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A Classification of Factors

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Abstract

A classification of factors is given. For every factor M we define an algebraic invariant $r_\infty(M)$, called the asymptotic ratio set, which is a subset of the nonnegative real numbers. For factors which are tensor products of type I factors, the set $r_\infty(M)$ must be one of the following sets: (i) the empty set. (ii) $\{0\}$. (iii) $\{1\}$. (iv) a one-parameter family of sets $\{0, x^n; n=0, \pm 1, \cdots\}$, $0<x<1$, (v) all nonnegative reals, (vi) $\{0, 1\}$. Case (i), (ii), (iii) occurs if and only if M is finite type I, I_∞, hyperfinite type II_1, respectively. Case (iv) contains one and only one isomorphic class for each x, and they are type III. The examples treated by Powers belong to case (iv). Case (v) contains only one isomorphic class and it is type III. Thus we have a complete classification of factors M which are tensor products of type I factors, $r_\infty(M) \neq \{0, 1\}$. Case (vi) contains $I_\infty \otimes$ hyperfinite II_1 and also nondenumerably many type III isomorphic classes.

Using the factors in the cases (ii), (iii), (iv) we define another algebraic invariant $\rho(M)$ which is able to distinguish nondenumerably many classes in case (vi).

1. Introduction

In the Murray-von Neumann classification of factors (Murray and von Neumann [11]) both the type II_1 and type III classes are known to contain nonisomorphic factors. In this paper we give a further isomorphic classification of factors on separable Hilbert spaces. This classification is based on a detailed study of factors constructed as infinite tensor products of factors of finite type I (hereafter referred to as ITPFI factors). Examples of ITPFI factors were first given by von Neumann [12]. Several authors (von Neumann [12], Pukanszky [14], Bures [6], Araki [1], Moore [10]) have determined the type of

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some of these factors in the Murray-von Neumann classification. Recently, Powers [13] has shown that these examples contain a one-parameter family of mutually nonisomorphic type III factors.

Sec. 2 contains some definitions and elementary lemmas concerning ITPFI factors. In Sec: 3 we define the asymptotic ratio set $\mathrm{r}_\infty(M)\subset[0, \infty)$ for ITPFI factors $M=\otimes M_\nu$ in terms of ratios of eigenvalues $\lambda_{\nu j}$ of density matrix states ω_ν on the component factors M_ν. We show that $\mathrm{r}_\infty(M)$ must be one of the following standard sets

$$S_0=\{0\}$$
$$S_1=\{1\}$$
$$S_x=\{0, x^n; n=0, \pm 1, \pm 2, \cdots\}, \quad 0<x<1$$
$$S_{01}=\{0, 1\}$$
$$S_\infty=[0, \infty).$$

We give some elementary properties of $\mathrm{r}_\infty(M)$, and discuss the one-parameter family of examples R_x, $0\leq x\leq 1$ given by von Neumann [12]. Sec. 4 consists of a basic technical lemma. In Sec. 5 we prove that $x\in\mathrm{r}_\infty(\otimes M_\nu)$ if and only if $\otimes M_\nu\sim R_x\otimes(\otimes M_\nu)$ and thus that $\mathrm{r}_\infty(\otimes M_\nu)$ is an algebraic invariant. Our method of proving that two factors are nonisomorphic is based entirely on the strong operator topology, in contrast to that of Powers [13] which uses C^*-algebra techniques. In Sec. 6 we use this result to define $\mathrm{r}_\infty(M)$ for arbitrary M by $x\in\mathrm{r}_\infty(M)$ if and only $M\sim M\otimes R_x$, $0\leq x\leq 1$ (if $x\in\mathrm{r}_\infty(M)$, $x\neq 0$ we include $x^{-1}\in\mathrm{r}_\infty(M)$ also). We give some elementary properties of $\mathrm{r}_\infty(M)$, including its relation to the Murray-von Neumann classification.

The remainder of the paper is devoted to a study of ITPFI factors. In Sec. 7 we prove that the class S_∞ contains one and only one isomorphic class and it is type III. Sec. 8 contains a number of technical lemmas which are needed for the classification of ITPFI factors belonging to the classes S_x, $0\leq x\leq 1$. In Sec. 9 we prove that $\mathrm{r}_\infty(M)=S_x$ if and only if $M\sim R_x$, $0\leq x\leq 1$. The factors R_x are type III if $0<x<1$, and they are the factors discussed by Powers

[13]. Thus, except for the class S_{01}, we give a complete classification of factors which are tensor products of type I factors. We also give some useful criteria for calculating $r_\infty(M)$ from the eigenvalue lists $\{\lambda_{\nu j};\ j=1,\cdots n_\nu,\ \nu=1,2,\cdots\}$. In particular a sufficient condition that $r_\infty(\otimes M_\nu)\neq S_{01}$ is that there exist subsequences $\nu(m), j_1(m), j_2(m)$ such that $\lambda_{\nu(m),j_i(m)}\to\lambda_i\neq 0$, $i=1,2$ and $\lambda_1/\lambda_2\neq 1$. In Sec. 10 we study factors $M=\otimes M_\nu$ belonging to the class S_{01} where M_ν is type I_2 for all ν. M is then either $I_\infty\otimes$ hyperfinite II_1 or type III. We construct a nondenumerable family of mutually nonisomorphic factors belonging to the class S_{01}. In Sec. 11 we define another algebraic invariant $\rho(M)$ by $x\in\rho(M)$ if and only if $M\otimes R_x\sim R_x$, $0\leqslant x\leqslant 1$. We construct factors in the class S_{01} which give a nondenumerable variety of $\rho(M)$. In Sec. 12 we apply our results to determine the isomorphic class of some factors which have been studied previously in the literature. In particular we show that certain ITPFI factors which occur in the quantum theories of infinite free Bose and Fermi systems at a finite density and finite temperature, belong to the class S_∞.

We shall use the following notation. If H is a Hilbert space, then $\mathscr{B}(H)$ denotes the set of all bounded linear operators on H, and **1** denotes the set of all multiples of the identity operator. All Hilbert spaces are separable. I_∞ denotes the set of all positive integers $\{1, 2, \cdots\}$. We shall also use I_∞ to denote a factor of type I_∞, but this should not lead to any confusion. We assume that the reader is familiar with the standard notation and terminology for von Neumann algebras (Dixmier [8]). If the von Neumann algebras $\mathfrak{A}$ and $\mathfrak{B}$ are algebraically isomorphic (unitarily equivalent) we write $\mathfrak{A}\sim\mathfrak{B}$ ($\mathfrak{A}\overset{u}{\sim}\mathfrak{B}$).

2. ITPFI Factors

This section contains some basic definitions and elementary lemmas concerning ITPFI factors. We discuss the notion of the eigenvalue list of a vector relative to a type I factor, and some related topics. We give a sufficient condition on the eigenvalue lists of the reference vectors for two ITPFI factors to be unitarily equivalent. We state some known results concerning the type of ITPFI factors in the Murray-von Neumann classification.

A family of matrix units on a Hilbert space H is a set of partial isometries e_{ij}, i, $j=1, \cdots n$ (n may be infinite) satisfying $e_{ij}^*=e_{ji}$, $e_{ij}e_{kl}=\delta_{jk}e_{il}$, and $\sum_{i=1}^{n} e_{ii}=1$. Any type I_n factor contains and is spanned by such a family of matrix units.

Let H be a Hilbert space, $M\subset\mathscr{B}(H)$ a type I factor. Then we can write $H=H_1\otimes H_2$ and $M=\mathscr{B}(H_1)\otimes\mathbf{1}$. If Ω is a vector in H then it defines a normal state on $\mathscr{B}(H_1)$ by

$$\Omega(A)=(\Omega, A\otimes 1\Omega). \tag{2.1}$$

Hence there exists a nonnegative trace class operator $\rho_\Omega\in\mathscr{B}(H_1)$ such that $\Omega(A)=\operatorname{Tr}\rho_\Omega A$. Let $\rho_\Omega=\sum\lambda_i P_i$ be a spectral decomposition of ρ_Ω where each P_i is one-dimensional, $\lambda_i\geq 0$ and $\sum\lambda_i=\|\Omega\|^2$. If Ω is a unit vector then ρ_Ω is a density matrix, that is $\operatorname{Tr}\rho_\Omega=1$.

Definition 2.1. Let $\Omega\in H_1\otimes H_2$, $M=\mathscr{B}(H_1)\otimes\mathbf{1}$. By the eigenvalue list of Ω relative to a type I factor M we mean the list $(\lambda_1, \lambda_2\cdots)$ of eigenvalues of the operator ρ_Ω in M defined by

$$\operatorname{Tr}\rho_\Omega A=(\Omega, A\otimes 1\Omega) \tag{2.2}$$

ordered so that $\lambda_1\geq\lambda_2\geq\cdots\geq 0$. We denote it by $\operatorname{Sp}(\Omega/M)$, or $\operatorname{Sp}\Omega$ if M is understood.

If some λ has multiplicity m then it occurs m times in $\operatorname{Sp}(\Omega/M)$. It should be noted that $\operatorname{Sp}(\Omega/M)$ and $\operatorname{Sp}(\Omega/M')$ are identical except that the zero eigenvalue can have different multiplicity.

Definition 2.2. Given $H=H_1\otimes H_2$, $M=\mathscr{B}(H_1)\otimes\mathbf{1}$, $\Omega\in H$. By a standard diagonal expansion of Ω relative to M we mean a choice of complete orthogonal bases ψ_{1i}, ψ_{2i} for H_1, H_2 respectively such that

$$\Omega=\sum\lambda_i^{1/2}\psi_{1i}\otimes\psi_{2i} \tag{2.3}$$

where $\lambda_1\geq\lambda_2\geq\cdots\geq 0$, ψ's in one of $\{\psi_{1i}\}$ and $\{\psi_{2i}\}$ are all normalized, ψ's in the other are normalized or 0, and $\lambda_i=0$ if ψ_{1i} or $\psi_{2i}=0$.

It is known that a standard diagonal expansion exists (see, e.g.

definition 2.1 of [2]). Note that the list of non-zero λ_i is identical with the non-zero part of $\mathrm{Sp}(\Omega/M)$.

Definition 2.3. Given $H=H_1\otimes H_2$, $M=\mathscr{B}(H_1)\otimes 1$, $\Omega\in H$. By a standard set of matrix units for M, M' relative to Ω we mean operators u_{ij}, v_{ij} defined by

$$u_{ij}\psi_{1k}\otimes\psi_{2l}=\delta_{jk}\psi_{1i}\otimes\psi_{2l} \tag{2.4}$$

$$v_{ij}\psi_{1k}\otimes\psi_{2l}=\delta_{jl}\psi_{1k}\otimes\psi_{2i} \tag{2.5}$$

where ψ_{1i}, ψ_{2i} is a choice of orthonormal bases for H_1, H_2 respectively for a standard diagonal expansion of Ω relative to M. If $\psi_{1i}=0$ for some i, we define u_{ij} and u_{ji} for such i and any j to be 0. If $\psi_{2i}=0$ for some i, we define v_{ij} and v_{ji} for such i and any j to be 0. We identify $\mathrm{Sp}(\Omega/M)$ with the set of λ_i for which $\psi_{1i}\neq 0$ and $\mathrm{Sp}(\Omega/M')$ with the set of λ_i for which $\psi_{2i}\neq 0$ in (2.3).

We now give a precise definition of an ITPFI factor. Let

$$H=\bigotimes_{\nu\in A}(H_\nu,\Omega_\nu)$$

be the incomplete tensor product space (ITPS) of the Hilbert spaces H_ν which contains the product vector $\Omega=\otimes\Omega_\nu$, $\Omega_\nu\in H_\nu$, $0<\Pi\|\Omega_\nu\|<\infty$. In this paper the index set A is always countable. If Ω and A are understood we just write $H=\otimes H_\nu$. We assume the reader is familiar with the standard properties of infinite tensor products (von Neumann [12]). We note that $\otimes\chi_\nu$ belongs to the same ITPS as $\otimes\Omega_\nu$ if and only if $\otimes\chi_\nu$ is in the strong equivalence class of $\otimes\Omega_\nu$, that is

$$\sum_{\nu\in A}\|\Omega_\nu-\chi_\nu\|<\infty. \tag{2.6}$$

This is equivalent to

$$\sum|1-(\chi_\nu,\Omega_\nu)|<\infty \quad\text{and}\quad \sum|1-\|\chi_\nu\||<\infty$$

In both cases, $0<\Pi\|\Omega_\nu\|<\infty$ is assumed.

Definition 2.4. We define a canonical mapping π from $\mathscr{B}(H_\nu)$ to $\mathscr{B}(H)$ by $\pi S=(\bigotimes_{\mu\neq\nu}1_\mu)\otimes S$ where $S\in B(H_\nu)$ and 1_μ is the identity operator on H_μ. If $\mathfrak{A}\subset\mathscr{B}(H_\nu)$ we define $\pi\,\mathfrak{A}=\{\pi S\colon S\in\mathfrak{A}\}$.

Definition 2.5. Given an ITPS $H=\bigotimes_{\nu\in A}(H_\nu, \Omega_\nu)$ and von Neumann algebras $\mathfrak{A}_\nu\subset\mathcal{B}(H_\nu)$ we define

$$\otimes\mathfrak{A}_\nu=\{\pi\mathfrak{A}_\nu;\ \nu\in A\}''.$$

If the $\mathfrak{A}_\nu$ are factors, then $\otimes\mathfrak{A}_\nu$ is a factor. In the following we will be concerned with factors $\otimes M_\nu$ where M_ν is type I. We shall denote these factors by $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu;\ \nu\in A)$ or $R(H_\nu, M_\nu, \Omega_\nu)$ or $\mathrm{R}(M_\nu, \Omega_\nu)$. Unless the contrary is stated explicitly, M_ν is type I_{n_ν}, $2\leq n_\nu<\infty$, and A is infinite. If $J\subset A$ we write $H(J)=\bigotimes_{\nu\in J}H_\nu$, $M(J)=\bigotimes_{\nu\in J}M_\nu$, $\Omega(J)=\bigotimes_{\nu\in J}\Omega_\nu$. If Ω_ν and ψ_ν are in the same strong equivalence class, then $\otimes\psi_\nu$ is in $\otimes(H_\nu, \Omega_\nu)$ and hence $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu)=\mathrm{R}(H_\nu, M_\nu, \psi_\nu)$. We shall use this repeatedly.

Definition 2.6. Any factor M which is unitarily equivalent to some $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu;\ \nu\in A)$ as given above where M_ν is a type I_{n_ν} factor, $2\leq n_\nu<\infty$ and A is infinite is called an ITPFI factor.

We recall that a von Neumann algebra M is called hyperfinite if it is generated by an increasing sequence $M_1\subset M_2\subset\cdots$ of finite type I factors, i.e.,

$$M=\{M_1, M_2, \cdots\}''.$$

An ITPFI factor is clearly a hyperfinite factor. It is known that all hyperfinite factors of type II_1 are isomorphic (Dixmier [8], theorem III. 7. 1). Since an ITPFI factor is not finite type I, it must either be infinite or (isomorphic to) the unique hyperfinite II_1 factor. We shall have several occasions to make use of this remark.

Lemma 2.7. Let $\psi\in H=\bigotimes_{\nu\in A}(H_\nu, \Omega_\nu)$. Given $\varepsilon>0$, there exists a finite $J\subset A$ and $\psi_J\in H(J)$ such that

$$\|\psi-\psi_J\otimes(\bigotimes_{\nu\in J^c}\Omega_\nu)\|<\varepsilon. \tag{2.7}$$

Proof. Araki and Woods [2], lemma 3.1.

Lemma 2.8. A countable tensor product of ITPFI factors is an ITPFI factor.

Proof. Let $M_\mu = \mathrm{R}(H_{\mu\nu}, M_{\mu\nu}, \Omega_{\mu\nu}; \nu\in A_\mu)$, $\mu\in A$ be ITPFI factors. Let $H_\mu = \bigotimes_{\nu\in A_\mu}(H_{\mu\nu}, \Omega_{\mu\nu})$. Let $\Phi_\mu = \bigotimes_\nu \Omega_{\mu\nu}$, and

$$H = \bigotimes_{\mu\in A}(H_\mu, \Phi_\mu)$$
$$M = \bigotimes_{\mu\in A} M_\mu .$$

Choose $\varepsilon_\mu > 0$, $\sum \varepsilon_\mu < \infty$. By lemma 2.5 there is a finite $J_\mu \subset A_\mu$ and $\psi_\mu \in H(J_\mu)$ such that

$$\|\Phi_\mu - \psi_\mu \otimes (\bigotimes_{\nu\in J^c_\mu} \Omega_{\mu\nu})\| < \varepsilon_\mu . \tag{2.8}$$

Thus $\bigotimes_{\mu\in A}[\psi_\mu \otimes (\bigotimes_{\nu\in J^c_\mu} \Omega_{\mu\nu})]$ is in the strong equivalence class of $\otimes \Phi_\mu$. It follows from the associative law for tensor products that H is (unitarily equivalent to) the ITPS

$$\{\bigotimes_{\mu\in A}(H(J_\mu), \psi_\mu)\} \otimes \{\bigotimes_{\mu\in A} \bigotimes_{\nu\in J^c_\mu}(H_{\mu\nu}, \Omega_{\mu\nu})\} . \tag{2.9}$$

Thus M is an ITPFI factor. Q.E.D.

Corollary 2.9. The factor $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu)$ where M_ν can be type I_∞ is an ITPFI factor.

Proof. Consider each type I_∞ factor M_ν as an ITPFI factor and apply lemma 2.8. Q.E.D.

Lemma 2.10. Given $H = H_1 \otimes H_2$, $M = \mathscr{B}(H_1) \otimes \mathbf{1}$. Then M has both cyclic vectors and separating vectors if and only if $\dim H_1 = \dim H_2$. If $\dim H_1 = \dim H_2 < \infty$, let $\Omega \in H$ have the standard diagonal expansion $\Omega = \sum \lambda_i^{1/2} \psi_{1i} \otimes \psi_{2i}$. Then the following three conditions are equivalent.

(i) Ω is cyclic for M

(ii) Ω is separating for (M)

(iii) no $\lambda_i = 0$.

Proof. Assume $\dim H_2 > \dim H_1$. Since M is spanned by $(\dim H_1)^2$ linearly independent elements, we have $\dim M\Omega \leq (\dim H_1)^2 < \dim H$ and Ω cannot be cyclic for M. Similarly, $\dim H_1 > \dim H_2$ implies Ω cannot be cyclic for M' and thus Ω is not separating for M. Thus the existence of both cyclic vectors and separating vectors for M implies that $\dim H_1 = \dim H_2$.

If $\dim H_1 = \dim H_2$, then we may label complete orthonormal bases $\{\psi_{1i}\}$ and $\{\psi_{2i}\}$ of H_1 and H_2 respectively by the same index i, and Eq. (2.3) with $\lambda_i > 0$, $\sum \lambda_i < \infty$ gives the cyclic and separating vector.

In the remainder of the proof we assume that $\dim H_1 = \dim H_2 < \infty$. If some $\lambda_j = 0$ then $H_1 \otimes \psi_{2j}$ is orthogonal to $M\Omega$ and Ω is not cyclic for M, hence (i)→(iii). Similarly, if some $\lambda_j = 0$ then $\psi_{1j} \otimes H_2$ is orthogonal to $M'\Omega$ and Ω is not cyclic for M'. It follows that Ω is not separating for M, and hence (ii)→(iii). If no $\lambda_j = 0$, let u_{ij} be a standard set of matrix units for M relative to the given standard diagonal expansion of Ω. Then $u_{ji}\Omega = \lambda_i^{1/2} \psi_{1j} \otimes \psi_{2i}$. Since the standard diagonal expansion of Ω must contain a complete basis for at least one of H_1, H_2, and we have $\dim H_1 = \dim H_2 < \infty$, it follows that $M\Omega$ contains a basis for H. Thus (iii)→(i). By a similar argument Ω is cyclic for M', hence separating for M and (iii)→(ii). Q.E.D.

It should be noted that if $\dim H_1 = \dim H_2 = \infty$, then the condition that Ω is cyclic for M is not equivalent to the condition that Ω is separating for M. To see this, let $\Omega = \sum \lambda_i^{1/2} \psi_{1i} \otimes \psi_{2i}$ where $\lambda_i \neq 0$ and ψ_{1i} is a complete basis for H_1, but the summation does not run over a complete basis for H_2. Then Ω is separating but not cyclic for M.

Corollary 2.11. Given $H = H_1 \otimes H_2$, $\dim H_1 = \dim H_2$, $M = \mathcal{B}(H_1) \otimes 1$. The set of all cyclic and separating unit vectors for M is dense in the set of all unit vectors.

Proof. Given $\varepsilon > 0$, $\psi \in H$, $\|\psi\| = 1$. Let $\psi = \sum_{i=1}^{N} \lambda_i^{1/2} \psi_{1i} \otimes \psi_{2i}$ be a standard diagonal expansion of ψ where $N = \dim H_1$. Choose $n < \infty$, $n \leq N$ so that $\sum_{i=n+1}^{N} \lambda_i < \varepsilon$, and let $\psi' = \sum_{i=1}^{n} \lambda_i^{1/2} \psi_{1i} \otimes \psi_{2i}$. Then $\|\psi - \psi'\|^2 < \varepsilon$. Now choose orthonormal bases Φ_{ki}, $i = 1, 2, \cdots$ for H_k, $k = 1, 2$ such that $\Phi_{ki} = \psi_{ki}$, $i = 1, \cdots n$. Let $\psi'' = \sum_{i=1}^{N} \lambda_i^{1/2} \Phi_{1i} \otimes \Phi_{2i}$, then $\|\psi' - \psi''\|^2 < \varepsilon$. Let

$$\Omega=\sum_{i=1}^{N}\mu_i^{1/2}\Phi_{1i}\otimes\Phi_{2i}$$

where

$$\mu_i=(\lambda_i+\varepsilon_i)(1+\varepsilon)^{-1},$$

$\varepsilon_i>0$, $\sum\varepsilon_i=\varepsilon$. Then $\|\Omega\|=1$ and by the same argument used in the proof of lemma 2.10 Ω is cyclic and separating for M. We have $\|\Omega-\psi''\|^2=\sum(\mu_i^{1/2}-\lambda_i^{1/2})^2<2\varepsilon$. Thus $\|\Omega-\psi\|\to 0$ as $\varepsilon\to 0$. Q.E.D.

Let $R\subset\mathcal{B}(H)$ be a von Neumann algebra. Let $P=EE'$ where E, E' are projections in R, R' respectively. Then

$$R_P=\{PAP;\ A\in R\} \tag{2.10}$$

is a von Neumann algebra on PH. In particular if R is a factor, $R_{E'}$ is isomorphic to R. Thus by using projections in this way, one can either shrink or enlarge R' without changing the isomorphic class of R.

Lemma 2.12. Given $\mathrm{R}(K_\nu, N_\nu, \psi_\nu)$ there exists $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu)\sim\mathrm{R}(K_\nu, N_\nu, \psi_\nu)$ such that Ω_ν is cyclic and separating for M_ν.

Proof. Write $K_\nu=K_{\nu1}\otimes K_{\nu2}$ where $N_\nu=\mathcal{B}(K_{\nu1})\otimes\mathbf{1}$. Let $n_\nu=\dim K_{\nu1}$, $J_+=\{\nu\colon \dim K_{\nu2}>n_\nu\}$, and $J_-=\{\nu\colon \dim K_{\nu2}<n_\nu\}$.

If $\nu\in J_+$ let $\psi_\nu=\sum\lambda_{\nu j}^{1/2}\psi_{1j}^{\nu}\otimes\psi_{2j}^{\nu}$ be a standard diagonal expansion of ψ_ν relative to N_ν. It follows that there is a projection $P_\nu\in N_\nu'$ such that $P_\nu\psi_\nu=\psi_\nu$ and $\dim P_\nu K_\nu=n_\nu^2$. Define $H_\nu=P_\nu K_\nu$, $M_\nu=(N_\nu)_{P_\nu}$ and $\omega_\nu=\psi_\nu$. Since $P_+=\prod_{\nu\in J_+}(\pi P_\nu)$ is a projection in $\mathrm{R}(K_\nu, N_\nu, \psi_\nu;\ \nu\in J_+)'$ it follows that $\mathrm{R}(K_\nu, N_\nu, \psi_\nu;\ \nu\in J_+)$ and $\mathrm{R}(H_\nu, M_\nu, \omega_\nu;\ \nu\in J_+)$ are isomorphic.

If $\nu\in J_-$, imbed $K_{\nu2}$ as a subspace of an n_ν-dimensional space $H_{\nu2}$. Define $H_\nu=K_{\nu1}\otimes H_{\nu2}$, $M_\nu=\mathcal{B}(K_{\nu1})\otimes\mathbf{1}$, $\omega_\nu=\psi_\nu$ (imbedded in H_ν). Let P_ν be the projection onto $K_{\nu1}\otimes K_{\nu2}$. Then $P_-=\prod_{\nu\in J_-}(\pi P_\nu)$ is a projection in $\mathrm{R}(H_\nu, M_\nu, \omega_\nu;\ \nu\in J_-)'$ and it follows that $\mathrm{R}(H_\nu, M_\nu, \omega_\nu;\ \nu\in J_-)$ is isomorphic to $\mathrm{R}(K_\nu, N_\nu, \psi_\nu;\ \nu\in J_-)$. For $\nu\notin J_+\cup J_-$ define $H_\nu=K_\nu$, $M_\nu=N_\nu$, $\omega_\nu=\psi_\nu$.

Thus we have $\mathrm{R}(H_\nu, M_\nu, \omega_\nu;\ \nu\in J)$ isomorphic to $\mathrm{R}(K_\nu, N_\nu, \psi_\nu;\ \nu\in J)$ where $H_\nu=H_{\nu1}\otimes H_{\nu2}$, $M_\nu=\mathcal{B}(H_{\nu1})\otimes\mathbf{1}$, $\|\omega_\nu\|=\|\psi_\nu\|$, $\dim H_{\nu1}=$

$\dim H_{\nu 2}$ for all $\nu \in J$. It follows from corollary 2.11 that we can choose vectors $\Omega_\nu \in H_\nu$ which are cyclic and separating for M_ν such that $\|\omega_\nu - \Omega_\nu\| < 2^{-\nu}$. Hence $\otimes \Omega_\nu$ and $\otimes \omega_\nu$ are in the same strong equivalence class (see Eq. (2.6)) and $\mathrm{R}(H_\nu, M_\nu, \omega_\nu) = \mathrm{R}(H_\nu, M_\nu, \Omega_\nu)$.

Q.E.D.

Given $H_\nu = H_{\nu 1} \otimes H_{\nu 2}$, $M_\nu = \mathscr{B}(H_{\nu 1}) \otimes \mathbf{1}$, let $\Omega_{\nu 1}$, $\Omega_{\nu 2}$ be unit vectors in H_ν. If there exist unitary operators $U_\nu = U_{\nu 1} \otimes U_{\nu 2}$ such that $U_\nu \Omega_{\nu 1}$ is in the strong equivalence class of $\Omega_{\nu 2}$ then $UR(M_\nu, \Omega_{\nu 1}) U^{-1} = \mathrm{R}(M_\nu, \Omega_{\nu 2})$ where $U = \otimes U_\nu$ (note that $\otimes U_\nu$ is not considered as an operator on the ITPS $\otimes(H_\nu, \Omega_{\nu 1})$, but as an operator from $\otimes(H_\nu, \Omega_{\nu 1})$ to $\otimes(H_\nu, \Omega_{\nu 2})$). The following lemma states this condition in terms of the eigenvalue lists.

Lemma 2.13. Given $H_\nu = H_{\nu 1} \otimes H_{\nu 2}$, $M_\nu = B(H_{\nu 1}) \otimes \mathbf{1}$ and unit vectors $\Omega_{\nu 1}$, $\Omega_{\nu 2} \in H_\nu$. Let $\mathrm{Sp}(\Omega_{\nu i}/M_\nu) = \{\lambda^i_{\nu j}\}$, $i=1, 2$. If

$$\sum_\nu [1 - \sum_j (\lambda^1_{\nu j} \lambda^2_{\nu j})^{1/2}] = \frac{1}{2} \sum_{\nu, j} ([\lambda^1_{\nu j}]^{1/2} - [\lambda^2_{\nu j}]^{1/2})^2 < \infty \tag{2.11}$$

then $\mathrm{R}(M_\nu, \Omega_{\nu 1})$ and $\mathrm{R}(M_\nu, \Omega_{\nu 2})$ are unitarily equivalent.

Proof. Let $\psi^{\nu i}_{1j}$ and $\psi^{\nu i}_{2j}$ be orthogonal vectors corresponding to $\lambda^i_{\nu j}$ in a standard diagonal expansion of $\Omega_{\nu i}$, $i=1, 2$. It is evidently possible to choose ψ's so that $\psi^{\nu 1}_{2j}$ and $\psi^{\nu 2}_{2j}$ are normalized or 0 simultaneously. We also supply, if necessary, additional indices so that $\psi^{\nu i}_{2j}$ are complete. Define unitary operators

$$U_{\nu k} \psi^{\nu 1}_{kj} = \psi^{\nu 2}_{kj}, \qquad k=1, 2$$
$$U_\nu = U_{\nu 1} \otimes U_{\nu 2}.$$

Then $U_\nu \Omega_{\nu 1}$ is in the strong equivalence class of $\Omega_{\nu 2}$ if

$$\infty > \Sigma_\nu |1 - (\Omega_{\nu 2}, U_\nu \Omega^1_{\nu 1})| = \sum_\nu |1 - \sum_j (\lambda^1_{\nu j} \lambda^2_{\nu j})^{1/2}|.$$

Q.E.D.

The following lemma gives some known results which we shall have occasion to use

Lemma 2.14. Given $M = \mathrm{R}(M_\nu, \Omega_\nu)$ where M_ν is type I_{n_ν},

$2 \leq n_\nu \leq \infty$, and $\mathrm{Sp}(\Omega_\nu/M_\nu) = \{\lambda_{\nu i}, i=1, 2, \cdots n_\nu\}$

1) M is type I if and only if

$$\sum_\nu |1-\lambda_{\nu 1}| < \infty \tag{2.12}$$

2) M is type II_1 if and only if $n_\nu < \infty$ for all ν and

$$\sum_{\nu, i} |(n_\nu)^{-1/2} - (\lambda_{\nu i})^{1/2}|^2 < \infty \tag{2.13}$$

3) If $\lambda_{\nu 1} \geq \delta$ for some $\delta > 0$ for all ν, then M is type III if and only if

$$\sum_{\nu, i} \lambda_{\nu i} \inf\{|(\lambda_{\nu 1}/\lambda_{\nu i}) - 1|^2, C\} = \infty \tag{2.14}$$

for some (and hence all) positive C.

Proof. For the type I conditions, see Araki [1] and Bures [6]. For the type II_1 and III conditions, see Pukanszky [14], Bures [6], and Moore [10]. Q.E.D.

The type I and II_1 conditions also follow from our results (see definition 8.2, lemmas 8.14, 8.15, 8.16 and theorem 9.1).

3. Asymptotic Ratio Set for ITPFI Factors

In this section we define the asymptotic ratio set for ITPFI factors and give some of its properties.

Consider $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu; \nu \in A)$, and a finite subset $I \subset A$. Let $\mathrm{Sp}(\Omega_\nu/M_\nu) = \{\lambda_{\nu j}\}$, then any $\lambda \in \mathrm{Sp}(\Omega(I)/M(I))$ is of the form $\lambda = \prod_{\nu \in I} \lambda_{\nu, k(\nu)}$ for some function $k(\nu)$.

Definition 3.1. Given $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu; \nu \in A)$ and a finite $I \subset A$, for any $K \subset \mathrm{Sp}(\Omega(I)/M(I))$ we define

$$\lambda(K) = \sum_{\lambda \in K} \lambda. \tag{3.1}$$

Definition 3.2. The asymptotic ratio set of $M = \mathrm{R}(M_\nu, \Omega_\nu)$, denoted by $\mathrm{r}_\infty(M, \Omega)$, is the set of all $x \in [0, \infty]$ such that there exists a sequence of mutually disjoint finite index sets $I_n \subset A$, $n \in I_\infty$, mutually disjoint subsets K_n^1, K_n^2 of $\mathrm{Sp}(\Omega(I_n)/M(I_n))$ for each n

such that $\lambda\in K_n^1$ implies $\lambda\neq 0$, and a bijection ϕ_n from K_n^1 to K_n^2 satisfying

$$\sum_n \lambda(K_n^1)=\infty \tag{3.2}$$

and

$$\lim_{n\to\infty}\max_{\lambda\in K_n^1}|x-\phi_n\lambda/\lambda|=0. \tag{3.3}$$

Such a sequence (I_n, K_n^i, ϕ_n) is called an x-sequence (K_n^1 and K_n^2 are to be regarded disjoint even if they contain the same eigenvalue as long as the total number does not exceed the multiplicity of the eigenvalue).

As defined here, $r_\infty(M, \Omega)$ could depend on the tensor product factorization $M=\otimes M_\nu$ as well as on Ω. However, it will be shown that $r_\infty(M, \Omega)$ is an algebraic invariant of M (Theorem 5.9) and therefore depends neither on the vector Ω nor on the factorization. Since we do not need to indicate explicitly the possible dependence on the factorization in the following, we shall not do so.

It should be noted that in definition 3.2, Ω_ν need not be a unit vector. Let $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_{\nu 1}, \cdots \lambda_{\nu n_\nu})$. Then $\sum_{i=1}^{n_\nu}\lambda_{\nu i}=\|\Omega_\nu\|^2\neq 1$ in general. However it follows from $0<\Pi\|\Omega_\nu\|^2<\infty$ that $\sum_{i=1}^{n_\nu}\lambda_{\nu i}\to 1$ sufficiently fast that

$$\sum_\nu |1-\sum_{i=1}^{n_\nu}\lambda_{\nu i}|<\infty.$$

Lemma 3.3. Given $\varepsilon_m>0$ and $x\in r_\infty(M, \Omega)$ there exists an x-sequence (I_m, K_m^i, ϕ_m) satisfying

$$|1-\lambda(K_m^1)-\lambda(K_m^2)|<\varepsilon_m. \tag{3.4}$$

Proof. Without loss of generality, we can assume $\lim \varepsilon_m=0$. Let (J_n, L_n^i, ψ_n) be an x-sequence. Let

$$Q_n=\|\Omega(J_n)\|^2.$$

Since

$$0<\Pi Q_n<\infty \tag{3.5}$$

we have

$$\sum |1-Q_n| < \infty$$

and Eq. (3.2) gives

$$\sum |1-Q_n+\lambda(L_n^1)| = \infty.$$

Thus

$$\prod_{n>N} [Q_n-\lambda(L_n^1)] = 0 \tag{3.6}$$

for arbitrary N. It follows from Eqs. (3.5), (3.6) that we can inductively choose mutually disjoint finite index sets A_m, $m \in I_\infty$ such that

$$|1-\prod_{n\in A_m} Q_n| < \varepsilon_m/2 \tag{3.7}$$

and

$$\prod_{n\in A_m} [Q_n-\lambda(L_n^1)] < \varepsilon_m/2. \tag{3.8}$$

Define $I_m = \bigcup_{n\in A_m} J_n$. For each $\lambda \in \mathrm{Sp}\, \Omega(I_m)$ we have $\lambda = \prod_{n\in A_m} \lambda(n)$ where $\lambda(n) \in \mathrm{Sp}\, \Omega(J_n)$. Define $n(\lambda)$ as the smallest $n \in J_m$ such that $\lambda(n) \in L_n^1 \cup L_n^2$ if such n exists, otherwise define $n(\lambda) = \infty$. Define

$$K_m^i = \{\lambda \in \mathrm{Sp}\, \Omega(I_m) : n(\lambda) \neq \infty,\ \lambda(n(\lambda)) \in L_n^i,\ \lambda(n) \neq 0 \text{ for } n \neq n(\lambda)\}.$$

For $\lambda \in K_m^1$ define

$$(\phi_m \lambda)(n) = \begin{cases} \lambda(n) & \text{if } n \neq n(\lambda),\ n \in A_m \\ \psi_{n(\lambda)} \lambda(n(\lambda)) & \text{if } n = n(\lambda). \end{cases}$$

Using Eq. (3.3) we have

$$\begin{aligned} &\lim_{m\to\infty} \max_{\lambda\in K_m^1} |x-\phi_m\lambda/\lambda| \\ &\qquad = \lim_{m\to\infty} \max_{n\in A_m} \max_{\lambda(n)\in L_n^1} |x-\psi_n\lambda(n)/\lambda(n)| = 0 \end{aligned}$$

thus

$$\lim_{m\to\infty} \max_{\lambda\in K_m^1} |x-\phi_m\lambda/\lambda| = 0. \tag{3.9}$$

If $\lambda \in \mathrm{Sp}\, \Omega(I_m)$ then $\lambda \notin K_m^1 \cup K_m^2$ only if $\lambda = 0$ or $\lambda(n) \notin L_n^1 \cup L_n^2$ for all $n \in A_m$. It follows that

$$
\begin{aligned}
(3.10)\qquad 0 &\le \prod_{n\in A_m} Q_n - \lambda(K^1_m) - \lambda(K^2_m)\\
&= \prod_{n\in A_m} [Q_n - \lambda(L^1_n) - \lambda(L^2_n)]\\
&\le \prod_{n\in A_m} [Q_n - \lambda(L^1_n)] < \varepsilon_m/2
\end{aligned}
$$

where we used Eq. (3.8). It follows from Eqs. (3.10), (3.7) that Eq. (3.4) is satisfied. It follows from Eq. (3.9) that

$$\lim_{m\to\infty} \lambda(K^2_m)/\lambda(K^1_m) = x.$$

Since $\lim \varepsilon_n = 0$, Eq. (3.10) then implies that $\lim \lambda(K^1_m) = (1+x)^{-1}$. Thus $\sum \lambda(K^1_m) = \infty$ and (I_m, K^i_m, ϕ_m) is an x-sequence. Q.E.D.

Corollary 3.4. If $x \in \mathrm{r}_\infty(M, \Omega)$ there exists an x-sequence (I_m, K^i_m, ϕ_m) satisfying

$$(3.11)\qquad \lim_{m\to\infty} \lambda(K^1_m) = (1+x)^{-1}$$

$$
\begin{aligned}
(3.12)\qquad \lim_{m\to\infty} \lambda(K^2_m) &= x/(1+x)\\
&= (1+x^{-1})^{-1} \quad \text{if } x \ne 0.
\end{aligned}
$$

Lemma 3.5. Given $\mathrm{R}(H_\nu, M_\nu, \Omega_\nu)$, $\mathrm{Sp}(\Omega_\nu/M_\nu) = \{\lambda_{\nu j}\}$. If there are subsequences $\nu(m)$, $j_1(m)$, $j_2(m)$ such that $\lambda_{\nu(m), j_1(m)} \to \lambda_1 \ne 0$ and $\lambda_{\nu(m), j_2(m)} \to \lambda_2$, then $\lambda_2/\lambda_1 \in \mathrm{r}_\infty(M, \Omega)$.

Proof. Let $I_m = \{\nu(m)\}$, $K^i_m = \{\lambda_{\nu(m), j_i(m)}\}$, $i = 1, 2$ and $\phi_m \lambda_{\nu(m), j_1(m)} = \lambda_{\nu(m), j_2(m)}$. Clearly (I_m, K^i_m, ϕ_m) is a (λ_2/λ_1)-sequence. Q.E.D.

Lemma 3.6. $\mathrm{r}_\infty(M, \Omega) - \{0\}$ is a multiplicative subgroup of $(0, \infty)$.

Proof. Let $x \in \mathrm{r}_\infty(M, \Omega) - \{0\}$. Choose an x-sequence $(I_n, K^1_n, K^2_n, \phi_n)$ as in Corollary 3.4. Then $(I_n, K^2_n, K^1_n, \phi^{-1}_n)$ is an x^{-1}-sequence.

Let $x, y \in \mathrm{r}_\infty(M, \Omega) - \{0\}$. Choose x and y-sequences $(I_n, K^{xi}_n, \phi^x_n)$ and $(I^y_n, K^{yi}_n, \phi^y_n)$ as in Corollary 3.4. Choose subsequences $p_n, q_n, n \in I_\infty$ such that $I^x_{p_n}$ and $I^y_{q_n}$ are mutually disjoint. Define $I_n = I_{p_n} \cup I_{q_n}$, $K^i_n = \{\lambda_x \lambda_y : \lambda_x \in K^{xi}_{p_n}, \lambda_y \in K^{yi}_{q_n}\}$, $\phi_n = \phi^x_{p_n} \phi^y_{q_n}$. Then

$$(3.13)\qquad \lim_{n\to\infty} \max_{\lambda \in K^1_n} |xy - \phi_n \lambda/\lambda| = 0.$$

Also

(3.14) $$\lim_{n\to\infty}\lambda(K_n^1)=(1+x)^{-1}(1+y)^{-1}$$

thus $\sum\lambda(K_n^1)=\infty$, and (I_n, K_n^i, ϕ_n) is an xy-sequence. Q.E.D.

Lemma 3.7. $r_\infty(M, \Omega)$ is closed.

Proof. Let $x_p\in r_\infty(M, \Omega)$, $x=\lim x_p$. Without loss of generality we can assume $x_p<y$ for some $y<\infty$ and all p. It then follows from Corollary 3.4 that there exist x_p-sequences $(I_n^p, K_n^{pi}, \phi_n^p)$ such that

(3.15) $$\lambda(K_n^{p1})>\frac{1}{2}(1+y)^{-1}>0$$

for all p, n. For each $p\in I_\infty$ choose n_p inductively such that $I_{n_p}^p$ is disjoint from $I_{n_q}^q$ for $q<p$, and such that

(3.16) $$\lim_{p\to\infty}\max_{\lambda\in K_{n_p}^{pl}}|x_p-\phi_{n_p}^p\lambda/\lambda|=0.$$

It follows from Eq. (3.15) that $\sum_{p=1}^{\infty}\lambda(K_{n_p}^p)=\infty$, hence $(I_{n_p}^p, K_{n_p}^{pi}, \phi_{n_p}^p; p\in I_\infty)$ is an x-sequence. Q.E.D.

Lemma 3.8. Given $R(M_\nu, \Omega_\nu)$ where M_ν is type I_{n_ν}, and $Sp(\Omega_\nu/M_\nu)=\{\lambda_{\nu j}, j=1, \cdots n_\nu\}$. If $\sum|1-\lambda_{\nu 1}|<\infty$ then $r_\infty(M, \Omega)=\{0\}$. If $\sum|1-\lambda_{\nu 1}|=\infty$ then $1\in r_\infty(M, \Omega)$.

Proof. We have $0<\Pi\|\Omega_\nu\|<\infty$ and thus $\sum_{j=1}^{n_\nu}\lambda_{\nu j}=\|\Omega_\nu\|=1+\delta_\nu$ where $\sum|\delta_\nu|<\infty$. By lemma 3.14 (which depends only on definition 3.2) $r_\infty(M)$ is unaffected by the change $\lambda_{\nu j}\to\lambda_{\nu j}/\|\Omega_\nu\|$. Since the condition $\sum|1-\lambda_{\nu 1}|<\infty$ is also unaffected by this change, we can assume $\sum\lambda_{\nu j}=1$ for all ν.

If $\sum|1-\lambda_{\nu 1}|<\infty$ then $\lambda_{\nu 1}\to 1$, hence $\lambda_{\nu 2}\to 0$ and $0\in r_\infty(M, \Omega)$ by Lemma 3.5. Let (I_m, K_m^i, ϕ_m) be an x-sequence for (M, Ω). Since $\lambda_{\nu 1}\neq 0$ and $\sum|1-\lambda_{\nu 1}|<\infty$ we have $\Pi\lambda_{\nu 1}>0$. Let $\lambda_1^{(m)}=\prod_{\nu\in I_m}\lambda_{\nu 1}$, then $\Pi_m\lambda_1^{(m)}>0$ which implies that $\sum_m(1-\lambda_1^{(m)})<\infty$ and thus $\sum\lambda(K_m^1)=\infty$ if and only if $\lambda_1^{(m)}\in K_m^1$ for infinitely many m. Since

$\Pi_m \lambda_1^{(m)} > 0$ implies that $\lambda_1^{(m)} \to 1$, it follows that $x=0$, hence $r_\infty(M, \Omega) = \{0\}$.

The second part of the lemma is more difficult, and we consider separately three different cases for $\{\lambda_{\nu j}\}$. Case (i), $\lambda_{\nu 2}$ has an accumulation point $\lambda_2 > 0$. Let $\lambda_{\nu(k),2} \to \lambda_2$ as $k \to \infty$. Then $\lambda_{\nu(k),1}$ must have an accumulation point $\lambda_1 \geq \lambda_2$ and $\lambda_2/\lambda_1 \in r_\infty(M, \Omega)$ by lemma 3. 5. It then follows from lemma 3. 6 that $1 \in r_\infty(M, \Omega)$.

Case (ii), $\lambda_{\nu 2} \to 0$, $\lambda_{\nu 1} \to 1$, $\sum \lambda_{\nu 2} = \infty$. We can remove all ν with $\lambda_{\nu 2} = 0$ and then reorder the remaining ones so that we have

$$\lambda_{12} \geq \lambda_{22} \geq \cdots \geq 0 \tag{3.17}$$

which implies that

$$\sum \lambda_{2\nu,2} = \infty. \tag{3.18}$$

For any $\varepsilon > 0$ let

$$I_\varepsilon = \{\nu : \lambda_{2\nu+1,2}/\lambda_{2\nu,2} > 1 - \varepsilon\}. \tag{3.19}$$

It follows from Eq. (3. 17), (3. 19) and the ratio test that

$$\sum_{\nu \notin I_\varepsilon} \lambda_{2\nu,2} < \infty. \tag{3.20}$$

Thus we can inductively choose mutually disjont finite sets J_n, $n \in I_\infty$ such that

$$\sum_{\nu \in J_n} \lambda_{2\nu,2} > 1 \tag{3.21}$$

and

$$|1 - \lambda_{2\nu+1,2}/\lambda_{2\nu,2}| < \varepsilon_n \tag{3.22}$$

for all $\nu \in J_n$, where $\varepsilon_n \to 0$. In this way we obtain a subsequence $\nu(j)$ such that

$$\sum \lambda_{2\nu(j),2} = \infty \tag{3.23}$$

and

$$\lim_{j \to \infty} \lambda_{2\nu(j)+1,2}/\lambda_{2\nu(j),2} = 1. \tag{3.24}$$

Let $I_j = \{2\nu(j), 2\nu(j)+1\}$, $K_j^1 = \{\lambda_{2\nu(j),2} \lambda_{2\nu(j)+1,1}\}$, $K_j^2 = \{\lambda_{2\nu(j),1} \lambda_{2\nu(j)+1,2}\}$ and ϕ_j the unique bijection from K_j^1 to K_j^2. Since $\lambda_{\nu 1} \to 1$ it follows that

$\sum \lambda(K_j^1)=\infty$ and (I_j, K_j^i, ϕ_j) is a 1-sequence, and $1\in r_\infty(M, \Omega)$.

Case (iii), $\lambda_{\nu 2}\to 0$ and either $\sum \lambda_{\nu 2}<\infty$ or $\lambda_{\nu 1}$ has an accumulation point $\lambda_1\neq 1$. It follows in either case that

$$\sum_\nu \sum_{j=3}^{n_\nu} \lambda_{\nu j}=\infty. \tag{3.25}$$

Let

$$x_{\nu k}=\begin{cases}\lambda_{\nu k}/\lambda_{\nu, k-1} & \text{if } \lambda_{\nu, k-1}\neq 0\\ 0 & \text{if } \lambda_{\nu, k-1}=0\end{cases} \tag{3.26}$$

$$P_\nu=\sum_{j=3}^{n_\nu} \left(\prod_{k=3}^{j} x_{\nu k}\right) \tag{3.27}$$

$$\Lambda_\nu=\sum_{j=3}^{n_\nu} \lambda_{\nu j}=\lambda_{\nu 2} P_\nu. \tag{3.28}$$

If $\sum \lambda_{\nu 2}<\infty$ it follows from Eqs. (3.25) and (3.28) that for all $N<\infty$ we have

$$\sum_{P_\nu>N} \Lambda_\nu=\infty. \tag{3.29}$$

If $\lambda_{\nu 1}$ has an accumulation point $\lambda_1\neq 1$ then there is a subsequence $\nu(j)$ such that $\lambda_{\nu(j),1}\to\lambda_1$, and hence $\Lambda_{\nu(j)}\to 1-\lambda_1$, and Eq. (3.29) holds. It follows from

$$\sum_{m=0}^{\infty} (1-\varepsilon)^m=\varepsilon^{-1},\ 0<\varepsilon<1 \tag{3.30}$$

that

$$\sum_{x_{\nu j}<1-\varepsilon} \left(\prod_{k=3}^{j} x_{\nu k}\right)<\varepsilon^{-1} \tag{3.31}$$

for a fixed ν. Hence

$$\sum_{x_{\nu j}\geqq 1-2P_\nu^{-1}} \left(\prod_{k=3}^{j} x_{\nu k}\right)>\frac{1}{2} P_\nu \tag{3.32}$$

which implies that for each ν there are disjoint pairs $(\lambda_{\nu k}, \lambda_{\nu, k+1})$ such that

$$\lambda_{\nu, k+1}/\lambda_{\nu, k}\geqq 1-2P_\nu^{-1} \tag{3.33}$$

and

$$\sum_k \lambda_{\nu k}>\frac{1}{4} P_\nu \lambda_{\nu 2}=\frac{1}{4}\Lambda_\nu. \tag{3.34}$$

Let $I_\nu=\{\nu\}$ and let K_ν^1, K_ν^2 contain the first and second members respectively of these disjoint pairs with $\phi_\nu\lambda_{\nu k}=\lambda_{\nu,k+1}$. It follows from Eqs. (3.29) and (3.34) that we can inductively choose mutually disjoint finite subsets L_n such that

(3.35) $$\sum_{\nu\in L_n}\lambda(K_\nu^1)>1$$

and

(3.36) $$1\geqq\phi_\nu\lambda/\lambda>1-2/n \text{ if } \lambda\in K_\nu^1,\ \nu\in L_n$$

which implies that $1\in \mathrm{r}_\infty(M, \Omega)$. Q.E.D.

Theorem 3.9. Let $\mathrm{R}(M_\nu, \Omega_\nu)$ be an ITPFI factor. Then $\mathrm{r}_\infty(M, \Omega)$ must be one of the following sets: S_x, $0\leq x\leq 1$, S_{01}, S_∞.

Proof. The sets S_x, $0\leq x\leq 1$, S_{01}, S_∞ are consistent with lemmas 3.6 and 3.7. By lemma 3.8. $\mathrm{r}_\infty(M, \Omega)$ is nonempty. If $\mathrm{r}_\infty(M, \Omega)$ is not one of the sets S_0, S_1, S_{01} consider the set of all l such that $e^l\in\mathrm{r}_\infty(M, \Omega)-\{0\}$. By lemma 3.6 this set must be of the form $\{nl, n=0, \pm 1, \cdots\}$ for some $0<l<\infty$, or be dense. In the former case we have $\mathrm{r}_\infty(M, \Omega)=S_x$, $x=e^{-l}$. If the latter case holds then $\mathrm{r}_\infty(M, \Omega)=[0,\infty)=S_\infty$ by lemma 3.7. Q.E.D.

We now discuss some standard cases which have received some attention in the literature (von Neumann [12], Powers [13]). The following definition introduces our notation for these examples.

Definition 3.10. Let $M=\mathrm{R}(H_\nu, M_\nu, \Omega_\nu)$ be an ITPFI factor where $\dim H_\nu=4$, M_ν is type I_2, and $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_1, \lambda_2)$ independent of ν. We denote the factor M by R_x where $x=\lambda_2/\lambda_1$.

Clearly $\mathrm{r}_\infty(R_x)=S_x$. By lemma 2.14, R_0 is type I_∞, R_1 is hyperfinite II_1, and R_x, $x\neq 0, 1$ is type III. Powers [13] has shown that R_x is nonisomorphic for different x. In the following we shall rederive this result in a more general context.

We now give some elementary properties of tensor products of ITPFI factors. More detailed results will be given later. Let A_1, A_2 be disjoint index sets and let $A=A_1\cup A_2$. Consider the ITPFI factors $M_i=\mathrm{R}(H_\nu, M_\nu, \Omega_\nu;\ \nu\in A_i)$, $i=1, 2$. Then $M_1\otimes M_2=\mathrm{R}(H_\nu, M_\nu, \Omega_\nu;\ \nu\in A)$.

Clearly $x\in r_\infty(M_1, \Omega)$ implies $x\in r_\infty(M_1\otimes M_2, \Omega)$. Thus we obtain

Lemma 3.11. Given $R(M_\nu, \Omega_\nu)$, $R(N_\alpha, \psi_\alpha)$. Then

$$r_\infty(M\otimes N, \Omega\otimes\psi)\supset r_\infty(M, \Omega)\cup r_\infty(N, \psi).$$

If either $r_\infty(M,\Omega)$ or $r_\infty(N, \psi)$ is S_∞, then

$$r_\infty(M\otimes N, \Omega\otimes\psi)=S_\infty.$$

Definition 3.12. Given $0\leq l_1, l_2<\infty$. If l_1/l_2 is rational, we define $(l_1, l_2)=l$ where l is the largest number such that both l_1 and l_2 are integer multiples of l.

Lemma 3.13. Given $0<x_1, x_2\leq 1$, $R(M_\nu, \Omega_\nu)$, $R(N_\alpha, \psi_\alpha)$ and $r_\infty(M, \Omega)=S_{x_1}$, $r_\infty(N, \psi)=S_{x_2}$. Let $x_1=e^{-l_1}$, $x_2=e^{-l_2}$. If l_1/l_2 is irrational then

$$r_\infty(M\otimes N, \Omega\otimes\psi)=S_\infty. \tag{3.37}$$

If l_1/l_2 is rational then

$$r_\infty(M\otimes N, \Omega\otimes\psi)\supset S_x$$

where $x=e^{-(l_1,l_2)}$.

Proof. Follows from lemmas 3.6, 3.7, and 3.11. Q.E.D.

The following lemma proves that we can always take Ω_ν to be a unit vector.

Lemma 3.14. Given $R(M_\nu, \Omega_\nu)$. Let $\Omega^a_\nu=\Omega_\nu/\|\Omega_\nu\|$. Then $\otimes\Omega^a_\nu\sim\otimes\Omega_\nu$ and $r_\infty(M, \Omega^a)=r_\infty(M, \Omega)$.

Proof. Since

$$0<\Pi\|\Omega_\nu\|<\infty \tag{3.38}$$

we have

$$\sum|1-(\Omega^a_\nu, \Omega_\nu)|=\sum|1-\|\Omega_\nu\||<\infty \tag{3.39}$$

and thus $\otimes\Omega^a_\nu\sim\otimes\Omega_\nu$. Let $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu j}\}$, then $\mathrm{Sp}(\Omega^a_\nu/M_\nu)=\{\alpha_{\nu j}\}$ where

$$\alpha_{\nu j}=\lambda_{\nu j}/\|\Omega_\nu\|^2. \tag{3.40}$$

Let (I_n, K^i_n, ϕ_n) be an x-sequence for $R(M_\nu, \Omega_\nu)$. The one-to-one map

$\lambda_{\nu j}\to\alpha_{\nu j}$ defines a sequence $(I_n, K_n^{ai}, \phi_n^a)$. Eq. (3.38) gives

(3.41) $$\lim_{m\to\infty}\prod_{\nu>m}\|\Omega_\nu\|^2=1$$

and thus $\sum\lambda(K_n^1)=\infty$ implies that

(3.42) $$\sum\lambda(K_n^{al})=\infty.$$

Eqs. (3.40), (3.41) imply that

(3.43) $$\phi_n\mu/\mu=\phi_n^a\mu^a/\mu^a$$

where $\mu\to\mu^a$, $\phi_n\mu\to\phi_n^a\mu^a$. Thus $(I_n, K_n^{ai}, \phi_n^a)$ is an x-sequence for $\mathrm{R}(M_\nu, \Omega_\nu^a)$. Since the argument is reversible, $\mathrm{r}_\infty(M, \Omega)=\mathrm{r}_\infty(M, \Omega^a)$.

Q.E.D.

Lemma 3.15. Given $\mathrm{R}(N_\nu, \psi_\nu)$ there exists $\mathrm{R}(M_\nu, \Omega_\nu)\sim\mathrm{R}(N_\nu, \psi_\nu)$ such that Ω_ν is cyclic and separating for M_ν and $\mathrm{r}_\infty(M, \Omega)=\mathrm{r}_\infty(N, \psi)$.

Proof. Construct M_ν, ω_ν as in lemma 2.12. Since $\mathrm{Sp}(\omega_\nu/M_\nu)=\mathrm{Sp}(\psi_\nu/N_\nu)$ we have $\mathrm{r}_\infty(M, \omega)=\mathrm{r}_\infty(N, \psi)$. Let $\mathrm{Sp}(\omega_\nu/M_\nu)=\{\lambda_{\nu i}^a\}$ and let m_ν be the number of $\lambda_{\nu i}^a=0$. If $m_\nu=0$ let

(3.44) $$\lambda_{\nu i}^b=\lambda_{\nu i}^a.$$

If $m_\nu\neq 0$ let

(3.45) $$\lambda_{\nu i}^b=\begin{cases}(1-\varepsilon_\nu)\lambda_{\nu i}^a & \text{if } \lambda_{\nu i}^a\neq 0\\ \varepsilon_\nu/m_\nu & \text{if } \lambda_{\nu i}^a=0\end{cases}$$

where

(3.46) $$\varepsilon_\nu=2^{-\nu}\min_k\{\lambda_{\nu k}^a;\ \lambda_{\nu k}^a\neq 0\}.$$

Now choose Ω_ν as in lemma 2.12 where $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu i}^b\}$. Let $(I_n, K_n^{ai}, \phi_n^a)$ be an x-sequence for $\mathrm{R}(M_\nu, \omega_\nu)$. The one-to-one map $\lambda_{\nu j}^a\to\lambda_{\nu j}^b$ defines a sequence $(I_n, K_n^{bi}, \phi_n^b)$. It follows from Eq. (3.46) that

(3.47) $$\lim_{m\to\infty}\prod_{\nu>m}(1-\varepsilon_\nu)=1$$

and thus $\sum\lambda(K_n^{a1})=\infty$ implies that

(3.48) $$\sum\lambda(K_n^{b1})=\infty.$$

Since $\mu^a \in K_n^{a1}$ implies $\mu^a \neq 0$ it follows from Eqs. (3.44–47) that

$$\lim_{n\to\infty} \max_{\mu^a \in K_n^{a1}} |\phi_n^a \mu^a/\mu^a - \phi_n^b \mu^b/\mu^b| = 0 \tag{3.49}$$

where $\mu^a \to \mu^b$, $\phi_n^a \mu^a \to \phi_n^b \mu^b$. It follows that $(I_n, K_n^{bi}, \phi_n^b)$ is an x-sequence for $\mathrm{R}(M_\nu, \Omega_\nu)$. Conversely let $(I_n, K_n^{bi}, \phi_n^b)$ be an x-sequence for $\mathrm{R}(M_\nu, \Omega_\nu)$. Let K be the set of all $\mu^b \in K_n^{b1}$, $n=1, 2, \cdots$ which contain $\lambda_{\nu j}^b$ as a factor where $\lambda_{\nu j}^a = 0$. It follows from Eqs. (3.45–46) that

$$\sum_{\mu^b \in K} \mu^b < \infty$$

and thus we can remove all $\mu^b \in K$ from K_n^{b1}, $n \in I_\infty$. The above argument can then be reversed and the sequence $(I_n, K_n^{ai}, \phi_n^a)$ defined by $\lambda_{\nu j}^b \to \lambda_{\nu j}^a$ is an x-sequence for $\mathrm{R}(M_\nu, \omega_\nu)$. Q.E.D.

4. Basic Technical Lemma

In this section we prove a basic technical lemma which is concerned with the following situation. Let M be a type I_m factor on a Hilbert space H and let Ω be a vector in H. Let $(\lambda_1, \cdots \lambda_n)$ be a possible eigenvalue list, that is $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$. Let $(\mu_1, \cdots \mu_l)$ be another possible eigenvalue list. Suppose the list of products $\{\lambda_i \mu_j\}$ approximates in some suitable way the eigenvalue list $\mathrm{Sp}(\Omega/M)$. Then it should be possible to find a type I_n factor M_1 such that (M, Ω) is in some sense approximated by $(M_1 \otimes M_2, \Phi_1 \otimes \Phi_2)$ where $\mathrm{Sp}(\Phi_1/M_1) = (\lambda_1, \cdots \lambda_n)$.

Definition 4.1. Let M be a factor on a Hilbert space H, N a type I factor, $N \subset M$. We say that $\Phi \in H$ factorizes N in M if

$$H = H_1 \otimes H_2$$
$$H_1 = H_{11} \otimes H_{12}$$
$$\Phi = \Phi_1 \otimes \Phi_2, \quad \Phi_i \in H_i$$
$$N = \hat{N} \otimes \mathbf{1}$$
$$\hat{N} = \mathcal{B}(H_{11}) \otimes \mathbf{1}$$
$$M = \hat{N} \otimes M_2.$$

Lemma 4. 2. Given $0<\varepsilon<1$, a Hilbert space H, a type I factor $M\subset\mathcal{B}(H)$, a unit vector $\Omega\in H$, $(\lambda_1,\cdots\lambda_n)$ satisfying $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n\geq 0$ and $\sum\lambda_j=1$. Let $K_1,\cdots K_n$ be disjoint subsets of $\mathrm{Sp}(\Omega/M)$ such that each K_j contains k elements and $\lambda\in K_1$ implies $\lambda\neq 0$. Let Φ_j be a bijection from K_1 to K_j, $j=2,\cdots n$. Let $L=\mathrm{Sp}(\Omega/M)-\bigcup_{j=1}^{n}K_j$. Let $\varepsilon'=\min(\varepsilon, \lambda_j/\lambda_1 \text{ for } \lambda_j\neq 0)$. If

$$\max_{j=2}^{n}\max_{\lambda\in K_1}|(\lambda_j/\lambda_1)^{1/2}-(\phi_j\lambda/\lambda)^{1/2}|<\varepsilon' \tag{4.1}$$

and

$$\lambda(L)<\varepsilon \tag{4.2}$$

then the following situation holds. There exist projections $P\in M$, $P'\in M'$, a type I_n factor $N\subset M_{PP'}$, and a unit vector $\Phi\in PP'H$ such that

$$\|(1-PP')\Omega\|^2<n\varepsilon \tag{4.3}$$

$$\|PP'\Omega-\Phi\|<c_n\varepsilon \tag{4.4}$$

where c_n depends only on n, Φ factorizes N in $M_{PP'}$ and $\mathrm{Sp}(\Phi/N)=\{\lambda_1,\cdots\lambda_n\}$.

Proof. Let $\Omega=\sum\omega_\alpha^{1/2}\psi_{1\alpha}\otimes\psi_{2\alpha}$ be a standard diagonal expansion of Ω (relative to M). We reindex the $\omega_\alpha\in\bigcup_{j=1}^{n}K_j$ as follows. Order the elements of K_1 by $\omega_{11}\geq\omega_{12}\geq\cdots\geq\omega_{1k}$, and let $\omega_{jl}=\phi_j\omega_{1l}$, $j=2,\cdots n$. Let $u_{\alpha\beta}$, $v_{\alpha\beta}$ be the standard matrix units for M, M' associated with this expansion. Define

$$P=\sum_{i=1}^{n}\sum_{j=1}^{k}u_{ij,ij}. \tag{4.5}$$

If $\lambda_n=0$ choose p so that $\lambda_p\neq 0$, $\lambda_{p+1}=0$, otherwise choose $p=n$. If $1\leq i\leq p$ then Eq. (4.1) implies that $\omega_{ij}\neq 0$ since by assumption $\omega_{1j}\neq 0$. Hence $v_{ij,i'j'}\neq 0$ for $i, i'=1\cdots p$. Since $\{\omega_{jl}\}$ is a subset of $\mathrm{Sp}(\Omega/M)$, $u_{ij,i'j'}\neq 0$ for $i, i'=1\cdots n$. (See definition 2. 3.) Define

$$P'=\sum_{i=1}^{p}\sum_{j=1}^{k}v_{ij,ij}. \tag{4.6}$$

Then $u_{ij,i'j'}$, $i, i'=1,\cdots n$ and $v_{ij,i'j'}$, $i, i'=1,\cdots p$ are matrix units for

$M_{PP'}$ and $M'_{PP'}$ respectively in $PP'H$. Define matrix units $e^{\nu q}_{ij}$, ν, $q=1, 2$ by

$$(4.7)\qquad \begin{array}{ll} e^{11}_{ij}=\sum_{l=1}^{k} u_{il,jl} & e^{12}_{ij}=\sum_{l=1}^{k} v_{il,jl} \\ e^{21}_{ij}=\sum_{l=1}^{n} u_{li,lj} & e^{22}_{ij}=\sum_{l=1}^{p} v_{li,lj}. \end{array}$$

One can easily verify that these four families of matrix units commute and that they are irreducible on $PP'H$. Thus we obtain

$$(4.8)\qquad PP'H=\bigotimes_{\nu=1}^{2}(H_{\nu 1}\otimes H_{\nu 2})$$

where $e^{\nu q}_{ij}$ are matrix units for $\mathscr{B}(H_{\nu q})\otimes 1$. Thus $\dim H_{11}=n$, $\dim H_{12}=p$, $\dim H_{21}=\dim H_{22}=k$. We have

$$(4.9)\qquad PP'\Omega=\sum_{i=1}^{p}\sum_{j=1}^{k}\omega_{ij}^{1/2}\psi_{1,ij}\otimes\psi_{2,ij}$$

where $\psi_{q,ij}\in H_{1q}\otimes H_{2q}, q=1, 2$. Thus

$$(4.10)\qquad \|(1-PP')\Omega\|^2=\omega(L)+\sum_{i=p+1}^{n}\sum_{j=1}^{k}\omega_{ij}.$$

For $i>p$ Eq. (4.1) gives $\omega_{ij}<(\varepsilon')^2\omega_{1j}<\varepsilon'\omega_{1j}$ since $\varepsilon'\leq\varepsilon<1$. Since $p\geq 1$ and $\sum\omega_{1j}\leq 1$ we get

$$(4.11)\qquad \|(1-PP')\Omega\|^2<\varepsilon+(n-1)\varepsilon'\leq n\varepsilon.$$

Now choose orthonormal bases $\psi^{\nu q}_i$ for $H_{\nu q}$ such that $\psi_{q,ij}=\psi^{1q}_i\otimes\psi^{2q}_j$. Define

$$(4.12)\qquad \Phi_1=\sum_{i=1}^{p}\lambda_i^{1/2}\psi^{11}_i\otimes\psi^{12}_i$$

$$(4.13)\qquad \Phi'_2=\sum_{j=1}^{k}(\omega_{1j}/\lambda_1)^{1/2}\psi^{21}_j\otimes\psi^{22}_j$$

$$(4.14)\qquad \Phi_2=\Phi'_2/\|\Phi'_2\|.$$

We have

$$\|PP'\Omega-\Phi_1\otimes\Phi'_2\|^2=\sum_{i=1}^{p}\sum_{j=1}^{k}(\omega_{ij}^{1/2}-(\lambda_i\omega_{1j}/\lambda_1)^{1/2})^2$$

Eq. (4.1) gives

$$(4.15)\qquad |\omega_{ij}^{1/2}-(\lambda_i\omega_{1j}/\lambda_1)^{1/2}|<\varepsilon'\omega_{1j}^{1/2}$$

hence

$$\|PP'\Omega-\Phi_1\otimes\Phi_2'\|^2<p(\varepsilon')^2\leq n\varepsilon^2. \tag{4.16}$$

Since $\|\Phi_1\|^2=\sum\lambda_i=1$, it follows from Eqs. (4.13), (4.14) that

$$\|\Phi_1\otimes\Phi_2-\Phi_1\otimes\Phi_2'\|=\|\Phi_2-\Phi_2'\|=|1-\{\sum_{j=1}^{k}(\omega_{1j}/\lambda_1)\}^{1/2}|. \tag{4.17}$$

Eq. (4.15) gives

$$|\sum_{i=1}^{n}\sum_{j=1}^{k}\omega_{ij}-\sum_{i=1}^{n}\lambda_i\sum_{j=1}^{k}(\omega_{1j}/\lambda_1)|<2n\varepsilon'\leq 2n\varepsilon \tag{4.18}$$

where we used $\sum\omega_{1j}\leq 1$, $\sum_j\omega_{ij}\leq 1$ and $\varepsilon'\leq\varepsilon<1$. Since $\sum\lambda_i=1$ and $|1-\sum\omega_{ij}|<\varepsilon$ we get

$$|1-\sum(\omega_{1j}/\lambda_1)|<(2n+1)\varepsilon. \tag{4.19}$$

Eqs. (4.16), (4.17), (4.19) now give

$$\|PP'\Omega-\Phi_1\otimes\Phi_2\|<c_n\varepsilon \tag{4.20}$$

where

$$c_n=n^{1/2}+2n+1 \tag{4.21}$$

Q.E.D.

We remark that if the dimensions of M, M' are consistent with setting $P=P'=1$ in lemma 4.2, then it is possible to choose $P=P'=1$ in lemma 4.2. However we shall not make any use of this fact.

5. Algebraic Invariance of $\mathrm{r}_\infty(M)$

In this section we prove that $\mathrm{r}_\infty(M,\Omega)$ given by definition 3.2 for ITPFI factors $M=\mathrm{R}(M_\nu,\Omega_\nu)$ is an algebraic invariant.

We note that given an ITPFI factor $\mathrm{R}(M_\nu,\Omega_\nu)$, by lemma 3.14 we can assume $\|\Omega_\nu\|=1$. Unless stated explicitly to the contrary, we shall always take $\|\Omega_\nu\|=1$ in this section.

Lemma 5.1. Given $M=\mathrm{R}(M_\nu,\Omega_\nu;\ \nu\in A)$, $x\in r_\infty(M,\Omega)$, $x\leq 1$ and $0<\varepsilon<1$. Then there is a finite subset $I\subset A$, projections P, P' in $M(I)$, $M(I)'$ respectively, a unit vector $\Phi\in PP'H(I)$, and a type I_2 factor $N\subset M(I)_{PP'}$ such that

$$(5.1)\qquad \|(PP'-1)\Omega(I)\|<\varepsilon$$

$$(5.2)\qquad \|PP'\Omega(I)-\Phi\|<\varepsilon$$

$$(5.3)\qquad \mathrm{Sp}(\Phi/N)=(\lambda, 1-\lambda)$$

where $x=(1-\lambda)/\lambda$, and Φ factorizes N in $M(I)_{PP'}$.

Proof. If $x=0$ choose $\varepsilon'=\varepsilon$, $\varepsilon''=\varepsilon^2$. If $x\neq 0$ choose $\varepsilon'=\min(\varepsilon, x)$, $\varepsilon''=x^{1/2}\varepsilon'$. By Lemma 3.3 there exists (I, K^1, K^2, ϕ) such that

$$(5.4)\qquad \max_{\mu\in K^1}|x-\phi\mu/\mu|<\varepsilon''$$

and

$$(5.5)\qquad |1-\lambda(K^1)-\lambda(K^2)|<\varepsilon/c_2$$

where c_2 is given by lemma 4.2. Eq. (5.4) implies that

$$(5.6)\qquad \max_{\mu\in K^1}|x^{1/2}-(\phi\mu/\mu)^{1/2}|<\varepsilon'.$$

The result now follows from lemma 4.2. Q.E.D.

Lemma 5.2. Let M be a factor, P a projection in M. If M is infinite then $M\overset{u}{\sim}M_P\otimes I_\infty$. If P is infinite then $M\overset{u}{\sim}M_P$.

Proof. Two projections $E, F\in M$ are equivalent (with respect to M), $E\sim F$, if there is a partial isometry $U\in M$ such that $U^*U=E$, $UU^*=F$ (Murray and von Neumann [11]), which implies that $UM_EU^*=M_F$. Thus if P is infinite then $P\sim 1$ and $M_P\sim M$. If M is infinite, then for any $P\in M$ there exists a family of projections $P_i\in M$, $i\in I_\infty$ such that $\sum P_i=1$, $P_iP_j=0$ for all $i\neq j$, and $P_i\sim P$ for all i. Let $M_{ij}=\{P_iAP_j;\ A\in M\}=\{U_{ij}B;\ B\in M_{P_j}\}$ where $U_{ij}\in M$, $U_{ij}^*=U_{ji}$, $U_{ij}U_{kl}=\delta_{jk}U_{il}$, $U_{ii}=P_i$. Then $M=\otimes M_{ij}$ which is unitarily equivalent to $M_P\otimes I_\infty$. Q.E.D.

Definition 5.3. Given $M=\mathrm{R}(M_\nu, \Omega_\nu)$ where $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu j}\}$. We define

$$\mathfrak{r}(M, \Omega)=\{\lambda_{\nu i}/\lambda_{\nu j};\ \lambda_{\nu j}\neq 0\}.$$

Lemma 5.4. Given $M=\mathrm{R}(H_\nu, M_\nu, \Omega_\nu;\ \nu\in A)$, $N=\mathrm{R}(N_\alpha, \psi_\alpha)$ where all N_α are type I_2. If $\mathfrak{r}(N, \psi)\subset\mathfrak{r}_\infty(M, \Omega)$ then $M\sim M\otimes N$.

Proof. We shall use lemma 5.1 to construct a projection $P\in M$ such that M_P is an ITPFI factor and $M_P\sim M_a\otimes(\otimes N)^{\infty}$ where $(\otimes N)^{\infty}$ means the tensor product of N with itself infinitely many times. Let $\mathrm{Sp}(\psi_\alpha/N_\alpha)=(\lambda_\alpha, 1-\lambda_\alpha)$, $x_\alpha=(1-\lambda_\alpha)/\lambda_\alpha\leq 1$. By corollary 3.4 there exist x_α-sequences $(\alpha, n)=(I_n^\alpha, K_n^{\alpha i}, \phi_n^\alpha)$ for $M=\otimes M_\nu$ such that $\lambda(K_n^{\alpha i})>1/3$ for all (α, n). Let $l(\alpha, k)$ be a one-to-one map of the pairs of positive integers (α, k) onto the positive integers. In the order of increasing $l(\alpha, k)$ we inductively select a subsequence $n(\alpha, k)$, $\alpha, k\in I_\infty$ such that $I^\alpha_{n(\alpha,k)}\cap I^{\alpha'}_{n(\alpha',k')}=\phi$ for all (α', k') with $l(\alpha', k')<l(\alpha, k)$. Now reorder the sequence $k=1, 2, \cdots$ into a double sequence $k=l(i, j)$, $i, j\in I_\infty$ where $l(i, j)$ was introduced above. Let $M_{\alpha i}=\bigotimes_{j=1}^{\infty} M(I_{ij}^\alpha)$, $\Omega_{\alpha i}=\bigotimes_{j=1}^{\infty}\Omega(I_{ij}^\alpha)$ where $I_{ij}^\alpha=I^\alpha_{n(\alpha,l(i,j))}$. By construction $x_\alpha\in \mathrm{r}_\infty(M(I_{ij}^\alpha), \Omega(I_{ij}^\alpha); j\in I_\infty)$ for each α and i. Now choose $\varepsilon_{\alpha i}>0$, $\sum\varepsilon_{\alpha i}<\infty$. For each (α, i), it follows from lemma 5.1 that there is a set $I_{\alpha i}$ which is a finite union of the I_{ij}^α, $j\in I_\infty$, projections $P_{\alpha i}\in M(I_{\alpha i})$, $P'_{\alpha i}\in M(I_{\alpha i})'$, a unit vector $\Phi_{\alpha i}\in P_{\alpha i}P'_{\alpha i}H(I_{\alpha i})$ such that

$$\|\Omega_{\alpha i}-P_{\alpha i}P'_{\alpha i}\Omega_{\alpha i}\|<\varepsilon_{\alpha i} \tag{5.7}$$

$$|1-(\Phi_{\alpha i}, \hat{\Omega}_{\alpha i})|<\varepsilon_{\alpha i} \tag{5.8}$$

where

$$\hat{\Omega}_{\alpha i}=P_{\alpha i}P'_{\alpha i}\Omega_{\alpha i}/\|P_{\alpha i}P'_{\alpha i}\Omega_{\alpha i}\|. \tag{5.9}$$

Also there is a type I_2 factor $N_{\alpha i}\subset(M_{\alpha i})_{P_{\alpha i}P'_{\alpha i}}$, such that $\Phi_{\alpha i}$ factorizes $N_{\alpha i}$ in $(M_{\alpha i})_{P_{\alpha i}P'_{\alpha i}}$, and $\mathrm{Sp}(\Phi_{\alpha i}/N_{\alpha i})=(\lambda_\alpha, 1-\lambda_\alpha)$. Thus $(M_{\alpha i})_{P_{\alpha i}P'_{\alpha i}}=M_{\alpha i}^b\otimes N_{\alpha i}$ where $\Phi_{\alpha i}=\Phi_{\alpha i}^b\otimes\Phi_{\alpha i}^N$. Let $P=\Pi P_{\alpha i}$, $P'=\Pi P'_{\alpha i}$. It follows from Eq. (5.7) that $P, P'\neq 0$. Let $K=A-\cup I_{\alpha i}$. Since $\Phi_{\alpha i}$ is in the strong equivalence class of $\hat{\Omega}_{\alpha i}$ we have

$$\begin{aligned} M_{PP'}&=M(K)\otimes(\otimes M_{\alpha i})_{P_{\alpha i}P'_{\alpha i}}) \\ &=M(K)\otimes(\otimes M_{\alpha i}^b)\otimes\{\bigotimes_{i=1}^{\infty}(\bigotimes_{\alpha=1}^{\infty}(N_{\alpha i}, \Phi_{\alpha i}^N))\}. \end{aligned} \tag{5.10}$$

By construction we have $\bigotimes_{\alpha=1}^{\infty}(N_{\alpha i}, \Phi_{\alpha i}^N)\sim N$ for each i and hence $\otimes_i(\otimes_\alpha N_{\alpha i})\sim(\otimes N)^{\infty}\sim\otimes_i(\otimes_\alpha N_{\alpha i})\otimes N$. Since $M_{PP'}\sim M_P$ we have $M_P\sim M_P\otimes N$. If M is infinite, then by lemma 5.2 $M\sim M_P\otimes I_\infty$ and thus $M \sim M \otimes N$. If M is hyperfinite II_1 then M_P is also hyperfinite II_1, and $M \sim M_P$ hence $M \sim M \otimes N$. Q.E.D.

Corollary 5.5. If $x \in \mathfrak{r}_\infty(M, \Omega)$ then $M \sim M \otimes R_x$.

Lemma 5.6. Given $0 < x_1, x_2 \leq 1$. Let $x_1 = e^{-l_1}$, $x_2 = e^{-l_2}$. If l_1/l_2 is rational then

$$R_{x_1} \otimes R_{x_2} \sim R_x$$

where $x = e^{-(l_1, l_2)}$ and (l_1, l_2) is given by definition 3.12.

Proof. By lemma 3.13 $x \in \mathfrak{r}_\infty(R_{x_1} \otimes R_{x_2})$, hence by corollary 5.5 $R_{x_1} \otimes R_{x_2} \sim (R_{x_1} \otimes R_{x_2}) \otimes R_x$. Since $\mathfrak{r}(R_{x_1} \otimes R_{x_2}) \subset S_x$, it follows from lemma 5.4 that $R_x \sim R_x \otimes (R_{x_1} \otimes R_{x_2})$. Q.E.D.

The converse result to corollary 5.5 will be proved in lemma 5.8. For this purpose we need the following rather lengthy lemma.

Lemma 5.7. Given a finite type I factor M, a unit vector Ω, $\frac{1}{2} < \lambda \leq 1$, $\varepsilon > 0$, and operators $e_{12} \in M$, $f_{12} \in M'$, $\|e_{12}\|$, $\|f_{12}\| \leq 2$. Let $(\lambda_1, \lambda_2) = (\lambda, 1-\lambda)$, $x = \lambda_2/\lambda_1$, $e_{21} = e_{12}^*$, $f_{21} = f_{12}^*$. Suppose that either of the following conditions hold:

(i) $\lambda = 1$, $\varepsilon < \frac{1}{4}$ and

$$\|e_{21}\Omega\|^2 > 1 - \varepsilon \tag{5.11}$$

$$\|e_{12}\Omega\|^2 < \varepsilon \tag{5.12}$$

(ii) $\lambda \neq 1$ and

$$\|e_{21}\Omega\|^2 > \lambda_1 - \varepsilon \tag{5.13}$$

$$\|\lambda_j^{-1/2} e_{ij}\Omega - \lambda_i^{-1/2} f_{ji}\Omega\|^2 < \varepsilon, \quad (ij) = (12) \text{ or } (21) \tag{5.14}$$

and ε is sufficiently small that

$$(1-\delta)^3 > x^{1/2} \tag{5.15}$$

and

$$\begin{aligned} \Delta &= \lambda - \varepsilon - 9\varepsilon\lambda/\delta^2 \\ &= (\lambda - \varepsilon)/2 > \frac{1}{4} \end{aligned} \tag{5.16}$$

where

(5.17) $$\delta=(18\lambda\varepsilon)^{1/2}(\lambda-\varepsilon)^{-1/2}<1.$$

Then there exist disjoint subsets K^1, $K^2\subset \mathrm{Sp}(\Omega/M)$ and a bijection ϕ from K^1 to K^2 such that

(5.18) $$\max_{\mu\in K1}|x^{1/2}-(\phi\mu/\mu)^{1/2}|<a\varepsilon^{1/2}$$

(5.19) $$\mu(K^1)>b$$

where a, b are positive constants depending only on λ.

Proof. Let $H=H_1\otimes H_2$ where $M=\mathcal{B}(H_1)\otimes 1$. Then $e_{ij}=\hat{e}_{ij}\otimes 1$, $f_{ij}=1\otimes\hat{f}_{ij}$. Let

$$\Omega=\sum\rho_\alpha^{1/2}\Omega_{1\alpha}\otimes\Omega_{2\alpha}$$

be a standard diagonal expansion of Ω, and let $u_{\alpha\beta}$, $v_{\alpha\beta}$ be the associated standard matrix units for M, M' respectively.

We consider first the case $\lambda\neq 1$. Since $u_{\alpha\alpha}v_{\beta\beta}$ are a complete set of orthogonal projections, Eq. (5.14) gives

(5.20) $$\begin{aligned}\varepsilon&>\sum_{\alpha,\beta}\|u_{\alpha\alpha}v_{\beta\beta}(\lambda_j^{-1/2}e_{ij}-\lambda_i^{-1/2}f_{ji})\Omega\|^2\\ &=\sum|(\rho_\beta/\lambda_j)^{1/2}(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})-(\rho_\alpha/\lambda_i)^{1/2}(\Omega_{2\beta},\hat{f}_{ji}\Omega_{2\alpha})|^2.\end{aligned}$$

Interchanging (β, j) and (α, i), taking the complex conjugate, and using $(\Omega_{1\beta},\hat{e}_{ji}\Omega_{1\alpha})^*=(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})$ we get

(5.21) $$\sum|(\rho_\alpha/\lambda_i)^{1/2}(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})-(\rho_\beta/\lambda_j)^{1/2}(\Omega_{2\beta},\hat{f}_{ji}\Omega_{2\alpha})|^2<\varepsilon.$$

Eqs. (5.20), (5.21) and the triangle inequality $\|x\|+\|y\|\geq\|x+y\|$ give

(5.22) $$\{\sum|(\rho_\alpha/\lambda_i)^{1/2}+(\rho_\beta/\lambda_j)^{1/2}|^2|(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})-(\Omega_{2\beta},\hat{f}_{ji}\Omega_{2\alpha})|^2\}^{1/2}<2\varepsilon^{1/2}$$

and thus

(5.23) $$\sum(\rho_\beta/\lambda_j)|(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})-(\Omega_{2\beta},\hat{f}_{ij}\Omega_{2\alpha})|^2<4\varepsilon.$$

Eq. (5.21) can be rewritten as

(5.24) $$\begin{aligned}&\sum|\{(\rho_\alpha/\lambda_i)^{1/2}-(\rho_\beta/\lambda_j)^{1/2}\}(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})\\ &\quad+(\rho_\beta/\lambda_j)^{1/2}\{(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})-(\Omega_{2\beta},\hat{f}_{ij}\Omega_{2\alpha})\}|^2<\varepsilon.\end{aligned}$$

Eqs. 5. (5.23), (5.24) and the triangle inequality give

(5.25) $$\sum\{(\rho_\alpha/\lambda_i)^{1/2}-(\rho_\beta/\lambda_j)^{1/2}\}^2|(\Omega_{1\alpha},\hat{e}_{ij}\Omega_{1\beta})|^2<9\varepsilon.$$

Define

(5.26) $$L=\{(\alpha,\beta):\ |1-(\rho_\alpha\lambda_1/\rho_\beta\lambda_2)^{1/2}|<\delta,\ \rho_\alpha\neq0,\ \rho_\beta\neq0\}.$$

Eq. (5.25) now gives

(5.27) $$\sum_{(\alpha,\beta)\notin L}\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2<9\varepsilon\lambda_1/\delta^2.$$

Since $\sum\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2=\|e_{21}\Omega\|^2$, Eqs. (5.27), (5.16) give

(5.28) $$\sum_{(\alpha,\beta)\in L}\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2>\Delta.$$

Let

$$L'=\{\beta:\ (\alpha,\beta)\in L\ \text{ for some }\alpha\}.$$

For $\beta\in L'$ we define

(5.29) $$\alpha(\beta)=\{\alpha:\ (\alpha,\beta)\in L\}.$$

Let

(5.30) $$L''=\{\beta\in L':\ \sum_{\alpha\in\alpha(\beta)}|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2\geq\Delta/2\}.$$

Using Eqs. (5.28), (5.30) and $\sum\rho_\beta=1$ we get

(5.31) $$\sum_{\beta\in L''}\sum_{\alpha\in\alpha(\beta)}\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2>1/2\Delta.$$

From $\|\hat{e}_{21}\|\leq2$, $\sum_\alpha|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2=\|\hat{e}_{21}\Omega_{1\beta}\|^2\leq4$ and Eq. (5.31) we obtain

(5.32) $$\sum_{\beta\in L''}\rho_\beta\geq\frac{1}{4}\sum_{\beta\in L''}\sum_\alpha\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2>\frac{\Delta}{8}.$$

Let

(5.33) $$K_n=\{\beta\in L'':\ (1-\delta)^n<\rho_\beta^{1/2}\leq(1-\delta)^{n-1}\},\ n\in\mathrm{I}_\infty.$$

Define N by

(5.34) $$(1-\delta)^{N+1}<x^{1/2}\leq(1-\delta)^N.$$

Let

(5.35) $$\hat{K}_n=\{\alpha\in\alpha(\beta):\ \beta\in K_n\}.$$

It follows from Eqs. (5.26), (5.29) that $\alpha\in\alpha(\beta)$ implies

$$|\rho_\alpha^{1/2}-x^{1/2}\rho_\beta^{1/2}|<x^{1/2}\rho_\beta^{1/2}\delta.$$

Since $0<\delta<1$ we have $(1+\delta)<(1-\delta)^{-1}$ and thus

$$(1-\delta)x^{1/2}\rho_\beta^{1/2}<\rho_\alpha^{1/2}<(1+\delta)x^{1/2}\rho_\beta^{1/2}<(1-\delta)^{-1}x^{1/2}\rho_\beta^{1/2}.$$

It now follows from Eqs. (5.33), (5.34) that

$$\widehat{K}_n\subset\bigcup_{m=-2}^{+2}K_{n+N+m}. \tag{5.36}$$

Consider an arbitrary $K\subset L''$. Let $\widehat{K}=\{\alpha\in\alpha(\beta):\beta\in K\}$. Let $N(K)$, $N(\widehat{K})$ be the number of elements in K, $\widehat{K}$ respectively. Using Eq. (5.30) and $\|\hat{e}_{21}\|\leq 2$ we get

$$\begin{aligned}\frac{1}{2}\Delta N(K)&\leq\sum_{\beta\in K}\sum_{\alpha\in\alpha(\beta)}|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2\\&\leq\sum_{\beta\in K}\sum_{\alpha\in\widehat{K}}|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2\\&\leq\sum_{\alpha\in\widehat{K}}\|\hat{e}_{21}^*\Omega_{1\alpha}\|^2\leq 4N(\widehat{K}).\end{aligned} \tag{5.37}$$

We now construct subsets $K_n^1\subset K_n$, $K_n^2\subset\widehat{K}_n$ and a bijection ϕ_n from K_n^1 to K_n^2 such that $\phi_n\beta\in\alpha(\beta)$. If K_n is nonempty, we order the elements in K_n, $\widehat{K}_n$ respectively by

$$\rho_{\beta_{n1}}\geq\rho_{\beta_{n2}}\geq\cdots\geq\rho_{\beta_{n,N(K_n)}} \tag{5.38}$$

and

$$\rho_{\alpha_{n1}}\geq\rho_{\alpha_{n2}}\geq\cdots\geq\rho_{\alpha_{n,N(\widehat{K}_n)}}. \tag{5.39}$$

Since $\alpha_{n1}\in\widehat{K}_n$ we have $\alpha_{n1}\in\alpha(\beta_{nk})$ for some $\beta_{nk}\in K_n$. Since Eq. (5.24) implies that $\alpha(\beta_{n1})$ is nonempty, there exists some $\alpha_{nj}\in\alpha(\beta_{n1})$. Thus

$$\rho_{\alpha_{n1}}/\rho_{\beta_{nk}}\geq\rho_{\alpha_{n1}}/\rho_{\beta_{n1}}\geq\rho_{\alpha_{nj}}/\rho_{\beta_{n1}} \tag{5.40}$$

and it follows from Eq. (5.26) that $\alpha_{n1}\in\alpha(\beta_{n1})$. We include $\beta_{n1}\in K_n^1$ and define $\phi_n\beta_{n1}=\alpha_{n1}$. Let m be the smallest integer such that $m\geq 8/\Delta$. Let p_n be the largest integer such that $p_nm<N(K_n)$. If $p_n\leqslant 0$ we are done. If $p_n>0$ we proceed as follows. Assume that $\alpha_{nj}\notin\alpha(\beta_{n,m+1})$ for any $j>1$. Since $\alpha(\beta_{n,m+1})$ is nonempty by Eq.

(5. 30), we must have $\alpha(\beta_{n,m+1})=\{\alpha_{n1}\}$. It follows from Eq. ;(5. 37) applied to $K=\{\beta_{n1}, \beta_{n2}, \cdots \beta_{n,m+1}\}$ that $\hat{K}$ must contain at least two elements. Thus there exists some $\alpha_{nj}\in\alpha(\beta_{n,k})$ for some $j>1$ and $k\leq m$. We have

$$\rho_{\alpha_{n1}}/\rho_{\beta_{n,m+1}}\geq\rho_{\alpha_{nj}}/\rho_{\beta_{n,m+1}}\geq\rho_{\alpha_{nj}}/\rho_{\beta_{nk}} \tag{5. 41}$$

and it follows from Eq. (5. 26) that $\alpha_{nj}\in\alpha(\beta_{n,m+1})$ which is a contradiction. Thus $\alpha(\beta_{n,m+1})$ must contain some $\alpha_{nj}, j>1$. Let $j(n, 1)$ be the smallest $j>1$ such that $\alpha_{nj}\in\alpha(\beta_{n,m+1})$. By a repetition of this argument we obtain a sequence

$$1<j(n, 1)<j(n,2)<\cdots<j(n, p_n) \tag{5. 42}$$

such that

$$\alpha_{n,j(n,k)}\in\alpha(\beta_{n,km+1}),\ k=1, \cdots p_n. \tag{5. 43}$$

We include $\beta_{n,m+1}, \beta_{n,2m+1}, \cdots \beta_{n,p_nm+1}\in K_n^1$ and define

$$\phi_n\beta_{n,km+1}=\alpha_{n,j(n,k)}, k=1, \cdots p_n. \tag{5. 44}$$

It follows from Eq. (5. 26) that

$$\max_{\beta\in K_n^1}|x^{1/2}-(\rho_\beta/\rho_{\phi_n\beta})^{1/2}|<\delta x^{1/2}. \tag{5. 45}$$

By construction K_n^1 contains at least $N(K_n)/m$ elements and

$$\sum_{\beta\in K_n^1}\rho_\beta\geq m^{-1}\sum_{\beta\in K_n}\rho_\beta. \tag{5. 46}$$

We now proceed to define the desired sets K^1, K^2. Since the K_n defined by Eq. (5. 33) are mutually disjoint, it follows that the K_n^1 are mutually disjoint. We note that Eq. (5. 15) implies that $N\geq 3$ where N is defined by Eq. (5. 34). It then follows from Eq. (5. 36) that $\hat{K}_n$ and K_p are disjoint if $p\leq n$. By induction on n, we define K^1 to contain all ρ_β, $\beta\in K_n^1$, $n\in I_\infty$ such that for any $\beta'\in K_p^1$, $p<n$ and $\beta'\in K^1$ the conditions

$$\beta\neq\phi_p\beta' \tag{5. 47}$$

$$\phi_n\beta\neq\phi_p\beta' \tag{5. 48}$$

are satisfied. We define $\phi\rho_\beta=\rho_{\phi_n\beta}$. Eq. (5. 47) eliminates at most one

β for each β' taken into K^1 and thus at most half of the β which would be otherwise available. It follows from Eq. (5.36) that Eq. (5.48) eliminates at most 4 of the available ρ_β for each $\rho_{\beta'}$ taken into K^1 and hence at most 4/5 of the ρ_β which would be otherwise available. The net effect of these two conditions is to reduce the total number of available ρ_β by a factor of at most 10. Since the eliminated ρ_β are never larger than the $\rho_{\beta'}$ taken into K^1, it follows from Eq. (5.32) that

$$\sum_{\rho_\beta\in K^1}\rho_\beta>(\Delta/8)m^{-1}/10.$$

Since $\Delta>1/4$ by Eq. (5.16) we have $m\leq 32$ and thus

$$\sum_{\rho_\beta\in K^1}\rho_\beta>1/10240. \tag{5.49}$$

Thus Eqs. (5.18), (5.19) are satisfied by Eqs. (5.45), (5.49) and the proof is complete for the case $\lambda\neq 1$.

We now consider the case $\lambda=1$. Let

$$L=\{(\alpha,\beta):\rho_\alpha\leq 2\varepsilon\rho_\beta,\ \rho_\beta\neq 0,\ \Omega_{1\alpha}\neq 0\}. \tag{5.50}$$

Using Eq. (5.12) we have

$$\varepsilon>\|e_{12}\Omega\|^2=\sum\rho_\alpha|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2$$

and thus

$$\varepsilon>\sum_{(\alpha,\beta)\notin L}\rho_\alpha|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2>2\varepsilon\sum_{(\alpha,\beta)\in L}\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2.$$

Using Eq. (5.11) we have

$$1-\varepsilon<\|e_{21}\Omega\|^2=\sum\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2$$

and it follows that

$$\sum_{(\alpha,\beta)\in L}\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2>\gamma=1/2-\varepsilon>1/4. \tag{5.51}$$

Let

$$\begin{aligned}&L'=\{\beta:(\alpha,\beta)\in L\text{ for some }\alpha\}\\&L''=\{\beta\varepsilon L':\sum_{\alpha\in\alpha(\beta)}|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2\geq\gamma/2\}.\end{aligned} \tag{5.52}$$

Using Eqs. (5.51), (5.52) and $\sum\rho_\beta=1$ we get

$$\sum_{\beta\in L''}\sum_{\alpha\in\alpha(\beta)}\rho_\beta|(\Omega_{1\alpha},\hat{e}_{21}\Omega_{1\beta})|^2>\gamma/2$$

and thus

(5. 53) $$\sum_{\beta\in L''}\rho_\beta>\gamma/8$$

since $\|\hat{e}_{21}\|\leq 2$. Given any $K\subset L''$, let $\hat{K}=\{\alpha\in\alpha(\beta):\beta\in K\}$ and let $N(K)$ be the number of elements in K. By the same argument used to derive Eq. (5. 37) it follows that $\hat{K}$ contains at least $\gamma N(K)/8$ elements. Let m be the smallest integer such that $m\geq 8\gamma^{-1}$. We order the elements in L'' by $\rho_{\beta_1}\geq\rho_{\beta_2}\geq\cdots$. We define K^0 to contain $\rho_{\beta_1}, \rho_{\beta_{m+1}}, \rho_{\beta_{2m+1}}, \cdots$. By the same argument which follows Eq. (5. 39) we can obtain elements $\alpha_n\in\alpha(\beta_{nm+1})$ such that $\alpha_n\neq\alpha_j$ for any $j<n$. We define $\phi\rho_{\beta_{nm+1}}=\rho_{\alpha_n}$. By induction on n, we define K^1 to contain all $\rho_{\beta_{nm+1}}$ such that $\rho_{\beta_{nm+1}}\neq\phi\rho_{\beta_{jm+1}}$ in Sp (Ω/K) for any $j<n$. This condition eliminates at most one ρ_β for each $\rho_{\beta'}$ taken into K^1, and thus at most half of the ρ_β which would be otherwise available. Since the eliminated ρ_β are never larger than the $\rho_{\beta'}$ taken into K^1, it follows from Eq. (5. 53) that

(5. 54) $$\sum_{\rho\in K^1}\rho\geq m^{-1}\gamma/16>1/2048$$

(where we used Eq. (5. 51) to obtain $m\leq 32$). Thus Eq. (5. 19) is satisfied. We define $K^2=\{\phi\rho:\rho\in K^1\}$. It follows from Eq. (5. 50) that

(5. 55) $$\max_{\rho\in K^1}\phi\rho/\rho\leq 2\varepsilon$$

and thus Eq. (5. 18) is satisfied. Q. E. D.

Lemma 5.8. Given $M=\mathrm{R}(M_\nu,\Omega_\nu)$. Then $M\sim M\otimes R_x$, $0\leq x\leq 1$ implies that $x\in \mathrm{r}_\infty(M,\Omega)$.

Proof. By lemma 3. 15 we can assume that M has a cyclic and separating vector. Since R_x has a cyclic and separating vector, we can assume that $M\otimes R_x$ has a cyclic and separating vector. Then $M\sim M\otimes R_x$ implies that $M\overset{u}{\sim}M\otimes R_x$ and we have

(5. 56) $$H=H_M\otimes H_R$$

(5. 57) $$H_R=\bigotimes_{n=1}^{\infty}(H_{n1}\otimes H_{n2},\Phi_n)$$

(5. 58) $$R=(\bigcup_{n=1}^{\infty}R_{n1})''$$

(5.59) $$R_{ni}=\mathbf{1}_M\otimes[(\bigotimes_{m\neq n}\mathbf{1}_m)\otimes(\mathscr{B}(H_{ni})\otimes\mathbf{1})],\ i=1,2$$

where $\dim H_{n1}=\dim H_{n2}=2$, $\mathrm{Sp}\,(\Phi_n/\mathscr{B}(H_{n1})\otimes\mathbf{1})=(\lambda, 1-\lambda)$ and $x=(1-\lambda)/\lambda$. Let

(5.60) $$\Phi_n=\lambda^{1/2}\Phi_{11}^n\otimes\Phi_{21}^n+(1-\lambda)^{1/2}\Phi_{12}^n\otimes\Phi_{22}^n$$

be a standard diagonal expansion of Φ_n, and let u_{ij}^n, v_{ij}^n be the associated standard matrix units for R_{n1}, R_{n2} respectively. Choose some unit vector $\Phi_M\in H_M$ and let $\Phi=\Phi_M\otimes(\bigotimes_{n=1}^{\infty}\Phi_n)$. It follows from lemma 2.7 that for any $Q_n\in\{R_{n1}, R_{n2}\}''$, $\|Q_n\|<N$ for some fixed $N<\infty$, that

(5.61) $$\lim_{n\to\infty}\{\|Q_n\Omega\|-\|Q_n\Phi\|\}=0.$$

Case (i), $x\neq 0, 1$. Define

(5.62) $$Q_{ij}^n=\lambda_j^{-1/2}u_{ij}^n-\lambda_i^{-1/2}v_{ji}^n,\ (ij)=(12)\text{ or }(21)$$

where $(\lambda_1, \lambda_2)=(\lambda, 1-\lambda)$. Then

(5.63) $$Q_{ij}^n\Phi=0.$$

Also

(5.64) $$\|u_{21}^n\Phi\|^2=\lambda.$$

Let $\varepsilon_m>0, \varepsilon_m\to 0$. It follows from Eqs. (5.61–64) that we can choose a subsequence $n(m)$, $m\in I_\infty$, such that

(5.65) $$\|Q_{ij}^{n(m)}\Omega\|<\varepsilon_m$$

(5.66) $$|\lambda-\|u_{21}^{n(m)}\Omega\|^2|<\varepsilon_m.$$

We have $M=\bigotimes_{\nu\in A}M_\nu$, $M'=\bigotimes_{\nu\in A}M'_\nu$. We will prove that there exist mutually disjoint finite subsets $J_m\subset A$, and $e_{ij}^m\in M(J_m)$, $f_{ij}^m\in M'(J_m)$, $\|e_{ij}^m\|, \|f_{ij}^m\|\leq 2$ satisfying

(5.67) $$\|(e_{ij}^m-u_{ij}^{n(m)})\Omega\|<\varepsilon_m$$

(5.68) $$\|(f_{ij}^m-v_{ij}^{n(m)})\Omega\|<\varepsilon_m$$

where $(ij)=(12)$, (21) and $(e_{ij}^m)^*=e_{ji}^m$, $(f_{ij}^m)^*=f_{ji}^m$. For $m=1$, this follows from Kaplansky's density theorem[1] applied to the hermitean and antihermitean parts of $u_{ij}^{n(m)}, v_{ij}^{n(m)}$. Assume that J_m, e_{ij}^m, f_{ij}^m exist for

1) Dixmier [8], Sec. I.3.

$m=1, \cdots k-1$. Let $J^k=\cup_{m=1}^{k-1} J_m$. Then we have $M=M(A-J^k)\otimes M(J^k)$ where $M(J^k)$ is a finite type I factor. If M is infinite then $M\sim M(A-J^k)$ by lemma 5.2. If M is finite then both M and $M(A-J^k)$ must be hyperfinite II_1 and again we have $M\sim M(A-J^k)$. By assumption M and $M(A-J^k)$ have a cyclic and separating vector, hence $M\overset{u}{\sim}M(A-J^k)$. Thus we can repeat the above argument with M replaced by $M(A-J^k)$ to obtain J_k, e_{ij}^k, f_{ij}^k. It follows from Eqs. (5.65–68) that

(5.69) $$\|(\lambda_j^{-1/2}e_{ij}^m-\lambda_i^{-1/2}f_{ji}^m)\Omega\|<\varepsilon_m[1+\lambda^{-1/2}+(1-\lambda)^{-1/2}]$$

(5.70) $$|\lambda-\|e_{21}^m\Omega\|^2|<2\varepsilon_m.$$

Since $\varepsilon_m\to 0$, it follows from lemma 5.7 that for $m>N$ there exist $K_m^1, K_m^2\subset \mathrm{Sp}(\Omega(J_m)/M(J_m))$ and a bijection ϕ_m from K_m^1 to K_m^2 such that $(I_m, K_m^i, \phi_m, m>N)$ is an x-sequence.

Case (ii), $x=0$. Retain Eq. (5.64) and replace Eq. (5.63) by

(5.71) $$u_{12}^n\Phi=0.$$

Then choose a subsequence $n(m)$ such that Eq. (5.66) holds and

(5.72) $$\|u_{12}^{n(m)}\Omega\|<\varepsilon_m.$$

The remainder of the argument is a straightforward repetition of the above argument, and we omit the details.

Case (iii), $x=1$. Since R_1 is type II_1, $M\sim M\otimes R_1$ implies that M is not type I. Lemmas 2.14 and 3.8 then imply that $1\in r_\infty(M)$.

Q. E. D.

Theorem 5.9. The asymptotic ratio set $r_\infty(M)$ given by definition 3.2 for ITPFI factors M, is an algebraic invariant of M.

Proof. By corollary 5.5 and lemma 5.8, $x\in r_\infty(M)$, $0\leq x\leq 1$ if and only if $M\sim M\otimes R_x$. By lemma 3.6, $x\in r_\infty(M)$, $x>1$ if and only if $x^{-1}\in r_\infty(M)$. Q. E. D.

We recall that $M=\mathrm{R}(M_\nu, \Omega_\nu)$ where M_ν is type $I_{n_\nu}, 2\leq n_\nu\leq\infty$ is an ITPFI factor (corollary 2.9). We now show that all non-zero $x\in r_\infty(M)$ can be calculated directly from definition 3.2 even if some of the n_ν are infinite. It should be noted that a direct application of definition 3.2 may fail to give a 0-sequence, even though $0\in r_\infty(M)$. Consider $M=I_\infty\otimes R_1$, then $0\in r_\infty(M)$ but the only x-sequences allowed are $x=1$. However this is not a problem since if some $n_\nu=\infty$, then we necessarily have $0\in r_\infty(M)$ since M is infinite.

Lemma 5.10. Given $M=R(M_\nu, \Omega_\nu)$ where M_ν is type I_{n_ν}, $2\leq n_\nu \leq\infty$ and $n_\nu=\infty$ for some ν. Then $r_\infty(M)$ contains 0 and all $x\in(0, \infty)$ for which there exists an x-sequence satisfying the conditions of definition 3.2.

Proof. By assumption $n_\nu=\infty$ for some ν and M is infinite. By lemma 5.2 $M\sim M\otimes R_0$ and $0\in r_\infty(M)$. It remains to show that $x\in r_\infty(M)$, $x\neq 0$ if and only if there exists an x-sequence for M.

Let $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu j}\}$. By lemma 2.14, M is type I_∞ if and only if $\sum(1-\lambda_{\nu 1})<\infty$. By lemma 3.8, $r_\infty(M)=S_0$ if and only if M is type I_∞ if M is an ITPFI factor. Thus $\sum(1-\lambda_{\nu 1})=\infty$ if and only if $1\in r_\infty(M)$. Since the proof of lemma 3.8 remains valid even if $n_\nu=\infty$ is allowed, $\sum(1-\lambda_{\nu 1})=\infty$ if and only if there is a 1-sequence for M. Thus it remains only to consider $x\neq 0, 1$.

We construct a projection $P\in M$ such that $M_P=R(M_\nu, \Omega'_\nu)$ has x-sequences if and only if $R(M_\nu, \Omega_\nu)$ has x-sequences, $x\neq 0, 1$. For each ν choose $m_\nu<\infty$, $m_\nu\leq n_\nu$ such that

$$\sum_{j=m_\nu+1}^{n_\nu}\lambda_{\nu j}<2^{-(\nu+2)}\|\Omega_\nu\|. \tag{5.73}$$

Let

$$\Omega_\nu=\sum\lambda_{\nu j}^{1/2}\psi_{\nu 1j}\otimes\psi_{\nu 2j}$$

be a standard diagonal expansion of Ω_ν. Define $P_\nu\otimes 1\in M_\nu$ by

$$P_\nu\psi_{\nu 1j}=\begin{cases}\psi_{\nu 1j} & \text{if } j\leq m_\nu\\ 0 & \text{if } j>m_\nu\end{cases}$$

and let $P=\otimes_\nu(P_\nu\otimes 1)\in M$. We have

$$\|P\Omega\|^2=\Pi\,[\|\Omega_\nu\|^2-\sum_{j=m_\nu+1}^{n_\nu}\lambda_{\nu j}]>0$$

where we used Eq. (5.73). We have $M_P=R(M_\nu, \Omega'_\nu)$ where $\mathrm{Sp}(\Omega'_\nu/M_\nu)=(\lambda_{\nu 1}, \cdots\lambda_{\nu m_\nu})$. Let $(I_n K_n^i, \phi_n)$ be an x-sequence for $R(M_\nu, \Omega'_\nu)$. Clearly

(I_n, K_n^i, ϕ_n) is also an x-sequence for $\mathrm{R}(M_\nu, \Omega_\nu)$. Conversely let (I_n, K_n^i, ϕ_n) be an x-sequence for $\mathrm{R}(M_\nu, \Omega_\nu)$, $x \neq 0$. It follows from definition 3.2 and Eq. (5.73) that we can omit all $\mu \in K_n^i, i=1, 2$, $n \in I_\infty$ which contain some $\lambda_{\nu j}, j > m_\nu$ as a factor. The argument is then reversible, and there exists an x-sequence for $\mathrm{R}(M_\nu, \Omega'_\nu)$. By lemma 5.2, $M \sim M_P \otimes R_0$. Hence $\mathrm{r}_\infty(M_P) \subset \mathrm{r}_\infty(M)$ and $x \notin \mathrm{r}_\infty(M)$ implies that $\mathrm{R}(M_\nu, \Omega'_\nu)$ does not have an x-sequence. Conversely, if $x \in \mathrm{r}_\infty(M)$, $x \neq 0, 1$, then M cannot be $R_1 \otimes R_0$. Hence M_P cannot be a finite factor and by lemma 5.2 $M \sim M_P \otimes R_0 \sim M_P$, which implies that $M_P = \mathrm{R}(M_\nu, \Omega'_\nu)$ has an x-sequence.

6. Asymptotic Ratio Set for Arbitrary Factors

In the preceding section we proved that for ITPFI factors M, $x \in \mathrm{r}_\infty(M)$, $0 \leq x \leq 1$ if and only if $M \sim M \otimes R_x$. In this section we use this result to extend the definition of $\mathrm{r}_\infty(M)$ to arbitrary M. We give some properties of $\mathrm{r}_\infty(M)$ for the general case.

Definition 6.1. Let M be any factor. We define $\mathrm{r}_\infty(M)$ by $x \in \mathrm{r}_\infty(M)$, $0 \leq x \leq 1$ if and only if $M \sim M \otimes R_x$. If $x \in \mathrm{r}_\infty(M)$, $x \neq 0$ we include $x^{-1} \in \mathrm{r}_\infty(M)$.

We shall need the following result which is due to Sakai [16].

Lemma 6.2. Let M, N be factors. If M is type III then $M \otimes N$ is a type III factor.

Lemma 6.3. M is finite if and only if $\mathrm{r}_\infty(M) = \phi$ or S_1. M is type III if $x \in \mathrm{r}_\infty(M)$ for some $0 < x < 1$.

Proof. If $\mathrm{r}_\infty(M) = \phi$ or S_1 then $0 \notin \mathrm{r}_\infty(M)$ and M is finite by lemma 5.2. Since R_x is infinite if $x \neq 1$, M finite implies that $M \nsim M \otimes R_x$ if $x \neq 1$ and hence $\mathrm{r}_\infty(M) = \phi$ or S_1. If $x \in \mathrm{r}_\infty(M)$ for some $0 < x < 1$ then we have $M \sim M \otimes R_x$ where R_x is type III. It follows from lemma 6.2 that M is type III. Q. E. D.

Schwartz [17] has given a type II_1 factor M such that M is non-isomorphic to $M \otimes R_1$. Thus there exists a type II_1 factor M with

$r_\infty(M)=\phi$.

Lemma 6.4. $r_\infty(M\otimes N)\supset r_\infty(M)\cup r_\infty(N)$

Proof. $M\sim M\otimes R_x$ implies $M\otimes N\sim M\otimes N\otimes R_x$. Q. E. D.

Lemma 6.5. If $r_\infty(M)-\{0\}$ is nonempty, then it is a multiplicative subgroup of $(0, \infty)$.

Proof. By definition $x^{-1}\in r_\infty(M)-\{0\}$ if and only if $x\in r_\infty(M)-\{0\}$. For $x\in(0, \infty)$ define

$$\alpha(x)=\begin{cases} x & 0<x\leq 1 \\ x^{-1} & 1<x<\infty. \end{cases}$$

Let $x, y\in r_\infty(M)-\{0\}$. Then we have $M\sim M\otimes R_{\alpha(x)}$, $M\sim M\otimes R_{\alpha(y)}$ which implies that $M\sim M\otimes(R_{\alpha(x)}\otimes R_{\alpha(y)})$. Since $x, y\in r_\infty(R_{\alpha(x)}\otimes R_{\alpha(y)})$ it follows from lemma 3.6 that $xy\in r_\infty(R_{\alpha(x)}\otimes R_{\alpha(y)})$. Thus $M\sim M\otimes R_{\alpha(xy)}$ and $xy\in r_\infty(M)-\{0\}$. Q. E. D.

Lemma 6.6. Given $x_1, x_2\in r_\infty(M)$, $0<x_1, x_2<1$. Let $x_i=e^{-l_i}$, $i=1, 2$. If l_1/l_2 is irrational then $r_\infty(M)=S_\infty$. If l_1/l_2 is rational then $e^{-(l_1, l_2)}\in r_\infty(M)$ (see definition 3.12).

Proof. If $x_1, x_2\in r_\infty(M)$ then $M\sim M\otimes R_{x_1}\sim M\otimes R_{x_1}\otimes R_{x_2}$. By lemmas 3.13 and 5.6 we have $r_\infty(R_{x_1}\otimes R_{x_2})=S_x$ where $x=\infty$ if l_1/l_2 is irrational and $x=e^{-(l_1, l_2)}$ otherwise. We have $R_{x_1}\otimes R_{x_2}\sim R_{x_1}\otimes R_{x_2}\otimes R_y$ for any $y\in S_x$. It follows that $M\sim M\otimes R_{x_1}\otimes R_{x_2}\otimes R_y\sim M\otimes R_y$ and hence $y\in r_\infty(M)$. Q. E. D.

Definition 6.7. Given $0<x<1$. Let K denote some sequence $\{k_\nu; \nu\in I_\infty\}$ where k_ν is either a positive integer or ∞. Let $p_1<p_2<\cdots$ be the set of all prime numbers. We define $S(x, K)$ as the subset of $[0, \infty)$ containing 0 and $x^{n/m}$ for all integers n and all integers $m=\Pi_\nu p_\nu^{n_\nu}$ where $0\leq n_\nu<k_\nu$.

Corollary 6.8. For any factor M, $r_\infty(M)$ is one of the following sets: the empty set ϕ, S_0, S_{01}, S_1, S_∞ or $S(x, K)$ for some $0<x<1$ and some K.

Proof. Follows immediately from lemmas 6.5 and 6.6. Q. E. D.

We note that if $K=\{k_\nu\}$ where $k_\nu=1$ for all ν, then $S(x, K)=S_x$. If it can be proved that $r_\infty(M)$ is closed, then $r_\infty(M)$ must be one of the sets ϕ, S_x $(0\leq x\leq 1), S_\infty, S_{01}$. The following lemma was obtained as a result of our effort, as yet unsuccessful, to prove that $r_\infty(M)$ is closed.[2)]

Lemma 6.9. Let M be a factor and $x_n\in r_\infty(M), n=1, 2, \cdots\cdots, 0< x_n<1, \lim x_n=x$. Then there exist subfactors M_n of M and N_n of M' and a vector χ such that

(i) $M_n\sim R_{x_n}\sim M_n$.

(ii) $M_a=(\bigcup_n M_n)''$ and $N_a=(\bigcup_n N_n)''$ are factors isomorphic to $\bigotimes_n R_{x_n}$.

(iii) $H_a\equiv\overline{M_a\chi}=\overline{N_a\chi}$.

(iv) χ is separating for M and M'.

(v) Restricted to H_a, $\{H_a, \chi, M_n, N_n, n=1, 2, \cdots\}$ are unitarily equivalent to $\{(\bigotimes H_m), (\bigotimes \chi_m), R_{x_n}, R'_{x_n}, n=1, 2, \cdots\}$ where R'_{x_n} is the commutant of R_{x_n} in H_n and χ_n is a cyclic and separating vector of R_{x_n} in H_n.

(vi) Let $M_b=M'_a\cap M$, $N_b=N'_a\cap M'$. $M_0\equiv(M_a\cup M_b)''$ and $N_0\equiv(N_a\cup N_b)''$ have the property that $M=M_0\oplus M_c$ and $M'=N_0\oplus N_c$ where M_c and N_c brings $\overline{M_0N_0\chi}$ to its orthogonal complement.

(vii) Restricted to $\overline{M_0N_0\chi}$, $M_0=R_a\otimes R_b$, $N_0=R'_a\otimes R'_b$, $M_a=R_a\otimes 1$, $M_b=1\otimes R_b$, $N_a=R'_a\otimes 1$, $N_b=1\otimes R'_b$.

(viii) $x\in r_\infty(M_a)$.

Proof. We first construct M_n and N_n, $n=1, 2\cdots$ which satisfy (i), (iii), and (v). Let $\varepsilon_j>0$, $\sum\varepsilon_j<\infty$, $R_{x_j}=R(H_\nu^{(j)}, R_\nu^{(j)}, \Phi_\nu^{(j)})$, $Sp(\Phi_\nu^{(j)}/R_\nu^{(j)})=\{\lambda_j, 1-\lambda_j\}$, $\lambda_j=(1+x_j)^{-1}$ and $\Phi_\nu^{(j)}$ be cyclic (and separating) for $R_\nu^{(j)}$.

Since R_{x_1} is type III, $M\sim M\otimes R_{x_1}$ implies $M\overset{u}{\sim}M\otimes R_{x_1}$. Thus we can write

$$H=H_{r1}\otimes H_1$$

2) The authors are indebted to Dr. D. J. C. Bures for pointing out a loophole in an earlier version. For the rest of the paper, the reader can skip this lemma.

$$M=M_{r1}\otimes M_{(1)}$$
$$M'=M'_{r1}\otimes M'_{(1)}$$

where $M_{r1}\overset{u}{\sim}M$, $H_1=\otimes H_\nu^{(1)}$, $M_{(1)}=R_{x_1}$. Choose a unit vector ψ_{r1} in H_{r1} and let

$$\Phi_1=\psi_{r1}\otimes\chi_1,\ \chi_1\equiv\otimes\Phi_\nu^{(1)}.$$

We set $M_1=\mathbf{1}\otimes M_{(1)}$, $N_1=\mathbf{1}\otimes M'_{(1)}$.

Since $M_{r1}\overset{u}{\sim}M\overset{u}{\sim}M\otimes R_{x_2}$, we have $M_{r1}\overset{u}{\sim}M_{r1}\otimes R_{x_2}$ and we can write

$$H_{r1}=K_2\otimes(\otimes H_\nu^{(2)})$$
$$M_{r1}=R^{(2)}\otimes R_{x2}$$

where $R^{(2)}\overset{u}{\sim}M_{r1}\overset{u}{\sim}M$.

By lemma 2.5, there exists a finite subset J_1 of indices ν and a unit vector ψ_{r2} of $H_{r2}\equiv K_2\otimes(\underset{\nu\in J_1}{\otimes}H_\nu^{(2)})$

$$\|\psi_{r2}\otimes\chi_2-\psi_{r1}\|<\varepsilon_1$$
$$\chi_2\equiv\underset{\nu\notin J_1}{\otimes}\Phi_\nu^{(2)}.$$

Let

$$\Phi_2=\psi_{r2}\otimes\chi_2\otimes\chi_1$$
$$H_2=\underset{\nu\notin J_1}{\otimes}H_\nu^{(2)},\ M_{(2)}=\underset{\nu\notin J_1}{\otimes}R_\nu^{(2)},\ M_{r2}=R^{(2)}\otimes(\underset{\nu\in J_1}{\otimes}R_\nu^{(2)}).$$

Then

$$H=H_{r2}\otimes H_2\otimes H_1$$
$$M=M_{r2}\otimes M_{(2)}\otimes M_{(1)},$$

$M_{r2}\overset{u}{\sim}R^{(2)}\overset{u}{\sim}M$, Φ_2 is a product vector and

$$\|\Phi_2-\Phi_1\|<\varepsilon_1.$$

We set

$$M_2=\mathbf{1}\otimes M_{(2)}\otimes\mathbf{1}$$
$$N_2=\mathbf{1}\otimes M'_{(2)}\otimes\mathbf{1}.$$

By repetition of this argument we obtain a sequence of Hilbert spaces H_j, H_{rj}, factors $M_{(j)}$, M_{rj}, and vectors $\chi_j\in H_j$, $\Phi_j\in H$, such that

$$H=H_{rj}\otimes(\underset{n\leq j}{\otimes}H_n)$$

$$M = M_{rj} \otimes (\bigotimes_{n \le j} M_{(n)})$$

$$\Phi_j = \psi_{rj} \otimes (\bigotimes_{n \le j} \chi_n)$$

$$\|\Phi_j - \Phi_{j-1}\| < \varepsilon_{j-1}$$

where

$$H_n = \bigotimes_{\nu \notin J_{n-1}} (H_\nu^{(n)}, \Phi_\nu^{(n)})$$

$$M_{(n)} = \mathrm{R}(H_\nu^{(n)}, R_\nu^{(n)}, \Phi_\nu^{(n)};\ \nu \notin J_{n-1}) \sim R_{x_n}$$

$$\chi_n = \bigotimes_{\nu \notin J_{n-1}} \Phi_\nu^{(n)}.$$

We then set

$$M_n = \mathbf{1}_{rj} \otimes M_{(n)} \otimes (\bigotimes_{j<n} \mathbf{1}_j)$$

$$N_n = \mathbf{1}_{rj} \otimes M'_{(n)} \otimes (\bigotimes_{j<n} \mathbf{1}_j)$$

which satisfies (i) by construction.

Since

$$\|\Phi_{j+k} - \Phi_j\| < \sum_{i=j}^{j+k-1} \varepsilon_i \to 0 \quad (j \to \infty)$$

$\{\Phi_j\}$ is a Cauchy sequence. Let

$$\chi = \lim_{j \to \infty} \Phi_j .$$

By construction $\|\chi\| = \|\Phi_j\| = 1$ and

$$Q_n \equiv (M_n \cup N_n)''$$

is a type I factor. Let the minimal projection of Q_n defined by χ_n be E_n. Then $E_n \Phi_j = \Phi_j$ for $j \ge n$ and hence $E_n \chi = \chi$. Namely, $\chi = \chi_{rn} \otimes (\bigotimes_{j \le n} \chi_j)$ and Q_n is irreducible on

$$\overline{Q_n \chi} = \chi_{rn} \otimes H_n \otimes (\bigotimes_{j<n} \chi_j)$$

$$= \overline{M_n \chi} = \overline{N_n \chi}.$$

From this, it follows $(\prod_{n \le N} M_n)\chi$ and $(\prod_{n \le N} N_n)\chi$ span the same space $H^{(N)} \equiv \chi_r{}^N \otimes (\bigotimes_{n \le N} H_n)$ and hence (iii) follows. Furthermore, χ considered as a state on $M_n, N_n, n = 1, 2, \cdots$ is the same as the product state of χ_j and hence M_a, N_a and χ restricted to H_a is unitarily equivalent to $\mathrm{R}(H_n, M_{(n)}, \chi_n)$, $\mathrm{R}(H_n, M'_{(n)}, \chi_n)$ and $\otimes \chi_n$ on $\otimes H_n$. This proves (v).

Now we proceed to the other conditions. Let E be the projection on $\overline{M'\chi}$. By (iii), $\overline{M_a M'\chi} = \overline{M'\chi}$ and hence E commutes with both M_a and N_a. Since M is a factor of type III and $E \in M$, $M \sim M_E$ and we consider M_E on EH instead of M on H. Then $(M_n)_E$, $(N_n)_E$ and $E\chi = \chi$ replace the rôle of M_n, N_n and χ. Obviously (i), (iii) and (v) are satisfied. In addition, χ is now cyclic for M' and hence separating for M. Henceforth, we drop $(\ )_E$. A similar procedure using $\overline{M\chi}$ makes χ cyclic for M without losing its cyclicity for M'.

If χ is separating for M, then $M_a | H_a \sim M_a$ and hence $M_a \sim \bigotimes_n R_{x_n}$. Similarly $N_a \sim \bigotimes_n R_{x_n}$. Thus (iv) and (ii) are proved. (viii) is then immediate.

We consider $R_\nu^{(n)}$ in $M_{(n)} = \bigotimes R_\nu^{(n)}$ and denote the corresponding type I_2 factor in M_n by $M_{n\nu}$. Similarly we write $N_{n\nu}$ for $R_\nu^{(n)\prime}$. We now proceed to (vi) and (vii). Let $u_{ij}^{n\nu}$ and $v_{ij}^{n\nu}$ be standard matrix units of $M_{n\nu}$ and $N_{n\nu}$ relative to $\Phi_\nu^{(n)}$. Let $l(n, \nu)$ be a one-to-one map of (n, ν) to the natural numbers and let

$$\tau_{n\nu}^{ij} A = \sum_k u_{ki}^{n\nu} A u_{jk}^{n\nu}$$

$$\tau_{n\nu} A = \lambda_1 \tau_{n\nu}^{11} A + (1-\lambda_1) \tau_{n\nu}^{22} A$$

$$\tau_N^L A = \Big(\prod_{N < l(n,\nu) \leq L} \tau_{n\nu} \Big) A.$$

We now verify the following properties for $A \in M$.

(α) $\tau_{n\nu}^{ij} A \in M \cap M'_{n\nu}$

(β) $\|\tau_{n\nu} A\| \leqq \|A\|$

(γ) $(C_2\chi, (\tau_{n\nu}A) C_1\chi) = (C_2\chi, AC_1\chi)$ for any $C_j \in (M_{n\nu} \cup N_{n\nu})'$

(δ) If $A \in M'_{n\nu}$, then $\tau_{n\nu} A = A$.

The property (α) is easily checked by calculating $[u_{ij}^{n\nu}, \tau_{n\nu}^{kl} A]$. ($\beta$) follows from

$$\begin{aligned} \|\sum_k u_{ki}^{n\nu} A u_{jk}^{n\nu} \psi\|^2 &= \sum_k \|u_{ki}^{n\nu} A u_{jk}^{n\nu} \psi\|^2. \\ &\leqq \sum_k \|A u_{jk}^{n\nu} \psi\|^2 \\ &\leqq \|A\|^2 \sum_k \|u_{jk}^{n\nu} \psi\|^2 = \|A\|^2 \|\psi\|^2. \end{aligned}$$

(γ) follows from $\lambda_j^{1/2} u_{jk}^{n\nu} \chi = \lambda_k^{1/2} v_{kj}^{n\nu} \chi$, $[C_j, u_{ik}^{n\nu}] = [C_j, v_{ki}^{n\nu}] = [A, v_{ki}^{n\nu}] = 0$, $\sum_i v_{ik}^{n\nu} v_{ki}^{n\nu}$

$=1$ and $\sum \lambda_k = 1$. (δ) follows from $\sum_k u^{n\nu}_{ki} u^{n\nu}_{ik} = 1$.

Let $\tau_N A$ be any operator in $\bigcap_G (\bigcup_{L>G} \tau^L_N A)^{\mathrm{w}}$, where w denotes the weak closure. Since $\tau^L_N A$ is bounded by (β), this set is non-empty by the weak compactness. By (α), (β), (γ) and (δ), we have

(α') $\tau_L A \in M \cap M'_{n\nu}$ for all $l(n, \nu) > L$

(β') $\|\tau_L A\| \leqq \|A\|$

(γ') $(C_2 \chi, (\tau_L A) C_1 \chi) = (C_2 \chi, A C_1 \chi)$

for any $C_j \in Q^{(L)} \equiv \bigcap_{l(n\nu)>L} (M_{n\nu} \cup N_{n\nu})'$

(δ') If $A \in M'_a$, $\tau_L A = A$.

For each sequence $I = \{i_{n\nu}, j_{n\nu}\}$, $l(n, \nu) \leq L$, let

$$\tau_L(I) A = \Big(\prod_{l(n\nu)\leq L} \tau^{i_{n\nu} j_{n\nu}}_{n\nu}\Big)(\tau_L A).$$

Then

$$\tau_L A = \sum_I \Big(\prod_{l(n\nu)\leq L} u^{n\nu}_{i_{n\nu} j_{n\nu}}\Big) \tau_L(I) A$$

because $\sum_{ij} u^{n\nu}_{ij} \tau^{ij}_{n\nu} A = \sum_{ij} u^{n\nu}_{ii} A u^{n\nu}_{jj} = A$. Furthermore $\tau_L(I) A \in M_b$ by (α') and (α). Hence $\tau_L A \in M_0$.

Let P be the projection on the subspace spanned by $\bigcup_L Q^{(L)} \chi$. Then (γ') implies that $M_0 = PMP$ on PH. Exactly the same argument can be done for N. Since PH obviously contains $\overline{M_0 N_0 \chi}$, (vi) follows.

Since χ is separating for M and N, we have only to prove (vii) on restrictions of the relevant algebra to the state given by χ. Since χ gives a product state for $(M_b \cup N_b)''$, $(M_{n\nu} \cup N_{n\nu})''$, n, $= 1, 2, \cdots$, we have

$$M_0 = R_a \otimes R_b, \quad N_0 = R'_a \otimes S_b,$$

$$M_a = R_a \otimes \mathbf{1}, \quad M_b = \mathbf{1} \otimes R_b$$

$$N_a = R'_a \otimes \mathbf{1}, \quad N_b = 1 \otimes S_b.$$

We now prcve that $M'_0 = N_0$ on $\overline{M_0 N_0 \chi}$, which proves $S_b = R'_b$.

Let F be any projection in the commutant of M_0 on $\overline{M_0 N_0 \chi}$. Let F_1 be the projection on $\overline{MFH}$. Since

$$MFH = M_0FH + M_cFH = FH + M_cFH$$

where $M_cFH \perp \overline{M_0N_0\chi}$. Furthermore, $F_1 \in M'$. Hence, by (vi), we have $F \in N_0$. Q. E. D.

The following lemma will be used in lemma 6.11 where it is needed only for the case that all $\mathfrak{A}_\nu$ are type I factors. For the case where all $\mathfrak{A}_\nu$ except one are type I factors, it has been proved by Araki ([3], lemma 5). We give here a stronger lemma, based on the result of Tomita [19] that $(\mathfrak{A}\otimes\mathfrak{B})' = \mathfrak{A}'\otimes\mathfrak{B}'$. If either $\mathfrak{A}$ or $\mathfrak{B}$ is a finite type I factor then one can easily prove that $(\mathfrak{A}\otimes\mathfrak{B})' = \mathfrak{A}'\otimes\mathfrak{B}'$. The following proof then provides an alternate proof of lemma 5 of [3] (without recourse to Tomita's general result), because a type I_∞ factor is an ITPFI factor.

Lemma 6.10. Given the ITPS $H = \bigotimes_{\nu=1}^{\infty}(H_\nu, \Omega_\nu)$ and von Neumann algebras $\mathfrak{A}_\nu \subset \mathcal{B}(H_\nu)$. Then $(\otimes\mathfrak{A}_\nu)' = \otimes\mathfrak{A}'_\nu$.

Proof. Let $\mathfrak{A} = \otimes\mathfrak{A}_\nu$. Clearly $\mathfrak{A}' \supset \otimes\mathfrak{A}'_\nu$. For any $T \in \mathfrak{A}'$ and any finite index set J define $T_J \in \mathcal{B}(H(J))$ by

$$(\psi_1^J,\ T_J\psi_2^J) = (\psi_1^J \otimes \Omega(J^c),\ T\psi_2^J \otimes \Omega(J^c)).$$

Then

$$\|T_J\| \leq \|T\|$$

and

$$T_J \in (\bigotimes_{\nu\in J}\mathfrak{A}_\nu)' = \bigotimes_{\nu\in J}\mathfrak{A}'_\nu.$$

Now let

$$N = \{S:\ |(\Phi_j^1,\ (S-T)\Phi_j^2)| < \varepsilon,\ j = 1, \cdots n\}$$

be a weak neighbourhood of $T \in \mathfrak{A}'$. We construct $S \in N$, $S \in \otimes\mathfrak{A}'_\nu$ as follows. We can assume $\|\Phi_j^i\| \leq 1, j = 1, \cdots n, i = 1, 2$. By lemma 2.7 there exists a finite set J and vectors $\psi_j^i \in H(J)$, $\|\psi_j^i\| \leq 1$ such that

$$\|\psi_j^i \otimes \Omega(J^c) - \Phi_j^i\| < (1/4)\varepsilon\|T\|^{-1}.$$

Then for any S we have

$$|(\Phi_j^1,\ (S-T)\Phi_j^2) - (\psi_j^1 \otimes \Omega(J^c),\ (S-T)\psi_j^2 \otimes \Omega(J^c))| \\ < (1/2)\varepsilon(\|S\| + \|T\|)\|T\|^{-1}.$$

It follows that $S=T_J\otimes 1$ gives the desired S. Thus we have $\mathfrak{A}'\subset\otimes\mathfrak{A}'_\nu$.
Q. E. D.

Lemma 6.11. If M has a cyclic and separating vector, then $\mathrm{r}_\infty(M)=\mathrm{r}_\infty(M')$.

Proof. Since M, R_x have cyclic and separating vectors so does $M\otimes R_x$, thus $M\sim M\otimes R_x$ if and only if $M\overset{u}{\sim}M\otimes R_x$, and similarly for M'. Using lemma 6.10 and $R'_x\overset{u}{\sim}R_x$ it follows that $M\overset{u}{\sim}M\otimes R_x$ if and only if $M'\overset{u}{\sim}M'\otimes R_x$. Q. E. D.

7. Classification of ITPFI Factors—The Class S_∞

In this section we prove that all ITPFI factors in the class S_∞ are isomorphic. This result is obtained by generalizing lemma 5.4 to arbitrary N_α. For this purpose we introduce the notion of an $(x_2,\cdots x_p)$-sequence.

Definition 7.1. $(I_n, K_n^1, K_n^2, \cdots K_n^p; \phi_n^2\cdots\phi_n^p)$, $n=1,2,\cdots$ is called an $(x_2,\cdots x_p)$ sequence for $\mathrm{R}(M_\nu,\Omega_\nu)$ if $K_n^1,\cdots K_n^p$ are mutually disjoint subsets of $\mathrm{Sp}(\Omega(I_n)/M(I_n))$ and $(I_n, K_n^1, K_n^j, \phi_n^j)$, $j=2,\cdots p$ is an x_j-sequence for $\mathrm{R}(M_\nu,\Omega_\nu)$.

Lemma 7.2. Given $\mathrm{R}(M_\nu,\Omega_\nu)$ and $x_2,\cdots x_p\in\mathrm{r}_\infty(M,\Omega)$. Then there exists an $(x_2,\cdots x_p)$-sequence for $\mathrm{R}(M_\nu,\Omega_\nu)$.

Proof. Let $(\overline{I}_{jk}, \overline{K}_{jk}^i, \overline{\phi}_{jk})$, $k\in I_\infty$ be x_j-sequences, $j=2,\cdots p$. By corollary 3.4 we can assume that $\lambda(\overline{K}_{jk}^1)>1/2(1+x_j)^{-1}$ for all j,k. We can inductively choose subsequences $k(j,n)$, $n=1,2,\cdots$ such that $\overline{I}_{j,k(j,n)}$ are mutually disjoint. Define $I_{jn}=\overline{I}_{j,k(j,n)}$ and let

$$I_n=\bigcup_{j=2}^{p} I_{jn}. \tag{7.1}$$

For all $\lambda\in\mathrm{Sp}(\Omega(I_n)/M(I_n))$ we have

$$\lambda=\prod_{j=2}^{p}\lambda(j),\ \lambda(j)\in\mathrm{Sp}(\Omega(I_{jn})/M(I_{jn})). \tag{7.2}$$

Define

$$K_n^1=\{\lambda\in\mathrm{Sp}\Omega(I_n):\lambda(j)\in K_{jn}^1 \text{ for all } j=2,\cdots p\} \tag{7.3}$$

(7.4) $$(\phi_n^j\lambda)(j') = \begin{cases} \lambda(j') & \text{if } j' \neq j \\ \phi_{jn}\lambda(j) & \text{if } j' = j \end{cases}$$

(7.5) $$K_n^j = \phi_n^j K_n^1$$

for $j=2, \cdots, p$. By construction, $(I_n, K_n^1, K_n^j, \phi_n^j)$ satisfies Eq. (3.3) for each $j=2, \cdots p$. We have

(7.6) $$\lambda(K_n^1) = \prod_{j=2}^{p} \lambda(K_{jn}^1) > 2^{-p} \prod_{j=2}^{p} (1+x_j)^{-1} > 0$$

which implies $\sum \lambda(K_n^1) = \infty$. Q. E. D.

The following three lemmas are straightforward generalizations of lemmas 3.3, 5.1, and 5.4 respectively.

Lemma 7.3. Given $\mathrm{R}(M_\nu, \Omega_\nu)$, $x_2 \cdots x_p \in \mathrm{r}_\infty(M, \Omega)$, and $\varepsilon_n > 0$. Then there exists an $(x_2, \cdots x_p)$-sequence $(I_n, K_n^1, K_n^j, \phi_n^j; j=2, \cdots p)$ satisfying

(7.7) $$|1 - \sum_{i=1}^{p} \lambda(K_n^i)| < \varepsilon_n.$$

Proof. The proof is essentially identical to that for lemma 3.3 with $L_n^i, K_m^i, i=1, 2$ replaced by L_n^i, K_m^i, $i=1, \cdots p$ and ψ_n, ϕ_m replaced by $\psi_n^j, \phi_m^j, j=2, \cdots p$. Q. E. D.

Lemma 7.4. Given $M = \mathrm{R}(H_\nu, M_\nu, \Omega_\nu; \nu \in A)$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p \geq 0$, $\sum_{i=1}^{p} \lambda_i = 1$, and $1 > \varepsilon > 0$. Let $x_j = \lambda_j / \lambda_1$, $j=2, \cdots p$. If $x_j \in \mathrm{r}_\infty(M, \Omega)$ for all j, then there is a finite subset $I \subset A$, projections P, P' in $M(I)$, $M(I)'$ respectively, a unit vector $\Phi \in PP'H(I)$, and a type I_p factor $N \subset M(I)_{PP'}$ such that $\|(1-PP')\Omega(I)\| < \varepsilon$, $\|\Phi - PP'\Omega(I)\| < \varepsilon$, $\mathrm{Sp}(\Phi/N) = (\lambda_1, \cdots \lambda_p)$, and Φ factorizes N in $M(I)_{PP'}$.

Proof. Define

(7.8) $$\varepsilon' = \min\{\varepsilon/(c_p + p),\ x_j \neq 0\}$$

(7.9) $$\varepsilon_j'' = \begin{cases} x_j^{1/2}\varepsilon' & \text{if } x_j \neq 0 \\ \varepsilon'^2 & \text{if } x_j = 0. \end{cases}$$

By lemma 7.3 there exists a finite $I \subset A$, disjoint sets $K^1, \cdots K^p \subset \mathrm{Sp}(\Omega(I)/M(I))$ and bijections ϕ^j from K^1 to K^j, $j=2, \cdots p$ satisfying

(7.10) $$\max_{\mu \in K^1} |x_j - \phi^j\mu/\mu| < \varepsilon_j'', j=2, \cdots p$$

and

(7.11) $$|1-\sum_{i=1}^{p}\lambda(K^i)|<\varepsilon/(c_p+p)$$

where c_p is given by lemma 4.2. Eq. (7.10) implies that

$$\max_{\mu\in K^1}|x_j^{1/2}-(\phi^j\mu/\mu)^{1/2}|<\varepsilon'.$$

The result now follows from lemma 4.2. Q. E. D.

Lemma 7.5. Given $M=\mathrm{R}(M_\nu, \Omega_\nu)$, $N=\mathrm{R}(N_\alpha, \psi_\alpha)$. If $\hat{\mathrm{r}}(N, \psi)\subset \mathrm{r}_\infty(M, \Omega)$ then $M\sim M\otimes N$.

Proof. The proof is essentially identical to that of lemma 5.4 (with lemma 5.1 replaced by lemma 7.4). Q. E. D.

Theorem 7.6. Given ITPFI factors $M=\mathrm{R}(M_\nu, \Omega_\nu)$, $N=\mathrm{R}(N_\alpha, \psi_\alpha)$. If $\mathrm{r}_\infty(M)=\mathrm{r}_\infty(N)=S_\infty$, then $M\sim N$.

Proof. For any ITPFI factor $M=\mathrm{R}(M_\nu, \Omega_\nu)$ we have $\hat{\mathrm{r}}(M, \Omega)\subset S_\infty$. Thus it follows from lemma 7.5 that $\mathrm{r}_\infty(N)=S_\infty$ implies $N\sim N\otimes M$. Conversely $\mathrm{r}_\infty(M)=S_\infty$ implies $M\sim M\otimes N$, and thus $M\sim N$. Q. E. D.

Definition 7.7. We shall denote the ITPFI factor M with $\mathrm{r}_\infty(M)=S_\infty$ by R_∞.

8. Some Technical Lemmas

This section contains a number of technical lemmas which are devoted to proving the result that for ITPFI factors M, $\mathrm{r}_\infty(M)=S_x$ if and only if $M\sim R_x$ (theorem 9.1). The basic idea is to exploit the condition for unitary equivalence given by lemma 2.13. The main results of this section are given in lemmas 8.3, 8.11, 8.14, 8.16.

Definition 8.1. Given a type I_n factor M, a unit vector Ω, $\mathrm{Sp}(\Omega/M)=\{\lambda_1, \cdots\lambda_n\}$. We define

$$\delta_0(M, \Omega)=(\lambda_1^{1/2}-1)^2+\sum_{j=2}^{n}\lambda_j$$

$$\delta_1(M, \Omega)=\sum_{j=1}^{n}(\lambda_j^{1/2}-n^{-1/2})^2$$

and

$$(8.1)\qquad \delta_x(M,\Omega)=\min_{(m_1,\cdots m_n)}\sum_{j=1}^{n}[\lambda_j^{1/2}-(x^{m_j}/\sum_{i=1}^{n}x^{m_j})^{1/2}]^2 \quad 0<x<1$$

where the minimum is taken over all n-tuples of integers $(m_1, \cdots m_n)$.

Note that the expression on the right-hand side of Eq. (8.1) does not change when $m_j \to m_j+m$. Thus the n-tuple of integers $(m_1, \cdots m_n)$ which gives the minimum in Eq. (8.1) is determined only up to an additive integer. This ambiguity could be removed by requiring that $m_1=0$, but this is unnecessary for our purposes.

Definition 8.2. Given $0\leq x\leq 1$, $M=\mathrm{R}(M_\nu, \Omega_\nu)$. We define

$$(8.2)\qquad \mathrm{d}_x(M, \Omega)=\sum_\nu \delta_x(M_\nu, \Omega_\nu)$$

where $\delta_x(M_\nu, \Omega_\nu)$ is given by definition 8.1.

Lemma 8.3. Given $0\leq x\leq 1$, $M=\mathrm{R}(M_\nu, \Omega_\nu)$, $x\varepsilon \mathrm{r}_\infty(M)$, $\mathrm{d}_x(M, \Omega)<\infty$. Then $M\sim R_x$.

Proof. Let $M_1=\mathrm{R}(M_\nu, \Omega'_\nu)$ where $\mathrm{Sp}(\Omega'_\nu/M_\nu)=\{\alpha_{\nu j};\, j=1, \cdots n_\nu\}$ and $\alpha_{\nu j}$ are defined as follows. If $x=0$ let $\alpha_{\nu 1}=1$, $\alpha_{\nu j}=0$, $j=2, \cdots n_\nu$. If $x=1$ let $\alpha_{\nu j}=n_\nu^{-1}$, $j=1, \cdots n_\nu$. If $0<x<1$ let

$$\alpha_{\nu j}=x^{m_{\nu j}}/\sum_{i=1}^{n_\nu}x^{m_{\nu i}}$$

where $(m_{\nu 1}, \cdots m_{\nu n_\nu})$ gives the minimum for $\delta_x(M_\nu, \Omega_\nu)$ in Eq. (8.1). By construction $\mathrm{f}(M_\nu, \Omega'_\nu)\subset S_x$ and thus $R_x\sim R_x\otimes M_1$ by lemma 7.5. By lemma 2.13, $\mathrm{d}_x(M, \Omega)<\infty$ implies that $M\overset{u}{\sim}M_1$. Thus $x\in \mathrm{r}_\infty(M_1)$ and $M_1\sim M_1\otimes R_x$ by corollary 5.5. Thus $M\sim M_1\sim M_1\otimes R_x\sim R_x$.

Q. E. D.

The following lemmas are devoted to proving the converse result, namely that $\mathrm{d}_x(M, \Omega)=\infty$ implies that $\mathrm{r}_\infty(M)\neq S_x$. The basic idea is to use the central limit theorem to obtain the existence of some $y\in \mathrm{r}_\infty(M)$, $y\notin S_x$ (except for $x=0$ where we use lemma 3.8).

Lemma 8.4. Given $X_\nu=\{\lambda_{\nu 1}, \cdots \lambda_{\nu n_\nu}\}$, $\lambda_{\nu i}\geq 0$ and $\sum_{i=1}^{n_\nu}\lambda_{\nu i}=1$. Let μ_ν be the probability measure on X_ν defined by $\mu_\nu(\{\lambda\})=\lambda$, $\lambda\in X_\nu$. Let K_ν^1, K_ν^2 be disjoint subsets of X_ν and let ϕ_ν be a bijection from K_ν^1 to K_ν^2. Assume $0\notin K_\nu^i$. Let

(8.3) $$\eta(\lambda)=\log(\phi_\nu\lambda/\lambda),\ \lambda\in K_\nu^1.$$

Let s_ν be the random variable defined by

(8.4) $$s_\nu(\lambda)=\begin{cases}\eta(\lambda) & \text{if } \lambda\in K_\nu^1\\ -\eta(\phi^{-1}_\nu\lambda) & \text{if } \lambda\in K_\nu^2\\ 0 & \text{if } \lambda\notin K_\nu^1\cup K_\nu^2.\end{cases}$$

Let

(8.5) $$Y_N=\sum_{\nu=1}^N s_\nu$$

and

(8.6) $$\delta_\nu=\max_{\lambda\in K_\nu^1}|\eta(\lambda)|.$$

If

(8.7) $$\lim_{\nu\to\infty}\delta_\nu=0$$

and

(8.8) $$\sum_\nu\sum_{\lambda\in K_\nu^1}\lambda\eta(\lambda)^2=\infty$$

then for any fixed $0<a<\infty$ we have

(8.9) $$\lim_{N\to\infty}(\prod_{\nu=1}^N\mu_\nu)(X_N(a))=0$$

where $X_N(a)$ is the subset of $\prod_{\nu=1}^N X_\nu$ defined by $|Y_N|\leq a$.

Proof. The mean of s_ν is

(8.10) $$\langle s_\nu\rangle=\sum_{\lambda\in K_\nu^1}\eta_\nu(\lambda)(\lambda-\phi_\nu\lambda)=\sum\lambda\eta_\nu(\lambda)(1-e^{\eta_\nu(\lambda)})$$

$$=-\sum\lambda\eta_\nu(\lambda)^2[1+0(\delta_\nu)].$$

Since $\sum_{\lambda\in K_\nu^1}\lambda\leq 1$ and $|\eta_\nu(\lambda)|\leq\delta_\nu$ we have

$$\langle s_\nu\rangle=0(\delta_\nu^2).$$

Thus $\langle s_\nu\rangle$ is bounded. The variance of s_ν is

$$\sigma_\nu=\sum_{\lambda\in K_\nu^1}[\lambda\eta_\nu(\lambda)^2+(\phi_\nu\lambda)\eta_\nu(\lambda)^2]-\langle s_\nu\rangle^2$$

$$=\sum_{\lambda\in K_\nu^1}\lambda\eta_\nu(\lambda)^2[1+e^{\eta_\nu}]-\langle s_\nu\rangle^2$$

$$=2\sum_{\lambda\in K_\nu^1}\lambda\eta_\nu(\lambda)^2[1+0(\delta_\nu)]-\langle s_\nu\rangle^2.$$

Using $\langle s_\nu\rangle=0(\delta_\nu^2)$ for one factor of $\langle s_\nu\rangle^2$ and Eq. (8.10) for another, we get

$$\sigma_\nu=2\sum_{\lambda\in K_\nu^1}\lambda\eta_\nu(\lambda)^2[1+0(\delta_\nu)]. \tag{8.11}$$

It follows that $\sigma_\nu\to 0$ and $\sum_\nu\sigma_\nu=\infty$. It now follows from the central limit theorem for bounded variances (Loève [9]) that

$$\sum_{\nu=1}^{N}(s_\nu-\langle s_\nu\rangle)/\{\sum_{\nu=1}^{N}\sigma_\nu\}^{1/2}$$

approaches a normal distribution as $N\to\infty$. Since $\sum\sigma_\nu=\infty$, the finite interval $[-a,a]$ gives a vanishing contribution as $N\to\infty$. Q. E. D.

Lemma 8.5. Given $R(M_\nu,\Omega_\nu;\nu\in A)$. Let $X_\nu=\mathrm{Sp}(\Omega_\nu/M_\nu)$. Let K_ν^1, K_ν^2, ϕ_ν, $\eta(\lambda)$, δ_ν be as in lemma 8.4, except K_ν^2 may contain 0's. If

$$\lim_{\nu\to\infty}\delta_\nu=0 \tag{8.12}$$

and

$$\sum_\nu\sum_{\lambda\in K_\nu^1}\lambda\eta(\lambda)^2=\infty \tag{8.13}$$

then $\mathrm{r}_\infty(M)=S_\infty$. Here the terms with $\eta(\lambda)=-\infty$ are excluded from the sum in (8.13).

Proof. We will use lemma 8.4 to prove that $e^{-l}\in\mathrm{r}_\infty(M)$ for arbitrary $0<l<\infty$, which implies that $\mathrm{r}_\infty(M)=S_\infty$. By (8.12), we can restrict ν such that $0\notin K_\nu^2$.

Let $0<l<\infty$ be given. Let μ_ν be the probability measure and let s_ν be the random variable defined in lemma 8.4. Let I be any finite subset of A, and define

$$Y(I)=\sum_{\nu\in I}s_\nu. \tag{8.14}$$

Let $X(I,a)$ be the subset of $\prod_{\nu\in I}\mathrm{Sp}(\Omega_\nu/M_\nu)=\mathrm{Sp}(\Omega(I)/M(I))$ defined by $|Y(I)|\leq a$. It follows from lemma 8.4 that there exist mutually disjoint subsets $I_n\subset A$, $n\in I_\infty$ such that

(8.15) $$(\prod_{\nu\in I_n}\mu_\nu)(X(I_n, 1/2l))<1/2.$$

We now construct an e^l-sequence (I_n, L_n^i, ψ_n) as follows.

For $\rho\in X^{(n)}=\mathrm{Sp}(\Omega(I_n)/M(I_n))$ the equation

(8.16) $$\rho=\prod_{\nu\in I_n}\lambda_\nu(\rho),\ \lambda_\nu(\rho)\in\mathrm{Sp}(\Omega_\nu/M_\nu)$$

defines the function $\lambda_\nu(\rho)$, $\nu\in I_n$. Let

(8.17) $$y(I_n, m, \rho)=\sum_{\substack{\nu\leq m\\ \nu\in I_n}} s_\nu(\lambda_\nu(\rho))$$

and define

(8.18) $$\alpha(I_n, \rho)=\begin{cases}\text{minimum } \alpha\in I_n \text{ with } |y(I_n, \alpha, \rho)|\geq 1/2l\\ \infty \text{ if } |y(I_n, \alpha, \rho)|<1/2l \text{ for all } \alpha\in I.\end{cases}$$

Let

(8.19) $$L_n^1=\{\rho: \alpha(I_n, \rho)<\infty,\ y(I_n, \alpha(I_n, \rho), \rho)\geq 1/2l,\ \lambda_\nu(\rho)\neq 0\}$$

(8.20) $$L_n^2=\{\rho: \alpha(I_n, \rho)<\infty,\ y(I_n, \alpha(I_n, \rho), \rho)\leq -1/2l,\ \lambda_\nu(\rho)\neq 0\}.$$

We define a bijection ψ_n from L_n^1 to L_n^2 as follows. If $\nu\leq\alpha(I_n, \rho)$, $\nu\in I_n$ define

(8.21) $$\lambda_\nu(\psi_n\rho)=\begin{cases}\phi_\nu\lambda_\nu(\rho) & \text{if } \lambda_\nu(\rho)\in K_\nu^1\\ \phi_\nu^{-1}\lambda_\nu(\rho) & \text{if } \lambda_\nu(\rho)\in K_\nu^2\\ \lambda_\nu(\rho) & \text{if } \lambda_\nu(\rho)\notin K_\nu^1\cup K_\nu^2\end{cases}$$

where ϕ_ν^{-1} is the inverse of the bijection ϕ_ν. If $\nu>\alpha(I_n, \rho)$, $\nu\in I_n$ define

(8.22) $$\lambda_\nu(\psi_n\rho)=\lambda_\nu(\rho).$$

Let

$$\delta(I_n)=\sup_{\nu\in I_n}\delta_\nu.$$

By construction we have

(8.23) $$e^{-l-2\delta(I_n)}>\psi_n\rho/\rho\geq e^{-l},\ \rho\in L_n^1.$$

Since $\delta(I_n)\to 0$ we have

(8.24) $$\lim_{n\to\infty}\max_{\rho\in L_n^1}|e^{-l}-\psi_n\rho/\rho|=0.$$

Also by construction we have

(8.25) $$1-\lambda(L_n^1)-\lambda(L_n^2)\leq(\prod_{\nu\in I_n}\mu_\nu)(X(I_n, 1/2l))<1/2.$$

It follows from Eqs. (8.24), (8.25) that

$$\sum_n \lambda(L_n^1) = \infty.$$

Thus (I_n, L_n^i, ψ_n) is an e^{-l}-sequence. Q. E. D.

Lemma 8.6. Given $0 < a < \infty$, $M = \mathrm{R}(M_\nu, \Omega_\nu; \nu \in A)$, disjoint subsets $K_\nu^1, K_\nu^2 \subset \mathrm{Sp}(\Omega_\nu/M_\nu)$, and a bijection ϕ_ν from K_ν^1 to K_ν^2. If

$$\sum_\nu \sum_{\lambda \in K_\nu^1} [\lambda^{1/2} - (\phi_\nu\lambda)^{1/2}]^2 = \infty \tag{8.26}$$

then there exists some $x \in \mathrm{r}_\infty(M)$, $x \neq 1$. If we also have

$$|\log(\phi_\nu\lambda/\lambda)| \leq a \tag{8.27}$$

for all $\lambda \in K_\nu^1$ and all ν, then there exists some $x \in \mathrm{r}_\infty(M)$, $e^{-a} \leq x < 1$.

Proof. First we throw out from K_ν^1 and K_ν^2 all λ and $\phi_\nu\lambda$ for which $\lambda = \phi_\nu\lambda = 0$. Since Eqs. (8.26), (8.27) are unaffected by the interchange of λ and $\phi_\nu\lambda$, we can assume $\phi_\nu\lambda \leq \lambda$ and thus $\phi_\nu\lambda/\lambda \in [0, 1]$ for all $\lambda \in K_\nu^1$ and all ν. We define a subset S of $[0, 1]$ as follows. Let $\varepsilon_m > 0$, $\varepsilon_m \to 0$, $m \in I_\infty$. Given $\alpha < \beta$, let

$$\Sigma(\alpha, \beta) = \sum_\nu \sum_{\substack{\lambda \in K_\nu^1 \\ \phi_\nu\lambda/\lambda \in (\alpha, \beta)}} [\lambda^{1/2} - (\phi_\nu\lambda)^{1/2}]^2. \tag{8.28}$$

Define

$$S = \{x : \Sigma(x - \varepsilon_m, x + \varepsilon_m) = \infty \text{ for all } m\}. \tag{8.29}$$

We now use the fact that $[0, 1]$ is compact to prove that S is nonempty. If $x \notin S$ then there is some fiinite integer $m(x)$ such that

$$\Sigma(x - \varepsilon_{m(x)}, x + \varepsilon_{m(x)}) < \infty. \tag{8.30}$$

If S is empty then we have a covering of $[0, 1]$ by the open sets $I(x, m(x))$, $x \in [0, 1]$. It follows that there is a finite collection $I(x_1, m(x_1))$, $\cdots I(x_p, m(x_p))$ which covers $[0, 1]$. Using Eqs. (8.30), (8.28) we get

$$\sum_\nu \sum_{\lambda \in K_\nu^1} [\lambda^{1/2} - (\phi_\nu\lambda)^{1/2}]^2 \leq \sum_{j=1}^{p} \Sigma(x_j - \varepsilon_{m(x_j)}, x_j + \varepsilon_{m(x_j)}) < \infty \tag{8.31}$$

which contradicts Eq. (8.26). Thus S is nonempty.

Let $x \in S$. It follows from Eq. (8.29) that we can inductively choose subsets $L_\nu^1(x) \subset K_\nu^1$ such that

$$\lim_{\nu \to \infty} \max_{\lambda \in L_\nu^1(x)} |x - \phi_\nu\lambda/\lambda| = 0 \tag{8.32}$$

and

$$\sum_{\nu}\sum_{\lambda\in L_\nu^1(x)}[\lambda^{1/2}-(\phi_\nu\lambda)^{1/2}]^2=\infty. \tag{8.33}$$

Let

$$L_\nu^2(x)=\{\phi_\nu\lambda:\lambda\in L_\nu^1(x)\}.$$

If $x\neq 1$ it follows from Eqs. (8.32), (8.33) that

$$\sum_{\nu}\lambda(L_\nu^1(x))=\infty. \tag{8.34}$$

It follows from Eqs. (8.32), (8.34) that $(\nu, L_\nu^i(x), \phi_\nu)$ is an x-sequence. If $x=1$, then $\mathrm{R}(M_\nu, \Omega_\nu)$, $L_\nu^1(x)$, $L_\nu^2(x)$ and ϕ_ν satisfy the conditions of lemma 8.5 and we have $\mathrm{r}_\infty(M)=S_\infty$. If Eq. (8.27) is satisfied then $S\subset[e^{-a}, 1]$ and thus $x\in\mathrm{r}_\infty(M)$ for some $e^{-a}\leq x<1$. Q. E. D.

Lemma 8.7. Given $0<a<\infty$, $\alpha_{\nu j}>0$, $-a\leq\eta_{\nu j}\leq a, j=1, \cdots n_\nu$, $\nu\in I_\infty$. Then the following statements hold.

(i) The conditions

$$\sum_{\nu,j}\alpha_{\nu j}(e^{\eta_{\nu j}/2}-1)^2=\infty \tag{8.35}$$

$$\sum_{\nu,j}\alpha_{\nu j}(e^{\eta_{\nu j}}-1)^1=\infty \tag{8.36}$$

are equivalent.

(ii) The conditions

$$\sum_{\nu}\sum_{i<j}\alpha_{\nu i}\alpha_{\nu j}(e^{\eta_{\nu i}}-e^{\eta_{\nu j}})^2=\infty \tag{8.37}$$

$$\sum_{\nu}\sum_{i<j}\alpha_{\nu i}\alpha_{\nu j}(e^{\eta_{\nu i}/2}-e^{\eta_{\nu j}/2})^2=\infty \tag{8.38}$$

$$\sum_{\nu}\sum_{i<j}\alpha_{\nu i}\alpha_{\nu j}(\eta_{\nu i}-\eta_{\nu j})^2=\infty \tag{8.39}$$

are equivalent.

Proof. Let $f(\eta)=e^\eta-1$. For any $\eta\in[-a, a]$ we have

$$0<f'(-a)|\eta|<|e^\eta-1|<f'(a)|\eta|<\infty$$

It follows that for $\eta_{\nu j}\in[-a, a]$, there exist positive constants C_1, C_2 such that

$$C_1|e^{\eta_{\nu j}/2}-1|\geq|e^{\eta_{\nu j}}-1|\geq C_2|e^{\eta_{\nu j}/2}-1|$$

from which (i) follows. Statement (ii) follows from a similar argument.

Q. E. D.

Lemma 8.8. Given $\alpha_1, \cdots \alpha_n > 0, \sum \alpha_j = 1$, and $-\infty < X_j < \infty, j = 1, \cdots n$ such that

(8.40) $$\sum \alpha_j X_j = 0.$$

Then

(8.41) $$\sum_{\substack{i,j=1 \\ i<j}}^{n} \alpha_i \alpha_j (X_i - X_j)^2 = \sum_{i=1}^{n} \alpha_i X_i^2.$$

Proof. The left hand side of Eq. (8.41) is

$$(1/2) \sum \alpha_i \alpha_j (X_i - X_j)^2 = (\sum \alpha_i X_i^2)(\sum \alpha_j) - (\sum \alpha_i X_i)^2.$$

By the assumption, this is the same as the right hand side of (8.41).

Lemma 8.9. Given $0 < x < 1, \lambda_1, \cdots \lambda_n > 0, \sum_{i=1}^{n} \lambda_i = 1$. Then there exists an n-tuple of integers $(m_1, \cdots m_n)$ such that

(8.42) $$|\eta_i - \eta_j| < |\log x|, i, j = 1, \cdots n$$

where η_j is defined by

(8.43) $$\lambda_j = e^{\eta_j} (x^{m_j} / \sum_{i=1}^{n} x^{m_i}), j = 1, \cdots n.$$

Also

(8.44) $$\max |\eta_j| < |\log x|.$$

Proof. For any $-\infty < \alpha < \infty$ we define integers $m_j(\alpha), j = 1, \cdots n$ by

(8.45) $$e^{\alpha} x^{m_j(\alpha)-1} > \lambda_j \geq e^{\alpha} x^{m_j(\alpha)}.$$

Define $\eta_j'(\alpha)$ by

(8.46) $$\lambda_j = e^{\eta_j'(\alpha)} e^{\alpha} x^{m_j(\alpha)}.$$

It follows from Eqs. (8.45), (8.46) that

(8.47) $$0 \leq \eta_j'(\alpha) < |\log x|.$$

Define $\eta_j(\alpha)$ by Eq. (8.43) with $m_j = m_j(\alpha)$. Then we have

(8.48) $$\eta_j(\alpha) = \eta_j'(\alpha) - \eta(\alpha)$$

where

$$\eta(\alpha) = -\log(\sum_j x^{m_j(\alpha)}) - \alpha.$$

Eq. (8.42) now follows from Eqs. (8.47), (8.48). Using Eq. (8.45) and $\sum\lambda_j=1$ we get

(8.49) $$x<e^{-\eta(\alpha)}\leq 1.$$

Eq. (8.44) now follows from Eqs. (8.47), (8.48), (8.49). Thus $m_j=m_j(\alpha)$ for any α satisfies the lemma. Q. E. D.

Lemma 8.10. Given $0<x<1$, $\lambda_1, \cdots\lambda_n>0$, $\sum\lambda_i=1$ and $\alpha_1, \cdots\alpha_n>0$. Then there exists an n-tuple of integers $(m_1, \cdots m_n)$ satisfying the conditions of lemma 8.9 and a subset I of $\{1, \cdots n\}$ such that

(8.50) $$\sup_{i,j\in I}|\eta_i-\eta_j|\leq(4/5)|\log x|$$

and

(8.51) $$\sum_{\substack{i,j\in I\\ i<j}}\alpha_i\alpha_j(\eta_i-\eta_j)^2>(1/9)\sum_{\substack{i,j=1\\ i<j}}^{n}\alpha_i\alpha_j(\eta_i-\eta_j)^2$$

where η_j is defined by Eq. (8.43).

Proof. Consider the $m_j(\alpha)$, $\eta_j(\alpha)$ given in the proof of lemma 8.9. We will show that one can choose α and I so that Eqs. (8.50), (8.51) are satisfied.

It follows from Eq. (8.48) that it is sufficient to prove Eqs. (8.50), (8.51) with $\eta_j(\alpha)$ replaced by $\eta'_j(\alpha)$. It follows from Eqs. (8.45), (8.46) that

(8.52) $$\eta'_j(\alpha+\beta)=\eta'_j(\alpha)-\beta(\text{modulo }|\log x|).$$

Thus we can consider the $\eta'_j(\alpha)$ as defined on a circle of circumference $|\log x|$. Choose the interval $\Delta(\eta_\alpha)=[\eta_\alpha, \eta_\alpha+(1/5)|\log x|)$ on this circle such that $\sum_{\eta'_j\in\Delta(\eta_\alpha)}\alpha_j$ is a minimum. We choose $\alpha=\eta_0$. Then $\eta_\alpha\equiv\eta_0-\alpha$ is 0 and this interval is $[0, (1/5)|\log x|)$. Let

(8.53) $$\Delta_k=\{j:\eta'_j(\alpha)\in[(k-1)|\log x|/5, k|\log x|/5)\},\ k=1, \cdots 5$$

and define

(8.54) $$\Sigma_{kl}=\sum_{\substack{i<j\\ i\in\Delta_k\\ i\in\Delta_l}}\alpha_i\alpha_j(\eta'_i-\eta'_j)^2.$$

By construction we have

(8.55) $$\sum_{i\in\Delta_1}\alpha_i\leqslant\sum_{i\in\Delta_k}\alpha_i,\quad k=2,3,4,5.$$

Now $i_1\in\Delta_1, i_2\in\Delta_2, i_5\in\Delta_5$ implies that

(8.56) $$|\eta'_{i_1}-\eta'_{i_5}|<2|\eta'_{i_2}-\eta'_{i_5}|.$$

It follows from Eqs. (8.55), (8.56) that

(8.57) $$\Sigma_{15}<4\Sigma_{25}.$$

By similar arguments we have

(8.58) $$\Sigma_{14}<9\Sigma_{24}$$

(8.59) $$\Sigma_{13}<9\Sigma_{35}$$

(8.60) $$\Sigma_{12}<\Sigma_{25}$$

(8.61) $$\Sigma_{11}<\Sigma_{25}.$$

Let $I=\bigcup_{k=2}^{5}\Delta_k$. It follows from Eq. (8.53) that Eq. (8.50) is satisfied. We have

(8.62) $$\sum_{\substack{i,j=1\\ i<j}}^{n}\alpha_i\alpha_j(\eta_i-\eta_j)^2=\sum_{k,l=1}^{5}\Sigma_{kl}$$

and

(8.63) $$\sum_{\substack{i,j\in I\\ i<j}}\alpha_i\alpha_j(\eta_i-\eta_j)^2=\sum_{k,l=2}^{5}\Sigma_{kl}.$$

It follows from Eqs. (8.57–63) that Eq. (8.51) is satisfied. Q. E. D.

Lemma 8.11. Given $0<x<1$, $M=\mathrm{R}(M_\nu, \Omega_\nu)$, $\mathrm{r}_\infty(M)=S_x$. Then $\mathrm{d}_x(M,\Omega)<\infty$.

Proof. We will use lemmas 8.7, 8.8 and 8.10 to translate the condition $\mathrm{d}_x(M,\Omega)=\infty$ into the conditions of lemma 8.6 with $|\log(\phi_\nu\lambda/\lambda)|\leq a<|\log x|$. It will then follow that there exists some $y\in\mathrm{r}_\infty(M)$, $x<y<1$ which contradicts $\mathrm{r}_\infty(M)=S_x$.

Let $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu1},\cdots\lambda_{\nu n_\nu}\}$. By lemma 3.15 we can assume $\lambda_{\nu j}>0$ for all ν and j. Use lemma 8.10 to choose integers $(m_{\nu1},\cdots m_{\nu n_\nu})$ and subsets $I_\nu\subset\{1,\cdots n_\nu\}$ such that $|\eta_{\nu i}|<|\log x|$,

(8.64) $$\sup_{i,j\in I_\nu}|\eta_{\nu i}-\eta_{\nu j}|<(4/5)|\log x|$$

and

(8.65) $$\sum_{\substack{i,j\in I_\nu\\ i<j}} \alpha_{\nu i}\alpha_{\nu j}(\eta_{\nu i}-\eta_{\nu j})^2 > (1/9)\sum_{\substack{i,j=1\\ i<j}}^{n_\nu} \alpha_{\nu i}\alpha_{\nu j}(\eta_{\nu i}-\eta_{\nu j})^2$$

where $\eta_{\nu j}$ is defined by

(8.66) $$\lambda_{\nu j}=e^{\eta_{\nu j}}\alpha_{\nu j}$$

and

(8.67) $$\alpha_{\nu j}=x^{m_{\nu j}}/\sum_{i=1}^{n_\nu} x^{m_{\nu i}}.$$

Let $M_1=\mathrm{R}(M_\nu, \Omega'_\nu)$ where $\mathrm{Sp}(\Omega'_\nu/M_\nu)=\{\alpha_{\nu 1}, \cdots \alpha_{\nu n_\nu}\}$. By Eq. (8.67) $\mathfrak{r}(M_\nu, \Omega'_\nu)\subset S_x=\mathrm{r}_\infty(M)$ and thus $M\sim M\otimes M_1$ by lemma 7.5. We have $M\otimes M_1=\mathrm{R}(M_\nu\otimes M_\nu, \Omega_\nu\otimes\Omega'_\nu)$ where $\mathrm{Sp}(\Omega_\nu\otimes\Omega'_\nu/M_\nu\otimes M_\nu)=\{\lambda_{\nu i}\alpha_{\nu j}: i, j=1, \cdots n_\nu\}$. We define disjoint subsets K^1_ν, $K^2_\nu\subset\mathrm{Sp}(\Omega_\nu\otimes\Omega'_\nu/M_\nu\otimes M_\nu)$ and a bijection ϕ_ν from K^1_ν to K^2_ν by

(8.68) $$K^1_\nu=\{\lambda_{\nu i}\alpha_{\nu j}: i, j\in I_\nu \text{ and } i<j\}$$

(8.69) $$K^2_\nu=\{\lambda_{\nu i}\alpha_{\nu j}: i, j\in I_\nu \text{ and } i>j\}$$

(8.70) $$\phi_\nu\lambda_{\nu i}\alpha_{\nu j}=\lambda_{\nu j}\alpha_{\nu i} \quad (i<j).$$

For $\lambda=\lambda_{\nu i}\alpha_{\nu j}\in K^1_\nu$ let

(8.71) $$\eta(\lambda)=\log(\phi_\nu\lambda/\lambda)=\eta_{\nu j}-\eta_{\nu i}.$$

It follows from Eqs. (8.64), (8.71) that

(8.72) $$|\log(\phi_\nu\lambda/\lambda)|\leq(4/5)|\log x|$$

for all $\lambda\in K^1_\nu$ and all ν. Using Eqs. (8.66), (8.67) and definitions 8.1, 8.2 the condition $\mathrm{d}_\tau(M, \Omega)=\infty$ implies that

(8.73) $$\sum_\nu\sum_j \alpha_{\nu j}(e^{\eta_{\nu j}/2}-1)^2=\infty.$$

By lemma 8.7 this is equivalent to

(8.74) $$\sum\alpha_{\nu j}(e^{\eta_{\nu j}}-1)^2=\infty.$$

Since $\sum_j\lambda_{\nu j}=\sum_j\alpha_{\nu j}=1$, it follows from Eq. (8.66) that

(8.75) $$\sum\alpha_{\nu j}(e^{\eta_{\nu j}}-1)=0.$$

Using Eqs. (8.74), (8.75) and lemma 8.8 we get

(8.76) $$\sum_\nu\sum_{\substack{i,j=1\\ i<j}}^{n_\nu} \alpha_{\nu i}\alpha_{\nu j}(e^{\eta_{\nu i}}-e^{\eta_{\nu j}})^2=\infty.$$

By lemma 8.7 this is equivalent to

$$\sum \alpha_{\nu i}\alpha_{\nu j}(\eta_{\nu i}-\eta_{\nu j})^2=\infty.$$

Eq. (8.65) now gives

$$\sum_{\nu}\sum_{\substack{i,j\in I_\nu\\ i<j}} \alpha_{\nu i}\alpha_{\nu j}(\eta_{\nu i}-\eta_{\nu j})^2=\infty. \tag{8.77}$$

By lemma 8.7 this is equivalent to

$$\begin{aligned}\infty &= \sum\sum_{\substack{i,j\in I_\nu\\ i<j}} \alpha_{\nu i}\alpha_{\nu j}(e^{\eta_{\nu i}/2}-e^{\eta_{\nu j}/2})^2 \\ &= \sum_{\nu}\sum_{\lambda\in K_\nu^1} [\lambda^{1/2}-(\phi_\nu\lambda)^{1/2}]^2.\end{aligned} \tag{8.78}$$

It follows from Eqs. (8.72), (8.78) and lemma 8.6 that there exists some $e^l\in \mathrm{r}_\infty(M)$, $0<|l|\leq(4/5)|\log x|$. Q. E. D.

The result that $\mathrm{r}_\infty(M)=S_x$ implies that $\mathrm{d}_x(M,\Omega)<\infty$ for $x=0,1$ can be obtained directly from the known conditions for M to be type I, II_1 respectively which have been stated in lemma 2.14. However it seems worthwhile to use our techniques to give an independent derivation of these results.

In the proof of lemma 8.11 we made frequent use of the fact that the $\eta_{\nu j}$ were bounded, a condition that does not hold when $x=1$. Instead of modifying the proof of lemma 8.11, it seems simpler to use the following two lemmas.

Lemma 8.12. Let x_j, $j\in K$ be a finite set of real numbers such that

$$\sum_{j\in K} x_j=0. \tag{8.79}$$

Then there exist disjoint subsets $K^1, K^2\subset K$ and a bijection ϕ from K^1 to K^2 such that

$$\sum_{j\in K^1} (x_j-x_{\phi(j)})^2\geq\sum_{j\in K} x_j^2. \tag{8.80}$$

Proof. Order the index set K by $x_1\geq x_2\geq\cdots\geq x_N$ and choose m so that either $N=2m$ or $N=2m+1$. Let $\alpha=x_{m+1}$. Let $K^1=\{1,\cdots m\}$, $K^2=\{N-m+1,\cdots N\}$, and define $\phi(j)=N-m+j$, $j\in K^1$. Let

$$y_j=x_j-\alpha,\ j=1,\cdots N. \tag{8.81}$$

By construction, if $j \in K^1$ then y_j and $y_{\phi(j)}$ have the opposite sign. Thus we have

(8.82) $$\sum_{j\in K^1}(x_j - x_{\phi(j)})^2 = \sum_{j\in K^1}(y_j - y_{\phi(j)})^2. \geq \sum_{j\in K^1}(y_j^2 + y_{\phi(j)}^2) = \sum_{j=1}^{N} y_j^2$$

where we used the fact that $y_j = 0$ if $j \notin K^1 \cup K^2$. Since $\sum x_j = 0$ we have

(8.83) $$\sum_{j=1}^{N} y_j^2 = \sum_{j=1}^{N} x_j^2 + N\alpha^2 \geq \sum x_j^2.$$

Q. E. D.

Lemma 8.13. Given $\lambda_1, \cdots \lambda_n \geq 0$, $\sum \lambda_i = 1$. Then there exist disjoint subsets K^1, $K^2 \subset \{\lambda_1, \cdots \lambda_n\}$ and a bijection ϕ from K^1 to K^2 such that

(8.84) $$\sum_{\lambda\in K^1}[\lambda^{1/2} - (\phi\lambda)^{1/2}]^2 > 1/2\sum_{j=1}^{n}(\lambda_j^{1/2} - n^{-1/2})^2.$$

Proof. Let

(8.85) $$\delta_j = \lambda_j^{1/2} - n^{-1/2}.$$

Then

$$\lambda_j = \delta_j^2 + n^{-1} + 2n^{-1/2}\delta_j$$

and using $\sum \lambda_j = 1$ we get

(8.86) $$\bar{\delta} = n^{-1}\sum \delta_j = -1/2n^{-1/2}\sum \delta_j^2.$$

We also get

$$\delta_j^2 = \lambda_j + n^{-1} - 2\lambda_j^{1/2}n^{-1/2}$$

and thus

(8.87) $$\sum \delta_j^2 < 2.$$

Since

(8.88) $$\sum(\delta_j - \bar{\delta}) = 0$$

we have

$$\sum \delta_j^2 = \sum(\delta_j - \bar{\delta})^2 + n\bar{\delta}^2.$$

Using Eqs. (8.86), (8.87) we get

$$(8.89)\qquad \sum(\delta_j-\bar{\delta})^2=\sum\delta_j^2-1/4(\sum\delta_j^2)^2>1/2\sum\delta_j^2.$$

Since $\lambda_i^{1/2}-\lambda_j^{1/2}=\delta_i-\delta_j$, the lemma now follows from Eqs. (8.88), (8.89) and lemma 8.12. Q. E. D.

Lemma 8.14. Given $M=\mathrm{R}(M_\nu, \Omega_\nu)$, $\mathrm{r}_\infty(M)=S_1$. Then $\mathrm{d}_1(M, \Omega)<\infty$.

Proof. Assume $\mathrm{d}_1(M, \Omega)=\infty$. Then we have

$$\sum_\nu\sum_{j=1}^{n_\nu}(\lambda_{\nu j}^{1/2}-n_\nu^{-1/2})^2=\infty$$

where $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu 1}, \cdots\lambda_{\nu n_\nu}\}$. It follows from lemma 8.13 that there exist disjoint subsets $K_\nu^1, K_\nu^2\subset\mathrm{Sp}(\Omega_\nu/M_\nu)$ and a bijection ϕ_ν from K_ν^1 to K_ν^2 such that

$$(8.90)\qquad \sum_\nu\sum_{\lambda\in K_\nu^1}[\lambda^{1/2}-(\phi_\nu\lambda)^{1/2}]^2=\infty.$$

It now follows from lemma 8.6 that there exists some $x\in\mathrm{r}_\infty(M)$, $x\neq 1$ which is a contradiction. Q. E. D.

Finally, we consider the case $x=0$. We first prove

Lemma 8.15. Given $M=\mathrm{R}(M_\nu, \Omega_\nu)$, $\mathrm{Sp}(\Omega_\nu/M_\nu)=\{\lambda_{\nu 1}, \cdots\lambda_{\nu n_\nu}\}$. The conditions

$$(8.91)\qquad \sum_\nu|1-\lambda_{\nu 1}|=\infty$$

$$(8.92)\qquad \mathrm{d}_0(M, \Omega)=\infty$$

are equivalent, where $\|\Omega_\nu\|=1$.

Proof. We have

$$(8.93)\qquad \begin{aligned}\mathrm{d}_0(M, \Omega)&=\sum[(1-\lambda_{\nu 1}^{1/2})^2+\sum_{j=2}^{n_\nu}\lambda_{\nu j}]\\&=\sum[1+\lambda_{\nu 1}-2\lambda_{\nu 1}^{1/2}+(1-\lambda_{\nu 1})]\\&=2\sum[1-\lambda_{\nu 1}^{1/2}].\end{aligned}$$

Using $(1-\lambda^{1/2})^2\geq 0$ we obtain the inequality

$$(8.94)\qquad |1-\lambda^{1/2}|\leq|1-\lambda|\leq 2|1-\lambda^{1/2}|,\ 0\leq\lambda\leq 1.$$

It follows that

(8.95) $$d_0(M, \Omega) \leq 2\sum(1-\lambda_{\nu 1}) \leq 2d_0(M, \Omega)$$

Q. E. D.

Lemma 8.16. Given $M = R(M_\nu, \Omega_\nu)$, $r_\infty(M) = S_0$. Then $d_0(M, \Omega) < \infty$.

Proof. By lemma 3.8 $r_\infty(M) = S_0$ if and only if $\sum|1-\lambda_{\nu 1}| < \infty$. By lemma 8.15 $\sum|1-\lambda_{\nu 1}| < \infty$ if and only if $d_0(M, \Omega) < \infty$. Q. E. D.

All the discussions in this section are valid even if we allow $\|\Omega_\nu\|^2 = 1+\delta_\nu \neq 1$. This is due to the following situation. Let $(1+\delta_\nu) = \theta_\nu$, $\lambda'_{\nu j} = \theta^{-1}_\nu \lambda_{\nu j}$, $\Omega'_\nu = \Omega_\nu / \|\Omega_\nu\|$. Then $\mathrm{Sp}(\Omega'_\nu / M_\nu) = \{\lambda'_{\nu j}\}$ if $\mathrm{Sp}(\Omega_\nu / M_\nu) = \{\lambda_{\nu j}\}$. By Lemma 3.14, $r_\infty(M, \Omega') = r_\infty(M, \Omega)$. On the other hand, $0 < \Pi \|\Omega_\nu\| < \infty$ implies $\sum|\delta_\nu| < \infty$. If $\sum_j \alpha_{\nu j} = 1$, then

$$\sum_j \left| [\lambda'^{1/2}_{\nu j} - \alpha^{1/2}_{\nu j}]^2 - [\lambda^{1/2}_{\nu j} - \alpha^{1/2}_{\nu j}]^2 \right|$$

$$\leq \sum_j |\lambda'^{1/2}_{\nu j} - \lambda^{1/2}_{\nu j}| \cdot \{\lambda'^{1/2}_{\nu j} + \lambda^{1/2}_{\nu j} + 2\alpha^{1/2}_{\nu j}\}$$

$$\leq \{1 + \theta^{1/2}_\nu + 2\} \cdot |1 - \theta^{1/2}_\nu|.$$

Thus if we adopt the Definition 8.1 and 8.2 for $\|\Omega_\nu\| \neq 1$, then $d_x(M, \Omega) = \infty$ and $d_x(M, \Omega') = \infty$ are equivalent.

9. Classification of ITPFI Factors—The Classes S_x

In this section we apply the results of the preceding section. We prove that $r_\infty(M) = S_x$ if and only if $M \sim R_x$, $0 \leq x \leq 1$. We obtain some useful criteria for calculating $r_\infty(M)$.

Theorem 9.1. Given $M = R(M_\nu, \Omega_\nu)$, $r_\infty(M) = S_x$, $0 \leq x \leq 1$. Then $M \sim R_x$.

Proof. By lemmas 8.11, 8.14, 8.16 $r_\infty(M) = S_x$ implies that $d_x(M, \Omega) < \infty$. By lemma 8.3, $r_\infty(M) = S_x$ and $d_x(M, \Omega) < \infty$ implies that $M \sim R_x$. Q. E. D.

We remark that we can prove Theorem 9.1 for $x=1$ without making any use of the condition $d_1(M, \Omega) < \infty$. Namely, by lemma 5.2 M is infinite if and only if $0 \in r_\infty(M)$. Thus $r_\infty(M) = S_1$ implies that

M is finite. Since $1 \in r_\infty(M)$ implies that M cannot be type I, M must be type II_1. Since all hyperfinite type II_1 factors are isomorphic (Dixmier [8], Theorem III. 7. 1) we have $M \sim R_1$.

If M is an ITPFI factor, $r_\infty(M) \neq S_{01}$, then by Theorems 7. 6 and 9. 1 M must be isomorphic to one of the factors R_x, $0 \leq x \leq 1$ or R_∞. However, the calculation of $r_\infty(M)$ by a direct application of definition 3. 2 may be a nontrivial problem. The following two lemmas give some useful criteria for calculating $r_\infty(M)$. The first lemma is a straightforward variation of lemma 3. 5.

Lemma 9.2. Given $M = \mathrm{R}(M_\nu, \Omega_\nu), M_\nu$ type I_n for all ν, and $\mathrm{Sp}(\Omega_\nu/M_\nu) = \{\lambda_1, \cdots \lambda_n\}$ independent of ν. Let $\widehat{S}(\lambda_1, \cdots \lambda_n)$ be the intersection of all sets S such that

(i) S is one of the sets S_x, $0 \leq x \leq 1$, S_{01} and S_∞

(ii) $\lambda_j \neq 0$ implies that $\lambda_i/\lambda_j \in S$, $i, j = 1, \cdots n$.

Then $r_\infty(M) = \widehat{S}(\lambda_1, \cdots \lambda_n)$.

Proof. Since $\widehat{S}(\lambda_1, \cdots \lambda_n)$ is nonempty, it must be one of the sets S_x, S_{01}, S_∞. Thus $\widehat{S}(\lambda_1, \cdots \lambda_n)$ is the smallest asymptotic ratio set containing all $\lambda_i/\lambda_j, \lambda_j \neq 0$. It now follows from lemma 3. 5 that $\widehat{S}(\lambda_1, \cdots \lambda_n) \subset r_\infty(M)$. It follows from definition 3. 2 that $r_\infty(M) \subset \widehat{S}(\lambda_1, \cdots \lambda_n)$.

Q. E. D.

Lemma 9.3. Given $M = \mathrm{R}(M_\nu, \Omega_\nu)$, M_ν type I_n for all ν, $\mathrm{Sp}(\Omega_\nu/M_\nu) = \{\lambda_{\nu 1}, \cdots \lambda_{\nu n}\}$ and

$$\lim_{\nu \to \infty} \lambda_{\nu j} = \lambda_j, \ j = 1, \cdots n. \tag{9.1}$$

If

$$\Delta = \sum_\nu \sum_{j=1}^n (\lambda_{\nu j}^{1/2} - \lambda_j^{1/2})^2 < \infty \tag{9.2}$$

then $r_\infty(M) = \widehat{S}(\lambda_1, \cdots \lambda_n)$ (see lemma 9. 2 for the definition of $\widehat{S}(\lambda_1, \cdots \lambda_n)$). If $\widehat{S}(\lambda_1, \cdots \lambda_n) = S_{01}$ in addition then $M \sim R_0 \otimes R_1$. If $\Delta = \infty$ and $\lambda_j \neq 0$ for all $j = 1, \cdots n$ then $r_\infty(M) = S_\infty$.

Proof. By Eq. (9. 1) and $\|\Omega_\nu\| \to 1$, we have $\sum \lambda_j = 1$. By lemma

3.14 we can assume $\sum_{j=1}^{n} \lambda_{\nu j}=1$ (note that this does not affect whether or not Δ is finite). Let $M_1 = \mathrm{R}(M_\nu, \Omega'_\nu)$ where $\mathrm{Sp}(\Omega'_\nu/M_\nu) = \{\lambda_1, \cdots \lambda_n\}$.

Case (i), $\Delta < \infty$. By lemma 2.13 $M \sim M_1$. By lemma 9.2, $\mathrm{r}_\infty(M_1) = \widehat{S}(\lambda_1, \cdots \lambda_n)$. If $\widehat{S}(\lambda_1, \cdots \lambda_n) = S_{01}$ then $\hat{\mathrm{r}}(M_1) \subset S_{01}$ and by lemma 7.5 we have $R_0 \otimes R_1 \sim R_0 \otimes R_1 \otimes M_1$ since $\mathrm{r}_\infty(R_0 \otimes R_1) = S_{01}$. Since $0, 1 \in \mathrm{r}_\infty(M_1)$ we have $M_1 \sim M_1 \otimes R_0$, $M_1 \sim M_1 \otimes R_1$. Thus $M_1 \sim M_1 \otimes R_0 \otimes R_1 \sim R_0 \otimes R_1$.

Case (ii), $\Delta = \infty$. Consider $M \otimes M_1 = \mathrm{R}(M_\nu \otimes M_\nu,\ \Omega_\nu \otimes \Omega'_\nu)$. By lemma 3.5 $\hat{\mathrm{r}}(M_\nu, \Omega'_\nu) \subset \mathrm{r}_\infty(M)$ and thus $M \sim M \otimes M_1$ by lemma 7.5. We have $\mathrm{Sp}(\Omega_\nu \otimes \Omega'_\nu / M_\nu \otimes M_\nu) = \{\lambda_{\nu i}\lambda_j : i, j = 1, \cdots n\}$. Define

(9.3) $$K_\nu^1 = \{\lambda_{\nu i}\lambda_j : i < j\}$$

(9.4) $$K_\nu^2 = \{\lambda_{\nu i}\lambda_j : i > j\}$$

and a bijection ϕ_ν from K_ν^1 to K_ν^2 by

(9.5) $$\phi_\nu \lambda_{\nu i}\lambda_j = \lambda_{\nu j}\lambda_i, \ i < j.$$

Define $\eta_{\nu j}$ by

(9.6) $$\lambda_{\nu j} = e^{\eta_{\nu j}}\lambda_j.$$

It follows from Eq. (9.1) that

(9.7) $$\lim_{\nu \to \infty} \eta_{\nu j} = 0, j = 1, \cdots n.$$

We have

$$\infty = \Delta = \sum_\nu \sum_j \lambda_j (e^{\eta_{\nu j}/2} - 1)^2.$$

By lemma 8.7 this is equivalent to

(9.8) $$\sum_\nu \sum_j \lambda_j (e^{\eta_{\nu j}} - 1)^2 = \infty.$$

Since $\sum_j \lambda_{\nu j} = \sum_j \lambda_j = 1$ we have

(9.9) $$\sum_j \lambda_j (e^{\eta_{\nu j}} - 1) = 0.$$

It follows from Eqs. (9.8), (9.9) and lemma 8.8 that

$$\sum_\nu \sum_{i<j} \lambda_i \lambda_j (e^{\eta_{\nu i}} - e^{\eta_{\nu j}})^2 = \infty.$$

Using lemma 8.7 again this is equivalent to

$$\sum_\nu \sum_{i<j} \lambda_i \lambda_j (\eta_{\nu i} - \eta_{\nu j})^2 = \infty.$$

Since $\lambda_j \neq 0$ for all j, it follows from Eq. (9.1) that this is equivalent to

$$\sum_{\nu} \sum_{i<j} \lambda_{\nu i} \lambda_j (\eta_{\nu i} - \eta_{\nu j})^2 = \infty. \tag{9.10}$$

For $\lambda = \lambda_{\nu i} \lambda_j \in K_\nu^1$ let

$$\eta(\lambda) = \log(\phi_\nu \lambda / \lambda) = \eta_{\nu j} - \eta_{\nu i}. \tag{9.11}$$

It follows from Eqs. (9.7), (9.10), (9.11) that the conditions of lemma 8.5 are satisfied and we have $\mathrm{r}_\infty(M \otimes M_1) = S_\infty$. Q. E. D.

The statement that $\Delta < \infty$, $\hat{S}(\lambda_1, \cdots \lambda_n) = S_{01}$ implies $M \sim R_0 \otimes R_1$ is nontrivial since the class S_{01} contains more than one isomorphic class (see Sec. 10). If $\lambda_j = 0$ for some j then we can have $\Delta = \infty$ but $\mathrm{r}_\infty(M) \neq S_\infty$ (see lemma 9.4).

If M is an ITPFI factor, $\mathrm{r}_\infty(M) \neq S_{01}$, then by theorems 3.9, 7.6 and 9.1 M must be isomorphic to one of the factors R_x, $0 \leq x \leq 1$ or R_∞. The factors R_x can be obtained as tensor products of type I_2 factors (see definition 3.10). If $M = \mathrm{R}(M_\nu, \Omega_\nu)$, $\mathrm{Sp}(\Omega_\nu / M_\nu) = (\lambda_\nu, 1 - \lambda_\nu)$ and $\lambda_\nu \to 1$, then $0 \in \mathrm{r}_\infty(M)$ by lemma 3.5 and thus $M \nsim R_1$. However we have

Lemma 9.4. Let M be an ITPFI factor, $\mathrm{r}_\infty(M) \neq S_1$, S_{01}. Then M can be obtained as $M = \mathrm{R}(M_\nu, \Omega_\nu)$ where M_ν is type I_2, $\mathrm{Sp}(\Omega_\nu / M_\nu) = (\lambda_\nu, 1 - \lambda_\nu)$ and $\lambda_\nu \to 1$.

Proof. By theorems 3.9, 7.6 and 9.1 M must be one of the factors R_x, $0 \leq x < 1$ or R_∞. R_0 as given in definition 3.10 is already in the desired form. By lemma 3.13 and theorem 7.6 $R_x \otimes R_y \nsim R_\infty$ if $x, y \neq 0, 1$ and $\log x / \log y$ is irrational. Thus it remains only to prove the lemma for R_x, $0 < x < 1$.

Given $0 < x < 1$, choose integers N_j for each $j \in I_\infty$ such that $N_j x^{2j} > 1$. For each $\nu \in I_\infty$ satisfying

$$2 \sum_{j=1}^{n-1} N_j < \nu \leq 2 \sum_{j=1}^{n} N_j \tag{9.12}$$

let

$$\lambda_\nu = \begin{cases} (1 + x^{2n})^{-1} & \text{if } \nu \text{ is odd} \\ (1 + x^{2n+1})^{-1} & \text{if } \nu \text{ is even.} \end{cases} \tag{9.13}$$

Consider $M=\mathrm{R}(M_\nu, \Omega_\nu)$ where $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_\nu, 1-\lambda_\nu)$. Then $\hat{\mathrm{r}}(M, \Omega) \subset S_x$ and $R_x \sim M\otimes R_x$ by lemma 5.4. We construct an x-sequence for $\mathrm{R}(M_\nu, \Omega_\nu)$ as follows. Let $I_m=\{2m-1, 2m\}$, $m\in I_\infty$, $K_m^1=\{(1-\lambda_{2m-1})\lambda_{2m}\}$, $K_m^2=\{\lambda_{2m-1}(1-\lambda_{2m})\}$ and let ϕ_m be the unique bijection from K_m^1 to K_m^2. Then

$$\phi_m\lambda/\lambda=x \tag{9.14}$$

for $\lambda\in K_m^1$ and all m. If $\lambda\in K_m^1$ where $\nu=2m$ satisfies Eq. (9.12) then we have

$$\lambda=(1+x^{-2n})^{-1}(1+x^{2n+1})^{-1}>x^{2n}/4. \tag{9.15}$$

It follows from Eqs. (9.12), (9.13), (9.15) that

$$\sum_m \lambda(K_m^1)\geq \sum_j N_j x^{2j}/4=\infty. \tag{9.16}$$

It follows from Eqs. (9.14), (9.16) that (I_m, K_m^i, ϕ_m) is an x-sequence. Thus $x\in \mathrm{r}_\infty(M)$ and $M\sim M\otimes R_x \sim R_x$. Q. E. D.

In Sec. 10 we consider ITPFI factors $M=\mathrm{R}(M_\nu, \Omega_\nu)$ where M_ν is type I_2 and $\mathrm{r}_\infty(M)=S_{01}$. Lemma 10.1 is the analog of lemma 9.4 for these factors. However it is not known whether or not all ITPFI factors in the S_{01} class can be obtained as tensor products of type I_2 factors.

10. The Class S_{01}

In this section we give some elementary properties of tensor products $M=\otimes M_\nu$ of type I_2 factors M_ν where $\mathrm{r}_\infty(M)=S_{01}$. We prove (lemma 10.1) that M is either hyperfinite $II_1\otimes I_\infty$, or is type III with $\lambda_\nu\to 1$ where $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_\nu, 1-\lambda_\nu)$. We give some conditions that such factors are nonisomorphic. Theorem 10.10 gives explicitly a nondenumerable family of mutually nonisomorphic type III factors with $\mathrm{r}_\infty(M)=S_{01}$.

Lemma 10.1. Given $M=\mathrm{R}(M_\nu, \Omega_\nu)$, M_ν type I_2, $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_\nu, 1-\lambda_\nu)$, $\mathrm{r}_\infty(M)=S_{01}$. Then either $M\sim R_0\otimes R_1$ or $M\sim \mathrm{R}(M_\nu, \Omega'_\nu)$, $\mathrm{Sp}(\Omega'_\nu/M_\nu)=(\lambda'_\nu, 1-\lambda'_\nu)$ and $\lambda'_\nu\to 1$. In the latter case M is type III.

Proof. By lemma 3.5, $r_\infty(M)=S_{01}$ implies that the only allowed accumulation points of λ_ν are 1/2 and 1. If 1 is the only accumulation point the first part of the lemma is trivially satisfied. If 1/2 is the only accumulation point, then by lemma 9.3 $r_\infty(M)$ is either S_1 or S_∞ which is a contradiction. If 1/2 and 1 are accumulation points then we can write $M=M_0\otimes M_1$ where M_0, M_1 are tensor products of type I_2 factors such that $\lambda_\nu\to 1$, 1/2 respectively. By lemma 9.3, $r_\infty(M_1)$ is either S_1 or S_∞, hence we must have $r_\infty(M_1)=S_1$ and $M_1\sim R_1$ by theorem 9.1. $r_\infty(M_0)$ must be either S_0 or S_{01}. If $r_\infty(M_0)=S_0$ then $M_0 \sim R_0$ by theorem 9.1 and $M\sim R_0\otimes R_1$. If $r_\infty(M_0)=S_{01}$ then $M_0\sim M_0\otimes R_1$, hence $M\sim M_0$.

If $M\sim \mathrm{R}(M_\nu, \Omega'_\nu)$, $\mathrm{Sp}(\Omega'_\nu/M_\nu)=(\lambda'_\nu, 1-\lambda'_\nu)$, where $\lambda'_\nu\to 1$ then lemma 3.8 implies that $\sum(1-\lambda'_\nu)=\infty$. It then follows from lemma 2.14 that M is type III. Q. E. D.

Definition 10.2. Given $0\leq l_1<l_2<\cdots$, $l_j\to\infty$ and positive integers $N_1, N_2, \cdots$. Let

$$\lambda_\nu=(1+e^{-l_j})^{-1},\ N_1+\cdots+N_{j-1}<\nu\leq N_1+\cdots+N_j. \tag{10.1}$$

We denote the factor $M=\mathrm{R}(M_\nu, \Omega_\nu)$ where $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_\nu, 1-\lambda_\nu)$ by $M[l_1, N_1; l_2, N_2; \cdots]$ or $M[l_j, N_j]$.

Lemma 10.3. Given $M=\mathrm{M}[l_j, N_j]$. Then $r_\infty(M)=S_0$ if and only if $\sum N_j e^{-l_j}<\infty$.

Proof. We have $M=\mathrm{R}(M_\nu, \Omega_\nu)$ where $\mathrm{Sp}(\Omega_\nu/M_\nu)=(\lambda_\nu, 1-\lambda_\nu)$. By lemma 3.8 $r_\infty(M)=S_0$ if and only if $\sum(1-\lambda_\nu)<\infty$. Since

$$1-\lambda_\nu=(1+e^{l_j})^{-1},\ N_1+\cdots+N_{j-1}<\nu\leq N_1+\cdots+N_j \tag{10.2}$$

we have

$$\sum_j N_j e^{-l_j}>\sum_\nu(1-\lambda_\nu)>1/2\Sigma N_j e^{-l_j}. \tag{10.3}$$

Q. E. D.

Lemma 10.4. Given $M=\mathrm{M}[l_j, N_j]$ where $l_{j+1}>(N_j+1)l_j$. If $\sum N_j e^{-l_j}=\infty$ then $r_\infty(M)=S_{01}$.

Proof. Consider any finite set I containing only $\nu>N_1+\cdots+N_j$.

Then the interval between different points $\log \mu$, $\mu \in \mathrm{Sp}(\Omega(I)/M(I))$ is at least l_{j+1}. Since $\lim l_j = \infty$ it follows that $\mathrm{r}_\infty(M) \subset S_{01}$. Since S_0 is excluded by lemma 10.3 we have $\mathrm{r}_\infty(M) = S_{01}$. Q. E. D.

It seems likely that lemma 10.4 should still hold if the condition $l_{j+1} > (N_j+1)l_j$ is replaced by

$$\lim_{j\to\infty} l_j N_j^{1/2}/l_{j+1} = 0. \tag{10.4}$$

We remark that one can obtain sequences (l_j, N_j) satisfying the conditions of lemma 10.4 by choosing N_j larger than e^{l_j} and choosing l_{j+1} larger than $l_j(N_j+1)$ for each j.

Lemma 10.5. Given $l>0$, $M = \mathrm{M}[n_j l, N_j]$ where the n_j are integers. Then $M \otimes R_x \sim R_x$ where $x = e^{-l}$.

Proof. Clearly $\hat{\mathrm{r}}(M) \subset S_x$ and the result follows from lemma 5.4.

Q. E. D.

Lemma 10.6. Given $0 < l, l' < \infty$, $M = \mathrm{R}(M_\nu, \Omega_\nu) = \mathrm{M}[n_j l, N_j]$ where the n_j are integers. Let $x = e^{-l}$, $x_k = e^{-l'/k}$, $k \in I_\infty$. If $\mathrm{d}_{x_k}(M) = \infty$ (see definition 8.2) then $M \otimes R_{x_1} \nsim R_{x_k}$. If this holds for all $k \in I_\infty$, then $M \otimes R_{x_1} \sim R_\infty$. Otherwise let K be the minimum k such that $\mathrm{d}_{x_k}(M) < \infty$, then $M \otimes R_{x_1} \sim R_{x_K}$.

Proof. Since $x_1 \in \mathrm{r}_\infty(M \otimes R_{x_1})$ it follows that $\mathrm{r}_\infty(M \otimes R_{x_1})$ is either S_∞ or S_{x_k} for some $k \in I_\infty$. By lemma 8.11 $\mathrm{r}_\infty(M \otimes R_{x_1}) = S_{x_k}$ implies that $d_{x_k}(M \otimes R_{x_1}) = \mathrm{d}_{x_k}(M) < \infty$. First two conclusions then follows. Conversely, if $\mathrm{d}_{x_k}(M) < \infty$ then $M \otimes R_{x_1} \otimes R_{x_k} \sim R_{x_k}$ by lemma 8.3. Hence $\mathrm{r}_\infty(M) \subset \mathrm{r}_\infty(R_{x_k}) = S_{x_k}$. Since $S_{x_k} \subset S_{x_{k'}}$ implies $k \le k'$, $\mathrm{r}_\infty(M \otimes R_{x_1}) = S_{x_k}$ for the minimum k with finite $\mathrm{d}_{x_k}(M)$ and hence $M \otimes R_{x_1} \sim R_{x_k}$.

Corollary 10.7. Given $0 < l$, $l' < \infty$, $M_1 = \mathrm{M}[n_j l, N_j]$, $M_2 = \mathrm{M}[n'_j l', N'_j]$ where the n_j, n'_j are integers. If $\mathrm{d}_x(M_2) = \infty$, $x = e^{-l}$ then $M_1 \nsim M_2$.

Proof. We have $\mathrm{r}_\infty(M_1 \otimes R_x) = S_x$. By lemma 10.6, $\mathrm{d}_x(M_2) = \infty$ implies that $\mathrm{r}_\infty(M_2 \otimes R_x) \neq S_x$. Q. E. D.

We remark that given M_1, M_2 as in corollary 10.7 where $l \neq l'$, we can obtain $\mathrm{r}_\infty(M_1) = \mathrm{r}_\infty(M_2) = S_{01}$ and also $\mathrm{d}_x(M_2) = \infty$ by taking N'_j

sufficiently large for each j.

Lemma 10.8. Given $M_1=\mathrm{M}[l_j, N_j]$, $M_2=\mathrm{M}[l'_j, N_j]$ where $\sum N_j e^{-l'_j}=\infty$. If $l'_j-l_j\to k$ as $j\to\infty$ then $e^k\in\mathrm{r}_\infty(M_1\otimes M_2)$.

Proof. We have $M_i=\mathrm{R}(M_{i\nu}, \Omega_{i\nu})$, i=1, 2. Let $M_\nu=M_{1\nu}\otimes M_{2\nu}$, $\Omega_\nu=\Omega_{1\nu}\otimes\Omega_{2\nu}$. Then $M_1\otimes M_2=\mathrm{R}(M_\nu, \Omega_\nu)$. Let

$$\mu_{\nu 1}=(1+e^{-l_j})^{-1}(1+e^{l'_j})^{-1} \tag{10.5}$$

$$\mu_{\nu 2}=(1+e^{l_j})^{-1}(1+e^{-l'_j})^{-1}=e^{l'_j-l_j}\mu_{\nu 1} \tag{10.6}$$

where $N_1+\cdots+N_{j-1}<\nu\leq N_1+\cdots N_j$. Then $\mu_{\nu 1}, \mu_{\nu 2}\in\mathrm{Sp}(\Omega_\nu/M_\nu)$. Let $I_\nu=\{\nu\}$, $K^i_\nu=\{\mu_{\nu i}\}$, $i=1, 2$ and $\phi_\nu\mu_{\nu 1}=\mu_{\nu 2}$. We have

$$\sum_\nu\lambda(K^1_\nu)=\sum\mu_{\nu 1}>(1/4)\sum N_j e^{-l'_j}=\infty. \tag{10.7}$$

It follows from Eq. (10.6) that

$$\lim_{\nu\to\infty}|e^k-\mu_{\nu 2}/\mu_{\nu 1}|=0.$$

Thus $(I_\nu, K^i_\nu, \phi_\nu)$ is an e^k-sequence for $M_1\otimes M_2$. Q. E. D.

Corollary 10.9. Given $0<l<\infty$, $M_1=\mathrm{M}[n_j l, N_j]$, $M_2=\mathrm{M}[n'_j l, N'_j]$ where n_j, n'_j are integers and $\mathrm{r}_\infty(M_1)=\mathrm{r}_\infty(M_2)=S_{01}$. If

$$n_{j+1}>(N'_j+1)(n'_j+1) \tag{10.8}$$

$$n'_j+1>(N_j+1)n_j \tag{10.9}$$

then $M_1\not\sim M_2$.

Proof. Let $M_3=M[(n'_j+1)l, N'_j]$. By lemma 10.8, $e^l\in\mathrm{r}_\infty(M_2\otimes M_3)$. We can write $M_1\otimes M_3=\mathrm{M}[n_1 l, N_1;\ (n'_1+1)l, N'_1;\ n_2 l, N_2;\ \cdots]$. It follows from Eqs. (10.8), (10.9) and lemma 10.4 that $\mathrm{r}_\infty(M_1\otimes M_3)=S_{01}$. Q. E. D.

Theorem 10.10. There exist nondenumerably many mutually non-isomorphic factors M with $\mathrm{r}_\infty(M)=S_{01}$.

Proof. Let $M_k=M[l_j+k, N_j]$, $0\leq k\leq 1$ where $l_{j+1}>2(N_j+1)(l_j+1)$ and $\sum N_j e^{-l_j}=\infty$ (this last condition can be achieved by choosing N_j sufficiently large for each j). By lemma 10.4, $\mathrm{r}_\infty(M_k)=\mathrm{r}_\infty(M_k\otimes M_k)=S_{01}$. By lemma 10.8, $e^{k-k'}\in\mathrm{r}_\infty(M_k\otimes M_{k'})$. Thus $k\neq k'$ implies that

$M_k \nsim M_{k'}$. Q. E. D.

11. Another Algebraic Invariant for ITPFI Factors

In this section we define a second algebraic invariant $\rho(M)$ for ITPFI factors M, and use it to analyze further the S_{01} class.

Definition 11.1. Let M be an ITPFI factor. We define the algebraic invariant $\rho(M)$ as the set of all $x \in [0, 1]$ such that $R_x \sim R_x \otimes M$.

Given $M = \mathrm{R}(H_\nu, M_\nu, \Omega_\nu)$ we note that $\mathrm{d}_x(M, \Omega)$ as given in definition 8.2 does depend on the vector $\Omega = \otimes \Omega_\nu$. However if $\psi = \otimes \psi_\nu \in \otimes(H_\nu, \Omega_\nu)$ then $\mathrm{d}_x(M, \Omega) = \infty$ if and only if $\mathrm{d}_x(M, \psi) = \infty$. Thus by a slight abuse of notation we can write $\mathrm{d}_x(M) = \infty$ if $\mathrm{d}_x(M, \psi) = \infty$ for any (and thus all) $\psi = \otimes \psi_\nu \in \otimes(H_\nu, \Omega_\nu)$.

Lemma 11.2. Given $M = \mathrm{R}(M_\nu, \Omega_\nu)$. Then

$$\rho(M) = \{x \in [0, 1] : \mathrm{d}_x(M) < \infty\}.$$

Proof. Assume $\mathrm{d}_x(M) < \infty$. Since $\mathrm{d}_x(R_x) < \infty$ it follows that $\mathrm{d}_x(M \otimes R_x) < \infty$. Since $x \in \mathrm{r}_\infty(M \otimes R_x)$ we have $R_x \sim R_x \otimes M$ by lemma 8.3. Conversely, by lemmas 8.11, 8.14, 8.16 $R_x \sim R_x \otimes M$ implies that $\mathrm{d}_x(M \otimes R_x) < \infty$ and thus $\mathrm{d}_x(M) < \infty$. Q.E.D.

Lemma 11.3. Given $0 < x < 1$, $M = \mathrm{R}(M_\nu, \Omega_\nu)$, $x \in \rho(M)$. Then $x^{1/n} \in \rho(M)$, $n \in I_\infty$.

Proof. By definitions 8.1 and 8.2, $\mathrm{d}_{x^{1/n}}(M, \Omega) \leq \mathrm{d}_x(M, \Omega)$. The result now follows from lemma 11.2. Q.E.D.

Lemma 11.4. Given ITPFI factors M, N. Then

$$\rho(M \otimes N) = \rho(M) \cap \rho(N).$$

Proof. We have $\mathrm{d}_x(M \otimes N) < \infty$ if and only if $\mathrm{d}_x(M) < \infty$ and $\mathrm{d}_x(N) < \infty$. The result now follows from lemma 11.2. Q.E.D.

Lemma 11.5.
$$\rho(R_0) = [0, 1)$$
$$\rho(R_1) = (0, 1]$$
$$\rho(R_0 \otimes R_1) = (0, 1)$$

$$\rho(R_x)=\{x^{1/n};\ n\in I_\infty\} \qquad 0<x<1$$
$$\rho(R_\infty)=\phi.$$

Proof. We have $x\in\rho(R_y)$ if and only if $R_x\sim R_x\otimes R_y$ which is the case if and only if $y\in r_\infty(R_x)=S_x$. This argument gives $\rho(R_y)$, $0\leq y\leq 1$. $\rho(R_0\otimes R_1)$ now follows from lemma 11.4. Since $R_x\otimes R_\infty\sim R_\infty\nsim R_x$ for any $x\in[0,1]$ we have $\rho(R_\infty)=\phi$. Q.E.D.

Lemma 11.6. Let M be an ITPFI factor. Then $0\in\rho(M)$ if and only if $M\sim R_0$, and $1\in\rho(M)$ if and only if $M\sim R_1$.

Proof. By lemma 11.5, $0\in\rho(R_0)$. Conversely, if $0\in\rho(M)$ then $R_0\sim R_0\otimes M$ and it follows that M must be type I since R_0 is type I_∞. Since the definition of an ITPFI factor excludes finite type I, we have $M\sim R_0$.

By lemma 11.5, $1\in\rho(R_1)$. Conversely. if $1\in\rho(M)$ then $R_1\sim R_1\otimes M$ and it follows that M must be finite since R_1 is type II_1. Since M cannot be finite type I, and all hyperfinite II_1 factors are isomorphic, we have $M\sim R_1$. Q.E.D.

In the remainder of this section we consider tensor products $M=\otimes M_\nu$ of type I_2 factors M_ν.

Lemma 11.7. Given $0<l,\ k<\infty$, $M=\mathrm{R}(M_\nu,\ \Omega_\nu)=\mathrm{M}[n_jl,\ N_j]$ where the n_j are integers (see definition 10.2). For each j choose an integer p_j so that $|\delta_j|$ is a minimum where

$$\delta_j=p_jk-n_jl. \tag{11.1}$$

Let $y=e^{-k}$. Then $d_y(M)<\infty$ if and only if

$$\sum_{j=1}^{\infty}N_je^{-n_jl}\delta_j^2<\infty. \tag{11.2}$$

Proof. For each j, choose m_j so that $(0,\ m_j)$ gives the minimum for $\delta_y(M_\nu,\ \Omega_\nu)$ in Eq. (8.1) where $\sum_1^{j-1}N_i<\nu\leq\sum_1^{j}N_i$. Let

$$\delta_j'=m_jk-n_jl. \tag{11.3}$$

Since $1<(1+e^{m_jk})/(1+e^{(m_j+1)k})<e^k$ and a similar inequality holds for $-k$, $|\eta_j'|$ is bounded by k. Hence we have

$$\delta_y(M_\nu,\ \Omega_\nu)=[(1+e^{-n_jl})^{-1/2}-(1+e^{-n_jl-\delta_j'})^{-1/2}]^2 \tag{11.4}$$

$$+[(1+e^{n_jl})^{-1/2}-(1+e^{n_jl+\delta'_j})^{-1/2}]^2$$
$$=e^{-n_jl}(1-e^{-\delta'_j/2})^2(1+O[e^{-n_jl}])$$

where the second term is $(1+e^{n_jl})^{-1}(1+e^{n_jl}e^{\delta'_j})^{-1}e^{2n_jl}(e^{\delta'_j}-1)^2[(1+e^{n_jl}e^{\delta'_j})^{1/2}+(1+e^{n_jl})^{1/2}]^{-2}$ and yields the main contribution. Thus

$$\text{(11.5)}\quad \mathrm{d}_y(M_\nu,\Omega_\nu)=\sum_\nu \delta_y(M_\nu,\Omega_\nu)=\sum_j N_je^{-n_jl}(1-e^{\delta'_j})^2[1+0(e^{-n_jl})].$$

Since $n_j\to\infty$ (see definition 10.2) it follows that $\mathrm{d}_y(M)=\infty$ if and only if

$$\text{(11.6)}\qquad \sum_j N_je^{-n_jl}(1-e^{-\delta'_j/2})^2=\infty.$$

Since $|\delta'_j|\leq k$ it follows from the same argument used to prove lemma 8.7 that Eq. (11.6) is equivalent to

$$\text{(11.7)}\qquad \sum_j N_je^{-n_jl}(\delta'_j)^2=\infty.$$

Since $n_j\to\infty$, it follows from definition 8.1 and Eqs. (11.1), (11.3), (11.4) that there is some finite J and some fixed $\varepsilon>0$ such that for all $j>J$, if either $\delta_j<\varepsilon$ or $\delta'_j<\varepsilon$ then $m_j=p_j$ and $\delta_j=\delta'_j$. Since we also have $|\delta_j|$, $|\delta'_j|\leq\frac{1}{2}k$, it follows that there exist positive constants C_1, C_2 such that

$$\text{(11.8)}\qquad C_1|\delta_j|\geq|\delta'_j|\geq C_2|\delta_j|,\quad j>J.$$

It follows from Eq. (11.8) that Eq. (11.7) is equivalent to Eq. (11.2). Q.E.D.

Lemma 11.8. Given l, $k_1,\cdots k_n\in(0,\infty)$ such that k_i/l is irrational, $i=1,\cdots n$. Then there exists an ITPFI factor M such that $e^{-jl}\in\rho(M)$, $j\in\mathrm{I}_\infty$ and $e^{-k_i}\notin\rho(M)$, $i=1,\cdots n$.

Proof. Consider $M=\mathrm{R}(M_\nu,\Omega_\nu)=\mathrm{M}[(j!)l, N_j]$ where we choose N_j as follows. Define

$$\text{(11.9)}\qquad \varepsilon_{ji}=\min_{m_i}|m_ik_i-(j!)l|,\quad i=1,\cdots n$$

where the minimum is taken over integers m_i. Since k_i/l is irrational we have $\varepsilon_{ji}>0$ and it follows that we can choose N_j sufficiently large for each j such that

(11.10) $$\sum N_j e^{-(j!)\varepsilon_{ji}^2} = \infty, \ i=1, \cdots n.$$

It follows from lemma 11.7 that $e^{-k_i} \notin \rho(M)$, $i=1, \cdots n$. By construction we have

(11.11) $$\delta_{e^{-jl}}(M_\nu, \Omega_\nu) = 0 \quad \text{if} \quad \nu > \sum_{i=1}^{j-1} N_i$$

and thus

$$d_{e^{-jl}}(M, \Omega) = \sum_\nu \delta_{e^{-jl}}(M_\nu, \Omega_\nu) < \infty$$

for all $j \in I_\infty$. It follows from lemma 11.2 that $e^{-jl} \in \rho(M)$, $j \in I_\infty$.
Q.E.D.

Corollary 11.9. The ITPFI factors constructed in lemma 11.8 belong to the class S_{01}.

Proof. The algebraic invariant $\rho(M)$ is not one of the sets given in lemma 11.5. Q.E.D.

We note that since the ε_{ji} defined by Eq. (11.9) are bounded, it follows from Eq. (11.10) that

(11.12) $$\sum N_j e^{-(j!)l} = \infty.$$

If the condition given in lemma 10.4 were satisfied we would have $N_j < j$, which contradicts Eq. (11.12). Furthermore, since Eq. (11.10) is the only condition the N_j must satisfy, they can be made arbitrarily large. Thus Eq. (10.4) is not a necessary condition that $r_\infty(M) = S_{01}$.

We now use lemma 11.7 and some results from number theory concerning the approximation of irrationals by rationals to construct more examples of ITPFI factors M in the class S_{01}. Given $0 < k$, $l < \infty$ and an integer n. Choose an integer m such that $\delta = |mk - nl|$ is a minimum. We have

$$\delta = nk|(l/k) - (m/n)|.$$

We recall that a real number ξ is said to be approximable by rationals to order p if there exists a positive constant c depending only on ξ such that the inequality

(11.13) $$|\xi - m/n| < c/n^p$$

has infinitely many rational solutions m/n with $n>0$. It is known that all irrational numbers are approximable to order 2, and that irrational number ξ whose continued fraction has bounded partial quotients cannot be approximated to any order higher than 2. The set of all irrationals with bounded partial quotients has measure zero, but it has the cardinal number of the continuum. It is an easy matter to construct irrational numbers which can be approximated to any degree $p \geq 2$.

Lemma 11.10. Let ξ^{-1} be a positive irrational number which is approximable by rationals to order $p=2+\varepsilon$, $\varepsilon>0$. Given $0<l<\infty$ there exists an ITPFI factor M such that e^{-l}, $e^{-\xi l} \in \rho(M)$ and $e^{-\theta l} \notin \rho(M)$ where θ^{-1} is any irrational with bounded partial quotients.

Proof. There is a positive constant c and an infinite sequence of integers m_j, $n_j>0$, $j \in \mathrm{I}_\infty$ such that

(11.14) $$|\xi^{-1} - m_j/n_j| < c/n_j^p.$$

Since $n_j>0$ we can order the n_j so that they are increasing. Consider $M = \mathrm{M}[n_j l, N_j]$ where the N_j will be chosen later. By construction $\mathrm{d}_{e^{-l}}(M)<\infty$ and thus $e^{-l} \in \rho(M)$. By lemma 11.7 $e^{-\xi l} \in \rho(M)$ if and only if $\sum N_j e^{-n_j l} \delta_j^2 < \infty$ where

(14.15) $$\delta_j < n_j \xi l (c/n_j^p) = c\xi l n_j^{-1-\varepsilon}.$$

Thus we have $e^{-\xi l} \in \rho(M)$ if

(11.16) $$\sum N_j e^{-n_j l} n_j^{-2-2\varepsilon} < \infty.$$

Now let θ be any positive irrational number with bounded partial quotients. Then there exists a positive constant γ such that

(11.17) $$\min_{m'_j} |\theta^{-1} - m'_j/n_j| > \gamma / n_j^{2+\frac{1}{2}\varepsilon}$$

where the minimum is taken over all integers m'_j. By lemma 11.7 $e^{-\theta l} \notin \rho(M)$ if and only if $\sum N_j e^{-n_j l} \delta_j(\theta)^2 = \infty$ where

(11.18) $$\delta_j(\theta) > \gamma\theta l n_j^{-1-\frac{1}{2}\varepsilon}.$$

Thus $e^{-\theta l} \notin \rho(M)$ if

(11.19) $$\sum N_j e^{-n_j l} n_j^{-2-\varepsilon} = \infty.$$

Choose N_j so that

(11.20) $$2j^{-1} > N_j e^{-n_j l} n_j^{-2-\varepsilon} > j^{-1}$$

then Eq. (11.19) is satisfied. Since the n_j are strictly increasing we have $n_j \geq j$. Thus

$$N_j e^{-n_j l} n_j^{-2-2\varepsilon} < 2j^{-1-\varepsilon}$$

and Eq. (11.16) is satisfied. Q.E.D.

It is not clear whether or not the algebraic invariant $\rho(M)$ will prove to be a useful tool for the program of classifying all ITPFI factors. Thus it is not known whether or not $\rho(M_1) = \rho(M_2)$ implies $M_1 \sim M_2$, or even whether $\rho(M) = \rho(R_x)$ implies $M \sim R_x$, $0 < x < 1$ (if $x = 0, 1$ see lemma 11.6). Furthermore it is not clear whether or not all sets $\rho(M)$ allowed by lemma 11.3 actually occur for some M, although lemmas 11.8 and 11.10 suggest that lemma 11.3 may be the only simple general property of $\rho(M)$.

For further classification of an ITPFI M, we may use $\mathrm{r}_\infty(M \otimes N)$, where N runs over all ITPFI. Again we do not know whether $\mathrm{r}_\infty(M_1 \otimes N) = \mathrm{r}_\infty(M_2 \otimes N)$ implies $M_1 = M_2$.

12. Some Applications

In this section we determine the isomorphic class of some factors which have been studied previously in the literature [1], [3], [4], [5], [7], [15], [18]. In particular we show that certain factors occurring in the quantum theories of infinite free Bose and Fermi systems at a finite temperature are isomorphic to the factor R_∞.

We consider first some factors associated with the Fock representation of the canonical commutation relations (CCR's). Let K be a real Hilbert space and let $\mathrm{H}_\mathrm{F}(K)$ be the complex Hilbert space on which the Fock representation $U_F(f)$, $V_F(g)$, f, $g \in K$ of the CCR's over K is defined. Let K_1, K_2 be subspaces (closed linear subsets) of K. The von Neumann algebra

$$(12.1) \qquad R(K_1, K_2/K) = \{U_F(f), V_F(g); f \in K_1, g \in K_2\}''$$

was introduced by Araki [1]. In the following we assume the reader is familiar with the results and notation of [1]. Given K_1, K_2 we define

$$K_4 = K_1 \cap K_2^\perp$$
$$K_5 = K_2 \cap K_1^\perp$$
$$K_6 = K_1 \cap K_2$$
$$K_7 = K_4^\perp \cap K_1 \cap K_6^\perp$$
$$K_8 = K_5^\perp \cap K_2 \cap K_6^\perp$$
$$K_9 = K_4^\perp \cap K_5^\perp \cap K_6^\perp.$$

The commutant of $R(K_1, K_2/K)$ is $R(K_2^\perp, K_1^\perp/K)$ and its center is $R(K_4, K_5/K)$. Furthermore it is unitarily equivalent to the tensor product of a maximal abelian algebra $R(K_4, K_5/K_4 \oplus K_5)$, a type I factor $R(K_6, K_6/K_6)$, and a factor $R(K_7, K_8/K_9)$. Therefore we are interested in the factor $R(K_1, K_2/K)$ when any two of K_1, K_2, $K_1^\perp$, $K_2^\perp$ have zero intersection. In this case there exists a unique closed linear operator ϕ from a dense set in K_1 into $K_1^\perp$ which is defined by the requirement that the graph of ϕ is K_2 in $K = K_1 \oplus K_1^\perp$. It follows from Theorem 2′ of [1] that $R(K_1, K_2/K)$ is then determined up to unitary equivalence by the spectral measure and multiplicity function of the nonnegative selfadjoint operator $\alpha = \phi^* \phi$ on K_1. If the operator α has only a discrete spectrum, then $R(K_1, K_2/K)$ can easily be constructed as an ITPFI factor. It is known that $R(K_1, K_2/K)$ is type I if and only if α is a trace class operator, and that otherwise it is type III [3]. If the spectrum of α is continuous then $R(K_1, K_2/K)$ can be considered as the analog of an ITPFI factor for the continuous tensor product introduced in [2]. In the following we show how the factor $R(K_1, K_2/K)$ can be obtained as the factor generated by a certain reducible representation of the CCR's.

Let W be a real Hilbert space and let $K=W\oplus W$. The Fock representation of the CCR's over K is given by the equations

(12. 2) $$\mathrm{H_F}(K)=\mathrm{H_F}(W)\otimes\mathrm{H_F}(W)$$

(12. 3) $$\mathrm{U_F}(f_1\oplus f_2)=\mathrm{U_F}(f_1)\otimes\mathrm{U_F}(f_2)$$

(12. 4) $$\mathrm{V_F}(g_1\oplus g_2)=\mathrm{V_F}(g_1)\otimes\mathrm{V_F}(g_2).$$

Let ρ be a (possibly unbounded) selfadjoint non-negative operator on W. Then the equations

(12. 5) $$\mathrm{U}_\rho(f)=\mathrm{U_F}([1+\rho]^{1/2}f)\otimes\mathrm{U_F}(\rho^{1/2}f)$$

(12. 6) $$\mathrm{V}_\rho(g)=\mathrm{V_F}([1+\rho]^{1/2}f)\otimes\mathrm{V_F}(-\rho^{1/2}g)$$

define a reducible representation of the CCR's over the domain D of the operator $\rho^{1/2}$ on W. The operator algebra

(12. 7) $$\mathrm{R}(\rho)=\{U_\rho(f),\ V_\rho(g): f,\ g\in D\}''$$

is a factor (see Sec. 4 of [4]). If we define subspaces of K by

(12. 8) $$K_1=\{f\oplus\rho^{1/2}(1+\rho)^{-1/2}f:\ f\in W\}$$

(12. 9) $$K_2=\{f\oplus-\rho^{1/2}(1+\rho)^{-1/2}f:\ f\in W\}$$

then

(12. 10) $$R(\rho)=R(K_1,\ K_2/K).$$

It follows from a straightforward calculation that the operator ϕ from K_1 to $K_1^\perp$ discussed above is given by

(12. 11) $$\phi(f\oplus\rho^{1/2}(1+\rho)^{-1/2}f)=2\rho f\oplus-2\rho^{1/2}(1+\rho)^{1/2}f$$

where f is in the domain of ρ, and that

(12. 12) $$\alpha=\phi^*\phi=4\rho(1+\rho).$$

It now follows from the above discussion that any factor $R(K_1,\ K_2/K)$ can be obtained as $\mathrm{R}(\rho)$ for some ρ.

If the spectrum of ρ is discrete and is given by $\{\lambda_n;\ n\in\mathrm{I}_\infty\}$ then $\mathrm{R}(\rho)$ can be constructed as an ITPFI factor $\mathrm{R}(M_n,\ \Omega_n)$ where M_n is type I_∞ and $\mathrm{Sp}(\Omega_n/M_n)=\{x_n^k(1-x_n)^{-1}:\ k=0, 1, 2, \cdots\}$ where $x_n=\lambda_n(1+\lambda_n)^{-1}$ (this follows either from Eq. (10. 52) of [1] or Eq. (A17) of [4]). It follows from lemma 5.10 and definition 3.2 that if λ is an accumulation point for the eigenvalues λ_n then $x \in r_\infty(R(\rho))$ where $x=\lambda(1+\lambda)^{-1}$. Dell'Antonio [7] has shown that any $R(\rho)$ is unitarily equivalent to some $R(\rho_d)$ where ρ_d has a discrete spectrum only. In the construction of [7], ρ_d satisfies the condition that the operator

$$\rho^{1/2}(1+\rho)^{-1/2}-\rho_d^{1/2}(1+\rho_d)^{-1/2}$$

is Hilbert-Schmidt. It follows that any point λ in the continuous spectrum of ρ will be an accumulation point for the eigenvalues of ρ_d. Thus ρ having a continuous spectrum is a sufficient condition that $R(\rho)\sim R_\infty$.

The representation of the CCR's describing a nonrelativistic infinite free Bose gas at a finite density and finite temperature with no macroscopic occupation of the ground state is of the form $U_\rho(f)$, $V_\rho(g)$ where the operator ρ has a continuous spectrum (see Eqs. (4.10–13), (5.2) of [4]). Thus the von Neumann algebra $\{U_\rho(f), V_\rho(g)\}''$ in this case is the factor R_∞.

Let $U(f)$, $V(g)$ be the representation of the CCR's describing a relativistic free Bose field where f, g are suitable functions defined on R^3. Let Δ be any open region in R^3. In the local observables approach to quantum field theory one is interested in the von Neumann algebras

$$R(\Delta)=\{U(f), V(g): \text{ support } f\subset\Delta\}''. \tag{12.13}$$

We now construct $R(\Delta)$ as $R(K_1, K_2/K)$ where K is the real Hilbert space $L^2(R^3)$. We define an unbounded nonnegative selfadjoint operator ω on K by

$$(\omega f)(\boldsymbol{k})=(k^2+m^2)^{1/2}\tilde{f}(\boldsymbol{k}) \qquad (m>0) \tag{12.14}$$

where $\tilde{f}(\boldsymbol{k})$ is the Fourier transform of $f(x)$. The operators $U(f)$, $V(g)$ are defined on the Fock space $H_F(K)$ by

$$U(f)=U_F(\omega^{1/2}f),\ f\in D \tag{12.15}$$

$$V(g)=V_F(\omega^{-1/2}g) \tag{12.16}$$

where D is the domain of $\omega^{1/2}$. Given $\Delta\subset R^3$ we define

(12.17) $$K_0=\{f\in K\colon \text{ support } f\subset \Delta\}$$

(12.18) $$K_1=\{\omega^{1/2}f\colon f\in K_0\cap D\}$$

(12.19) $$K_2=\{\omega^{-1/2}f\colon f\in K_0\}.$$

Then $\mathrm{R}(\Delta)=R(K_1, K_2/K)$. The operator ϕ for this case is

$$\phi=(\omega^{-1/2}P\omega^{1/2}-1)P_1$$

where P is the orthogonal projection on K_0, and P_1 is the orthogonal projection on K_1. Thus $\mathrm{R}(\Delta)$ is determined by the spectral properties of the operator

(12.20) $$\alpha=\phi^*\phi=P_1(\omega^{1/2}P\omega^{-1/2}-1)(\omega^{-1/2}P\omega^{1/2}-1)P_1.$$

While we have not been able to determine the spectrum of α, it seems a reasonable conjecture that $\mathrm{R}(\Delta)\sim R_\infty$ for any $\Delta\neq\phi$, R^3.

We now consider the factors defined by some representations of the canonical anticommutation relations (CAR's) analogous to the representations of the CCR's defined by Eqs. (12.5), (12.6). We follow the notation of [5]. Let K be a real Hilbert space and let $\mathrm{H}_{\mathrm{JW}}(K)$ be the complex Hilbert space on which the no-particle representation of the CAR's over K is defined. Let ρ be a self-adjoint operator on K satisfying $0\leq\rho\leq 1$. We consider the representation of the CAR's defined by the equations

(12.21) $$H=\mathrm{H}_{\mathrm{JW}}(K)\otimes\mathrm{H}_{\mathrm{JW}}(K)$$

(12.22) $$\psi_\rho(f)=\psi_{\mathrm{JW}}([1-\rho]^{1/2}f)\otimes 1+\theta_{\mathrm{JW}}\otimes\psi_{\mathrm{JW}}(\rho^{1/2}f)^*$$

(Araki and Wyss [5], Shale and Stinespring [18]). The operator algebra

(12.23) $$\mathrm{R}_A(\rho)=\{\psi(f), \psi(f)^*\colon f\in K\}''$$

is a factor. If the spectrum of ρ is discrete and is given by $\{\lambda_n\colon n\in \mathrm{I}_\infty\}$, then $\mathrm{R}_A(\rho)$ can be constructed as an ITPFI factor $\mathrm{R}(M_n, \Omega_n)$ where M_n is type I_2 and $\mathrm{Sp}(\Omega_n/M_n)=(\lambda_n, 1-\lambda_n)$. It follows that if λ is an accumulation point for the eigenvalues λ_n, then $x\in \mathrm{r}_\infty(R_A(\rho))$ where $x=\lambda(1-\lambda)^{-1}$. Dell'Antonio [7] and Rideau [15] have shown that any $\mathrm{R}_A(\rho)$ is unitarily equivalent to some $\mathrm{R}_A(\rho_d)$

where ρ_d has a discrete spectrum only. In the construction of [7], ρ_d satisfies the condition that the operator

$$\rho^{1/2}(1-\rho)^{-1/2}-\rho_d^{1/2}(1-\rho_d)^{-1/2}$$

is Hilbert-Schmidt. It follows that if ρ has a continuous spectrum the factor $R_A(\rho)$ is the factor R_∞.

The representation of the CAR's describing a nonrelativistic infinite free Fermi gas at a finite density and finite temperature is of the form $\psi_{F_\rho}(f)$ where the operator ρ has a continuous spectrum (see Sec. 12 of [5]). Thus the von Neumann algebra $R_A(\rho)$ in this case is again the factor R_∞.

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