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Wave Operators for $-\Delta$ in a Domain with Non-Finite Boundary

Dedicated to Professor Atuo Komatu in honor of his 60th birthday

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§1. Introduction

Let $\Omega\subset\boldsymbol{R}^n$, $n\geqslant 2$ be a domain (open connected set) exterior to obstacles such that the obstacles, not necessarily finite in number, form a closed set enclosed in a *cylinder* $S_{r_0}=\{x=(x_1,\cdots,x_n)=(\tilde{x}, x_n)\in\boldsymbol{R}^n: |x|\leqslant r_0, r_0>0\}$. The complement of S_{r_0} is, therefore, contained in Ω. We consider the differential operator $-\Delta$ on $C_0^\infty(\Omega)$,[1] which will be denoted by A. It is easy to see that A is a well-defined, non-negative definite operator in the Hilbert space $L_2(\Omega)$, so that it has at least one self-adjoint extension. Let H be *any* such extension.[2] We are to compare H with the operator H_0 in $L_2(\boldsymbol{R}^n)$ defined as follows: $D(H_0)$[3] $=\{u\in L_2(\boldsymbol{R}^n): |\xi|^2\hat{u}(\xi)\in L_2(\boldsymbol{R}^n)\}$, $(H_0u)^\wedge(\xi)=|\xi|^2\hat{u}(\xi)$ for $u\in D(H_0)$, where $\hat{u}$ denotes the Fourier transform of u, i.e.,

$$\hat{u}(\xi)=(2\pi)^{-n/2}\,\text{l.i.m.}\int e^{-i\xi\cdot x}u(x)dx.\text{[4]} \tag{1.1}$$

H_0 is also known to be the unique self-adjoint extension of the negative Laplacian defined on $C_0^\infty(\boldsymbol{R}^n)$.

Let J be the bounded linear map: $L_2(\boldsymbol{R}^n)\to L_2(\Omega)$ defined by

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1) $C_0^\infty(\Omega)$ is the set of all infinitely differentiable functions with compact support in Ω.
2) We may say that different boundary conditions give rise to different H.
3) $D(A)$ denotes the domain of A.
4) l.i.m. $\int\cdots dx$ = limit in the mean for $R\to\infty$ of $\int_{|x|\leqslant R}\cdots dx$.

(1.2) $$(Ju)(x) = u(x), \qquad x \in \Omega.$$

Then the wave operator $W_{\pm} = _{\pm}(H, H_0; J)$ for the pair (H, H_0) and the *identification operator* J is defined to be the strong limit

(1.3) $$W_{\pm}(H, H_0; J) = s-\lim_{t\to\pm\infty} e^{itH} J e^{-itH_0}$$ [5]

if it exists. Now we assert the following

Theorem. *The wave operators $W_{\pm}$ exist and are isometries.*

The existence of the isometric wave operators $W_{\pm}$ implies that there is a subspace M in $L_2(\Omega)$ reducing H such that the part of H in M is unitarily equivalent with H_0 (see Kato [2]). Consequently, the absolutely continuous spectrum of *any* self-adjoint extension of A is never empty and contains at least $[0, \infty)$, since H_0 is known to have the absolutely continuous spectrum $[0, \infty)$. This property is thus independent of whatever (homogeneous) boundary condition may be attached to $-\Delta$ in Ω.[6]

In closing this Introduction we mention that the existence and some related properties of the wave operators have been obtained for a bounded (set of) obstacle(s) (see, e.g., Ikebe [1], Lax-Phillips [3] and Shenk [4]).

§ 2. A Decay Principle

If $\varphi(x)$ is a (measurable) function defined on R^n or Ω, let us denote by φ the operator of multiplication by $\varphi(x)$.

Lemma 2.1. *Let $\varphi(x)$ be a bounded function on $\boldsymbol{R}^n$ such taat* supp (φ)[7] $\subset S_r$ *for an $r > 0$. Then for any $u \in L_2(\boldsymbol{R}^n)$ we have*

(2.1) $$\|\varphi e^{-itH_0} u\|_{L_2(R^n)} \to 0 \qquad (t \to \pm\infty).$$ [8]

Proof. In order to show (2.1) it is sufficient to prove that (2.1) holds for u in a fundamental set D, since the operator norm of

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5) Cf. Kato [2].
6) See footnote 2).
7) supp(f)=support of $f(x)$.
8) The norm of a Hilbert space X is designated by $\|\ \|_X$.

$\varphi \exp(-itH_0)$ is uniformly bounded in t. Let D be the totality of such functions u that $u(x)=f(\tilde{x})g(x_n)$ with $f\in C_0^\infty(\boldsymbol{R}^{n-1})$ and $g\in C_0^\infty(\boldsymbol{R}^1)$. For $u=f\cdot g\in D$, we have

$$(e^{-itH_0}u)^\wedge(\xi) = e^{-it|\xi|^2}\hat{f}(\tilde{\xi})\hat{g}(\xi_n), \tag{2.2}$$

which implies

$$e^{-itH_0}u(x) = (2\pi)^{-n/2}\int_{R^{n-1}} e^{i\tilde{x}\cdot\tilde{\xi}-it|\tilde{\xi}|^2}\hat{f}(\tilde{\xi})d\tilde{\xi}\times \times\int_{R_1} e^{ix_n\xi_n-it|\xi_n|^2}\hat{g}(\xi_n)d\xi_n. \tag{2.3}$$

Fixing $\tilde{x}$ and integrating with respect to x_n we get

$$\int_{R_1}|\varphi e^{-itH_0}u(\tilde{x}, x_n)|^2dx_n \leqslant \text{const.}\,\tilde{\varphi}(\tilde{x})\|g\|^2_{L_2(R^1)}F(\tilde{x}, t), \tag{2.4}$$

where $\tilde{\varphi}(\tilde{x})=\sup\{|\varphi(\tilde{x}, x_n)|: x_n\in R^1\}$ and

$$F(\tilde{x}, t) = \left|\int_{R^{n-1}} e^{i\tilde{x}\cdot\tilde{\xi}-it|\tilde{\xi}|^2}\hat{f}(\tilde{\xi})d\tilde{\xi}\right|^2. \tag{2.5}$$

Consequently, noting that $\tilde{\varphi}(\tilde{x})$ is bounded with compact support in $\boldsymbol{R}^{n-1}$, we obtain

$$\|\varphi e^{-itH_0}u\|^2_{L_2(R^n)} \leqslant \text{const.}\int_{\text{supp}(\tilde{\varphi})} F(\tilde{x}, t)d\tilde{x}. \tag{2.6}$$

By the Riemann-Lebesgue lemma $F(\tilde{x}, t)$ tends to 0 as $|t|$ goes to infinity, and this convergence is uniform in $\tilde{x}\in\text{supp}(\tilde{\varphi})$. Hence we have the right side of (2.6) tending to 0 in view of the bounded convergence theorem. Q. E. D.

§ 3. Proof of the Theorem

We shall consider W_+ alone, for W_- can be handled quite similarly.

Let $\eta(x)$ be a smooth function on $\boldsymbol{R}^n$ satisfying the following conditions: $0\leqslant\eta(x)\leqslant 1$; $\eta(x)=1$ in a neighborhood of the boundary of Ω; $\text{supp}(\eta)\subset S_r$ for a sufficiently large r. Put $\zeta(x)=1-\eta(x)$. Then $W(t)=\exp(-itH)J\exp(-itH_0)$ can be written

$$W(t) = W_1(t)+W_2(t) \tag{3.1}$$

with

$$W_1(t) = e^{itH}J\eta e^{-itH_0}, \quad W_2(t) = e^{itH}J\zeta e^{-itH_0}. \tag{3.2}$$

Since we have

$$||W_1(t)u||_{L_2(\Omega)} \leqslant ||\eta e^{-itH_0}u||_{L_2(\Omega)} \leqslant ||\eta e^{-itH_0}u||_{L_2(R^n)}, \tag{3.3}$$

it follows from Lemma 2.1 with $\varphi=\eta$ that for $u \in L_2(\boldsymbol{R}^n)$

$$||W_1(t)u||_{L_2(\Omega)} \to 0 \qquad (t \to \infty). \tag{3.4}$$

In order to show the strong convergence of $W_2(t)$, we first differentiate $W_2(t)u$, $u \in D(H_0)$, obtaining

$$dW_2(t)u/dt = ie^{itH}(HJ\zeta - J\zeta H_0)e^{-itH_0}u. \tag{3.5}$$

Now (3.5) makes sense. Indeed, $\exp(-itH_0)u \in D(H_0)$ and $\zeta(x)$ is smooth and bounded, and hence $\zeta \exp(-itH_0)u(x)$ is twice strongly differentiable. Since in addition $\zeta(x)$ vanishes identically near the boundary of Ω, the application of J to $\zeta \exp(-itH_0)u$ does not affect the differentiability, and thus $J\zeta \exp(-itH_0)u \ni D(H)$. On the other hand, $J\zeta H_0 \exp(-itH_0)u$ is meaningful, for J and ζ are bounded operators. Thus (3.5) holds for $u \in D(H_0)$. Now since

$$(HJ\zeta - J\zeta H_0)v = -2J(\text{grad}\,\zeta)\cdot(\text{grad}\,v) - J(\Delta\zeta)v \tag{3.6}$$

for $v(=\exp(-itH_0)u) \in D(H_0)$, we have on integrating (3.5)

$$W_2(t)u - J\zeta u = -2i\int_0^t e^{isH}J(\text{grad}\,\zeta)\cdot(\text{grad}\,e^{-isH_0}u)ds - \tag{3.7}$$
$$-i\int_0^t e^{isH}J(\Delta\zeta)e^{-isH_0}u\,ds.$$

If we can show that

$$\int_0^\infty ||(\text{grad}\,\zeta)\cdot(\text{grad}\,e^{-itH_0}u)||dt < \infty, \tag{3.8}$$

$$\int_0^\infty ||(\Delta\zeta)e^{-itH_0}u||dt < \infty \tag{3.9}$$

for u in a fundamental set $D \subset D(H_0)$, then the existence of the strong limit W_+ will be concluded in virtue of the uniform boundedness in t of the operator norm of $W_2(t)$, and of (3.4), (3.1).

As D we take all functions $u_a(x)$ for which

$$(3.10)\qquad \hat{u}_a(\xi) = (\prod_{i=1}^{n} \xi_i) \exp(-|\xi|^2 - i\xi\cdot a), \qquad a\in \boldsymbol{R}^n .$$

Obviously $D\subset D(H_0)$. That D is fundamental follows from a theorem of Wiener [5] in view of the fact that $u_a(x)=u(x-a)$, where $u(x)$ is a constant multiple of $\prod_{i=1}^{n} x_i \exp(-|x|^2/4)$ which is $\neq 0$ almost everywhere. If we put

$$(3.11)\qquad v(x, t;\ a) = \exp[-|x-a|^2/(4+4it)],$$

we have

$$(3.12)\qquad e^{-itH_0}u_a(x) = \text{const.}\,(1+it)^{-3n/2} \prod_{i=1}^{n} (x_i-a_i)v(x, t;\ a),$$

$$(3.13)\qquad (\text{grad}\, e^{-itH_0}u_a(x))_j = \text{const.}\,(1+it)^{-3n/2}[\prod_{i\neq j} (x_i-a_i)v(x, t;\ a) - (2+2it)^{-1}(x_j-a_j)v(x, t;\ a)].$$

A straightforward computation shows that

$$(3.14)\qquad |(\Delta\zeta)e^{-itH_0}u_a(x)| \leqslant \text{const.}\, |1+it|^{-3(n-1)/2}(\Delta\zeta)^{\sim}(\tilde{x})|\tilde{x}-\tilde{a}|^{n-1} \times |1+it|^{-3/2}|x_n-a_n|\,|v(x_n, t;\ a_n)|,$$ [9]

$$(3.15)\qquad |(\text{grad}\,\zeta)\cdot(\text{grad}\, e^{-itH_0}u_a)(x)| \leqslant \text{const.}\, |1+it|^{-3(n-1)/2}(\text{grad}\,\zeta)^{\sim}(\tilde{x})|\tilde{x}-\tilde{a}|^{n-2} \times |1+it|^{-3/2}(1+|1+it|^{-1}|\tilde{x}-\tilde{a}|)(|\tilde{x}-\tilde{a}| + |x_n-a_n|)|v(x_n, t;\ a_n)|.$$

Noting the inequality

$$(3.16)\qquad \int_{-\infty}^{\infty} |1+it|^{-2p}|x_n|^{2m}|v(x_n, t;\ a_n)|^2 dx_n \leqslant \text{const.}, \quad m\geqslant 0,$$

where $p\geqslant m+1/2$ and the constant is independent of t, we obtain (3.8) and (3.9) in view of the facts that $\text{supp}\,(\Delta\zeta)^{\sim}))$ and $\text{supp}\,((\text{grad}\,\zeta)^{\sim})$ are bounded in $\boldsymbol{R}^{n-1}$, and that we have the factor $|1+it|^{-3(n-1)/2}$ on the right-hand side of both (3.14) and (3.15). This completes the proof of the existence of W_+.

It remains to verify the isometry of W_+. Let $\chi_S(x)$ denote the characteristic function of S, and let CS be the complement of S. Then

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9) For the definition of the $\sim$ operation see just below (2.4).

$$(3.17)\qquad \|W(t)u\|^2_{L_2(\Omega)} = \|Je^{-itH_0}u\|^2_{L_2(\Omega)} = \|\chi_\Omega e^{-itH_0}u\|^2_{L_2}(\boldsymbol{R}^n)$$
$$= \|u\|^2_{L_2}(\boldsymbol{R}^n) - \|\chi_{C\Omega} e^{-itH_0}u\|^2_{L_2}(\boldsymbol{R}^n)\,.$$

The last term tends to 0 as $t\to\infty$ by Lemma 2.1, which proves the desired isometry. Q. E. D.

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