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On Some Generalized Compactness Properties*

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1. Introduction

Since Dieudonné [3] first defined paracompactness in 1944, various authors have introduced, studied and related to one another a number of properties similar in description to paracompactness. Examples are metacompactness [1], hypocompactness [9], countable paracompactness [4] and countable metacompactness [6]. The purpose of this paper is to present some examples which clarify the extent to which Hausdorff spaces with various combinations of these properties can exist. Since terminology varies considerably in the literature, some preliminary definitions are advisable. Terms not defined herein may be found in [5].

Definition 1. Suppose C is an open cover of a space X. Then C is

(i) *point-finite* (*point-countable*) if and only if each element of X belongs to only finitely (countably) many elements of C.

(ii) *locally-finite* (*locally-countable*) if and only if each element of X belongs to some open set which intersects only finitely (countably) many elements of C.

(iii) *star-finite* (*star-countable*) if and only if each element of C intersects only finitely (countably) many elements of C.

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Definition 2. A topological space X is

(i) *(countably) metacompact* (**CMC** or **MC**) if and only if each (countable) open cover of X has a point-finite, open refinement.

(ii) *(countably) paracompact* (**CPC** or **PC**) if and only if each (countable) open cover of X has a locally-finite, open refinement.

(iii) *(countably) hypocompact* (**CHC** or **HC**) if and only if each (countable) open cover of X has a star-finite, open refinement.

(iv) *(countably) compact* (**CC** or **C**) if and only if each (countable) open cover of X has a finite, open refinement.

(v) *Lindelöf* (**L**) if and only if each open cover of X has a countable, open refinement.

For convenience in presenting diagrams, the abbreviations shown in **boldface** type will be used occasionally in lieu of the terms themselves. The hypoLindelöf (**HL**) property is defined below in Section 3. The implications shown in Diagram 1 are either obvious or follow from Morita's theorem [16, Th. 6] that every paracompact Hausdorff space is countably hypocompact.

(Hausdorff Spaces)

$$\begin{array}{ccccc} \mathbf{L} & \longleftarrow & \mathbf{C} & \longrightarrow & \mathbf{CC} \\ \downarrow & & \downarrow & & \downarrow \\ \mathbf{HL} & \longleftarrow & \mathbf{HC} & & \mathbf{CHC} \\ & & \downarrow & \nearrow & \downarrow \\ & & \mathbf{PC} & & \mathbf{CPC} \\ & & \downarrow & & \downarrow \\ & & \mathbf{MC} & \longrightarrow & \mathbf{CMC} \end{array}$$

Diagram 1

2. Lindelöf Spaces

In Lindelöf spaces, there is an obvious equivalence between each countable compactness property and the corresponding compactness property. Thus, Diagram 1 reduces to the following:

(Lindelöf, Hausdorff Spaces)

$$\mathbf{C} \leftrightarrow \mathbf{CC} \rightarrow \mathbf{HC} \leftrightarrow \mathbf{CHC} \leftrightarrow \mathbf{PC} \leftrightarrow \mathbf{CPC} \rightarrow \mathbf{MC} \leftrightarrow \mathbf{CMC}$$

Diagram 2

The following examples show that there are no additional implications between these properties in Lindelöf, Hausdorff spaces.

Example A. A Lindelöf, Hausdorff space which is not countably metacompact.

Construction. Let X be the set of real numbers, let A be the set of rationals, let $B=X-A$, let $\mathcal{E}$ be the Euclidean topology for X and let $\mathcal{T}=\{U-A'\colon U\in\mathcal{E}, A'\subset A\}$. The fact that $(X, \mathcal{E})$ is a Lindelöf, Hausdorff space makes it easy to see that $(X, \mathcal{T})$ is also.

Let C be the countable, open cover $\{(X-A)\cup\{a\} : a\in A\}$ and suppose there exists a point-finite, open refinement $\mathcal{R}$ of C. If $x\in B$, then x belongs to only finitely many elements of $\mathcal{R}$ and so there exists $\varepsilon_x>0$ such that $\{y: |x-y|<\varepsilon_x, y\in B\}\subset R$ whenever $x\in R\in\mathcal{R}$. By the Baire Category Theorem, there exist $\varepsilon>0$ and an open interval Q in $(X, \mathcal{E})$ such that $D=\{x\colon \varepsilon_x\geq\varepsilon, x\in B\cap Q\}$ is dense in Q. Without loss of generality, one may assume Q is of length less than ε. Then each element of $\mathcal{R}$ which intersects D must contain $B\cap Q$. For each $a\in A\cap Q$, let R_a be such that $x\in R_a\in\mathcal{R}$; let $\mathcal{G}=\{R_a\colon a\in A\cap Q\}$. Then $\mathcal{G}$ is an infinite subcollection of $\mathcal{R}$. Also, if $a\in A\cap Q$, then R_a intersects D and so R_a contains $B\cap Q$. Therefore each point of $B\cap Q$ belongs to all elements of $\mathcal{G}$ and thus $\mathcal{G}$, and hence $\mathcal{R}$, is not point-finite. This is a contradiction and so $(X, \mathcal{T})$ is not countably metacompact.

Example B. A Lindelöf, Hausdorff space which is metacompact but not countably paracompact.

Construction. Let X be the set of real numbers, let $A=\{1/n\colon n=1, 2, \cdots\}$, let $\mathcal{E}$ be the Euclidean topology for X and let $\mathcal{T}=\{U-B\colon B\subset A\}$. As in Example A, note that $(X, \mathcal{T})$ is Lindelöf and Hausdorff since $(X, \mathcal{E})$ has these properties. For each $V\in\mathcal{T}$, let V^* be an open set in $\mathcal{E}$ such that $V=V^*-B$ for some $B\subset A$.

Let C be the countable open cover $\{(X-A)\cup\{a\} : a\in A\}$, and let $\mathcal{R}$ be any open refinement of C. If $0\in V\in\mathcal{T}$, then infinitely many elements of A belong to V^* and so V intersects infinitely many elements of $\mathcal{R}$. Thus $\mathcal{R}$ is not locally-finite and so $(X, \mathcal{T})$ is not countably paracompact.

Now let $\mathcal{U}$ be any open cover of $(X, \mathcal{T})$. Then $\mathcal{V}=\{U^*: U\in\mathcal{U}\}$ is an open cover of $(X, \mathcal{E})$ and hence, since $(X, \mathcal{E})$ is Lindelöf and metacompact, there exist a countable, open (in $\mathcal{E}$) refinement $\mathcal{R}_1$ of $\mathcal{V}$ and a point-finite, open (in $\mathcal{E}$) refinement $\mathcal{R}_2$ of $\mathcal{V}$. Let $\mathcal{R}_1'=\{R-A: R\in\mathcal{R}_1\}$ and $\mathcal{R}_2'=\{R-A: R\in\mathcal{R}_2\}$. For each positive integer n, let G_n be an element of $\mathcal{T}$ which is contained in some element of $\mathcal{U}$ and is such that $1/n\in G_n\subset\{x: 1/(n+1)<x<1/(n-\frac{1}{2})\}$. If $\mathcal{G}=\{G_n: n=1, 2, \cdots\}$, then $\mathcal{R}_1'\cup\mathcal{G}$ and $\mathcal{R}_2'\cup\mathcal{G}$ are countable and point-finite, respectively, open refinements of $\mathcal{U}$ and so $(X, \mathcal{T})$ is both Lindelöf and metacompact.

By a theorem of Morita [16, Th. 10], every regular Lindelöf space is hypocompact. Thus no regular space exists with the properties of Example A or Example B. On the other hand, by a theorem of Dieudonné [3], every paracompact Hausdorff space is normal. Thus only normal spaces exist with the properties listed in Example C or Remark D.

Example C. A Lindelöf, Hausdorff space which is hypocompact but not countably compact.

Construction. Any separable metric space which is not compact has the required properties. A nonmetrizable example is furnished by the set of real numbers with the lower-limit topology.

Remark D. There exist compact, and hence Lindelöf, Hausdorff spaces.

3. HypoLindelöf Spaces

The property of hypo Lindelöfness is introduced now because of its usefulness in Theorem 1.

Definition 3. A space X is *hypoLindelöf* if and only if each open cover of X has a star-countable, open refinement.

Theorem 1. Let X be a hypoLindelöf space. Then

(i) X is metacompact if and only if X is countably metacompact.

(ii) X is paracompact if and only if X is countably paracompact.

(iii) X is hypocompact if and only if X is countably hypocompact.

(iv) X is compact if and only if X is countably compact.

Proof. Let $\mathcal{C}$ be an open cover of X. Since X is hypoLindelöf, there exists a star-countable, open refinement $\mathcal{R}$ of $\mathcal{C}$. Let $\{\mathcal{R}_\alpha: \alpha \in \Lambda\}$ be the collection of components of $\mathcal{R}$ and, for each $\alpha \in \Lambda$, let $X_\alpha = \bigcup \mathcal{R}_\alpha$. Suppose $\beta \in \Lambda$. Then it follows by a simple modification in the proof of a theorem of Lefschetz [13, p. 15] that $\mathcal{R}_\beta$ is countable. Also, it is easily seen that $\{X_\alpha: \alpha \in \Lambda\}$ is a pairwise-disjoint collection of open and closed sets whose union is X. Thus $\mathcal{R}_\beta \cup \{X - X_\beta\}$ is a countable open cover of X.

According as X is countably metacompact, countably paracompact, or countably hypocompact, let $\mathcal{R}'_\beta$ be an open refinement of $\mathcal{R}_\beta \cup \{X - X_\beta\}$ which is point-finite, locally-finite, or star-finite, respectively. Let $\mathcal{R}''_\beta$ consist of those elements of $\mathcal{R}'_\beta$ which do not intersect $X - X_\beta$. Then $\mathcal{R}''_\beta$ is a collection of open sets which is point-finite, locally-finite, or star-finite, respectively. Since $\bigcup \mathcal{R}''_\beta = X_\beta$ and since $\{X_\alpha: \alpha \in \Lambda\}$ is a pairwise-disjoint open cover of X, then $\bigcup \{\mathcal{R}''_\alpha: \alpha \in \Lambda\}$ is an open refinement of $\mathcal{C}$ which is point-finite, locally-finite, or star-finite, respectively, according as X is countably metacompact, countably paracompact, or countably hypocompact. Finally, if X is countably compact, then the above proof shows that X is metacompact. A theorem of Arens and Dugundji [1] states that every countably compact, metacompact T_1-space is compact. Iséki [11, p. 41], without use of his assumption of the T_1-axiom, has shown that every point-finite, open cover of a space S has a subcover $\mathcal{U}$ which contains no subcover of S different from $\mathcal{U}$. When S is countably compact, it is easily seen that $\mathcal{U}$ must be finite and so the Arens-Dugundji Theorem does not require the T_1-hypothesis. Thus X is compact, if X is countably compact.

By virtue of Theorem 1, Diagram 1 reduces to Diagram 3 for hypoLindelöf Hausdorff spaces which are not Lindelöf spaces. Examples

E, F and G show that no additional implications exist.

(HypoLindelöf, NonLindelöf, Hausdorff Spaces)

HC $\leftrightarrow$ CHC $\leftrightarrow$ PC $\leftrightarrow$ CPC $\rightarrow$ MC $\leftrightarrow$ CMC

Diagram 3

Example E. A hypocompact, and hence hypoLindelöf, Hausdorff space which is not Lindelöf.

Construction. The set of real numbers with the discrete topology has the requisite properties.

Definition 4. A *disjoint union* of two spaces $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ is a space $(Y_1 \cup Y_2, \mathcal{T})$ where $(Y_1, \mathcal{U}_1)$ and $(Y_2, \mathcal{U}_2)$ are homeomorphic to $(X_1, \mathcal{T}_1)$ and $(X_2, \mathcal{T}_2)$ respectively, $\mathcal{T} = \{U_1 \cup U_2 : U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2\}$ and $Y_1 \cap Y_2 = \emptyset$.

It is easily seen that a disjoint union of spaces X and Y has one of the 10 compactness (and Lindelöfness) properties under discussion if and only if each of X and Y has that property.

Example F. A metacompact, hypoLindelöf Hausdorff space which is not countably paracompact and not Lindelöf.

Construction. A disjoint union of the spaces of Examples B and E has the requisite properties.

Example G. A hypoLindelöf, Hausdorff space which is not countably metacompact and not Lindelöf.

Construction. A disjoint union of the spaces of Examples A and E has the requisite properties.

By Dieudonné's theorem cited above, every space with the properties of Example E must be normal. However, Смирнов [17, p. 256] has shown that every regular hypoLindelöf space must be hypocompact and so no regular space exists with the properties of Example F or Example G.

4. NonhypoLindelöf Spaces

By the Arens-Dugundji Theorem, a nonhypoLindelöf, metacompact, Hausdorff space must fail to be countably compact. Thus Diagram 1 reduces to the following diagrams:

(NonhypoLindelöf, Metacompact, Hausdorff Spaces)

PC → CHC → CPC

Diagram 4

(NonhypoLindelöf, Nonmetacompact, Hausdorff Spaces)

CC → CHC → CPC → CMC

Diagram 5

Isêki [10] has shown that countably paracompact, normal Hausdorff spaces must be countably hypocompact, but the author knows of no solution to the following problem:

Problem 1. Does there exist a countably paracompact Hausdorff space which is not countably hypocompact? If so, can such a space be metacompact?

Except for the examples needed to answer affirmatively the above questions, the following examples show that no additional implications exist between the properties of Diagram 4 or Diagram 5. Note that a space with the properties of Example H must be normal by Dieudonné's theorem (see Section 1).

Example H (Morita). A nonhypoLindelöf, paracompact, Hausdorff space.

Construction. Let $I=\{x: 0\leq x\leq 1\}$ and let X consist of the union of uncountably many distinct copies of I, with the point 0 on each copy identified. Define a metric ρ for X by $\rho(x, y)=|x-y|$, if x and y belong to the same copy of I, and $\rho(x, y)=x+y$, otherwise. Then the metric space (X, ρ) is paracompact by a theorem of Stone [18]. Also, (X, ρ) is easily seen to be a regular, connected, non-Lindelöf space and thus, by a corollary of Morita [16, p. 66], is not hypocompact. The theorem of Смирнов quoted in Section 3 shows that X is not hypoLindelöf.

Example I_1 (Bing and Michael). A nonhypoLindelöf, metacompact, nonparacompact, countably hypocompact, normal Hausdorff space.

Construction. Let P be the set of real numbers, let Q be the set of all subsets of P and, for each $p \in P$, let f_p be the function from Q into $\{0, 1\}$ such that $f_p(q)=1$ if and only if $p \in q$. Let $F^* = \{f_p : p \in P\}$ and let F be the set of functions from Q into $\{0, 1\}$ such that $f(q)=0$ except for finitely many $q \in Q$. Let $G = F^* \cup F$ and let $\mathcal{T}$ be that topology for G which has as a base the collection consisting of (i) $\{f\}$ whenever $f \in F - F^*$ and (ii) $\{f : f \in G, f(q)=f_p(q)\}$ whenever $p \in P$ and r is a finite subset of Q. Michael [15, Example 2], through modification of an example of Bing [2], has shown that the space $(G, \mathcal{T})$ is a metacompact, nonparacompact, normal Hausdorff space. Dowker [4] and Katětov [12], independently, have shown that countable metacompactness and countable paracompactness are equivalent in normal spaces. Thus $(G, \mathcal{T})$ is countably paracompact and hence, by the theorem of Iséki [10] cited above, $(G, \mathcal{T})$ is countably hypocompact. That $(G, \mathcal{T})$ is not hypoLindelöf follows from Theorem 1.

Example J_1 (Heath). A nonhypoLindelöf, metacompact, non-countably-paracompact, regular Hausdorff space.

Construction. Let X consist of all points in the xy-plane such that $y \geq 0$ and let $\mathcal{T}$ be that topology for X which has as a base the collection consisting of (i) $\{(x, y)\}$ whenever $y > 0$ and (ii) the set $N_p^\varepsilon = \{(x, y) : 0 \leq x-p=y \leq \varepsilon\} \cup \{(x, y) : 0 \leq p-x=y \leq \varepsilon\}$ whenever p is real and $\varepsilon > 0$. Heath [8, p. 765] has observed that this space is a metacompact, regular Hausdorff space. Suppose $L = \{(x, 0) : x \text{ is real}\}$ and $Q = \{(x, 0) : x \text{ is rational}\}$. Category arguments similar to those on p. 69 of [5] show that $\{N_p^1 : p \in Q\} \cup \{X-Q\}$ is a countable open cover of X with no locally-finite, open refinement and $\{N_p^1 : p \in L\} \cup \{X-L\}$ is an open cover of X with no star-countable, open refinement. Thus $(X, \mathcal{T})$ is neither countably paracompact nor hypoLindelöf.

By the theorem of Dowker and Katětov quoted in the discussion of Example I_1, no normal space can exist with the properties stated for Example J_1. A nonregular space is described in Example J_2.

Example J_2. A nonhypoLindelöf, metacompact, non-countably-paracompact, Hausdorff space.

Construction. A disjoint union of the spaces of Examples B and H has the desired properties.

Example K_0. A nonhypoLindelöf, nonmetacompact, countably compact, normal Hausdorff space.

Construction. The set of ordinals less than the first uncountable ordinal with the order topology has the requisite properties. All properties stated are well-known except perhaps for the first one which follows from Theorem 1.

Example L_0. A nonhypoLindelöf, nonmetacompact, countably hypocompact, non-countably-compact, normal Hausdorff space.

Construction. A disjoint union of the spaces of Examples E and K_0 will suffice.

Example M_0. A nonhypoLindelöf, nonmetacompact, countably metacompact, non-countably-paracompact, regular Hausdorff space.

Construction. A disjoint union of the spaces of Examples J_1 and K_0 suffices.

Examples N_0, N_1, N_2. A nonhypoLindelöf, non-countably-metacompact Hausdorff space.

Construction. For Examples N_0, N_1 and N_2 take a disjoint union of the spaces of Examples A and K_0, A and J_1 or A and H, respectively.

No space with the properties stated for Example M_0 can be normal by the theorem of Dowker and Katětov cited above. The spaces of Examples N_0, N_1 and N_2 are not regular and in this connection there are outstanding unsolved problems.

Problem 2 (Dowker-Katětov). Is every normal Hausdorff space countably paracompact? [4, p. 221], [12, p. 90].

Problem 3 (Hayashi). Is every regular Hausdorff space countably metacompact? [6, p. 164].

Note that, by the Dowker-Katětov theorem, an affirmative answer to Problem 3 would automatically provide an affirmative answer to Problem 2. Since posing problem 3, Hayashi [7] has shown that one obtains a negative answer if one assumes the generalized hypothesis of the continuum.

Younglove [19] attributes to Dowker the conjecture that every countably paracompact Hausdorff space is normal and cites [4] as his authority. This author can find no evidence that Dowker made such a conjecture in [4]. At any rate, let Ω be the first uncountable ordinal and let X be the set of ordinals $\leq\Omega$ with the order topology. Mack and Johnson [14, p. 240] have shown that the subspace $\{(x, y): x\leq y, x<\Omega\}$ of $X\times X$ is a countably compact, completely regular Hausdorff space which is not normal.

In imitation of Definition 3, one can also define the *paraLindelöf* and *metaLindelöf* properties. Some readers will undoubtedly wish to know which of the nonhypoLindelöf spaces in Examples H through N_2 have these properties. We thus state without proof that Examples H, J_2 and N_2 are paraLindelof, Examples I_1, J_1 and N_1 are metaLindelöf but not paraLindelöf and Examples K_0, L_0, M_0 and N_0 are not metaLindelöf.

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