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Wave and Scattering Operators for Second-Order Elliptic Operators in $\boldsymbol{R}^3$

Dedicated to Professor Atuo Komatu in honor of his 60th birthday

By

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§1. Introduction

Our concern in this paper will be with the existence and completeness of wave operators intertwining the negative Laplacian $L_0=-\Delta$ and the second-order elliptic differential operator

$$L=\sum_{j,k=1}^{3}(D_j+b_j(x))\,a_{jk}(x)(D_k+b_k(x))+q(x) \tag{1.1}$$

in the three-dimensional Euclidean space $\boldsymbol{R}^3$, where $D_j=-i\partial/\partial x_j$. In a suitable sense and under appropriate conditions on the coefficient functions $a_{jk}(x)$, $b_j(x)$, and $q(x)$, L_0 and L may be regarded as self-adjoint operators defined in the Hilbert space L_2, square integrable functions on $\boldsymbol{R}^3$. The wave operators $W_\pm$ are the strong limits for $t\to\pm\infty$ of $\exp(itL)\exp(-itL_0)$, and then the scattering operator S is defined as $S=W_+^*W_-$ (* denotes the adjoint of an operator). The wave operator $W_\pm$ maps isometrically *into* the scattering states for L, in other words, the complement of the bound states, or more precisely the subspace of absolute continuity for L, but it does not necessarily map *onto* this subspace. If it does, $W_\pm$ is called complete. The completeness of $W_\mp$ implies that S is unitary, which physicists expect, or sometimes believe, to hold in most problems.

We shall prove that $W_\pm$ exist and are complete when L is asymptotically, that is, as $|x|$ tends to infinity, equal to L_0. The

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point of the proof is to set up conditions on the coefficients of L so that abstract criteria expounded in Kato's book [3] may become applicable. It should be noted here that in most cases the existence of the wave operators is proved under an assumption milder than one for their completeness (see [3] for examples of this situation, especially, for Schrödinger operators). This is also the case with our L_0 and L. This seems, at present, to be a weak point inherent with the time-dependent theory of scattering. However, a stationary approach to scattering theory, in which $W_{\pm}$ are constructed, roughly speaking, from the limit of the resolvent $R(z)$ of L for z tending to real spectral points, has been successful for the Schrödinger operator $-\Delta+q(x)$ (see, e.g., Ikebe [1], Kuroda [4], and Thoe [5]). Recently, Kuroda and Thoe's result has been extended to the Schrödinger operator with external magnetic field by Ushijima [6]. (In [4], [5] and [6] are considered the Schrödinger operators in n dimensions, while we have restricted ourselves in this paper to the case $n=3$.) Our result, however, differs from [1], [4], [5] and [6] in that $L-L_0$, the perturbation, is a second-order differential operator, whereas it is at most a first-order one in the latter works.

In §§2 and 3 some fundamental properties of L_0 and L will be mentioned. In §4 will be formulated conditions to guarantee the existence of the wave operators. Some stronger conditions will be proposed in §5 to assure their completeness. In the final §6 we shall try to improve in a certain direction the result of §5.

§2. The Operator L_0 Associated with $-\Delta$ in $\boldsymbol{R}^3$

In this brief section we collect some known properties of the differential operator $[L_0]u(x)=-\Delta u(x)$, $x\in\boldsymbol{R}^3$.

Consider first the operator $[L_0]_0$ defined on $C_0^\infty=C_0^\infty(\boldsymbol{R}^3)$ (infinitely many times differentiable functions with compact support) by $[L_0]_0u=[L_0]u$. $[L_0]_0$ is well-defined as a linear operator acting in the Hilbert space $L_2=L_2(\boldsymbol{R}^3)$, and, moreover, it is densely defined, symmetric and non-negative definite. It is also known that $[L_0]_0$ has a unique self-adjoint extension which is the closure $[L_0]_0^{\sim}$ of $[L_0]_0$, i.e., $[L_0]_0$ is essentially self-adjoint. We set $L_0=[L_0]_0^{\sim}$. A useful characterization of L_0 is provided in terms of Fourier transforms: The Fourier transform of an L_2-function $u(x)$ is given by

$$\hat{u}(\xi) = (2\pi)^{-3/2} \text{ l.i.m.} \int e^{-i\xi\cdot x} u(x)\, dx\,, \tag{2.1}$$

where l.i.m. $\int \cdots dx$ means that one first integrates over $|x| \leq R$ and then takes the limit in the mean for $R\to\infty$. The Fourier transformation is a unitary operator on L_2. We have

$$D(L_0) = \{u\in L_2 : |\xi|^2\hat{u}(\xi)\in L_2\}\,, \tag{2.2}$$

$$(L_0 \hat{u})(\xi) = |\xi|^2 \hat{u}(\xi) \qquad \text{for} \quad u\in D(L_0)\,, \tag{2.3}$$

where $D(L_0)$ denotes the domain of L_0. If z is in the resolvent set of L_0, that is, in our case, if z is in the complement of $[0, \infty)$, then the resolvent

$$R_0(z) = R(z\,;L_0) = (L_0 - z)^{-1} \tag{2.4}$$

of L_0 is a bounded integral operator

$$R_0(z)u(x) = \frac{1}{4\pi}\int \frac{e^{i\sqrt{z}|x-y|}}{|x-y|} u(y)\, dy\,, \tag{2.5}$$

$\sqrt{z}$ designating the branch of the square root of z with non-negative imaginary part. Here and in the sequel, an integral with integration domain unspecified is agreed to be extended over all of $\boldsymbol{R}^3$.

§ 3. The Operator L Associated with $\sum(D_j+b_j)a_{jk}(D_k+b_k)+q$

In this section we shall be concerned with the differential operator

$$[L]u(x) = \sum_{j,k=1}^{3} (D_j+b_j(x))\, a_{jk}(x)(D_k+b_k(x))\, u(x) + q(x)u(x)\,. \tag{3.1}$$

We assume that

(C. I) $[a_{jk}(x)]$ *is a real, positive-definite-matrix-valued smooth function*; $b_j(x)$ *are real, smooth functions*; $q(x)$ *is locally square integrable, i.e.* $q\in L_{2,loc}$.

Thus $[L]$ is an elliptic operator, and the operator $[L]_0$ with $D([L]_0)$

$=C_0^\infty$ is symmetric in L_2 and is known to be essentially self-adjoint under considerably general conditions on the coefficients $a_{jk}(x)$, $b_j(x)$ and $q(x)$ (see, e.g., Ikebe-Kato [2]). However, we are to impose some stronger restrictions on the coefficients, or, more explicitly, we are to deal with the case when $[L]$ equals $[L_0]=-\Delta$ asymptotically, which will give us not only the essential self-adjointness of $[L]_0$ but also the identity of $D([L]_0^\sim)$ and $D(L_0)$.

Lemma 3.1. *Assume in addition to* (C. I) *that*

(C. II) *$[a_{jk}(x)]$ tends to $[\delta_{jk}]$ uniformly as $|x|\to\infty$, and $D_j a_{jk}(x)$ are bounded, where δ_{jk} denotes Kronecker's delta.*

Let

$$[M]u(x)=\sum_{j,k=1}^{3} D_j a_{jk}(x) D_k u(x)\,, \tag{3.2}$$

$$[M]_0 u(x)=[M]u(x) \qquad for \quad u\in D([M]_0)=C_0^\infty\,. \tag{3.3}$$

Then $[M]_0$ has a unique self-adjoint extension $M=[M]_0^\sim$, and, moreover, $D(M)=D(L_0)=H_2$, which implies in view of the closed graph theorem that

$$\text{const.}\,\|u\|_2\leq\|Mu\|+\|u\|\leq\text{const.}\,\|u\|_2 \tag{3.4}$$

for $u\in D(M)$ with positive constants. Here $\|u\|$ denotes the L_2-norm of u, H_2 is the totality of L_2-functions whose distribution derivatives up to the second order are square integrable, and $\|u\|_2$ is the H_2-norm of u:

$$\|u\|_2=[\|u\|^2+\sum_{j=1}^{3}\|D_j u\|^2+\sum_{j,k=1}^{3}\|D_j D_k u\|^2]^{1/2}\,. \tag{3.5}$$

Proof. Under conditions (C. I) and (C. II) it is known that $[M]_0$ is essentially self-adjoint (see, e.g., Ikebe-Kato [2]). It is, therefore, sufficient to show $D(M)=H_2$.

Since a straightforward computation with (C. II) in regard shows that the second inequality of (3.4) holds for $u\in H_2=D(L_0)$, we have only to prove that $D(M)\subset H_2$. According to a result of [2], $u\in D(M)$ if and only if $u\in H_{2,\text{loc}}$ and $[M]u\in L_2$. Let us choose a smooth function $\varphi(x)$ such that $\varphi(x)=1$ for $|x|\leq R$ $(R>0)$ and

$\varphi(x)=0$ for $|x|\geq R+1$. Now let $u\in D(M)$. Since obviously $\varphi u\in H_2$, it suffices to show $(1-\varphi)u\in H_2$. For this it is enough to have $D(M(1-\varphi))=D(L_0(1-\varphi))$. But we have, taking R sufficiently large and noting (C. II),

$$(3.6)\qquad ||(M(1-\varphi)n-L_0(1-\varphi)u|| \leq a||L_0(1-\varphi)u||+\text{const.}\,||u||$$

for $u\in D(L_0(1-\varphi))$ with $a<1$, where we have made use of the facts that $||L_0u||+||u||$ is equivalent with $||u||_2$ and that $||D_ju||$ is bounded by $c||L_0u||+d||u||$, where c can be chosen arbitrarily small (while d may have to be large). By a well-known theorem (see Kato [3], page 190) (3.6) implies $D(M(1-\varphi))=D(L_0(1-\varphi))$ if we note that $M(1-\varphi)$ and $L_0(1-\varphi)$ are closed operators. Q. E. D.

Now we further impose restrictions on the coefficients b_j and q:

(C. III) *D_jb_k are bounded; $q(x)=q_1(x)+q_2(x)$, where $q_1\in L_2$, and q_2 is essentially bounded ($\in L_\infty$).*

Theorem 3. 2. *Assume* (C. I), (C. II) *and* (C. III). *Then $[L]_0$ is essentially self-adjoint, $L=[L]_0^\sim$ is lower semi-bounded, and $D(L)=H_2$.*

Proof. The differential operator $[L]$ can be written as

$$(3.7)\qquad [L]=[M]+[T],$$

where $[M]$ is given by (3. 2) in the preceding lemma, and

$$(3.8)\qquad [T]=2\sum_{j=1}^{3}\beta_j'(x)D_j+\gamma'(x),$$

$$(3.9)\qquad \beta_j'(x)=\sum_{k=1}^{3}a_{jk}(x)b_k(x),$$

$$(3.10)\qquad \gamma'(x)=\sum_{j=1}^{3}D_j\beta_j'(x)+\sum_{j,k=1}^{3}a_{jk}(x)b_j(x)b_k(x)+q(x).$$

Conditions (C. I) through (C. III) imply in view of (3. 9) and (3. 10) that $\beta_j'(x)$ and $\gamma'(x)-q(x)$ are bounded. As already mentioned in the proof of Lemma 3. 1, we have

$$(3.11)\qquad ||D_ju||\leqq a||L_0u||+b||u|| \quad \text{with small enough } a>0$$

for $u\in D(L_0)$. Concerning $q(x)$ it is known ([3], pages 302, 303) that

$$(3.12)\qquad ||qu||\leq a||L_0u||+b||u|| \quad \text{with small enough } a>0$$

for $u\in D(L_0)$. These facts together with Lemma 3.1 yield

$$\|[T]u\| \leq a\|Mu\|+b\|u\| \quad \text{for} \quad u\in D(M)=D(L_0) \tag{3.13}$$

with $a<1$, whence follows that $[L]_0=[M]_0+[T]_0$ has a unique self-adjoint extension L, and $D(L)=D(L_0)=H_2$. That L is lower semi-bounded follows from (3.13) and the fact that M is non-negative definite, which can be easily verified (cf. [3], page 291). Q. E. D.

§4. Existence of the Wave Operators

Consider two unitary groups $U_0(t)$ and $U(t)$, $t\in \boldsymbol{R}^1$, associated with L_0 and L:

$$U_0(t) = e^{-itL_0}, \qquad U(t) = e^{-itL}. \tag{4.1}$$

If the strong limits

$$W_\pm = W_\pm(L, L_0) = s-\lim_{t\to\pm\infty} U(-t)U_0(t) \tag{4.2}$$

exist, $W_\pm$ are called the wave operators for the pair (L, L_0), and then the scattering operator S for the same pair of self-adjoint operators is defined to be

$$S = S(L, L_0) = W_+^* W_-. \tag{4.3}$$

A useful criterion for the wave operators to exist is given by the following

Lemma 4.1. ([3], page 533) *Let L_0 and L be self-adjoint operators in a Hilbert space. Let there exist a fundamental set D of the subspace of absolute continuity for L_0 such that for each $u\in D$, $U_0(t)u \in D(L_0)\cap D(L)$ for $t\in[0, \infty)$, $(L-L_0)U_0(t)u$ is continuous in t, and $\|(L-L_0)U_0(t)u\|$ is integrable over $[0, \infty)$. Then $W_+(L, L_0)$ exists. Similarly for $W_-(L, L_0)$ with $[0, \infty)$ replaced by $(\infty, 0]$.*

In our case one obtains a second-order differential operator for $L-L_0$:

$$[L]-[L_0] = [M]+[V]-[L_0] = [A]+[B]+[C], \tag{4.4}$$

$$[A] = \sum_{j,k=1}^{3} \alpha_{jk}(x) D_j D_k, \quad \alpha_{jk}(x) = a_{jk}(x)-\delta_{jk}, \tag{4.5}$$

(4.6) $[B]=\sum_{j=1}^{3}\beta_j(x)D_j\,,\ \beta_j(x)=\sum_{k=1}^{3}D_k a_{jk}(x)+2\sum_{k=1}^{3}a_{jk}(x)b_k(x)\,,$

(4.7) $[C]=\gamma(x)=\sum_{j,k=1}^{3}D_j(a_{jk}(x)b_k(x))+\sum_{j,k=1}^{3}a_{jk}(x)b_j(x)b_k(x)+q(x)\,,$

where it should be noted that $\beta_j(x)$ and $\gamma(x)$ are different from those primed given in (3.9) and (3.10). [A], [B] and [C] are well-defined operators in L_2 with domain H_2, which will be denoted hereafter by A, B and C, respectively, and $A+B+C=L-L_0$ is symmetric and relatively bounded by L_0 (for the latter terminology see [3], page 190), as can be seen from the discussion of the preceding section.

In order to guarantee the existence of the wave operators we impose besides (C.I) through (C.III) the following restriction on $a_{jk}(x)$, $b_j(x)$ and $q(x)$:

(C.IV) *For* $f(x)=\alpha_{jk}(x)$, $\beta_j(x)$ *and* $\gamma(x)$ *the following inequality holds*:

$$\int(1+|x|)^{-1+h}|f(x)|^2dx<\infty\,,\qquad h>0\,.$$

Theorem 4.2. *Let* (C.I) *through* (C.IV) *be satisfied. Then the wave operators* $W_\pm=W_\pm(L,L_0)$ *exist and are isometries.*

Proof. The isometry follows immediately if one proves the existence. We shall show that

(4.8) $$\int_0^\infty\|PU_0(t)u\|\,dt<\infty$$

for $u\in D$ with $P=A$, B and C, where D is a fundamental subset of L_2 to be specified below, so that Lemma 4.1 is applicable to deduce that W_+ exists. Here one should note that the subspace of absolute continuity for L_0 is all of L_2, or L_0 is spectrally absolutely continuous, which can be easily checked by (2.3). A similar handling is possible for W_-.

Let

(4.9) $$D=\{u:u(x)=\exp(-|x-a|^2/2),\ a\in\boldsymbol{R}^3\}\,.$$

D is fundamental, because it is the set of all translations of a positive L_2-function. We have (see [3], page 534)

$$U_0(t)u(x) = (1+2it)^{-3/2}\exp(-|x-a|^2/(2+4it)), \quad u\in D. \tag{4.10}$$

It is proved in [3] that $U_0(t)u\in H_2=D(A)=D(B)=D(C)$ and that

$$\int_0^\infty \|CU_0(t)u\|\,dt < \infty. \tag{4.11}$$

Consequently one needs only to show (4.8) for $P=A, B$.

First take the case $P=B$. By (4.10) it follows that

$$D_j U_0(t)u(x) = i(1+2it)^{-5/2}(x_j-a_j)\exp(-|x-a|^2/(2+4it)), \tag{4.12}$$

which enables us to deduce the following estimate:

$$\begin{aligned} |BU_0(t)u(x)| &= \Big|\sum_{j=1}^{3}\beta_j(x)D_jU_0(t)u(x)\Big| \\ &\leq \text{const.}\,|1+2it|^{-1-(h/2)}|x-a|^{-(1-h)/2}p(x) \end{aligned} \tag{4.13}$$

(cf. [3], page 534), where $p(x)=\max_j|\beta_j(x)|$, and use has been made of the relation

$$\max[r^m e^{-r^2/(2s)};\ r\geq 0] = (sm)^{m/2}e^{-m/2} \qquad (m\geq 0,\ s>0). \tag{4.14}$$

By virtue of condition (C. IV), which is also satisfied by $f(x)=p(x)$, it follows that

$$\|BU_0(t)u\| \leq \text{const.}\,(1+4t^2)^{-(1+h')/2}, \qquad h'=h/2, \tag{4.15}$$

which yields (4.8) for $P=B$.

The case $P=A$ may be dealt with quite similarly: If one notes that

$$\begin{aligned} D_jD_kU_0(t)u(x) &= -(1+2it)^{-7/2}(x_j-a_j)(x_k-a_k) \\ &\quad\times\exp(-|x-a|^2/(2+4it)) \qquad (j\neq k), \end{aligned} \tag{4.16}$$

$$\begin{aligned} D_j^2U_0(t)u(x) &= (1+2it)^{-5/2}\exp(-|x-a|^2/(2+4it)) \\ &\quad -(1+2it)^{-7/2}(x_j-a_j)^2\exp(-|x-a|^2/(2+4it)), \end{aligned} \tag{4.17}$$

one can obtain an inequality corresponding to (4.13), which will lead to (4.8) with $P=A$. (Note that the contribution of the first term on the right of (4.17) to a result of type (4.15) gives a rate of decrease in t more than required, as is seen by comparing it with (4.10).)

Thus (4.8) is valid for $P=A$, B and C, and this completes the proof of our assertion. Q. E. D.

§5. Completeness of the Wave Operators

A consequence of the existence of $W_\pm$ proved in the preceding section is that the range $R(W_\pm)$ of $W_\pm$ is a reducing subspace for L which is contained in, but does not in general coincides with, the subspace M_{ac} of absolute continuity for L. If $R(W_\pm)=M_{ac}$, $W_\pm$ is said to be *complete*, in which case the absolutely continuous part of L is unitarily equivalent to L_0. If both W_+ and W_- are complete, then, as is obvious from its definition (4.3), the scattering operator S is unitary. A criterion for the completeness of $W_\pm$ which plays a fundamental rôle in this section is the following lemma stated in [3] (page 545):

Lemma 5.1. *Let L_0 and L be self-adjoint operators with positive lower bound. If $L^{-m}-L_0^{-m}$ is in trace class for some $m>0$, then $W_\pm$ exist and are complete.* (*For the definition of trace class see* [3], *page* 519.)

It should be remarked here that the above lemma answers not only the completeness problem but also the existence one. However, the lemma applied to our actual problem will require conditions more stringent than those formulated in Theorem 4.2. This is why we have discussed the existence of the wave operators separately in the preceding section.

Lemma 5.2. *Let* (C. I)–(C. III) *be satisfied* (*the assumption of Theorem* 3.2). *Then for z in the intersection of the resolvent sets of L_0 and L*

$$R_0(z)^2-R(z)^2=R_0(z)^2VR_0(z)F(z)+G(z)R_0(z)VR_0(z)^2F(z), \tag{5.1}$$

where $V=L-L_0=A+B+C$ in the notation of §4, *and $F(z)$ and $G(z)$ are bounded operators.*

Proof. Conditions (C. I)–(C. III) are just to ascertain by Theorem 3.2 that L is a well-defined self-adjoint operator with $D(L)=D(L_0)$, so that the second resolvent equation

$$(5.2)\qquad R_0(z)-R(z) = R_0(z)VR(z) = R(z)VR_0(z)$$

holds. By differentiation in z (or by multiplying by $R_0(z)$ and $R(z)$ and subtracting) one obtains

$$\begin{aligned}(5.3)\qquad R_0(z)^2-R(z)^2 &= R_0(z)^2VR(z)+R_0(z)VR(z)^2\\ &= R_0(z)^2VR_0(z)(L_0-z)R(z)+R(z)VR_0(z)R(z)\\ &= R_0(z)^2VR_0(z)F(z)+R(z)(L_0-z)R_0(z)VR_0(z)^2F(z)\\ &= R_0(z)^2VR_0(z)F(z)+G(z)R_0(z)VR_0(z)^2F(z),\end{aligned}$$

where $F(z)=(L_0-z)R(z)$ and $G(z)=$the closure of $R(z)(L_0-z)$. It now suffices to show that $F(z)$ and $G(z)$ are bounded.

$R(z)$ is an everywhere defined bounded operator with range $R(R(z))=D(L)=D(L_0)$, so that $F(z)$ is everywhere defined. Moreover, $F(z)$ is closed, since L_0-z is so. It follows by the closed graph theorem that $F(z)$ is bounded.

For $G(z)$ we have $(R(z)(L_0-\bar{z}))^{\sim}=(R(z)(L_0-z))^{**}=((L_0-z)^*R(z)^*)^*$ $=((L_0-\bar{z})R(\bar{z}))^*$, where we have used the fact that $(HK)^*=K^*H^*$ if H is bounded. $(L_0-\bar{z})R(\bar{z})$ is bounded as shown above, and so is $G(z)$. Q. E. D.

As (2. 5) shows, $R_0(z)$ is an integral operator whose kernel we shall denote by $R_0(x, y; z)$. Let $R_0^{(2)}(x, y; z)$ be the kernel of the integral operator $R_0(z)^2$:

$$(5.4)\qquad R_0^{(2)}(x, y; z) = \int R_0(x, s; z)R_0(s, y; z)\,ds.$$

Lemma 5. 3. *Concerning* $R_0(x, y; z)$ *and* $R_0^{(2)}(x, y; z)$ *the following estimates hold*:

$$(5.5)\qquad |R_0(x, y; z)| \leq \text{const.}\,|x-y|^{-1}e^{-a|x-y|},$$

$$(5.6)\qquad |R_0^{(2)}(x, y; z)| \leq \text{const.}\,e^{-a|x-y|},$$

$$(5.7)\qquad |D_jR_0^{(2)}(x, y; z)| \leq \text{const.}\,\big|\log|x-y|\big|e^{-a|x-y|},$$

$$(5.8)\qquad |D_jD_jR_0^{(2)}(x, y; z)| \leq \text{const.}\,|x-y|^{-1}e^{-a|x-y|},$$

where the constants ("const." *and* a) *depend only on* z, *and* $D_j=-i\partial/\partial x_j$.

Proof. Straightforward. It would be enough to remark that two integral kernels with singularities $|x-y|^{-m}$ and $|x-y|^{-n}$ will yield the iterated kernel with singularity $|x-y|^{3-m-n}$ if $m+n\neq 3$ and $\log|x-y|$ if $m+n=3$ $(0<m, n<3)$, and that because of the symmetry in x and y of $R_0(x, y; z)$ and $R_0^{(2)}(x, y; z)$, which follows from (2.5) and (5.4), the operation D_jD_j applied to the variable x may be interpreted as D_j acting on x and D_k on y.

Lemma 5.4. *If $\gamma(x)\in L_1$, then $R_0(z)^2CR_0(z)$ and $R_0(z)CR_0(z)^2$ are in trace class (see (4.7)).*

Proof. It suffices to prove the assertion for one of the two expressions, for the other is in the form of adjoint of the first. Consider, therefore, $R_0(z)CR_0(z)^2$. Let $C=C'C''$, where C' and C'' are the operators of multiplication by $\operatorname{sgn}\gamma(x)|\gamma(x)|^{1/2}$ and $|\gamma(x)|^{1/2}$, respectively ($\operatorname{sgn}\gamma(x)=\gamma(x)/|\gamma(x)|$ if $\gamma(x)\neq 0$ and $=0$ if $\gamma(x)=0$). Since a trace-class operator is the product of the operators of Hilbert-Schmidt type, we have only to show that $R_0(z)C'$ and $C''R_0(z)^2$ are (to determine operators) of Hilbert-Schmidt type. But this is obvious from the assumption of the lemma and (5.5) and (5.6) of Lemma 5.3; for instance,

$$(5.9)\qquad \iint |\operatorname{sgn}\gamma(x)|\gamma(x)|^{1/2}R_0(x, y; z)|^2 dx\,dy \leq \int|\gamma(x)|dx\int\frac{e^{-2a|x-y|}}{|x-y|^2}dy<\infty. \qquad \text{Q.E.D.}$$

Lemma 5.5. *If $\beta_j(x)\in L_1$, then $R_0(z)^2BR_0(z)$ and $R_0(z)BR_0(z)^2$ are in trace class (see (4.6)).*

Proof. It is enough to check the assertion for $R_0(z)B_jR_0(z)^2$, where B_j is the differential operator $\beta_j(x)D_j$ with domain H_2. B_j can be decomposed in the form $B_j=B'B''$ with $B'=\beta_j(x)|\beta_j(x)|^{1/2}$ and $B''=|\beta_j(x)|^{1/2}D_j$. Now one can prove, by using Lemma 5.3 ((5.5) and (5.7)) and the assumption that $\beta_j(x)\in L_1$, that $R_0(z)B'$ and $B''R_0(z)^2$ are of Hilbert-Schmidt type, the method employed being similar to that of the proof of the preceding lemma. Q.E.D.

Lemma 5.6. *If* $\alpha_{jk}(x) \in L_1$, *then* $R_0(z)^2AR_0(z)$ *and* $R_0(z)AR_0(z)^2$ *are in trace class* (*see* (4.5)).

Proof. Similar to the preceding two proofs. Note that in the present case use has to be made of (5.8) of Lemma 5.3.

Now we can prove the completeness of the wave operators. Let us assume in addition to (C.I)-(C.III) that

(C.V) $\alpha_{jk}(x)$, $\beta_j(x)$ *and* $\gamma(x)$, *as prescribed in* (4.5) *through* (4.7), *are integrable.*

Conditions (C.I) through (C.III) ensure by Theorem 3.2 that if r is sufficiently large, then $L+r$ is positive definite with domain $D(L_0)$, while L_0+r is positive definite as is obvious from (2.3). We have

$$(L+r)^{-2}-(L_0+r)^{-2} = R(-r)^2-R_0(-r)^2, \tag{5.10}$$

which is seen to belong to trace class in virtue of Lemmas 5.2, 5.4, 5.5 and 5.6. This shows that the conditions of Lemma 5.1 are satisfied with $m=2$ and with L and L_0 replaced by $L+r$ and L_0+r, respectively. Thus $W_\pm(L+r, L_0+r)$, which is obviously equal to $W_\pm(L, L_0)$, exists and is complete. We have thus proved the following

Theorem 5.7. *Assume that* (C.I), (C.II), (C.III) *and* (C.V) *hold. Then the wave operators* $W_\pm$ *exist and are complete, and the scattering operator* S *is unitary.*

§6. An Extension of Theorem 5.7

It is known (see Ikebe [1] and Kuroda [4]) that if $q(x)$ behaves asymptotically like $|x|^{-2-\varepsilon}$, and if $L=(-\Delta+q)_0^\sim$, then $W_\pm(L, L_0)$ exist and are complete, though the assumption on $q(x)$ is not precisely stated (see (C.VI) below). Note that the rate of decrease $O(|x|^{-2-\varepsilon})$ for $q(x)$ is in a sense milder than the requirement that $q \in L_1$, because the latter claims that $q(x)$ vanish like $|x|^{-3-\varepsilon}$ at ∞ if it decreases to some negative power of $|x|$. In this section we shall combine the above result of [1] with that of §5 to the effect that we can in a certain respect weaken (C. V) concerning $\gamma(x)$.

Let $\alpha_{jk}(x)$, $\beta_j(x)$ and $\gamma(x)$ be as given by (4.5), (4.6) and (4.7). In addition to (C. I), (C. II) and (C. III) let us assume that

(C. VI) $\alpha_{jk}(x)$, $\beta_j(x)$ *and* $\operatorname{Im}\gamma(x)$ *satisfy* (C. V), *i.e., they are integrable.* $\operatorname{Re}\gamma(x)$ *is locally Hölder continuous except possibly at a finite number of singularities, is in* L_2, *and has the asymptotic order* $O(|x|^{-2-\varepsilon})$, $\varepsilon>0$.

Let L be as before, and let $L'=(-\Delta+\operatorname{Re}\gamma(x))_0^{\sim}$. Then according to [1] there exist the complete wave operators $W_\pm(L', L_0)$, and $D(L')=D(L_0)=D(L)$. In order to prove the existence of the complete wave operators $W_\pm(L, L_0)$, it is sufficient, in view of the chain rule for wave operators ([3], page 532), to show that $W_\pm(L, L')$ exist and are complete. Here, the definition of the wave operators has to be modified, however. Namely, we define

$$W_\pm(L, L') = s-\lim_{t\to\pm\infty} U(-t)U'(t)P' , \tag{6.1}$$

if the strong limits exist, where $U'(t)$ is the unitary group associated with L', and P' is the orthogonal projection onto the subspace of absolute continuity for L'. Now our criterion, Lemma 5.1, on the existence of complete wave operators is still applicable to "generalized" wave operators defined by (6.1). Therefore, we shall examine whether or not the conditions of Lemma 5.1 are satisfied by L' and L, or what comes to the same thing, as remarked in §5, by $L'+r$ and $L+r$.

If we take r sufficiently large, $L'+r$ is positive definite, since L' is at any rate a special case of L formulated in Theorem 3.2. Thus the first condition of Lemma 5.1 is satisfied.

Next, we check if $(L'+r)^{-2}-(L+r)^{-2}$ is in trace class. Since $D(L')=D(L)$, $V'=L-L'$ is well-defined and symmetric on $D(L)=H_2$, and

$$V' = A+B+C' \quad \text{with} \quad C'u(x) = \operatorname{Im}\gamma(x)u(x) \quad \text{for} \quad u\in H_2 . \tag{6.2}$$

We note that C' enjoys the same properties as C in the preceding section. On the other hand, it has been established in [1] that $R(z; L')$, the resolvent of L', is an integral operator (of Carleman type) with kernel $R'(x, y; z)$, and the properties of $R'(x, y; z)$, investigated there, allow us to obtain the estimates (5.5) through (5.8) of Lemma 5.3 for $R'(x, y; z)$ just as for $R_0(x, y; z)$. This enables us to have Lemma 5.4, 5.5 and 5.6 valid with $R_0(z)$ and C replaced by $R(z; L')$ and C'. Putting these together, since Lemma 5.2 obviously holds good if $R(z; L')$ and V' take the place of $R_0(z)$ and V, we see that $(L'+r)^{-2}-(L+r)^{-2}$ is in trace class.

Thus the conditions of Lemma 5.1 are satisfied for the pair $(L'+r, L+r)$. Consequently, we have

Theorem 6.1. *Let* (C.I), (C.II), (C.III) *and* (C.VI) *be satisfied. Then the wave operators* $W_{\pm}(L, L_0)$ *exist and are complete.*

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