Publ. RIMS, Kyoto Univ. Ser. A
Vol. 4 (1968), pp. 387-412

On the Diagonalization of a Bilinear Hamiltonian by a Bogoliubov Transformation

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Abstract

Under the condition that a certain hermitian operator has a self-adjoint extension a necessary and sufficient condition that a bilinear Fermion Hamiltonian can be diagonalized by a Bogoliubov transformation is obtained. Under the same assumption, any bilinear Fermion Hamiltonian can be diagonalized in a slightly extended sense by an extended Bogoliubov transformation. The meaning of this diagonalization from the view point of the Clifford C^* algebra is discussed. It is shown that a parallel treatment is possible for a bilinear Boson Hamiltonian (with complications concerning unbounded operators and an indefinite metric) if a spectral theory of pseudo hermitian operator on a Hilbert space of an indefinite metric hold in parallel with that of definite metric Hilbert space.

§ 1. Introduction

Several authors have investigated the diagonalization of a general bilinear Hamiltonian by a Bogoliubov transformation [1], [2], [3], [7]. We shall present a complete solution of this problem for the case of canonical anticommutation relations (the Fermion case). We shall indicate a similar procedure for the Bose case, which is however quite incomplete due to the lack of a spectral theory of a pseudo hermitian operator on a Hilbert space of an indefinite metric.

In section 2, we shall discuss various view point on the Clifford algebra, which was the motivation for our treatment, though this section is logically unnecessary for the later sections. In section 3, we formulate the notion of Bogoliubov transformation in an abstract

Received July 11, 1968.

fashion. In section 4, we introduce a bilinear Hamiltonian as a derivation of a Clifford algebra, which is an infinitesimal generator of a one parameter group of automorphisms of the Clifford algebra if a certain operator has a selfadjoint extension. Then the problem of diagonalization is reduced to the problem of finding a projection operator satisfying a few properties and this problem is easily solved by a spectral theory of a selfadjoint operator. In section 5, the abstract language in preceding two sections are written out in the conventional notation and the main theorems are stated as Theorem 5.4 and Theorem 5.6.

In passing, it is shown that any automorphism defined by the bilinear Hamiltonian has an invariant state in which the canonically defined Hamiltonian is positive semidefinite. It is also pointed out that an infinite dimensional Clifford algebra is * isomorphic to C^* algebra obtained by adjoining evenoddness operator to the Clifford algebra.

In section 6, we indicate how a parallel treatment can be done for the Bose case to the extent that a Hilbert space of an indefinite metric can be treated in parallel with a Hilbert space of a definite metric.

§ 2. Alternative Definitions of the Clifford Algebra

We shall here collect various view points for Clifford algebra, of which we shall use one in later discussions.

A standard definition of the canonical anticommutation relations [4] is

Definition 2.1. Let K be a complex Hilbert space. A CAR algebra over K, denoted by $\mathfrak{A}_{\mathrm{CAR}}(K)$ is the quotient of free * algebra with complex coefficients, generated by symbols (a^*, f), (f, a) $(f \in K)$ and the identity $\mathbf{1}$, by (the two sided * algebra generated by) the following relations

$$(f, a)^* = (a^*, f)\,, \tag{2.1}$$

$$[(a^*, f),\ (a^*, g)]_+ = [(f, a),\ (g, a)]_+ = 0 \tag{2.2}$$

(2. 3) $$[(a^*, f), (g, a)]_+ = (g, f)\mathbf{1}$$

(2. 4) $$(a^*, c_1f_1+c_2f_2) = c_1(a^*, f_1)+c_2(a^*, f_2)$$

where

(2. 5) $$[A, B]_+ = AB+AB$$

A standard definition of the Clifford algebra [5] is

Definition 2. 2. Let H be a real Hilbert space. A Clifford algebra over H, denoted by $\mathfrak{A}_{\mathrm{CLI}}(H)$ is the quotient of the free * algebra with the complex coefficients, generated by the symbol $\phi(f)$ $(f\in H)$ and the identity $\mathbf{1}$, by the following relations

(2. 6) $$\phi(f)^* = \phi(f)$$

(2. 7) $$\phi(f)^2 = (f, f)\mathbf{1}$$

(2. 8) $$\phi(c_1f_1+c_2f_2) = c_1\phi(f_1)+c_2\phi(f_2)$$

where c_1 and c_2 in (2. 8) are now reals.

The two definitions are related by

Lemma 2. 3. Given K and $\mathfrak{A}_{\mathrm{CAR}}(K)$. Equip K with a real inner product

(2. 9) $$(f, g)_H = \mathrm{Re}\,(f, g)_K\,,$$

making K a real Hilbert space, which we denote by H. Then the mapping π_{21} defined by

(2. 10) $$\pi_{21}(a^*, f) = \frac{1}{2}(\phi(f)-i\phi(\beta f))$$

(2. 11) $$\pi_{21}(f, a) = \frac{1}{2}(\phi(f)+i\phi(\beta f))$$

(2. 12) $$\pi_{21}\mathbf{1} = \mathbf{1}$$

generates *a* * isomorphism of $\mathfrak{A}_{\mathrm{CAR}}(K)$ onto $\mathfrak{A}_{\mathrm{CLI}}(H)$. Conversely, let H and $\mathfrak{A}_{\mathrm{CLI}}(H)$ be given. Further, given an operator β in H such that

(2. 13) $$\beta^2 = -1\,, \quad \beta^* = -\beta\,.$$

Introduce a complex inner product into H by

(2.14) $$(f,g)_K = (f, g)_H - i(f, \beta g)_H .$$

This makes H a complex Hilbert space, which we denote by K.

Then the mapping π_{12} defined by

(2.15) $$\pi_{12}\phi(f) = (a^*, f) + (f, a)$$

(2.16) $$\pi_{12}\mathbf{1} = \mathbf{1}$$

generates a * isomorphism of $\mathfrak{A}_{\mathrm{CLI}}(H)$ onto $\mathfrak{A}_{\mathrm{CAR}}(K)$. π_{12} and π_{21} are the inverse of each other if β on H happens to coincide with the multiplication of i on K.

Proof. To show that π_{21} is a homomorphism, it is enough to prove that the images of the relations (2.1)—(2.4) are contained in the two sided * ideal generated by the relations (2.6)~(2.8). To show that π_{12} is a homomorphism, it is enough to prove that (2.1) ~(2.4) imply (2.6)~(2.8). To show that π_{21} is an onto isomorphism, construct H as indicated, define the operator β on H by $\beta(f) = (if)$, consider π_{12} for this H and β and prove that $\pi_{12}\pi_{21}$ and $\pi_{21}\pi_{12}$ are the identity mapping. This also shows that π_{12} is an onto isomorphism. The verification of these statements are straight-forward, among which we only mention

(2.17) $$[\pi_{12}\phi(f)]^2 = (a^*, f)^2 + (f, a)^2 + [(a^*, f), (f, a)]_+ = (f, f)\mathbf{1}$$

(2.18) $$[\phi(f), \phi(g)]_+ = \frac{1}{2}\{\phi(f+g)^2 - \phi(f-g)^2\} = 2(f, g)_H\mathbf{1}$$

(2.19) $$(f, \beta g) + (g, \beta f) = 0 \quad (\beta f, \beta g) = (f, g)$$

(2.20) $$\begin{aligned}&[\pi_{21}(a^*, f), \pi_{21}(g, a)]\\ &= \frac{1}{2}[(f, g) + (\beta f, \beta g) + i(f, \beta g) - i(\beta f, g)]\mathbf{1}\\ &= [(g, f) - i(g, \beta f)]\mathbf{1} = (g, f)_K\mathbf{1} .\end{aligned}$$

Q.E.D.

The CAR algebra and Clifford algebra can be defined even if we make K and H a Hilbert space with an indefinite metric.

Definition 2.4. Let K be a complex Hilbert space and γ be a linear operator on K such that

(2. 21) $$\gamma^2 = 1\,, \quad \gamma^* = \gamma\,.$$

An indefinite CAR algebra $\mathfrak{A}_{\mathrm{ICA}}(K, \gamma)$ is the quotient of the free * algebra, generated by the symbols (b^*, f), (f, b) $(f \in K)$ and the identity $\mathbf{1}$, by the following relations

(2. 22) $$(b^*, f)^* = (\gamma f, b)$$

(2. 23) $$[(b^*, f), (g, b)]_+ = (g, \gamma f)\mathbf{1}$$

(2. 24) $$(b^*, c_1 f_1 + c_2 f_2) = c_1(b^*, f_1) + c_2(b^*, f_2)\,.$$

Definition 2. 5. Let H be a real Hilbert space, γ be a linear operator on H such that

(2. 35) $$\gamma^2 = 1\,, \quad \gamma^* = \gamma\,.$$

An indefinite Clifford albebra $\mathfrak{A}_{\mathrm{ICL}}(H, \gamma)$ is the quotient of the free * algebra, generated by the symbol $\psi(f)$ and the identity, by the following relations

(2. 26) $$\psi(f)^* = \psi(\gamma f)$$

(2. 27) $$\psi(f)^2 = (f, \gamma f)\mathbf{1}$$

(2. 28) $$\psi(c_1 f_1 + c_2 f_2) = c_1\psi(f_1) + c_2\psi(f_2)$$

where c_1 and c_2 are reals.

Lemma 2. 6. Let E be any projection operator in H and $\gamma = (2E-1)$. Then the mapping π_{42} defined by

(2. 29) $$\pi_{42}\phi(f) \equiv \psi(Ef) + i\psi((1-E)f)$$

(2. 30) $$\pi_{42}\mathbf{1} = \mathbf{1}$$

generates *a* * isomorphism of $\mathfrak{A}_{\mathrm{CLI}}(H)$ onto $\mathfrak{A}_{\mathrm{ICL}}(H, \gamma)$. Conversely, given H, γ,

(2. 31) $$\pi_{24}\psi(f) = \phi([1+\gamma]f/2) - i\phi([1-\gamma]f/2)$$

(2. 32) $$\pi_{24}\mathbf{1} = \mathbf{1}$$

generate *a* * isomorphism of $\mathfrak{A}_{\mathrm{ICL}}(H, \gamma)$ onto $\mathfrak{A}_{\mathrm{CLI}}(H)$. The two mappings are inverse of each other.

Proof. The same as Lemma 2. 3, except for the calculational part, which is straightforward.

Lemma 2. 7. Let E be any projection on K and $\gamma=2E-1$. Then

$$\pi_{31}(a^*, f) = (a^*, Ef) - i(a^*, (1-E)f) \tag{2. 33}$$

$$\pi_{31}(f, a) = (Ef, a) + i((1-E)f, a) \tag{2. 34}$$

$$\pi_{31}\mathbf{1} = \mathbf{1} \tag{2. 35}$$

generate *a* * isomorphism of $\mathfrak{A}_{\mathrm{CAR}}(K)$ onto $\mathfrak{A}_{\mathrm{ICA}}(K, \gamma)$. (Converse mapping π_{13} can be defined similarly.)

Proof. The same as Lemma 2. 6.

Lemma 2. 8. If there exists β which satisfies (2. 13) and commutes with γ, then

$$\pi_{34}\psi(f) = (b^*, f) + (f, b) \tag{2. 36}$$

$$\pi_{34}\mathbf{1} = \mathbf{1} \tag{2. 37}$$

generates *a* * isomorphism of $\mathfrak{A}_{\mathrm{ICL}}(H, \gamma)$ onto $\mathfrak{A}_{\mathrm{ICA}}(K, \gamma)$ where K is related to (H, β) as in Lemma 2. 3. If (K, γ) is given first, then define (H, β) from (K, i) as in Lemma 2. 3, and there always exists *a* * isomorphism π_{43} of $\mathfrak{A}_{\mathrm{ICA}}(K, \gamma)$ onto $\mathfrak{A}_{\mathrm{ICL}}(H, \gamma)$ generated by

$$\pi_{43}(b^*, f) = \frac{1}{2}(\psi(f) - i\psi(\beta f)) \tag{2. 38}$$

$$\pi_{43}(f, b) = \frac{1}{2}(\psi(f) + i\psi(\beta f)) \tag{2. 39}$$

$$\pi_{43}\mathbf{1} = \mathbf{1} \tag{2. 40}$$

Given γ, the required β exists if and only if dimensions of the projections $(1+\gamma)/2$ and $(1-\gamma)/2$ are even.

Proof. The same as Lemma 2. 6.

§ 3. Bogoliubov Transformations

We now introduce a new view on CAR algebra which is a natural frame for the study of Bogoliubov transformations.

Definition 3. 1. Let K be a complex Hilbert space and Γ be an antiunitary operator satisfying

$$\Gamma^2 = 1\,, \quad \Gamma i = -i\Gamma \tag{3.1}$$

$$(\Gamma\psi_1, \Gamma\psi_2) = (\psi_2, \psi_1)\,.$$

A selfdual CAR algebra $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ is the quotient of the free * algebra, generated by (B^*, f), (f, B) $(f \in K)$and $\mathbf{1}$, by the following relations

$$(B^*, f)^* = (f, B) \tag{3.2}$$

$$[(B^*, f), (g, B)]_+ = (g, f)\mathbf{1} \tag{3.3}$$

$$(B^*, c_1 f_1 + c_2 f_2) = c_1(B^*, f_1) + c_2(B^*, f_2) \tag{3.4}$$

$$(B^*, h) = (\Gamma h, B)\,. \tag{3.5}$$

Remark 3. 2. The last relation for $\mathfrak{A}_{\mathrm{SDC}}$ replaces (2. 2) for $\mathfrak{A}_{\mathrm{CAR}}$. A useful relation is

$$(B^*, f)^2 = \frac{1}{2}(f, \Gamma f)\mathbf{1}\,. \tag{3.6}$$

Lemma 3. 3. If $\dim K =$ even or infinite, there exists a projection operator E such that $\Gamma E \Gamma = 1 - E$, and the following π_{15} generates *a* * isomorphism of $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ onto $\mathfrak{A}_{\mathrm{CAR}}(EK)$

$$\pi_{15}(B^*, f) = (a^*, Ef) + (\Gamma(1-E)f, a) \tag{3.7}$$

$$\pi_{15}(f, B) = (Ef, a) + (a^*, \Gamma(1-E)f) \tag{3.8}$$

$$\pi_{15}\mathbf{1} = \mathbf{1}\,. \tag{3.9}$$

Conversely given K_0, define $K = K_0 \oplus K_0$ $EK = K_0$. Choose a complex conjugation operator T on K_0 (namely any (antiunitary) operator satisfying $T^2 = 1$, $Ti = -iT$, $(Tf, Tg) = (g, f)$), and define $\Gamma(f \oplus g) = (Tg \oplus Tf)$. Then the mapping π_{51} defined by

$$\pi_{51}(a^*, f) = (B^*, f \oplus 0) \quad (= (0 \oplus Tf, B)) \tag{3.10}$$

$$\pi_{51}(f, a) = (B^*, 0 \oplus Tf) \quad (= (f \oplus 0, B)) \tag{3.11}$$

$$\pi_{51}\mathbf{1} = \mathbf{1} \tag{3.12}$$

generates *a* * isomorphism of $\mathfrak{A}_{\mathrm{CAR}}(K_0)$ onto $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ which is the inverse of π_{15}. The space K has either even or infinite dimension.

Proof. The same as Lemma 2. 6. We only mention the following calculations :

$$(3.13)\quad [\pi_{15}(B^*, f), \pi_{15}(g, B)] = \{(Eg, Ef)+(\Gamma(1-E)f, \Gamma(1-E)g)\}\mathbf{1} = \{(g, Ef)+(g, (1-E)f)\}\mathbf{1} = (g, f)\mathbf{1}$$

$$(3.14)\quad \pi_{15}(B^*, \Gamma f) = (a^*, E\Gamma f)+(\Gamma(1-E)\Gamma f, a) = (a^*, \Gamma(1-E)f)+(Ef, a) = \pi_{15}(f, B)\,.$$

The existence of the desired projection E for given Γ can be seen as follows. Since Γ is a complex conjugation, there exists a Γ invariant basis $\{f_n, n=1, 2,\cdots\}$ of K. ($f+\Gamma f$ and $i(f-\Gamma(f))$ are both Γ invariant and one of them is nonzero if $f\neq 0$. Thus one can choose successively an orthonormal Γ invariant basis vector f_k from the Γ invariant subspace $(f_1, f_2\cdots f_{k-1})^{\perp}$ of K.) Since K is even dimensional, we can pair f_{2n-1} and f_{2n}, $n=1, 2,\cdots$. Now define E as a projection on the subspace spanned by $2^{-1/2}(f_{2n-1}+if_{2n})$. It satisfies the required property $\Gamma E\Gamma=(1-E)$.

Remark 3.4. We may write $B^*=(a^*, a)$, $B=\begin{pmatrix}a\\a^*\end{pmatrix}$, where (a, g) is understood as (Tg, a) and (g, a^*) is understood as (a^*, Tg). When (K, Γ) is given, the operator T can be chosen to be any complex conjugation on EK. T on $(1-E)K$ is defined as $\Gamma T\Gamma(1-E)$. Then $T\Gamma=\gamma$ is a linear operator satisfying $\gamma^2=1$, $\gamma^*=\gamma$ and $(1-E)\gamma=\gamma E$, and the identification of $(1-E)K$ with EK is done by the unitary mapping γ. It is also possible to start from an arbitrary unitary mapping γ (identification) of EK onto $(1-E)K$, satisfying $\gamma\Gamma=\Gamma\gamma^*$. Then $T=\gamma\Gamma$ is a complex conjugation.

Definition 3.5. A projection E satisfying $\Gamma E\Gamma=1-E$ is called a basis projection of K. A unitary operator U is called a Bogoliubov transformation between two basis projections E and F if U commutes with Γ and $UEU^{-1}=F$.

Lemma 3.6. For any two basis projections E and F, there exists a Bogoliubov transformation between them. If a unitary operator U commutes with Γ and if E is a basis projection, then UEU^{-1} is also a basis projection.

Proof. Let $f_1f_2\cdots$ and $g_1g_2\cdots$ be a complete orthonormal basis for EK and FK respectively. Since $\dim EK = \dim FK = \frac{1}{2} \dim K$, we can use the same index set for f and g. Since Γ is antiunitary, Γf_i and Γg_i are complete orthonormal bases for $(1-E)K$ and $(1-F)K$ respectively. We define

$$U \sum_i (c_i f_i + d_i \Gamma f_i) = \sum_i (c_i g_i + d_i \Gamma g_i) .$$

Then U is unitary, commutes with Γ and $UEU^{-1} = F$.
The last half of the lemma follows from

$$\Gamma UEU^{-1}\Gamma = U\Gamma E\Gamma U^{-1} = U(1-E)U^{-1} = 1 - UEU^{-1} . \tag{3.15}$$

Lemma 3.7. If $\dim K$ is finite and odd, there exist mutually commuting projections E_1, E_2 and E_0 such that $\Gamma E_1 \Gamma = E_2$, $\Gamma E_0 \Gamma = E_0$, $E_1 + E_2 + E_0 = 1$, $\dim E_0 K = 1$. $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ is * isomorphic to a direct product of $\mathfrak{A}_{\mathrm{SDC}}((E_1+E_2)K, \Gamma)$ and a two dimensional abelian algebra $\{c_1 \mathbf{1} + c_2 x\}$ $x^2 = \mathbf{1}$, $x^* = x$. $\mathfrak{A}_{\mathrm{SDC}}((E_1+E_2)K, \Gamma)$ is * isomorphic to $\mathfrak{A}_{\mathrm{CAR}}(E_1 K)$.

Proof. There exists a Γ invariant orthonormal basis $f_1 \cdots f_{2n+1}$ of K where $\dim K = 2n+1$. Now we define $E_1 K$, $E_2 K$ and $E_0 K$ to be subspaces spanned by

$$\{f_1 + if_2, f_3 + if_4, \cdots f_{2n-1} + if_{2n}\}, \ \{f_1 - if_2, f_3 - if_4, \cdots f_{2n-1} - if_{2n}\}, \ \{f_{2n+1}\}.$$

Then the required properties hold. If we set

$$x = \sqrt{2}\,(2i)^n (B^*, f_1)(B^*, f_2) \cdots (B^*, f_{2n+1}) \tag{3.16}$$

then it commutes with all elements in $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ and $x^2 = \mathbf{1}$, $x^* = x$. Obviously, $\{c_1 \mathbf{1} + c_2 x\}$ and $\mathfrak{A}_{\mathrm{SDC}}((E_1+E_2)K, \Gamma)$ generate $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$. The last statement of the lemma follows from Lemma 3.3.

§ 4. Bilinear Hamiltonian

We now define a bilinear Hamiltonian as a generator of a one parameter automorphism group of CAR algebra. We need a C^* algebra view point for this purpose.

Lemma 4.1. If K has a finite even dimension, then all non-zero representation of $\mathfrak{A}_{\mathrm{CAR}}(K)$ is * isomorphic and defines a unique

C^* norm on $\mathfrak{A}_{\mathrm{CAR}}(K)$. If K is infinite dimensional, the C^* norm of $\mathfrak{A}_{\mathrm{CAR}}(K') \subset \mathfrak{A}_{\mathrm{CAR}}(K)$, dim K'=finite and even, defines a unique C^* norm on $\mathfrak{A}_{\mathrm{CAR}}(K)$. The completion $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K)$ of $\mathfrak{A}_{\mathrm{CAR}}(K)$ with respect to this norm is a C^* algebra.

Proof. Known.

Lemma 4.2. If S is a selfadjoint operator on K, the mapping $\tau(tS)$ defined by

$$\tau(tS)(a^*, f) = (a^*, e^{iSt} f) \tag{4.1}$$

$$\tau(tS)(f, a) = (e^{iSt} f, a) \tag{4.2}$$

generates a * automorphism of $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K)$, continuous in t. The infinitesimal generator $i^{-1}\mathrm{d}\tau(tS)/\mathrm{d}t \equiv \mathrm{d}\tau(S)$ is a densely defined derivation on $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K)$. In particular if f is in the domain of S.

$$\mathrm{d}\tau(S)(a^*, f) = (a^*, Sf) \tag{4.3}$$

$$\mathrm{d}\tau(S)(f, a) = -(Sf, a)\,. \tag{4.4}$$

Proof. Known.

Remark 4.3. The derivation $\mathrm{d}\tau(S)$ is often denoted by

$$\mathrm{d}\tau(S)A = [(a^*, Sa), A]\,. \tag{4.5}$$

This is due to the following situation. Let S be a trace class operator and

$$(x, Sy) = \sum \lambda_i (x, f_i)(f_i, y)\,. \tag{4.6}$$

Then $\mathrm{d}\tau(S)$ is an inner derivation and

$$(a^*, Sa) \equiv \sum \lambda_i (a^* f_i)(f_i a) \in \bar{\mathfrak{A}}_{\mathrm{CAR}}(K) \tag{4.7}$$

satisfies (4.5). By extending this notation, $\tau(tS)A$ is often written as

$$\tau(tS)A = e^{i(a^*, Sa)} \quad A\, e^{-i(a^*, Sa)} \tag{4.8}$$

though $e^{i(a^*, Sa)}$ is not an element of algebra for a general selfadjoint S.

It is also possible to write

(4.9) $$d\tau(S)A = -[(a, S^*a^*), A]$$

(4.10) $$\tau(tS)A = e^{-i(a,S^*a^*)} \quad A\, e^{i(a,S^*a^*)}$$

If S is in the trace class, and $(x, Sy) = \sum \lambda_i(x, f_i)(g_i, y)$, then

(4.11) $$(a, S^*a^*) = \sum \lambda_i(g_i, a)(a^*, f_i) \in \mathfrak{A}_{\mathrm{CAR}}(K)$$

(4.12) $$= (\mathrm{tr}\, S)\mathbf{1} - (a^*, Sa)\,.$$

Even if S is not in the trace class, it is conventional to say that $(a, S^*a^*) = -(a^*, Sa) +$constant and that the constant cancels out in (4.9) and (4.10). This language is made rigorous in the present discussion by using the notion of automorphisms and derivations of a C^* algebra.

We now consider a similar derivation on $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$.

Lemma 4.4. Let S be a trace class operator on K such that

(4.13) $$(x, Sy) = \sum \lambda_i(x, f_i)(g_i, y), \quad \sum |\lambda_i| \, ||f_i|| \, ||g_i|| < \infty\,.$$

Then

(4.14) $$(B^*, SB) \equiv \sum \lambda_i(B^* f_i)(g_i B) \in \mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$$

and

(4.15) $$\frac{1}{2}[(B^*, SB), (B^*, f)] = (B^*, \alpha(S)f)$$

(4.16) $$\alpha(S) = \frac{1}{2}(S - \Gamma S^* \Gamma)\,.$$

We have

(4.17) $$\Gamma \alpha(S) \Gamma = -\alpha(S)^*\,.$$

If $\Gamma S \Gamma = -S^*$, then $\alpha(S) = S$.

Proof. For $A \equiv (B^*, f)$, we have $A^*A + AA^* = ||f||^2 \mathbf{1}$ and hence $||A|| \leq 1$. Therefore

$$|| \sum_{i=n}^{N} \lambda_i(B^*, f_i)(g_i B) || \leq \sum_{i=n}^{N} |\lambda_i| \, ||f_i|| \, ||g_i||$$

and (4.14) converges in norm. (4.15) follows from

(4.18)
$$\frac{1}{2}[(B^*, SB), (B^*f)]$$
$$= \frac{1}{2}\sum \lambda_i\{(B^*, f_i)[(g_i, B), (B^*, f)]_+$$
$$-[(B^*, f_i), (B^*, f)]_+(g_i, B)\}$$
$$= (B^*, f')$$

where (B^*, f) in the second term is to be replaced by $(\Gamma f, B)$ and

(4.19)
$$f' = \frac{1}{2}\sum_i \lambda_i\{(g_i, f)f_i - (\Gamma f, f_i)\Gamma g_i\},$$
$$= \frac{1}{2}\sum_i \{\lambda_i(g_i, f)f_i - \Gamma[\lambda_i^*(f_i, \Gamma f)g_i]\}$$
$$= \alpha(S)f.$$

Lemma 4.5. There exists a maximal norm of all * representation of $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ by operators on a Hilbert space. The completion $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ of $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ by this norm is a C^* algebra.

Proof. If K has an infinite dimension, this follows from Lemma 3.6 and Lemma 4.1. If K has a finite dimension, the $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ has a finite dimension by Lemma 3.6 and by the known fact on $\mathfrak{A}_{\mathrm{CAR}}(K)$, the lemma holds for this case, too.

Lemma 4.6. Let S be a selfadjoint operator satisfying $\Gamma S\Gamma = -S$. Then the mapping

(4.20)
$$\tau(tS)(B^*, f) = (B^*, e^{iSt}f)$$

(4.21)
$$\tau(tS)\mathbf{1} = \mathbf{1}$$

generates a * automorphism of $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$, continuous in t.

Proof. Let

$$S = \int_{-\infty}^{\infty} \lambda dE(\lambda)$$

be the spectral decomposition of S. Then $\Gamma S\Gamma = -S$ implies $\Gamma E(\Delta)\Gamma = E(-\Delta)$. Sicne Γ is conjugate linear, we have

(4.22)
$$\Gamma e^{iSt}\Gamma = \int e^{-i\lambda t}\Gamma \mathrm{d}E(\lambda)\Gamma$$
$$= \int e^{-i\lambda t}\mathrm{d}E(-\lambda) = e^{iSt}.$$

Namely e^{iSt} commutes with Γ.

Since the quantity entering in the definition of $\mathfrak{A}_{\mathrm{SDC}}$ is an inner product (f, g) in K and the mapping $f \rightarrow \Gamma f$, and since e^{iSt} induces an isomorphism of (K, Γ) onto $(e^{iSt}K, \Gamma)=(K, \Gamma)$ with respect to this structure, we see that (4.20) and (4.21) induce a * isomorphism of $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ onto $\mathfrak{A}_{\mathrm{SDC}}(e^{iSt}K, \Gamma)=\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$. Thus it is an automorphism of $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ and hence an automorphism of its unique C^* extention $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$.

Definition 4.7. Let S be any linear operator on K. Then we use the notation on the left hand side of the following equation to denote the right hand side if f is in the domain of $\alpha(S)$.

$$\left[\frac{1}{2}(B^*, SB),\ (B^*, f)\right] = (B^*, \alpha(S)f)\,. \tag{4.23}$$

The notation is extended as a derivation on * algebra generated by (B^*, f), f in the domain of $\alpha(S)$.

If S is selfadjoint, the automorphism $\tau(t\alpha(S))$ is denoted by the following expression:

$$\tau(t\alpha(S))A = e^{i(B^*,SB)/2}\, A\, e^{-i(B^*,SB)/2}\,. \tag{4.24}$$

The symbol (B^*, SB) is called a bilinear Hamiltonian. It is said hermitian or selfadjoint if S is hermitian or self-adjoint, respectively.

Lemma 4.4 motivates this definition.

Lemma 4.8. Let S be a selfadjoint operator on K such that $\Gamma S\Gamma=-S$ with a spectral projections $E(\cdot)$ and let $E_+=E((0, \infty))$, $E_-=E((-\infty, 0))$, $E_0=E(\{0\})$.

If $\dim E_0K$ is even or infinite, then there exist a projection E on K, a selfadjoint operator S_0 on EK and a * isomorphism π from $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ onto $\bar{\mathfrak{A}}_{\mathrm{CAR}}(EK)$ such that $K=EK\oplus\Gamma EK$ and $\pi\tau(tS)=\tau(tS_0)\pi$.

Proof. From $\Gamma S\Gamma=-S$, we obtain $\Gamma E(\Delta)\Gamma=E(-\Delta)$ and hence $\Gamma E_+\Gamma=E_-$, $\Gamma E_0\Gamma=E_0$. Since $\dim E_0K$ is even or infinite, and since E_0K is Γ invariant, we can find a subspace $E_{01}K$ of E_0K such that

$\Gamma E_{01}\Gamma = E_0 - E_{01}$ as in the last part of the proof of Lemma 3.3. Set $E = E_+ + E_{01}$. $S_0 = SE$. By construction, $\Gamma E \Gamma = 1 - E$ and $[S, E] = 0$. We have

$$e^{itS} f = e^{itS} Ef + \Gamma e^{itS} \Gamma(1-E)f = e^{itS_0} Ef + \Gamma e^{itS_0} \Gamma(1-E)f \tag{4.25}$$

and e^{itS_0} is a one parameter unitary group on EK. We now use the mapping π_{15} from $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ onto $\mathfrak{A}_{\mathrm{CAR}}(EK)$ defined in Lemma 3.3. It can be extended to a * isomorphism of $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ onto $\bar{\mathfrak{A}}_{\mathrm{CAR}}(EK)$. (4.25) now shows that

$$\pi_{15}\tau(tS) = \tau(tS_0)\pi_{15} .$$

Q.E.D.

Lemma 4.9. Let $\mathfrak{A}_2$ be a * algebra consisting of $c_1\mathbf{1} + c_2x$ where $x^2 = \mathbf{1}$, $x^* = x$. The following semitensor product $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K) \otimes \mathfrak{A}_2$ of $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K)$ with $\mathfrak{A}_2$ defines a C^* algebra.

$$(c_1A_1 + c_2A_2) \otimes B = c_1(A_1 \otimes B) + c_2(A_2 \otimes B) \tag{4.26}$$

$$A \otimes (c_1B_1 + c_2B_2) = c_1(A \otimes B_1) + c_2(A \otimes B_2) \tag{4.27}$$

$$(cA) \otimes B = A \otimes (cB) = c(A \otimes B) \tag{4.28}$$

$$(A_1 \otimes x)(A_2 \otimes B) = \{A_1(\tau(-1)A_2)\} \otimes xB \tag{4.29}$$

$$(A_1 \otimes 1)(A_2 \otimes B) = A_1A_2 \otimes B . \tag{4.30}$$

There exists a * isomorphism of $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K) \otimes \mathfrak{A}_2$ onto $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K_0 \oplus K_0 \oplus K', \Gamma)$ where K' is one dimensional, $\Gamma(f \oplus g \oplus c) = (Tg \oplus Tf \oplus c^*)$, T is any complex conjugation on K_0 (i.e. $(Tf, Tg) = (g, f)$, $T^2 = 1$), and $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K_0)$ is mapped onto the subalgebra $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K_0 \oplus K_0 \oplus 0, \Gamma)$ of $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K_0 \oplus K_0 \oplus K', \Gamma)$.

Proof. The mapping π given by

$$\pi(B^*, f \oplus g \oplus c) = (B^*, f) + (Tg, B) + cx \tag{4.31}$$

$$\pi\mathbf{1} = \mathbf{1} \tag{4.32}$$

generaters a * isomorphism of $\mathfrak{A}_{\mathrm{SDC}}(K_0 \oplus K_0 \oplus K', \Gamma)$ onto $\mathfrak{A}_{\mathrm{CAR}}(K_0) \otimes \mathfrak{A}_2$, as is easily proved in the same way as Lemma 2.3. (The inverse map is

$$\pi^{-1}(a^*, f) = (B^*, f \oplus 0 \oplus 0), \ \pi^{-1}x = (B^*, 0 \oplus 0 \oplus c), \text{ etc.})$$

Therefore π can be extended to a unique C^* closure $\bar{\mathfrak{A}}_{\mathrm{SDC}}$. Its image is then the C^* closure of $\mathfrak{A}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$. Since $\mathfrak{A}_2$ is finite dimensional, it must be $\bar{\mathfrak{A}}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$.

Lemma 4.10. If $\dim E_0K$ is finite and odd in Lemma 4.8, then there exist projections E and F, a selfadjoint operator S_0 on EK and a * isomorphism π from $\bar{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ onto $\bar{\mathfrak{A}}_{\mathrm{CAR}}(EK)\otimes\mathfrak{A}_2$ such that $\dim FK=1$, $K=EK\oplus\Gamma EK\oplus FK$ and $\pi\tau(tS)=(\tau(tS_0)\otimes 1)\pi$.

Proof. In the proof of Lemma 4.8, we now have $\dim E_0K=$ odd. Hence we can find three mutually orthogonal subprojections E_{01}, E_{02} and F of E_0 such that $E_{01}+E_{02}+F=E_0$ and $\Gamma E_{01}\Gamma=E_{02}$, $\Gamma F\Gamma=F$, as in Lemma 3.7. We set $E=E_++E_{01}$ and $S_0=SE$. Then $\Gamma E\Gamma=E_-+E_{02}$, $E+\Gamma E\Gamma+F=1$, $[S, E]=0$, $SF=0$, and

$$e^{itS}f = e^{itS_0}Ef+\Gamma e^{itS_0}\Gamma(\Gamma E\Gamma)f+Ff\,. \tag{4.33}$$

The mapping is given as in Lemma 4.9 where $K_0=EK$, T is an arbitrary complex conjugation on EK, and the identification of $EK=K_0$ and $(1-E)K$ is done by the mapping $T\Gamma=\gamma$. Then the required properties are satisfied.

§ 5. Diagonalization of a Hermitian Bilinear Hamiltonian

Definition 5.1. A projection operator E diagonalizes a bilinear Hamiltonian $\frac{1}{2}(B^*, SB)$ if $\Gamma E\Gamma=1-E$ and $(1-E)SE=ES(1-E)=0$. E diagonalizes $\frac{1}{2}(B^*, SB)$ in an extended sense if $E\Gamma E=0$ and $S=ESE+(\Gamma E\Gamma)S(\Gamma E\Gamma)$.

Remark 5.2. Motivation of this definition is as follows. If and only if $\Gamma E\Gamma=1-E$, we have a * isomorphism of $\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$ onto $\mathfrak{A}_{\mathrm{CAR}}(EK)$, by Lemma 3. By this isomorphism, B^* and B are identified with (a^*, a) and $\begin{pmatrix} a \\ a^* \end{pmatrix}$ (Remark 3.4). Hence (B^*, SB) can be written in a matrix form

$$(a^*, a)\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}\begin{pmatrix} a \\ a^* \end{pmatrix} \tag{5.1}$$

where $S_{ij}=E_iSE_j$ with $E_1=E$, $E_2=1-E$. (B^*SB) is said diagonal in conventional terminology if it can be written as

$$(a^*S_{11}a)+(a,\ S_{22}a^*)\ . \tag{5.2}$$

This condition is expressed by $S_{12}=S_{21}=0$ namely $ES(1-E)=(1-E)SE=0$. This motivates the first definition. In the second case $K=EK\oplus\Gamma EK\oplus(1-E-\Gamma E\Gamma)K$. B^* and B are identified with (a^*, a, ϕ) and $\begin{pmatrix} a \\ a^* \\ \phi \end{pmatrix}$ where a^* and a part satisfies CARs whereas ϕ anticommutes with a, a^* and defines a certain $\mathfrak{A}_{\mathrm{SDC}}$ by itself. The stated condition then says that the (B^*, SB) is of the form

$$(a^*,\ a,\ \phi)\begin{pmatrix} S_{11} & 0 & 0 \\ 0 & S_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ a^* \\ \phi \end{pmatrix}. \tag{5.3}$$

To the extent that we forget about ϕ, it is again of the desired form (5.2).

If the dimension of $(1-E-\Gamma E\Gamma)K$ is even or infinite, then we can divide and absorb ϕ into a^* and a. Thus it is reduced to the first case. If the dimension is odd then we can make ϕ one dimensional. If we are allowed to add one more ϕ' anticommuting with a, a^* and ϕ, then the entire system is reduced to the first case.

As we have seen in Lemma 4.4, the derivation depends only on $\alpha(S)=(S-\Gamma S^*\Gamma)/2$ and hence we do not lose generality by assuming $\Gamma S^*\Gamma=-S$. Since $\Gamma=\begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$ for a complex conjugation T on EK in the matrix representation of (5.4), we have the requirement

$$-\begin{bmatrix} TS_{22}^*T & TS_{12}^*T \\ TS_{21}^*T & TS_{11}^*T \end{bmatrix}=\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}. \tag{5.4}$$

It is customary to write $TA^*T={}^tA$ and $TAT=\bar{A}$. Then we have the requirement

$$\begin{aligned} {}^tS_{12} &= -S_{12}\,, \quad {}^tS_{21} = -S_{21} \\ {}^tS_{11}+S_{22} &= 0 \end{aligned} \tag{5.5}$$

where

$$(B^*SB) = (a^*, S_{11}a) + (a, S_{22}a^*) + (a^*, S_{12}a^*) + (a, S_{21}a)\,. \tag{5.6}$$

We note that in ref. 2, the following matrix was considered instead of our S.

$$\begin{bmatrix} S_{21} & S_{22} \\ S_{11} & S_{12} \end{bmatrix}. \tag{5.7}$$

The hermiticity for S gives the requirement

$$S_{11}^* = S_{11}\,, \quad S_{22}^* = S_{22} \quad S_{12}^* = S_{21}\,. \tag{5.8}$$

Hence S can be written in terms of two operator $S_{11} \equiv R_1$ and $S_{12} = R_2$ as

$$\begin{bmatrix} R_1 & R_2 \\ -\bar{R}_2 & -\bar{R}_1 \end{bmatrix} \tag{5.9}$$

where R_1, R_2 satisfies

$$R_1^* = R_1\,, \quad {}^tR_2 = -R_2\,. \tag{5.10}$$

It is also customary to use the notation

$$(a^*, f) = \int a^*(x) f(x) dx \tag{5.11}$$

$$(f, a) = \int Tf(x) a(x) dx \tag{5.12}$$

$$(f, Sg) = \int Tf(x) S(x, y) g(y) dx dy \tag{5.13}$$

where $a^*(x)$, $a(x)$, $S(x, y)$, $R(x, y)$ can be taken in distribution sense or x can be taken as a discrete index variable, in which case $\int dx$ is replaced by a summation. With this notation, we may write

$$(a^*, Ra) = \int a^*(x) R(x, y) a(y) dx dy \tag{5.14}$$

and similar expressions for (a^*, Ra^*) and (a, Ra).

Remark 5.3. *Diagonalization problem.* If we start with $\mathfrak{A}_{\mathrm{CAR}}(K_0)$ and $\frac{1}{2}(B^*SB)$ in the form of (5.6), we introduce $K = K_0 \oplus K_0$

and $\Gamma=\begin{pmatrix}0 & T\\ T & 0\end{pmatrix}$ using the complex conjugation T which is used to define the matrix element $S(x, y)$ in (5.13). Then the projection operator F onto K_0 is a basis projection for (K, Γ), which however does not diagonalize the given (B^*, SB). We then look for another basis projection E which diagonalizes (B^*, SB). If E is found, then we denote the annihilation and creation operators in $\mathfrak{A}_{\mathrm{CAR}}(EK)$ by b and b^* instead of a and a^*, which are already used for $\mathfrak{A}_{\mathrm{CAR}}(K_0)$. Then we have

$$\frac{1}{2}(B^*SB) = \frac{1}{2}\{(b^*, Rb)-(b, {}^tRb^*)\} = (b^*, Rb) \tag{5.14}$$

where $R=R^*$ and last equality is in the sense described before.

The pair of b^* and b are related to a^* and a by a Bogoliubov transformation U, as was paoved in Lemma 3.6. The requirement that U is unitary is equivalent to the information that the mapping is one to one onto and the canonical anticommutation relations hold for b^* and b. The requirement that U commutes with Γ is equivalent to the information that the expression for b^* and b are adjoint of each other. Thus our definition 3.5 for a Bogoliubov transformation coincides with the ordinary definition.

From these two remarks and Lemma 4.8, we obtain

Theorem 5.4. Given the hermitian bilinear Hamiltonian

$$(a^*, a)\begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}\begin{pmatrix}a\\ a^*\end{pmatrix} \tag{5.15}$$

and assume that

$$\alpha(S) = \frac{1}{2}(S-\Gamma S\Gamma) \tag{5.16}$$

has a selfadjoint extension, where

$$S = \begin{bmatrix}S_{11} & S_{12}\\ S_{21} & S_{22}\end{bmatrix}, \qquad \Gamma = \begin{bmatrix}0 & T\\ T & 0\end{bmatrix} \tag{5.17}$$

and T is a complex conjugation operator. (The hermiticity of (5.15) means $S^*\supset S$.)

Then it can be diagonalized to the form

$$(b^*, Rb) \tag{5.18}$$

by a Bogoliubov tronsformation if and only if the dimension of the eigenspace belonging to 0 is either infinite or even.

Proof. The sufficiency is already proved. To see the necessity, we note that a basis projection E which diagonalizes (5.15) must satisfy $\Gamma E\Gamma = 1-E$ and $(1-E)\alpha(S)E = E\alpha(S)(1-E) = 0$. The last requirement implies $[E, \alpha(S)] = 0$ and hence E must commute with all spectral projections of $\alpha(S)$. From the property $\Gamma\alpha(S)\Gamma = -\alpha(S)$, and $\Gamma E\Gamma = 1-E$, it follows that $\Gamma EE(0)\Gamma = -(1-E)E(0)$, where $E(0)$ is the projection to the 0 eigensubspace of $\alpha(S)$. Therefore $\dim E(0)K$ must have either even or infinite dimension.

Definition 5.5. Let E be a basis projection, F be a projection such that $F(\Gamma F\Gamma) = (\Gamma F\Gamma)F = 0$. An isometric operator U commuting with Γ is called an extended Bogoliubov transformation if $UEU^* = F$. Given a bilinear Hamiltonian $\frac{1}{2}(B^*, SB)$ in the form (5.15) for $\mathfrak{A}_{\mathrm{CAR}}(K_0)$. An isometric operator U on $K_0 \oplus K_0$ diagonalizes $\frac{1}{2}(B^*, SB)$ if it is an extended Bogoliubov transformation from the projection onto K_0 to a projection F which diagonalizes $\frac{1}{2}(B^*, SB)$ in an extended sense (cf. Definition 5.1).

Theorem 5.6. Given the bilinear Hamiltonian (5.15) for which $\alpha(S)$ has a selfadjoint extension. Then it can always be diagonalized to a form of (5.18) by an extended Bogoliubov transformation.

Proof. This follows from Lemma 4.10.

We now add a few results concerning the significance of the diagonalization from C^* algebra point of view.

Theorem 5.7. Let φ be the Fock vacuum state for $\mathfrak{A}_{\mathrm{CAR}}(EK)$ and S be a selfadjoint operator on K such that $\Gamma S\Gamma = -S$. Then φ is invariant under $\tau(tS)$ if and only if $[E, S] = 0$, namely if and only if E diagonalizes S.

Proof. For the Fock vacuum state φ, we have

$$\varphi((f, B)(B^*, f)) = \|Ef\|^2. \tag{5.19}$$

If φ is $\tau(tS)$ invariant, we must have

$$e^{-itS} E e^{itS} = E. \tag{5.20}$$

Hence $[E, S]=0$, namely E diagonalizes S. Converse is known.

Remark 5.8. Thus the diagonalization of $\frac{1}{2}(B^*, SB)$ is possible if and only if there exists "a" Fock vacuum state which is invariant under $\tau(t\alpha(S))$.

Definition 5.9. Let φ be a Fock vacuum state of $\mathfrak{A}_{\mathrm{CAR}}(K_0)$. Let N be the number operator on H_φ. Then the representation $\hat{\pi}_\varphi$ of $\mathfrak{A}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$ generated by

$$\hat{\pi}_\varphi(A) = \pi_\varphi(A) \qquad \text{if} \quad A\in\mathfrak{A}_{\mathrm{CAR}}(K_0) \tag{5.21}$$

$$\hat{\pi}_\varphi(x) = (-1)^N \tag{5.22}$$

is called a pseudo Fock representation of $\mathfrak{A}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$, and the vector state defined by Ω_φ in this representation is called a pseudo Fock state.

Theorem 5.10. For any selfadjoint S on K, there exists either Fock or pseudo Fock state which is invariant under $\tau(tS)$.

Proof. This follows from Theorem 5.7 and Lemma 4.10.

Q.E.D.

Theorem 5.11. Let S be a selfadjoint operator on K, then there exists a state φ of $\mathfrak{A}_{\mathrm{SDC}}(K)$, invariant under $\tau(t\alpha(S))$ such that the operator H defined by

$$e^{iHt}\pi_\varphi(A)\Omega_\varphi = \pi_\varphi(\tau(t\alpha(S))A)\Omega_\varphi \tag{5.23}$$

is positive semidefinite.

Proof. We can reduce the problem, by the foregoing results, to the case where $\mathfrak{A}_{\mathrm{SDC}}(K)$ is identified with either $\mathfrak{A}_{\mathrm{DAR}}(K_0)$ or $\mathfrak{A}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$ and $\tau(t\alpha(S))$ is $\tau(tS_0)$ for a selfadjoint operator on K_0. Let E_+ and E_- be two projections such that $E_+ + E_- = 1$, $E_+S_0 \geqq 0$,

$E_{-}S_0 \leqq 0$. (It is essentially the spectral projection of S_0 for the positive and negative real axis, where 0 eigenvalue space is divided into E_{+} and E_{-} in an arbitrary manner.) Then

$$\varphi((E_{-}f, a)(a^*, E_{-}f)) = 0 \tag{5.24}$$

$$\varphi((a^*, E_{+}f)(E^{+}f, a)) = 0 \tag{5.25}$$

defines the state uniquely which is Fock for E_{+} and anti Fock for E_{-}. The associated state $\hat{\varphi}$ for $\mathfrak{A}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$ is constructed as before. Then it is known that φ and $\hat{\varphi}$ satisfies the required property for $\tau(tS_0)$. Q.E.D.

In passing, we mention the following remarkable fact. In the Fock representation, the evenoddness operator $(-1)^N$ is not contained in $\pi(\mathfrak{A}_{\mathrm{CAR}}(K_0))$ if K_0 is infinite, because any operator A in $\pi(\mathfrak{A}_{\mathrm{CAR}}(K_0))$ which commute with $(-1)^N$ satisfies

$$\lim_i \|[A, (a^*, f_i)]\| = 0 \tag{5.26}$$

for an orthonormal f_i, whereas $(-1)^N$ does not have this property. Thus if we adjoin $(-1)^N$ to $\pi(\mathfrak{A}_{\mathrm{CAR}}(K_0))$, they generate a C^* algebra $\mathfrak{B}$ which is larger than $\pi(\mathfrak{A}_{\mathrm{CAR}}(K_0))$. Since $\pi(\mathfrak{A}_{\mathrm{CAR}}(K_0))$ is a faithful representation of $\mathfrak{A}_{\mathrm{CAR}}(K_0)$ (the latter being simple), $\mathfrak{B}$ is a faithful representation of $\mathfrak{A}_{\mathrm{CAR}}(K_0)\otimes\mathfrak{A}_2$, which is isomorphic to $\mathfrak{A}_{\mathrm{CAR}}(K_0)$ by Lemma 4.9 and Lemma 3.3 if $\dim K_0$ is infinite. Thus $\mathfrak{A}_{\mathrm{CAR}}(K_0)$ is isomorphic to $\mathfrak{A}_{\mathrm{CAR}}(K_0)$ "plus" evenoddness operator.

§ 6. The Bose Case

We now briefly describe a similar analysis for canonical commutation relations. We omit the analysis in § 2 for this case a part of which is found in [6]. Here we shall be content to describe a program without caring for a mathematical rigour. Hence we shall use unbounded form using creation and annihilation operators. We shall also use unbounded operators without considering the domain questions.

We start from (a^*, f) and (g, a) which satisfies

$$(6.1)\qquad [(g, a), (a^*, f)]_- = (g, f)\mathbf{1}$$

$$(6.2)\qquad [(g, a), (f, a)]_- = [(a^*, g), (a^*, f)]_- = 0$$

where $[A, B]_- = AB - BA$ and $f \in K_0$ for a complex Hilbert space K_0. (a^*, f) is linear in f and $(f, a) = (a^*, f)^*$. We consider $K = K_0 \oplus K_0$, a projection operator E on $K_0 \oplus 0$, a complex conjugation T on K, $\Gamma = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix}$ and define

$$(6.3)\qquad (B^*, f) = (a^*, Ef) + (\Gamma(1-E)f, a)$$

$$(6.4)\qquad (f, B) = (Ef, a) + (a^*, \Gamma(1-E)f)$$

which satisfies

$$(6.5)\qquad [(g, B), (B^*, f)] = (g, \gamma f)1$$

$$(6.6)\qquad (B^*, f)^* = (f, B)$$

$$(6.7)\qquad (B^*, f) = (\Gamma f, B)$$

where $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2E - 1$ and we have used $\Gamma E \Gamma = 1 - E$. (B^*, f) is linear in f.

If F is a linear operator on K such that

$$(6.8)\qquad \Gamma F \Gamma = 1 - F, \quad F^2 = F, \quad \gamma F = F^* \gamma$$

$$(6.9)\qquad (f, \gamma f) > 0 \quad \text{for} \quad 0 \neq f \in FK$$

then we can obtain a new creation and annihilation operators b^* and b appropriate for F in the following manner. Let K_F be FK equipped with a new inner product

$$(6.10)\qquad (g, f)_{K_F} = (g, \gamma F f)(= (g, \gamma f))$$

(If F happens to be the original E, this coincides with (g, f)).
For $f \in K_F$, we define

$$(6.11)\qquad (b^*, f) = (B^*, f), \quad (f, b) = (f, B).$$

Then (6.1) and (6.2) for b^*, b can be checked by using (6.8) and (6.6). (Note that $F^* \gamma F = \gamma F$, $F^* \gamma (1-F) = (1-F)^* \gamma F = 0$.) Further, if we define B^* and B out of this b^* and b, we obtain the original one, because $(B^*, f) = (B^*, Ff) + (B^*, (1-F)f) = (B^*, Ff) + (\Gamma(1-F)$

f, B). Thus we call any F satisfying (6.8) and (6.9) as a basis (nonorthogonal) projection.

Let the completion of FK with respect to K_F norm be $\bar{K}_F$. Then the dimension of $\bar{K}_F$ and $K_0 = EK$ must be the same and there exists a unitary mapping U_0 of K_0 outo $\bar{K}_F$, which is densely defined operator from K_0 onto K_F. By construction, U_0 satisfies $U_0^* \gamma U_0 E = E$, $U_0 U_0^* = \gamma F^*$, $FU_0 E = U_0 E$, $EU_0^* F^* = U_0^* F^* = U_0^*$ where U_0^* is defined by $(f, U_0 g) = (U_0^* f, g)$ together with $U_0^* f \in EK$ for all $g \in EK$. We define $U = U_0 E + \Gamma U_0 E \Gamma$. Then U satisfies

$$\begin{aligned} &[\Gamma, U] = 0, \quad U^* \gamma U = \gamma, \quad U \gamma U^* = \gamma \\ &UE = FU. \end{aligned} \tag{6.12}$$

Conversely, if U is an operator satisfying (6.12), then $F \equiv UEU^* \gamma$ has the properties (6.8) and (6.9). Thus we shall call the operator U as the Bogoliubov transformation from a basis E to a basis F.

We now consider a bilinear form

$$(B^*, SB) = (a^*, a) \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{pmatrix} a \\ a^* \end{pmatrix} \tag{6.14}$$

which can be understood as describing the following derivation

$$\begin{aligned} &\frac{1}{2}[(B^*, SB), (B^*, f)] = (B^*, \alpha'(S) \gamma f) \\ &\qquad\qquad \alpha'(S) = \frac{1}{2}(S + \Gamma S^* \Gamma). \end{aligned} \tag{6.15}$$

It is then enough to consider those S satisfying $S = \alpha'(S)$ or

$$\Gamma S \Gamma = S^*. \tag{6.16}$$

The hermiticity requirement is

$$S^* = S. \tag{6.17}$$

In terms of S_{ij}, we have

$$S = \begin{bmatrix} R_1 & R_2 \\ \bar{R}_2 & \bar{R}_1 \end{bmatrix}, \quad R_1^* = R_1, \quad {}^t R_2 = R_2. \tag{6.18}$$

A basis projection F diagonalizes S if the transformation (6.15) brings a^* and a to b^* and b associated with F, respectively. Namely

(6.19) $$S\gamma = FS\gamma F + (1-F)S\gamma(1-F)\,.$$

For this, it is necessary and sufficient that

(6.20) $$FS\gamma(1-F) + (1-F)S\gamma F = 0$$

namely

(6.21) $$[F, S\gamma] = 0\,.$$

If F diagonalizes S, then in terms of b, b^* associated with F, we have

(6.22) $$\frac{1}{2}(B^*, SB) = (b^* S_0 b)\,, \quad S_0 = FS\,.$$

The operator $S_0\gamma$ satisfies the property

(6.23) $$(S_0\gamma)^*(\gamma F) = (\gamma F)(S_0\gamma)\,.$$

Namely it is hermitian relative to the inner product of the space $\bar{K}_F$. If $S_0\gamma$ has a selfadjoint extension, then we have a unitary operator $e^{iS_0\gamma t}$ on $\bar{K}_F$. We define $\tau(tS)$ by

(6.24) $$\tau(tS)(b^*, f) = (b^*, e^{iS_0\gamma t} f)$$

(6.25) $$\tau(tS)(f, b) = (e^{iS_0\gamma t} f, b)$$

which induces an "automorphism" of "CCR algebra". It is often written as

(6.26) $$\tau(tS)A = e^{i(B^*, SB)/2}\, A\, e^{-i(B^*, SB)/2}$$

of which the derivation $\frac{1}{2}(B^*, SB)$ is an infinitesimal generator $i^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\tau(tS)$.

As a result of the above formulation, the problem of the diagonalization of a bilinear Hamiltonian by a Bogoliubov transformation for the Bose case is reduced to the following:
Given S satisfying

(6.27) $$S^* = S\,, \quad \Gamma S \Gamma = S$$

where Γ is a complex conjugation. Find an operator F such that

(6.28) $$\Gamma F\Gamma = 1-F, \quad F^2 = F, \quad \gamma F^*\gamma = F, \quad \gamma F>0$$

(6.29) $$[F, S\gamma] = 0$$

where

(6.30) $$\gamma^2 = 1, \quad \gamma^* = \gamma, \quad \Gamma\gamma = -\gamma\Gamma.$$

This problem can be solved if we have the following type of spectral theory of pseudo hermitian operator on a Hilbert space of indefinite metric. Let γ and a complex conjugation Γ be given satisfying (6.30), on a Hilbert space of a definite metric. An operator H is called pseudohermitian with respect to the indefinite metric $(f, g)_\gamma \equiv (f, \gamma g)$ if

(6.31) $$(f, Hg)_\gamma = (Hf, g)_\gamma.$$

An operator E is called a pseudoprojection if it is pseudohermitian and

(6.32) $$E^2 = E.$$

We say that a pseudo spectral theory holds for pseudohermitian H if there exists a mutually commuting pseudoprojection valued measure $E(\Delta)$ on real line such that $E((-\infty, \infty))=1$ and

(6.33) $$H = \int \lambda \mathrm{d}E(\lambda).$$

To apply these notions to the problem under investigation, we note that $S\gamma$ is pseudohermitian. If a pseudo spectral theory holds for $S\gamma$, then we consider three spectral projections.

(6.34) $$E_+ = E((0, \infty)), \quad E_- = E((-\infty, 0)), \quad E_0 = E(\{0\}).$$

Since $\Gamma(S\gamma)\Gamma = -S\gamma$ we have

(6.35) $$\Gamma E_+\Gamma = E_-, \quad \Gamma E_0\Gamma = E_0.$$

We now want to take into account the condition (6.9) for a basis pseudoprojection and construct the desired E. This again hinges on the spectral theory which we do not have at the moment. We shall treat only the case where the spectrum of $S\gamma$ is discrete and the multiplicity is finite. Let E_λ be an eigenprojection belonging to $\lambda \neq 0$. If $f \in E_\lambda H$ satisfies $(g, \gamma f)=0$ for all $g \in E_\lambda H$, then setting $g=E_\lambda \psi$, $\psi \in H$, we have $(\psi, E\gamma f)=0$, namely $0=E_\lambda{}^* \gamma f=\gamma E_\lambda f=\gamma f$. Hence $f=\gamma(\gamma f)=0$. Thus γ restricted to $E_\lambda H$ is nonsingular and we can find a basis $f_1 \cdots f_n$ in $E_\lambda H$ such that $(f_j, \gamma f_k)=\varepsilon_j \delta_{jk}$, $\varepsilon_j = \pm 1$. Let $E_\lambda^\pm$ be defined by $E_\lambda^\pm f_j=\frac{1}{2}(1 \pm \varepsilon_j) f_j$. Then it satisfies $\gamma E^\pm=(E^\pm)^* \gamma$ and $\gamma E>0$. If $E_\lambda^\pm$ is chosen for a γ, then we choose $E_{-\lambda}^\pm=\Gamma E_\lambda^\mp \Gamma$. Because $\Gamma \gamma \Gamma=-\gamma$, this $E_{-\lambda}^\pm$ have the required property. For $\lambda=0$, we have $\Gamma E_0 \Gamma=E_0$. Hence $f \in E_0 H$ implies $\Gamma f \in E_0 H$. Now it is always possible to find a Γ invariant basis f_j of $E_0 H$: $\Gamma f_j=f_j$. From $\Gamma \gamma \Gamma=-\gamma$, it follows $(f_j \gamma f_j)=0$, and $(f_j \gamma f_k)$ is pure imaginary. Since γ is nonsingular, it is always possible to find an f_2 for given f_1 such that $(f_2 \gamma f_1) \neq 0$. Let σ be the sign of $\operatorname{Im}(f_2 \gamma f_1)$. Then $g_1^\pm=f_1 \pm i\sigma f_2$ satisfies $(g_1^+ \gamma g_1^-)=0$, $(g_1^+ \gamma g_1^+)>0$, $\Gamma g_1^+ = g_1^-$. We then modify the rest of the basis f_j to $f_j'=f_j-(g_1^+, \gamma f_j) g_1^+(g_1^+, \gamma g_1)^{-1}-(g_1^- \gamma f_j) g_1^-(g_1^- \gamma g_1^-)^{-1}$. We can apply the same procedure to $f_3' f_4' \cdots$. Proceeding successively in this way, we can exhaust $E_0 H$ and we obtain a basis $g_j^\pm$ which satisfies $(g_j^\sigma \gamma g_k^\tau)=\delta_{jk} \delta_{\sigma\tau} \sigma$, $\Gamma g_j^+=g_j^-$. We then define E_0^σ by $E_0^\sigma g_j^\tau=\delta_{\sigma\tau} g_j^\tau$. $F=\sum_\lambda E_\lambda^+$ satisfies the required property: $[F, H]=0$, $\gamma F=F^* \gamma$, $F^2=F$, $\gamma F>0$.

Because of the existence of Γ, $\dim E_0 H$ is always even in the present case.

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