfy the same recurrence formula and $\Delta_0^{(\nu)}=\widetilde{\Delta}_0^{(\nu)}=1$. Hence we have

$$\Delta_n^{(\nu)}=\widetilde{\Delta}_n^{(\nu)} \quad (n=0, 1, 2, \cdots).$$

The proof of Theorem 2.8 is completed.

Sometimes it is useful to write

$$(1-\lambda T)^{-1}=1+\lambda\Gamma(\lambda;\ T),$$
$$\Gamma(\lambda;\ T)=\frac{N_\nu(\lambda;\ T)}{\delta_\nu(\lambda;\ T)}.$$

The $C_p(\mathfrak{H})$-valved entire function $N_\nu(\lambda;\ T)$ can be expanded in an everywhere convergent power series

$$N_\nu(\lambda;\ T)=\sum_{n=0}^{\infty} N_n^{(\nu)}\lambda^n.$$

It is easy to see that

$$N_n^{(\nu)}=T\Delta_n^{(\nu)}=\Delta_n^{(\nu)}T \quad (n=0, 1, 2, \cdots).$$

7. DEPENDENCE OF $\Delta_\nu(\lambda;\ T)$ AND $N_\nu(\lambda;\ T)$ ON T.

Theorem 2.9. *Let $1\leq p\leq\infty$ and let ν be an integer not smaller than p. Suppose that U is an open set in C^m and let $T(z)\in\mathcal{A}(U;\ C_p(\mathfrak{H}))$. Then we have $\Delta_\nu(1;\ T(z))\in\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))$ and $N_\nu(1;\ T(z))\in\mathcal{A}(U;\ C_p(\mathfrak{H}))$.*

Proof. According to a result of Dunford-Schwartz [3], the generalized Carlemann's inequality holds for any $T\in C_p(\mathfrak{H})$,

$$\|\Delta_\nu(\lambda;\ T)\|\leq\exp(\Gamma|\lambda|^p\|T\|_p^p),$$

where Γ is a constant depending only on p. By means of this inequality we have

$$\|\Delta_n^{(\nu)}(T)\|\leq\frac{1}{r^n}\exp(\Gamma r^p\|T\|_p^p).$$

It follows immediately that

$$\|\Delta_n^{(\nu)}(T)\|\leq\left(\frac{ep}{n}\Gamma\|T\|_p^p\right)^{n/p}.$$

For an arbitrary compact set $K\subset U$, we put

$$\kappa=\sup_{z\in K}\|T(z)\|_p^p<\infty.$$

Hence we have

$$\|\Delta_n^{(\nu)}(T(z))\| \leq \left(\frac{ep}{n}\Gamma\kappa\right)^{n/p}.$$

Since $\sum_{n=0}^{\infty}(ep\Gamma\kappa/n)^{n/p}<\infty$, the series

$$\Delta_\nu(1;\ T(z))=\sum_{n=0}^{\infty}\Delta_n^{(\nu)}(T(z))$$

is convergent in $\mathscr{B}(\mathfrak{H})$ uniformly on K. Thus we have established that $\Delta_\nu(1;\ T(z))\in\mathscr{A}(U,\mathscr{B}(\mathfrak{H}))$. It readily follows that $N_\nu(1;\ T(z)) = T(z)\Delta_\nu(1;\ T(z))\in\mathscr{A}(U;\ \boldsymbol{C}_p(\mathfrak{H}))$. The proof of Theorem 2.9 is now complete.

Theorem 2.10. *Let $1\leq p\leq\infty$ and let ν be an integer not smaller than p. Suppose that U is an open set in $\boldsymbol{C}^m$ and let $T(z)$, $T_n(z)$ $(n=1, 2, \cdots)\in\mathscr{A}(U;\ \boldsymbol{C}_p(\mathfrak{H}))$. If $T_n(z)\to T(z)$ $(n\to\infty)$ in $\boldsymbol{C}_p(\mathfrak{H})$ uniformly on any compact set in U, then*

$$\Delta_\nu(1;\ T_n(z))\to\Delta_\nu(1;\ T(z))$$

in $\mathscr{B}(\mathfrak{H})$ and

$$N_\nu(1;\ T_n(z))\to N_\nu(1;\ T(z))$$

in $\boldsymbol{C}_p(\mathfrak{H})$, the convergence being uniform on any compact set in U.

Proof. The assertion of this theorem can be proved in a similar way as that of Theorem 2.9.

§3. Analytic properties of $T(z)^{-1}$

1. Definitions of $\mathscr{M}_0(U;\ \boldsymbol{C}_p(\mathfrak{H}))$, $\mathscr{M}_0(U;\ \mathscr{B}(\mathfrak{H}))$ and $\mathscr{M}_0(U;\ \mathscr{B}_0(\mathfrak{H}))$.

Let U be an open set in $\boldsymbol{C}^m$, and let $T(z)\in\mathscr{M}(U;\ \boldsymbol{C}_p(\mathfrak{H}))$, $1\leq p\leq\infty$, and ω_1 be the set of all singularities of $T(z)$. We say $T(z)\in\mathscr{M}_0(U;\ \boldsymbol{C}_p(\mathfrak{H}))$, if for each $a\in\omega_1$ there exists a neighbourhood $U(a)$ of a so that $T_0(z)\in\mathscr{A}(U(a);\ \boldsymbol{C}_p(\mathfrak{H}))$ and $T_1(z)\in\mathscr{M}_0(U(a);\ \boldsymbol{C}_0(\mathfrak{H}))$ exist and $T(z)=T_0(z)+T_1(z)$ for $z\in U(a)-\omega_1$. $\mathscr{M}_0(U;\ \mathscr{B}(\mathfrak{H}))$ is defined in a similar way. We define a subclass $\mathscr{B}_0(\mathfrak{H})$ of $\mathscr{B}(\mathfrak{H})$ by $\mathscr{B}_0(\mathfrak{H})=\mathscr{B}(\mathfrak{H})\cap\Phi_0(\mathfrak{H})$. Of course $\mathscr{B}_0(\mathfrak{H})$ is not a Banach space and it is an open set in $\mathscr{B}(\mathfrak{H})$ (Lemma 3.2). By $T(z)\in\mathscr{A}(U;\ \mathscr{B}_0(\mathfrak{H}))$ we mean that i) $T(z)\in\mathscr{A}(U;\ \mathscr{B}(\mathfrak{H}))$ and ii) $T(z)\in\mathscr{B}_0(\mathfrak{H})$ for each

$z \in U$. By $T(z) \in \mathcal{M}_0(U; \mathcal{B}_0(\mathfrak{H}))$ we mean that i) $T(z) \in \mathcal{M}(U; \mathcal{B}(\mathfrak{H}))$, ii) if ω_1 denotes the set of all singularities of $T(z)$, $T(z) \in \mathcal{B}_0(\mathfrak{H})$ for $z \in U-\omega_1$ and iii) for each $a \in \omega_1$ there exists a neighbourhood $U(a)$ of a so that $T_0(z) \in \mathcal{A}(U(a); \mathcal{B}_0(\mathfrak{H}))$ and $T_1(z) \in \mathcal{M}_0(U(a); \boldsymbol{C}_0(\mathfrak{H}))$ exist and $T(z) = T_0(z) + T_1(z)$ for $z \in U(a) - \omega_1$.

Theorem 3.1. (i) $\mathcal{A}(U; \mathcal{B}(\mathfrak{H})) \cap \mathcal{M}(U; \boldsymbol{C}_p(\mathfrak{H})) = \mathcal{A}(U; \boldsymbol{C}_p(\mathfrak{H}))$, $1 \leq p \leq \infty$.

(ii) *Let* $T(z) \in \mathcal{M}_0(U; \mathcal{B}(\mathfrak{H}))$ *and* ω_1 *be the singularities of* $T(z)$. *If* $T(z) \in \mathcal{M}(U-\omega_1; \boldsymbol{C}_p(\mathfrak{H}))$, $1 \leq p \leq \infty$, *then* $T(z) \in \mathcal{M}_0(U; \boldsymbol{C}_p(\mathfrak{H}))$.

Proof. (i) Let $T(z) \in \mathcal{A}(U; \mathcal{B}(\mathfrak{H})) \cap \mathcal{M}(U; \boldsymbol{C}_p(\mathfrak{H}))$. Take $a \in U$. Then there exist a neighbourhood $U(a)$ of a and $f(z) \in \mathcal{A}(U(a))$ such that $f(z) \not\equiv 0$ in $U(a)$ and $f(z)T(z) \in \mathcal{A}(U(a); \boldsymbol{C}_p(\mathfrak{H}))$. Let $\omega_0 = \{z \in U(a); f(z) = 0\}$. If $a \notin \omega_0$, $T(z) \in \mathcal{A}(V(a); \boldsymbol{C}_p(\mathfrak{H}))$ for a sufficiently small neighbourhood $V(a)$ of a. If $a \in \omega_0$, we take $b \in U(a) - \omega_0$. By translation and rotation of the coordinate we may assume that $a=0$ and $b=(0, \cdots, 0, b_m)$. Since $f(0, \cdots, 0, 0)=0$ and $f(0, \cdots, 0, b_m) \neq 0$, there exist $\varepsilon_m > 0$ and $\varepsilon > 0$ such that $f(z_1, \cdots, z_{m-1}, z_m) \neq 0$ for $z \in \gamma = \{z \in \boldsymbol{C}^m; |z_j| = \varepsilon \ (j=1, \cdots, m-1), |z_m| = \varepsilon_m\}$. Hence $\gamma \subset U(a) - \omega_0$, and $T(z) \in \mathcal{A}(V; \boldsymbol{C}_p(\mathfrak{H}))$ for some neighbourhood V of γ. Since $T(z) \in \mathcal{A}(U(a); \mathcal{B}(\mathfrak{H}))$, we have

$$T(z) = \frac{1}{(2\pi i)^m} \int \cdots \int_{\zeta \in \gamma} \frac{T(\zeta_1, \cdots, \zeta_m)}{(\zeta_1 - z_1) \cdots (\zeta_m - z_m)} d\zeta_1 \cdots d\zeta_m$$

for $z \in V(a) = \{z \in \boldsymbol{C}^m; |z_j| < \varepsilon_j \ (j=1, \cdots, m-1), |z_m| < \varepsilon_m\}$. The right hand side belongs to $\mathcal{A}(V(a); \boldsymbol{C}_p(\mathfrak{H}))$. Hence $T(z) \in \mathcal{A}(V(a); \boldsymbol{C}_p(\mathfrak{H}))$ and the proof is completed.

(ii) The assumtion and (i) implies $T(z) \in \mathcal{A}(U-\omega_1; \boldsymbol{C}_p(\mathfrak{H}))$. Let $a \in \omega_1$. Then for a neighbourhood $U(a)$ of a there exist $T_0(z) \in A(U(a); \mathcal{B}(\mathfrak{H}))$ and $T_1(z) \in \mathcal{M}_0(U(a); \boldsymbol{C}_0(\mathfrak{H}))$ so that $T(z) = T_0(z) + T_1(z)$ for $z \in U(a) - \omega_1$. Clearly $T_0(z) \in \mathcal{A}(U(a) - \omega_1; \boldsymbol{C}_p(\mathfrak{H}))$. We may assume that there exists $f(z) \in \mathcal{A}(U(a))$ such that $f(z) \not\equiv 0$ in $U(a)$, $f(z) T_1(z) \in \mathcal{A}_0(U(a); \boldsymbol{C}_0(\mathfrak{H}))$ and $\omega_1 \cap U(a) \subset \omega_0 = \{z \in U(a); f(z) = 0\}$. By the same method as in the proof of (i) we can show that $T_0(z)\in\mathcal{A}(V(a);\ \boldsymbol{C}_p(\mathfrak{H}))$ for some neighbourhood $V(a)$ of a. Hence $T(z)\in\mathcal{M}_0(U;\ \boldsymbol{C}_p(\mathfrak{H}))$.

Lemma 3.1. *Let U be a domain in $\boldsymbol{C}^m$. Let $T(z)\in\mathcal{M}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$ and ω_1 be the set of all singularities of $T(z)$. Suppose that there exists $z_0\in U-\omega_1$ such that $1\in\rho(T(z_0))$. Then $\det_1(1-T(z))\in\mathcal{M}(U)$ with singularities contained in ω_1, $(1-T(z))^{-1}\in\mathcal{M}_0(U;\ \mathcal{B}_0(\mathfrak{H}))$, and $\Gamma(T(z))\in\mathcal{M}_0(U;\ \boldsymbol{C}_0(\mathfrak{H}))$, where*

$$\Gamma(T(z))=-\{(1-T(z))^{-1}-1\}=-T(z)(1-T(z))^{-1}. \tag{3.1}$$

The singularities of $(1-T(z))^{-1}$ and $\Gamma(T(z))$ are contained in $\omega_1\cup\omega$, $\omega=\{z\in U-\omega_1;\ 1\in\sigma(T(z)),\ i.e.\ \det_1(1-T(z))=0\}$.

Proof. Take a point $a\in U$. Then $T(z)$ has the expression of the form (1.20) in a neighbourhood $U(a)$ of a. By virtue of Lemma 1.4, we may assume that there exists $P(z)\in\mathcal{A}(U(a);\ \boldsymbol{C}_0(\mathfrak{H}))$ with $P(z)^2=P(z)$ and $\mathcal{R}(T(z))\subset\mathcal{R}(P(z))$ for each $z\in U(a)-\omega_1$. We may also assume that $P(z)$ has the expression of the form (1.24). We put $P_0(z)=1-P(z)$, $z\in U(a)$. Then $P_0(z)\in\mathcal{A}(U(a);\ \mathcal{B}(\mathfrak{H}))$ and $P_0(z)^2=P_0(z)$. For $z\in U(a)-\omega_1$, $1\in\rho(T(z))$ if and only if $1\in\rho(T(z)P(z))$, and

$$\begin{aligned}(1-T(z))^{-1}&=P(z)(1-T(z)P(z))^{-1}(1+T(z)P_0(z))+P_0(z)\\&=P(z)\{P(z)(1-T(z)P(z))^{-1}(1+T(z)P_0(z))-1\}+1.\end{aligned} \tag{3.2}$$

We put

$$t_{ij}(z)=(T(z)\varphi_j(z),\ \psi_i(z))=(\varphi_j(z),\chi_i(z)),\quad i,j=1,\cdots,k, \tag{3.3}$$

using the expression (1.20) or (1.24). Clearly $t_{ij}(z)\in\mathcal{M}(U(a))$. Note that $1\in\rho(T(z)P(z))$ if and only if $\det(\delta_{ij}-t_{ij}(z))\neq0$. If we assume that $\det(\delta_{ij}-t_{ij}(z))\equiv0$ in $U(a)-\omega_1$, then by analytic continuation we have $1\notin\rho(T(z))$ for each $z\in U-\omega_1$, since $U-\omega_1$ is connected. This contradicts to that $1\in\rho(T(z_0))$. Hence $\det(\delta_{ij}-t_{ij}(z))\not\equiv0$ in $U(a)-\omega_1$. We put

$$(s_{ij}(z))=(\delta_{ij}-t_{ij}(z))^{-1}. \tag{3.4}$$

Then $s_{ij}(z)\in\mathcal{M}(U(a))$, and

(3. 5) $$P(z)(1-T(z)P(z))^{-1}=\Sigma\varphi_i(z)s_{ij}(z)\psi_j(z)^*.$$

Hence $P(z)(1-T(z)P(z))^{-1}\in\mathcal{M}_0(U(a)\,;\,C_0(\mathfrak{H}))$. This, together with (3. 2), completes the proof except that of the assertion on $\det_1(1-T(z))$.

By virtue of Theorem 2. 3 $\det_1(1-T(z))\in\mathcal{A}(U-\omega_1)$. Since $\det_1(1-T(z))=\det(\delta_{ij}-t_{ij}(z))$ in $U(a)$ by virtue of Lemma 2. 2, $\det_1(1-T(z))\in\mathcal{M}(U(a))$. Hence $\det_1(1-T(z))\in\mathcal{M}(U)$, and $\omega\cap U(a)=\{z\in U(a)-\omega_1;\ \det(\delta_{ij}-t_{ij}(z))=0\}$.

2. ANALYTIC PRORERTIES OF $T(z)^{-1}$. To consider the inverse of $T(z)\in\mathcal{A}(U;\ \mathcal{B}_0(\mathfrak{H}))$ (or $T(z)\in\mathcal{M}_0(U;\ \mathcal{B}_0(\mathfrak{H}))$) we need the following lemma.

Lemma 3. 2. ([5], [8], [12])

(i) *Let* $T\in\Phi(\mathfrak{H},\mathfrak{H}_1)$. *Then there exists a positive number* $\varepsilon(T)$ *with the following property: if* $S\in\mathcal{B}(\mathfrak{H},\mathfrak{H}_1)$ *and* $\|S\|<\varepsilon(T)$, *then* $T+S\in\Phi(\mathfrak{H},\mathfrak{H}_1)$ $(\mathcal{D}(T+S)=(\mathcal{D}(T))$, $\kappa(T+S)=\kappa(T)$ *and* $\alpha(T+S)\leq\alpha(T)$.

(ii) *Let* $T\in\Phi(\mathfrak{H},\mathfrak{H}_1)$ *and* $S\in C_\infty(\mathfrak{H},\mathfrak{H}_1)$. *Then* $T+S\in\Phi(\mathfrak{H},\mathfrak{H}_1)$ $(\mathcal{D}(T+S)=\mathcal{D}(T))$ *and* $\kappa(T+S)=\kappa(T)$.

Now we state our main results in this section.

Theorem 3. 2. *Let* U *be a domain in* C^m.

(i) *Let* $T(z)\in\mathcal{A}(U;\ \mathcal{B}_0(\mathfrak{H}))$ *and suppose that there exists* $z_0\in U$ *such that* $\alpha(T(z_0))=0$, *i.e.* $0\in\rho(T(z_0))$. *Then* $T(z)^{-1}\in\mathcal{M}_0(U;\ \mathcal{B}_0(\mathfrak{H}))$ *exists, and the singularities of* $T(z)^{-1}$ *coincides with* $\omega=\{z\in U;\ \alpha(T(z))>0\}$ (ω *is locally represented as the zeros of an analytic function*).

(ii) *Let* $T(z)$ *be as in* (i) *and* $T_j(z)\in\mathcal{A}(U;\ \mathcal{B}(\mathfrak{H}))$, $j=1,2,\cdots$. *Suppose* $T_j(z)\rightarrow T(z)$ *in* $\mathcal{B}(\mathfrak{H})$ $(j\rightarrow\infty)$ *uniformly on each compact set of* U. *Then on each compact set* K *of* U, $T_j(z)^{-1}\in\mathcal{M}_0(K;\ \mathcal{B}_0(\mathfrak{H}))$ *exists for sufficiently large* j (*i.e.* $j\geq N(K)$) *and converges to* $T(z)^{-1}$ *in* $\mathcal{B}(\mathfrak{H})$ *uniformly on each compact set of* $U-\omega$.

(iii) *Let* $T(z)$ *be as in* (i) *and* $m=1$. *Then* ω *is a discrete set of points in* U. *Each* $a\in\omega$ *is a pole of* $T(z)^{-1}\in\mathcal{M}_0(U;\ \mathcal{B}_0(\mathfrak{H}))$, *and Laurent's expansion of* $T(z)^{-1}$ *at* $z=a$ *has the form*

(3. 6) $$T(z)^{-1}=\frac{T_l}{(z-a)^l}+\cdots+\frac{T_1}{z-a}+T_0(z),$$

where $T_j\in \boldsymbol{C}_0(\mathfrak{H})$, $j=1, \cdots, l$, $T_0(z)\in\mathcal{A}(U(a)\,;\,\mathcal{B}_0(\mathfrak{H}))$ *for some neighbourhood* $U(a)$ *of* a, *and*

(3. 7) $$T_l=\sum_{j=1}^{k}\lambda_j(\ \ ,\psi_j)\varphi_j$$

with $\lambda_j>0$, $T(a)\varphi_j=0$, $T(a)^*\psi_j=0$, *and* $(\varphi_i,\varphi_j)=(\psi_i,\psi_j)=\delta_{ij}$.

Proof. First we prove (i). If $z_1\in U-\omega(\neq\phi)$, then $T(z_1)^{-1}\in\mathcal{B}_0(\mathfrak{H})$ exists. If we take a sufficiently small neighbourhood $V(z_1)$ of z_1, there holds $\|T(z)-T(z_1)\|<\frac{1}{2}\|T(z_1)^{-1}\|^{-1}$ for $z\in V(z_1)$. Hence we have

(3. 8)
$$\begin{aligned} T(z)^{-1}&=\{T(z_1)+(T(z)-T(z_1)\}^{-1}\\ &=T(z_1)^{-1}\{1-(T(z)-T(z_1))T(z_1)^{-1}\}^{-1}\\ &=T(z_1)^{-1}\sum_{j=0}^{\infty}\{(T(z)-T(z_1))T(z_1)^{-1}\}^j \end{aligned}$$

for $z\in V(z_1)$. Since $\|(T(z)-T(z_1))T(z_1)^{-1}\|<1/2$, the above series converges uniformly on $V(z_1)$. Thus $T(z)^{-1}\in\mathcal{A}(V(z_1)\,;\,\mathcal{B}_0(\mathfrak{H}))$. If $a\in\omega$, then $\alpha(T(a))=\beta(T(a))=n>0$. We take a base $\{u_j;\ j=1,\cdots,n\}$ of $\mathfrak{N}(T(a))$ and a base $\{\varphi_j;\ j=1,\cdots,n\}$ of a complementary subspace $\mathfrak{M}$ of $\mathcal{R}(T(a))$ (i.e. $\mathcal{R}(T(a))\dot{+}\mathfrak{M}=\mathfrak{H}$[7]). We take a biorthogonal base $\{\chi_j\}$ for $\{u_j\}$ and put

(3. 9) $$R=\sum_{j=1}^{n}(\ \ ,\chi_j)\varphi_j\,,$$

(3. 10) $$\widetilde{T}(z)=T(z)+R.$$

Then $\widetilde{T}(z)\in\mathcal{A}(U;\ \mathcal{B}_0(\mathfrak{H}))$ and $\widetilde{T}(a)^{-1}\in\mathcal{B}_0(\mathfrak{H})$ exists. Hence there exists a neighbourhood $V(a)$ of a such that $\widetilde{T}(z)^{-1}\in\mathcal{A}(V(a)\,;\,\mathcal{B}_0(\mathfrak{H}))$ exists. Clearly $T(z)=\widetilde{T}(z)-R=(1-R\widetilde{T}(z)^{-1})\widetilde{T}(z)$, $z\in V(a)$, and $0\in\rho(T(z))$ if and only if $1\in\rho(R\widetilde{T}(z)^{-1})$. Since $R\widetilde{T}(z)^{-1}\in\mathcal{A}_0(V(a)\,;\,\boldsymbol{C}_0(\mathfrak{H}))$, we have $\det_1(1-R\widetilde{T}(z)^{-1})\in\mathcal{A}(V(a))$. Clearly $1\in\rho(R\widetilde{T}(z)^{-1})$

7) $\mathfrak{M}_1\dot{+}\mathfrak{M}_2$ denotes the direct sum of the linear subspaces $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of $\mathfrak{H}$. Note that the closedness of $\mathfrak{M}_1$ and $\mathfrak{M}_2$ is always assumed in this definition.

if and only if $\det_1(1-R\widetilde{T}(z)^{-1})\neq 0$. If we assume $\det_1(1-R\widetilde{T}(z)^{-1})\equiv 0$ in $V(a)$, then $\alpha(T(z))>0$ for $z\in V(a)$. By analytic continuation we have $\alpha(T(z))>0$ for $z\in U$, since U is connected. This is a contradiction, and hence $\det_1(1-R\widetilde{T}(z)^{-1})\not\equiv 0$ in $V(a)$. Hence we have $(1-R\widetilde{T}(z)^{-1})^{-1}\in\mathscr{M}_0(V(a);\mathscr{B}_0(\mathfrak{H}))$ and $\Gamma(R\widetilde{T}(z)^{-1})=-\{(1-RT(z)^{-1})^{-1}-1\}\in\mathscr{M}_0(V(a);\boldsymbol{C}_0(\mathfrak{H}))$ by virtue of Lemma 3.1. Thus

$$
\begin{aligned}
T(z)^{-1} &= \widetilde{T}(z)^{-1}(1-R\widetilde{T}(z)^{-1})^{-1} \\
&= \widetilde{T}(z)^{-1}-\widetilde{T}(z)^{-1}\Gamma(R\widetilde{T}(z)^{-1}).
\end{aligned}
\tag{3.11}
$$

By Lemma 1.3 $\widetilde{T}(z)^{-1}\Gamma(R\widetilde{T}(z)^{-1})\in\mathscr{M}_0(V(a);\boldsymbol{C}_0(\mathfrak{H}))$. Hence $T(z)^{-1}\in\mathscr{M}_0(V(a);\mathscr{B}_0(\mathfrak{H}))$. Clearly $\omega\cap V(a)=\{z\in V(a);\det_1(1-R\widetilde{T}(z)^{-1})=0\}$.

Next we prove (ii). Fix a compact set K in U and take a compact neighbourhood V of K. Since $\mathscr{B}_1=\{T(z);z\in V\}\subset\mathscr{B}_0(\mathfrak{H})$ is compact in $\mathscr{B}(\mathfrak{H})$ and $\mathscr{B}_2=\mathscr{B}(\mathfrak{H})-\mathscr{B}_0(\mathfrak{H})$ is closed in $\mathscr{B}(\mathfrak{H})$, we have

$$\inf_{T_1\in\mathscr{B}_1,\ T_2\in\mathscr{B}_1}\|T_1-T_2\|=\delta_0>0.$$

By assumption there exists a positive number $N_0(V)$ such that $\|T_j(z)-T(z)\|\leq\frac{1}{2}\delta_0$ for $z\in V$ and $j\geq N_0(V)$. Then $\{T_j(z);z\in V\}\subset\mathscr{B}_0(\mathfrak{H})$, and hence $T_j(z)\in\mathscr{A}(V;\mathscr{B}_0(\mathfrak{H}))$ for $j\geq N_0(V)$. Take a compact set K_1 contained in $V-\omega$, and put

$$\delta_1=\inf_{z\in K_1}\|T(z)^{-1}\|^{-1}>0.$$

If we take a sufficiently large $N(V)$, we have $\|T_j(z)-T(z)\|\leq\frac{1}{2}\delta$, $\delta=\min(\delta_0,\delta_1)$, for $z\in V$ and $j\geq N(V)$. Since $\|(T_j(z)-T(z))T(z)^{-1}\|\leq 1/2$ for $z\in K_1$ and $j\geq N(V)$, $T_j(z)^{-1}\in\mathscr{B}_0(\mathfrak{H})$ exists for such z and j, and

$$
\begin{aligned}
T_j(z)^{-1} &= T(z)^{-1}\{1-(T_j(z)-T(z))T(z)^{-1}\}^{-1} \\
&= T(z)^{-1}\sum_{j=0}^{\infty}\{(T_j(z)-T(z))T(z)^{-1}\}^{j}.
\end{aligned}
\tag{3.12}
$$

Hence $T_j(z)^{-1}\in\mathscr{M}_0(V;\mathscr{B}_0(\mathfrak{H}))$ exists for $j\geq N(V)$ by virture of (i), and $T_j(z)^{-1}\to T(z)^{-1}$ in $\mathscr{B}(\mathfrak{H})$ uniformly on K_1.

Finally we prove (iii). Each $a\in\omega$ is a pole of $\Gamma(R\widetilde{T}(z)^{-1})$ in

(3. 11), and every coefficient of Laurent's expansion of $\widetilde{T}(z)^{-1}\Gamma(R\widetilde{T}(z)^{-1})$ at $z=a$ belongs to $\boldsymbol{C}_0(\mathfrak{H})$. This together with $T(a)T_l=T_lT(a)=0$ completes the proof.

Theorem 3. 3. *Let U be a domain in $\boldsymbol{C}^m$ and $T(z)\in\mathcal{M}_0(U; \mathcal{B}_0(\mathfrak{H}))$. Suppose that there exists $z_0\in U$ such that $\alpha(T(z_0))=0$, i.e. $0\in\rho(T(z_0))$. Then there exists $T(z)^{-1}\in\mathcal{M}_0(U; \mathcal{B}_0(\mathfrak{H}))$ (and $\alpha(T(z_0)^{-1})=0$).*

Proof. Let ω be the singularities of $T(z)$. Then $T(z)^{-1}\in\mathcal{M}_0(U-\omega; \mathcal{B}_0(\mathfrak{H}))$ exists by virtue of Theorem 3. 2 (i), since $U-\omega$ is a domain in $\boldsymbol{C}^m$. Let $a\in\omega$. By assumption for a neighbourhood $U(a)$ of a there exist $T_0(z)\in\mathcal{A}(U(a); \mathcal{B}_0(\mathfrak{H}))$ and $T_1(z)\in\mathcal{M}_0(U(a); \boldsymbol{C}_0(\mathfrak{H}))$ so that $T(z)=T_0(z)+T_1(z)$, $z\in U(a)-\omega$. With no loss of generality we may assume $\alpha(T_0(a))=0$. In fact, suppose that $\alpha(T_0(a))=\beta(T_0(a))=n>0$. By definition we may assume that $T_1(z)$ has the expression of the form (1.18) with linearly independent $\{\varphi_j(z); j=1, \cdots, k\}\subset\mathcal{A}(U(a), \mathfrak{H})$. Without any loss of generality we can assume that $\varphi_1(a), \cdots, \varphi_l(a)$ are linearly independent modulo $\mathcal{R}(T_0(a))$ and $\varphi_{l+1}(a), \cdots, \varphi_k(a)\in\mathcal{R}(T_0(a))$. Adding appropriate vectors $\varphi_{k+1}, \cdots, \varphi_{k+n-l}$ to $\{\varphi_i(a); i=1, \cdots, l\}$ (if necessary), we can make a base $\{\varphi_1(a), \cdots, \varphi_l(a), \varphi_{k+1}, \cdots, \varphi_{k+n-l}\}$ of a complementary subspace of $\mathcal{R}(T_0(a))$. We put $\varphi_j(z)=\varphi_j$ for $z\in U(a)$ and $j=k+1, \cdots, k+n-l$. Obviously $\{\varphi_j(a); j=1, \cdots, k+n-l\}$ are linearly independent, and hence $\{\varphi_j(z); j=1, \cdots, k+n-l\}$ are linearly independent in a neighbourhood $V_1(a)(\subset U(a))$ of a. Let $\{u_j; j=1, \cdots, n\}$ be a base of $\mathcal{N}(T_0(a))$ and $\{\chi_j; j=1, \cdots, n\}$ a biorthogonal base for $\{u_j\}$. We put

$$R(z)=\sum_{j=1}^{l}(\ ,\chi_j)\varphi_j(z)+\sum_{j=l+1}^{n}(\ ,\chi_j)\varphi_{k+j-l}(z),$$

$$\widetilde{T}_0(z)=T_0(z)+R(z)$$

for $z\in V_1(a)$. Clearly $\widetilde{T}_0(z)\in\mathcal{A}(V_1(a); \mathcal{B}_0(\mathfrak{H}))$ and $\alpha(\widetilde{T}_0(a))=0$. Since $T(z)=\widetilde{T}_0(z)+(T_1(z)-R(z))$ in $V(a)$ and $T_1(z)-R(z)\in\mathcal{M}_0(V(a); \boldsymbol{C}_0(\mathfrak{H}))$, the assumption $\alpha(T_0(a))=0$ is justified.

It follows from $\alpha(T_0(a))=0$ that there exists a neighbourhood $V(a)$ of a such that $T_0(z)^{-1}\in\mathcal{A}(V(a); \mathcal{B}_0(\mathfrak{H}))$ exists. Hence we have

$$T(z) = (1 + T_1(z) T_0(z)^{-1}) T_0(z), \quad z \in V(a).$$

Since $T_1(z) T_0(z)^{-1} \in \mathcal{M}_0(V(a); \boldsymbol{C}_0(\mathfrak{H}))$, the same argument as in the proof of Theorem 3.2 (i) shows that $(1 + T_1(z) T_0(z)^{-1})^{-1} \in \mathcal{M}_0(V(a); \mathcal{B}_0(\mathfrak{H}))$ exists, and $\Gamma(-T_1(z) T_0(z)^{-1}) = 1 - (1 + T_1(z) T_0(z)^{-1})^{-1} \in \mathcal{M}_0(V(a); \boldsymbol{C}_0(\mathfrak{H}))$ by virtue of Lemma 3.1. Hence $T(z)^{-1} = T_0(z)^{-1} \times (1 + T_1(z) T_0(z)^{-1})^{-1} = T_0(z)^{-1} - T_0(z)^{-1}\Gamma(-T_1(z) T_0(z)^{-1}) \in \mathcal{M}_0(V(a); \mathcal{B}_0(\mathfrak{H}))$ by Lemma 1.3. Thus we have completed the proof.

Theorem 3.4. (i) *Let U be a domain in $\boldsymbol{C}^m$, and let $T(z) \in \mathcal{M}_0(U; \boldsymbol{C}_p(\mathfrak{H}))$, $1 \leq p \leq \infty$. If there exists $z_0 \in U$ such that $1 \in \rho(T(z_0))$, then $(1 - T(z))^{-1} \in \mathcal{M}_0(U; \mathcal{B}_0(\mathfrak{H}))$ and $\Gamma(T(z)) \in \mathcal{M}_0(U; \boldsymbol{C}_p(\mathfrak{H}))$.*

(ii) *Let $T(z) \in \mathcal{M}_0(U; \boldsymbol{C}_1(\mathfrak{H}))$. Then* $\det_1(1 - T(z)) \in \mathcal{M}(U)$!

Proof. To prove (i) we have only to show $\Gamma(T(z)) \in \mathcal{M}_0(U; \boldsymbol{C}_p(\mathfrak{H}))$. But thit is evident from Theorem 3.3 and 3.1 (ii), since $\Gamma(T(z) = 1 - (1 - T(z))^{-1} = - T(z)(1 - T(z))^{-1}$.

(ii) Let ω be the singularities of $T(z)$. Then $\det_1(1 - T(z)) \in \mathcal{A}(U - \omega)$ by virtue of Theorem 2.3. Let $a \in \omega$. Then for a neighbourhood $U(a)$ of a there exists $T_0(z) \in \mathcal{A}(U(a); \boldsymbol{C}_1(\mathfrak{H}))$ and $T_1(z) \in \mathcal{M}_0(U(a); \boldsymbol{C}_0(\mathfrak{H}))$ so that $T(z) = T_0(z) + T_1(z)$ for $z \in U(a) - \omega$. By the same argument as in the proof of Theorem 3.3 we may assume that $\alpha(1 - T_0(a)) = 0$, i.e. $1 \in \rho(T_0(a))$. Hence $(1 - T_0(z))^{-1} \in \mathcal{A}(V(a); \mathcal{B}_0(\mathfrak{H}))$ in a neighbourhood $V(a)$ of a (Theorem 3.2). Thus $1 - T(z) = \{1 - T_1(z)(1 - T_0(z)^{-1}\}(1 - T_0(z))$, $z \in V(a)$, and $\det_1(1 - T(z)) = \det_1(1 - T_1(z)(1 - T_0(z))^{-1}) \det_1(1 - T_0(z))$ by (2.12). Since $T_1(z) \times (1 - T_0(z))^{-1} \in \mathcal{M}_0(V(a); \boldsymbol{C}_0(\mathfrak{H}))$, $\det_1(1 - T(z)) \in \mathcal{M}(V(a))$ by virtue of Lemma 3.1.

In addition we give a sufficient condition that makes $1 - T \in \mathcal{B}_0(\mathfrak{H})$ possible.

Lemma 3.3. *Let $T_0 \in \mathcal{B}(\mathfrak{H})$ with $\|T_0\| < 1$ and $T_1 \in \boldsymbol{C}_\infty(\mathfrak{H})$. If we put $T = T_0 + T_1$, then $1 \in \Phi_0(T)$. Indeed $\{\lambda \in \boldsymbol{C};\ |\lambda| \geq 1\} \subset \bar{\rho}(T)$.*

Remark. In the forthcoming papers, we shall give some applications of the results stated in this paper. We shall also give more detailed argument on the problem of §3 in the case $m = 1$. One of applications is a generalization of Weinstein-Aronszajn formula. Concerning Theorem 3.2 (iii), for example, we can prove that $a \in \omega$ is a simple pole of $T(z)^{-1}$ if and only if $\mathcal{R}(T(a)) \dot{+} T'(a) \times \mathfrak{N}(T(a)) = \mathfrak{H}$.

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