Publ. RIMS, Kyoto Univ. Ser. A
Vol. 4 (1968), pp. 361-371

Multiple Time Analyticity of a Quantum Statistical State Satisfying the KMS Boundary Condition

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Abstract

A multiple time expectation $\varphi(AB_1(t_1)\cdots B_n(t_n))$ in a stationary state φ satisfying the KMS boundary condition is studied. It is found to be holomorphic in a simplicial tube domain $0<\text{Im}\, t_1<\text{Im}\, t_2<\cdots<\text{Im}\, t_n<\beta$, continuous and bounded in the closure and the expectation of cyclic permutation of operators are obtained as its values on various distinguished boundaries of the domain.

§ 1. Introduction

The Gibbs ensemble in quantum statistical mechanics satisfies the Kubo-Martin-Schwinger (KMS) boundary condition and a general property of such a state has been discussed by several authors [1], [2], [3], [4]. In this paper we shall study the analyticity of $\varphi(AB_1(t_1)\cdots B_n(t_n))$ in $t_1\cdots t_n$. The main theorem is Theorem 3.1 and 3.3 of section 3.

In passing, it is shown by the analyticity method that the center of the representing algebra is time translation invariant. It is also pointed out that the KMS boundary condition holds for the weak closure, which will be used in [4].

§ 2. The KMS Boundary Condition and Analyticity

We shall discuss an analyticity tube domain for single time expectation function in this section. We also give a proof that the center of the representative algebra is elementwise time translation invariant.

Received July 9, 1968.

Let $\mathfrak{A}$ be a C^* algebra, $\tau(t)$ be a one parameter group of automorphisms of $\mathfrak{A}$, continuous in t, and φ be a state of $\mathfrak{A}$ invariant under $\tau(t)$:

$$\varphi(A)=\varphi(\tau(t)A), \quad A\in\mathfrak{A}. \tag{2.1}$$

Definition 2.1. φ satisfies the KMS boundary condition if

$$\int\varphi(A\tau(t)B)\tilde{f}_0(t)\mathrm{d}t=\int\varphi([\tau(t)B]A)\tilde{f}_\beta(t)\mathrm{d}t, \tag{2.2}$$

$$\tilde{f}_\alpha(t)=\int_{-\infty}^{\infty}f(p)\,e^{-ipt+\alpha p}\mathrm{d}p \tag{2.3}$$

for arbitrary two elements A and B of $\mathfrak{A}$ and for arbitrary function f in the class $\mathscr{D}$.

Lemma 2.2. If φ is $\tau(t)$ invariant and satisfies the KMS boundary condition, and A and B are elements of $\mathfrak{A}$, then there exists a function $\mathrm{F}(\zeta)$ of a complex variable ζ such that

(1) F is continuous and bounded for $0\leq\mathrm{Im}\,\zeta\leq\beta$.

(2) F is holomorphic for $0<\mathrm{Im}\,\zeta<\beta$.

(3) For real t,

$$\mathrm{F}(t)=\varphi(A\tau(t)B), \quad \mathrm{F}(t+i\beta)=\varphi([\tau(t)B]A). \tag{2.4}$$

Proof. We note that a representation π_φ of $\mathfrak{A}$ on a Hilbert space H_φ, a cyclic vector Ω_φ and a continuous one parameter group of unitary operator $U_\varphi(t)$ are uniquely determined by the relation

$$\varphi(A)=(\Omega_\varphi,\pi_\varphi(A)\Omega_\varphi) \tag{2.5}$$

$$U_\varphi(t)\pi_\varphi(A)\Omega_\varphi=\pi_\varphi(\tau(t)A)\Omega_\varphi. \tag{2.6}$$

In particular,

$$\varphi(A\tau(t)B)=(\Omega_\varphi,\pi_\varphi(A)U_\varphi(t)\pi_\varphi(B)\Omega_\varphi) \tag{2.7}$$

is a Fourier transform of a finite complex measure μ_0:

$$\varphi(A\tau(t)B)=\int e^{ipt}\,\mathrm{d}\mu_0(p). \tag{2.8}$$

Similarly

$$\varphi([\tau(t)B]A)=\int e^{ipt}\,\mathrm{d}\mu_\beta(p) \tag{2.9}$$

A complex finite measure can be considered as a dual to the Banach space C_0 of bounded continuous functions vanishing at infinity, in which $\mathscr{D}$ is dense. Hence (2.2) implies

$$\mathrm{d}\mu_0 = e^{\beta p}\,\mathrm{d}\mu_\beta\,. \tag{2.10}$$

Let χ be the characteristic function of $(0, \infty)$ and set

$$\mathrm{d}\mu = \chi \mathrm{d}\mu_0 + (1-\chi)\mathrm{d}\mu_\beta\,. \tag{2.11}$$

It is a finite complex measure. Let

$$g_\alpha(p) = e^{-\alpha p}\,\chi(p) + e^{(\beta-\alpha)p}\,(1-\chi(p))\,, \tag{2.12}$$

which is a bounded continuous function if $0 \leq \alpha \leq \beta$. Therefore

$$\mathrm{d}\mu_\alpha = g_\alpha \mathrm{d}\mu \tag{2.13}$$

is a finite complex measure and

$$\mathrm{F}(t+i\alpha) \equiv \int e^{ipt}\,\mathrm{d}\mu_\alpha(p) \tag{2.14}$$

is a bounded continuous function of t and α for $-\infty < t < +\infty$, $0 \leq \alpha \leq \beta$. From (2.10), (2.11), (2.12), we see that (2.4) is satisfied. For $0 < \alpha < \beta$, $e^{itp}\, g_\alpha(p)$ satisfies the Cauchy-Riemann relation with respect to $t+i\alpha$ in the topology of C_0 and hence $\mathrm{F}(t+i\alpha)$ is holomorphic in $t+i\alpha$ for $0<\alpha<\beta$.

Remark 2.3. The existence of F satisfying (1), (2), (3) is equivalent to the KMS boundary condition. This is known except that the boundedness of F in the tube has not been treated in the literature.

Lemma 2.4. Let $\mathfrak{A}_1 \equiv (\pi_\varphi(\mathfrak{A}))''$, $\varphi_1(A) = (\Omega_\varphi, A\Omega_\varphi)$, $\tau_1(t)A = U_\varphi(t)AU_\varphi(t)^{-1}$ $(A \in \mathfrak{A}_1)$. Then φ_1 satisfies the KMS boundary condition with respect to $\mathfrak{A}_1$ and τ_1, if φ satisfies the same with respect to $\mathfrak{A}$ and τ.

Proof. We prove (2.2) for φ_1, $A \in \pi_\varphi(\mathfrak{A})$ and $B \in \pi(\mathfrak{A})''$. A similar argument will then yield (2.2) for general A in $\pi_\varphi(\mathfrak{A})''$. Since $B=1$ obviously satisfies (2.2), we consider B in the weak closure of $\pi_\varphi(\mathfrak{A})$. By the density theorem, it is enough to consider

B in the weak closure of the unit ball of $\pi_{\varphi}(\mathfrak{A})$.

Let T be such that $\int_{|t|>T} |\tilde{f}_0(t)| \mathrm{d}t < \varepsilon$. Since $U_{\varphi}(t)$ is continous in t, we can find an open interval I_t containing t such that

$$||U_{\varphi}(t)^{-1}\pi_{\varphi}(A)^*\Omega_{\varphi} - U_{\varphi}(t')^{-1}\pi_{\varphi}(A)^*\Omega_{\varphi}|| < \varepsilon \tag{2.15}$$

for any $t' \in I_t$. A finite number of such $I_{t_1} \cdots I_{t_n}$ cover the compact interval $[-T, T]$. Let N be the weak neighbourhood of B defined by

$$N = \{B' ; \ |(U_{\varphi}(t_j)^{-1}\pi_{\varphi}(A)^*\Omega_{\varphi}, (B-B')\Omega_{\varphi})| < \varepsilon, j=1\cdots n\} . \tag{2.16}$$

Then we have for $B' \in N$, $||B'|| \leq 1$,

$$|\int \varphi_1(A\tau(t)(B-B'))\tilde{f}_0(t)\mathrm{d}t| \leq 2||A||\varepsilon + 3\varepsilon \int |\tilde{f}_0(t)| \mathrm{d}t . \tag{2.17}$$

We have a similar equation for the right hand side of (2.2). Since (2.2) holds for $B' \in \pi(\mathfrak{A})$, we have (2.2) for B in the weak closure of the unit ball of $\pi(\mathfrak{A})$. Q.E.D.

Corollary 2.5. The element of the center of $\pi_{\varphi}(\mathfrak{A})''$ is invariant under $\tau_1(t)$.

Proof. Since $U_{\varphi}(t)RU_{\varphi}(t)^{-1} = R$ holds for $R = \pi_{\varphi}(\mathfrak{A})$, it holds for $R = \pi_{\varphi}(\mathfrak{A})'$ and hence for $R = \pi_{\varphi}(\mathfrak{A})''$ and therefore for $R =$ the center of $\pi_{\varphi}(\mathfrak{A})''$. Thus

$$\varphi_1(A\tau_1(t)B) = \varphi_1([\tau_1(t)B]A) \tag{2.18}$$

if B is in the center of $\pi_{\varphi}(\mathfrak{A})''$. Lemma 2.2 implies the existence of a function $\mathrm{F}(\zeta)$ which is holomorphic for $0 < \mathrm{Im}\,\zeta < \beta$ and continuous for $0 \leq \mathrm{Im}\,\zeta \leq \beta$. (2.4) and (2.18) implies, due to the edge of wedge theorem, that $\mathrm{F}(\zeta)$ is an entire function with period $i\beta$. Since F is bounded, it must be a constant. Since $\tau_1(t)B$ is in the center of $\pi_{\varphi}(\mathfrak{A})''$, we see that $\varphi_1(A_1[\tau_1(t)B]A_2)$ is constant of t and hence $\tau_1(t)B = B$.

Remark 2.6. This corollary can be proved also from (2.10) directly. Namely, $\mu_0 = \mu_{\beta}$ and (2.10) imply $\mu_0 = c\delta(p)\mathrm{d}p$, from which it follows that $\varphi(A\tau(t)B)$ is independent of t.

This proof can be use to show that a state is invariant under $\tau(t)$, if it satisfies the KMS boundary condition (This is pointed out by H. Miyata).

The equation (2.10) at $p=0$ implies that Ω_φ is a trace vector for $E(\{0\})\pi_\varphi(\mathfrak{A})E(\{0\})$. This is used in [4].

§ 3. Analyticity of Multiple Time Expectation Values

Theorem 3.1. Let φ be a $\tau(t)$ invariant state of $\mathfrak{A}$ satisfying the KMS boundary condition. Let $A, B_1, \cdots, B_n$ be arbitrary $n+1$ elements of $\mathfrak{A}$ $(n=1, 2, \cdots)$. There exists a function $\mathrm{H}(\zeta_1, \cdots, \zeta_n)$ of n complex variables such that

(1) F is holomorphic for

$$0<\mathrm{Im}\,\zeta_1<\cdots<\mathrm{Im}\,\zeta_n<\beta \tag{3.1}$$

(2) The boundary value of F for $\mathrm{Im}\,\zeta_1=\cdots=\mathrm{Im}\,\zeta_j=0$, $\mathrm{Im}\,\zeta_{j+1}=\cdots=\mathrm{Im}\,\zeta_n=\beta$ in the distribution sense is the function

$$\varphi([\tau(t_{j+1})B_{j+1}]\cdots[\tau(t_n)B_n]A[\tau(t_1)B_1]\cdots[\tau(t_j)B_j]) \tag{3.2}$$

where $j=0, \cdots, n$ and $t_k=\mathrm{Re}\,\zeta_k$.

Proof. Let us consider the Fourier transform of (3.2) in distribution sense:

$$\begin{aligned} f_j(p_1, \cdots, p_n) = \int &\varphi([\tau(t_{j+1})B_{j+1}]\cdots[\tau(t_n)B_n]\times \\ &\times A[\tau(t_1)B_1]\cdots[\tau(t_j)B_j])\, e^{-i(p_1t_1+\cdots+p_nt_n)}\times \\ &\times \mathrm{d}t_1\cdots\mathrm{d}t_n/(2\pi)^n \end{aligned} \tag{3.3}$$

(2.2) implies

$$f_{j+1}(p_1, \cdots, p_n) = e^{\beta p_{j+1}} f_j(p_1, \cdots, p_n)\,. \tag{3.4}$$

Let χ_j be the characteristic function of the following region

$$\begin{aligned} B_j = \{(p_1, \cdots, p_n)\,;\ & p_k+p_{k+1}+\cdots+p_j>0\,,\quad k=1, \cdots, j \\ & p_{j+1}+\cdots+p_k<0,\quad k=j+1, \cdots, n\} \end{aligned} \tag{3.5}$$

where $j=0, \cdots, n$. Let g be a nonnegative function in the class $\mathscr{D}$ such that

$$\int g(p_1, \cdots, p_n)dp_1\cdots dp_n = 1 \tag{3.6}$$

and χ_j^g be the regularization of χ_j by g:

$$\chi_j^g(p_1, \cdots, p_n) = \int \chi_j(p_1 - r_1, \cdots, p_n - r_n)g(r_1, \cdots, r_n)\mathrm{d}r_1 \cdots \mathrm{d}r_n . \tag{3.7}$$

If we denote a vector with the first k components equal to 1 and the last $(n-k)$ components equal to 0 by q_k $(k=0, \cdots, n)$, then $\bar{B}_j = \{p\,;\ \max_k (p, q_k) = (p, q_j)\}$. From this we see that $\bigcup_{j=0}^{n} \bar{B}_j$ is the entire space and $\bar{B}_j \cap \bar{B}_k$ is in the plane orthogonal to $q_j - q_k$ if $j \neq k$, namely $\dim \bar{B}_j \cap \bar{B}_k < n$. Hence

$$\sum_{j=0}^{n} \chi_j^g = 1 . \tag{3.8}$$

Let us define the following distribution

$$\begin{aligned} &\mathrm{H}(p_1, \cdots, p_n\,;\ t_1 + i\alpha_1, \cdots, t_n + i\alpha_n) \\ &\quad = \sum \chi_j^g(p_1, \cdots, p_n) h_j(p_1, \cdots, p_n\,;\ \alpha_1, \cdots, \alpha_n) f_j(p_1, \cdots, p_n) \\ &\quad\quad \exp i \sum_{j=1}^{n} t_j p_j \end{aligned} \tag{3.9}$$

where

$$\begin{aligned} h_j(p_1 \cdots p_n\,;\ \alpha_1 \cdots \alpha_n) = \exp \Big\{ &\sum_{k=j+1}^{n} (\alpha_{k+1} - \alpha_k)(p_{j+1} + p_{j+2} + \cdots + p_k) \\ &- \sum_{k=1}^{j} (\alpha_k - \alpha_{k-1})(p_k + p_{k+1} + \cdots + p_j) \Big\} \end{aligned} \tag{3.10}$$

and $\alpha_0 \equiv 0$, $\alpha_{n+1} \equiv \beta$. If

$$0 < \alpha_1 < \cdots < \alpha_n < \beta , \tag{3.11}$$

then (3.10) implies that h_j decreases exponentially whenever $(p, q_j - q_l)$ tends to $+\infty$ for one l. On the other hand the part of B_j, in which $(p, q_j - q_l) < R$ for all l and a fixed $R > 0$, is compact. Hence

$$\exp i(p_1 t_1 + \cdots + p_n t_n) h_j(p_1 \cdots p_n\,;\ \alpha_1 \cdots \alpha_n) \chi_j^g(p_1 \cdots p_n)$$

is in the class $\mathcal{S}$ and satisfies the Cauchy-Riemann relation with respect to each $t_k + i\alpha_k$. We now define

$$\text{(3.12)} \qquad \mathrm{F}(\zeta_1 \cdots \zeta_n) = \int \mathrm{H}(p_1 \cdots p_n\,;\ \zeta_1 \cdots \zeta_n) \mathrm{d}p_1 \cdots \mathrm{d}p_n$$

which is holomorphic for ζ satisfying (3.1). Furthermore, the Fourier transform of the boundary value of F for $\mathrm{Im}\,\zeta_1 = \cdots = \mathrm{Im}\,\zeta_j = 0$, $\mathrm{Im}\,\zeta_{j+1} = \cdots = \mathrm{Im}\,\zeta_n = \beta$ becomes

$$\text{(3.13)} \qquad \sum \chi_k^g(p_1 \cdots p_n) h_k(p_1 \cdots p_n\,;\ \alpha_1 \cdots \alpha_n) f_k(p_1 \cdots p_n)$$

where $\alpha_1 = \cdots = \alpha_j = 0$, $\alpha_{j+1} = \cdots = \alpha_n = \beta$. From (5.10) we have

$$\text{(3.14)} \quad h_k(p_1 \cdots p_n\,;\ \alpha_1 \cdots \alpha_n) = \begin{cases} \exp \beta(p_{k+1} + \cdots + p_j) & \text{if } k<j \\ 1 & \text{if } k=j \\ \exp -\beta(p_{j+1} + \cdots + p_k) & \text{if } k>j \end{cases}$$

Hence, from (3.4), we have

$$\text{(3.15)} \qquad h_k(p_1 \cdots p_n\,;\ \alpha_1, \cdots \alpha_n) f_k(p_1 \cdots p_n) = f_j(p_1 \cdots p_n)$$

for all k. By using (3.8), we see that the boundary value in question is (3.2).

Remark 3.2. (i) The above theorem and its proof are stated in a form which holds for Wightman fields. The next theorem uses the fact that A and B_i are bounded operators. (ii) In the discussion of the analyticity, it is more symmetric to consider

$$\text{(3.16)} \qquad \varphi(A_1(t_1) \cdots A_n(t_n))$$

on the space $\{(t_1 \cdots t_n) \bmod (1, \cdots, 1)\}$. The step function χ_j can be written in terms of the edge vectors of the simplicial domain in question. For such a technique, see generalized θ function introduced in [5].

Theorem 3.3. The function F in Theorem 3.1 is continuous and bounded in the closure of the simplicial tube domain (3.1).

Proof. We investigate each summand more closely. By definition (3.3), we have

$$\text{(3.17)} \quad \begin{cases} \displaystyle\int f_k(p_1 \cdots p_n) \exp i\{\sum (t_l + s_l) p_l\} dp_1 \cdots dp_n \\ \quad = (\Omega_\varphi, Q_{k+1} U_\varphi(s_{k+2} - s_{k+1}) Q_{k+2} \cdots Q_n U_\varphi(-s_n) Q_0 U_\varphi(s_1) \\ \qquad Q_1 U_\varphi(s_2 - s_1) \cdots U_\varphi(s_k - s_{k-1}) Q_k \Omega_\varphi) \\ Q_l = \pi_\varphi[\tau(t_l) B_l],\ Q_0 = \pi_\varphi[A]\,. \end{cases}$$

On the other hand

$$(3.18)\quad \chi_k(p_1\cdots p_n)=\prod_{i=1}^{k}\theta(p_i+\cdots+p_k)\prod_{j=k+1}^{n}\theta(-(p_{k+1}+\cdots+p_j))$$

$$h_k(p_1\cdots p_n\,;\ \alpha_1\cdots\alpha_n)$$
$$=\prod_{i=1}^{k}e^{-(\alpha_i-\alpha_{i-1})(p_i+\cdots+p_k)}\prod_{j=k+1}^{n}e^{(\alpha_{j+1}-\alpha_j)(p_{k+1}+\cdots+p_j)}$$

where θ is the characteristic function for positive reals. We note that

$$(3.19)\quad \sum t_lp_l=t_1(p_1+\cdots+p_k)+(t_2-t_1)(p_2+\cdots+p_k)+\cdots$$
$$+(t_k-t_{k-1})p_k-(t_{k+2}-t_{k+1})p_{k+1}-(t_{k+3}-t_{k+2})(p_{k+1}+p_{k+2})$$
$$-\cdots-(t_n-t_{n-1})(p_{k+1}+\cdots+p_{n-1})+t_n(p_{k+1}+\cdots+p_n)\,.$$

If we set

$$(3.20)\quad \tilde{\theta}(z)=\frac{1}{2\pi}\int_0^\infty e^{-izp}\,\mathrm{d}p=\frac{1}{2\pi iz}$$

$$(3.21)\quad \tilde{g}(z_1\cdots z_n)=\int g(p_1\cdots p_n)\exp-i\sum_{l=1}^{n}z_lp_l\mathrm{d}p_1\cdots\mathrm{d}p_n$$

we have

$$(3.22)\quad \int h_k(p_1\cdots p_n\,;\ \alpha_1\cdots\alpha_n)\chi_k^g(p_1\cdots p_n)\exp-i(\sum s_lp_l)\mathrm{d}p_1\cdots\mathrm{d}p_n/(2\pi)^n$$
$$=\tilde{g}(z_1\cdots z_n)\tilde{\theta}(z_1)\tilde{\theta}(z_2-z_1)\cdots\tilde{\theta}(z_k-z_{k-1})\tilde{\theta}(z_{k+2}-z_{k+1})\cdots$$
$$\cdots\tilde{\theta}(-z_n)$$

where

$$(3.23)\quad z_l=s_l-i\alpha_l\,,\quad l=1\cdots k$$

$$(3.24)\quad z_l=s_l-i\alpha_l+i\beta\,,\quad l=k+1\cdots n.$$

Combining (3.17) and (3.22), we obtain the following expression for the integral of the k th term of (3.9).

$$(3.25)\quad \int\tilde{g}(z_1\cdots z_n)\mathrm{d}s_1\cdots\mathrm{d}s_n(\Omega_\varphi,\ Q_{k+1}X(z_{k+2}-z_{k+1})Q_{k+2}\cdots$$
$$\cdots Q_nX(-z_n)Q_0X(z_1)Q_1X(z_2-z_1)\cdots X(z_k-z_{k-1})Q_k\Omega_\varphi)$$

where

$$(3.26)\quad X(z)=U_\varphi(\mathrm{Re}\,z)\tilde{\theta}(z)\,.$$

For any testing function $\tilde{g}$ in the class $\mathcal{D}$, we have

$$\int g(t-\zeta)X(t-\zeta)\mathrm{d}t = \int \tilde{g}(q)\mathrm{d}qU^{+}(\zeta\,;\,q) \tag{3.27}$$

where $\operatorname{Im}\zeta\geqq 0$, $\operatorname{Re}\zeta=0$,

$$g(t) = \int e^{-itq}\,\tilde{g}(q)\mathrm{d}q\,, \tag{3.28}$$

$$U^{+}(\zeta\,;\,q) = \int_{q}^{\infty} e^{i\zeta\lambda}\,\mathrm{d}E(\lambda)\,, \tag{3.29}$$

E is the spectral projection of $U_{\varphi}(t)$.
The integral in (3.29) is ambiguous at the lower end but this ambiguity does not affect (3.27). If the lower end is $q\pm 0$, we denote $U^{+}_{\pm}$. We define (3.29) as an average of U^{+}_{+} and U^{+}_{-}.
The expression (3.25) is then equal to

$$\int g(p_1\cdots p_n)(\Omega_{\varphi},\, Q_{k+1}U^{+}(\zeta_{k+1}\,;\,q_{k+1})Q_{k+2} \\ Q_nU^{+}(\zeta_n\,;\,q_n)Q_0U^{+}(\zeta_0\,;\,q_1)Q_1\cdots U^{+}(\zeta_{k-1}\,;\,q_k)Q_k\Omega_{\varphi})\mathrm{d}p_1\cdots\mathrm{d}p_n \tag{3.30}$$

where

$$\zeta_l = i(\alpha_{l+1}-\alpha_l) \quad l = 0,\cdots,n \tag{3.31}$$

$$\alpha_{n+1} = \beta\,, \quad \alpha_0 = 0 \tag{3.32}$$

$$q_l = \begin{cases} p_l+\cdots+p_k & \text{if } l\leqq k & (3.33)\\ -(p_{k+1}+\cdots+p_l) & \text{if } l>k\,. & (3.34)\end{cases}$$

We now take the limit of sequence $g=g^{(\nu)}$ such that $\int g^{(\nu)}\mathrm{d}p_1\cdots\mathrm{d}p_n=1$, $g^{(\nu)}\geqq 0$; $g^{(\nu)}=0$ for $\sum p_l^2\geqq(1/\nu)$. Let $B(\sigma_1\cdots\sigma_n)$ be the region in which $\sigma_l q_l>0$ for all l, where $\sigma_l=\pm 1$. Assume that

$$\mu_k(\sigma_1\cdots\sigma_n) = \lim_{\nu}\int_{B(\sigma_1\cdots\sigma_n)} g^{(\nu)}\mathrm{d}p_1\cdots\mathrm{d}p_n\,. \tag{3.35}$$

Then the limit of (3.30) is

$$\sum_{\sigma_1\cdots\sigma_n}\mu_k(\sigma_1\cdots\sigma_n)(\Omega_{\varphi},\, Q_{k+1}U^{+}_{\sigma_{k+1}}(\zeta_{k+1}\,;\,0)Q_{k+2}\cdots \\ \cdots Q_nU^{+}_{\sigma_n}(\zeta_n\,;\,0)Q_0U^{+}_{\sigma_1}(\zeta_1\,;\,0)Q_1\cdots \\ \cdots U^{+}_{\sigma_k}(\zeta_{k-1}\,;\,0)Q_k\Omega_{\varphi}) \tag{3.36}$$

where $\mu\geqq 0$ and $\sum\mu(\sigma_1\cdots\sigma_n)=1$.

In obtaining (3.36) we have used the fact that $U^{+}(\zeta\,;\,q)-U^{+}_{\sigma}(\zeta\,;\,0)$ strongly tends to zero as $q\to 0$ with $\sigma q>0$, and that $\|U^{+}(\zeta\,;\,q)\|$ is bounded uniformly in q in the neighbourhood of 0.

Since

$$U_{\sigma}^{+}(\zeta\,;\,0)=\int_{\sigma 0}^{+} e^{i\lambda\zeta}\,\mathrm{d}E(\lambda) \tag{3.37}$$

is bounded and continuous for $\mathrm{Im}\,\zeta_l\geqq 0$ (and holomorphic for $\mathrm{Im}\,\zeta_l>0$), (3.36) is bounded and continuous in $\mathrm{Im}\,\zeta_l\geqq 0$.

Corollary 3.4. F is given by

$$\mathrm{F}(\zeta_1\cdots\zeta_n)=\sum \mathrm{F}_k(\zeta_1\cdots\zeta_n) \tag{3.38}$$

$$\begin{aligned}\mathrm{F}_k(\zeta_1\cdots\zeta_n)=\sum_{\sigma_1\cdots\sigma_n}\mu_k(\sigma_1\cdots\sigma_n)(\Omega_\varphi,\ \pi_\varphi(B_{k+1})U_{\sigma_{k+1}}^{+}(\zeta_{k+2}-\zeta_{k+1})\cdots\\ \cdots U_{\sigma_{n-1}}^{+}(\zeta_n-\zeta_{n-1})\pi_\varphi(B_n)U_{\sigma_n}^{+}(i\beta-\zeta_n)\pi_\varphi(A)U_{\sigma_1}^{+}(\eta_1)\\ \pi_\varphi(B_1)U_{\sigma_2}^{+}(\zeta_2-\zeta_1)\cdots U_{\sigma_k}^{+}(\zeta_k-\zeta_{k-1})\pi_\varphi(B_k)\Omega_\varphi)\,,\end{aligned} \tag{3.39}$$

where

$$U_{\sigma}^{+}(\zeta)=\int_{\sigma 0}^{\infty} e^{i\zeta\lambda}\,\mathrm{d}E(\lambda) \tag{3.40}$$

and $\mu_k(\sigma_1\cdots\sigma_n)$ is the volume of those part of the ball $p_1^2+\cdots+p_n^2=K$ of the unit volume which is defined by $\sigma_l q_l^{(k)}>0$, $q_l^{(k)}=p_l+\cdots+p_k$ for $l\leqq k$, $q_l^{(k)}=-(p_{k-1}+\cdots+p_l)$ for $l>k$.

Proof. This follows from (3.36) where we take as $g^{(\nu)}$ a function obtained by smoothly cutting off tails of

$$C(\nu)\exp -\nu\sum_{l=1}^{n}p_l^2\,.$$

Remark 3.5. If we insert a formal expression

$$U_{?}^{+}(\zeta)=\frac{1}{2\pi i}\int U_\varphi(t)\frac{\mathrm{d}t}{t-\zeta}\qquad(\mathrm{Im}\,\zeta>0)$$

into (3.38) and (3.39), we obtain an unsubtracted form of the Bergman Weil formula.

Remark 3.6. As a special case of $n=1$, we obtain

$$F(\zeta)=(\Omega_\varphi,\ Q_bU^{+}(i\beta-\zeta)Q_a\Omega_\varphi)+(\Omega_\varphi,\ Q_aU^{+}(\zeta)Q_b\Omega_\varphi) \tag{3.42}$$

where

$$U^{+}(\zeta)=\int_{+0}^{\infty} e^{i\zeta\lambda}\,\mathrm{d}E(\lambda)+\frac{1}{2}E(\{0\})\,. \tag{3.43}$$

By setting $\zeta=0$, we have

$$(\Omega_\varphi, Q_b U^+(i\beta) Q_a \Omega_\varphi) = (\Omega_\varphi, Q_a(1-U^+(0))Q_b\Omega_\varphi) . \tag{3.44}$$

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