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Note on the Spectrum of Some Schrödinger Operators*

By

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§ 0. Introduction

Recently S. T. Kuroda has developed a new stationary method of perturbation of continuous spectra using the technique of factorization of the perturbation term ([2], [3]). The object of this note is to show that his theory can be applied to n-dimensional Schrödinger operators which have first order differentiations with variable coefficients. Namely we consider the differential operator:

$$L = \sum_{j=1}^{n} \left(\frac{\partial}{i\partial x_j} + b_j(x)\right)^2 + q(x)$$

where b_j and q are real valued functions. The case of $n \geqslant 3$ will be treated in this note. This problem has been already treated by Kuroda ([4]) in the case of $b_j(x)=0$. In this case our work agrees with his result.

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§ 1. Statement of the Results

Let H_0 be the self-adjoint realization of $-\Delta$ in $L^2(R^n)$, where domain $\mathfrak{D}(H_0) = \mathcal{D}^2_{L^2}(R^n)$. We consider the following conditions:

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(I) *For some constants a, p_1 and p_2 satisfying*

$$a>\frac{n}{2},\ 2n>p_1>n \quad \text{and} \quad 2n>p_2>\max\left(2, \frac{n}{2}\right),$$

it holds that

$$\begin{cases} (1+|x|)^a b_j(x)\in L^{p_1}(R^n) & (1\leqslant j\leqslant n)\,, \\ (1+|x|)^a q'(x)\in L^{p_2}(R^n)\,, \end{cases}$$

where $b_j(x)$ are continuously differentiable functions and

$$q'(x) = \sum_{j=1}^{n}\left(\frac{\partial}{i\partial x_j}b_j(x)+b_j^2(x)\right)+q(x)\,.$$

(II) *For some p_3 with $n>p_3>2$, it holds that*

$$\begin{cases} (1+|x|)^a b_j(x)\in L^{p_3}(R^n) & (1\leqslant j\leqslant n)\,, \\ (1+|x|)^a q'(x)\in L^{p_3}(R^n)\,. \end{cases}$$

Theorem. **(a)** *Under assumption* (I), *we have*:

The restriction L on $\mathcal{S}$, the totality of rapidly decreasing functions, has a unique selfadjoint extension H_1, with $\mathfrak{D}(H_1)=\mathcal{D}^2_{L^2}(R^n)$. The absolutely continuous part of H_1 is unitarily equivalent to H_0.

(b) *Assuming further* (II), *we have*:

The singular spectrum of H_1 consists of non zero eigenvalues of finite multiplicity and possibly zero. Negative eigenvalues have not a finite limiting point. Zero is the only possible finite limiting point for positive eigenvalues if they exist.

Now we resume results of Kuroda. Let H_j $(j=0,1)$ be self-adjoint in a separable Hilbert space $\mathfrak{H}$, $R_j(z)\equiv(H_j-z)^{-1}$ be its resolvent for nonreal z. Let $\sigma(H_j)$ be its spectrum. Let $\mathfrak{H}_{j,ac}(\mathfrak{H}_{j,s})$ be the subspace of absolute continuity (of singularity) with respect to H_j. These concepts have been defined in Chapt. X of [1]. Then $\mathfrak{H}_{j,ac}$ and $\mathfrak{H}_{j,s}$ are closed linear subspaces of $\mathfrak{H}$, are orthogonal complements to each other and reduce H_j. If $\mathfrak{H}_{j,ac}=\mathfrak{H}$, H_j is said to be absolutely continuous. $H_{j,ac}(H_{j,s})$ be the restriction of H_j to $\mathfrak{H}_{j,ac}(\mathfrak{H}_{j,s})$. The set $\sigma(H_{j,ac})(\sigma(H_{j,s}))$ is the absolutely continuous (the singular) spectrum of H_j and is denoted by $\sigma_{ac}(H_j)(\sigma_s(H_j))$. We denote by $\mathfrak{B}$ the space of all bounded linear operators from $\mathfrak{H}$ into $\mathfrak{H}$ having the uniform operator topology. For any linear operator T its domain (or range) is represented as $\mathfrak{D}(T)$ (or $\mathfrak{R}(T)$).

Consider the following conditions on H_j.

(K. 1) $\mathfrak{D}(H_1)=\mathfrak{D}(H_0)=\mathfrak{D}$.

(K. 2) *There exist linear operators A and B such that: (a) A lies in $\mathfrak{B}$, and is invertible with the range being dense in $\mathfrak{H}$, (b) $\mathfrak{D}(B) \supset \mathfrak{R}(R_0(z)A)$ for any nonreal z, (c) $(H_1-H_0)u=ABu$ for $u\in\mathfrak{D}(B)\cap\mathfrak{D}$.*

(K. 3) *$S(z)\equiv A^*\{R_0(z)-R_0(\bar{z})\}A$ lies in $\mathfrak{B}$, $\lim_{\varepsilon\downarrow 0} S(\lambda+i\varepsilon)=S(\lambda)$ exists in $\mathfrak{B}$, and this convengence is locally uniform in λ of the real axis.*

(K. 4) *$Q(z)\equiv BR_0(z)A$ is in $\mathfrak{B}$, and completely continuous for nonreal z.*

(K. 5) *$\lim_{\varepsilon\downarrow 0} Q(\lambda\pm i\varepsilon)=Q(\lambda\pm i0)$ exists in $\mathfrak{B}$ for any real λ, and $Q(z)$ is a $\mathfrak{B}$-valued continuous function on either the upper or the lower half-plane, including the corresponding edge of the real axis.*

(K. 6) *The operator $\left\{\frac{1}{2\pi i}S(\lambda)\right\}^{1/2}$ is Hölder continuous with Hölder exponent $\theta>1/2$ on a closed interval I of real axis, and $Q(z)$ is also Hölder continuous with exponent θ on I^+ or I^-, where $I^+=\{z:\ \mathrm{Re}\, z\in I,\ \mathrm{Im}\, z\geqslant 0\}$ and $I^-=\{z:\ \mathrm{Re}\, z\in I,\ \mathrm{Im}\, z\leqslant 0\}$.*

We can deduce the following theorem as a corollary of Kuroda's theory ([2], [3]). The outline of the proof will be sketched in §3.

Theorem K. *Under conditions (K. 1) to (K. 5) we have:*

1° *H_0 is absolutely continuous.*

2° *$\sigma_{ac}(H_1)=\sigma(H_0)$, and $\sigma_s(H_1)$ is a closed null set.*

3° *There exist wave operators $W_\pm\in\mathfrak{B}$ such that: $W_\pm^* W_\pm=1$, $W_\pm W_\pm^*=P_1$, $H_1 W_\pm=W_\pm H_0$ and $W_\pm=\text{s-}\lim_{t\to\pm\infty} e^{itH_1}e^{-itH_0}$ where 1 is the identity operator, and where P_1 is the projection to $\mathfrak{H}_{1,ac}$.*

Assuming further (K. 6), we obtain:

4° *$\sigma_s(H_1)\cap I$ consists of at most a countable number of eigenvalues of finite multiplicity which have no accumulation point interior to I.*

§ 2. Proof of the Theorem

First we write

$$L = -\Delta + \alpha(x)\left(\sum_{j=1}^{n} \beta_j(x)\frac{\partial}{\partial x_j} + \beta_0(x)\right)$$

where $\alpha(x)\beta_0(x) = q'(x) = \sum_{j=1}^{n}\left(\frac{\partial}{i\partial x_j}b_j(x) + b_j(x)^2\right) + q(x)$ and $\alpha(x)\beta_j(x) = -2ib_j(x)$ $(1 \leqslant j \leqslant n)$. Define $(Af)(x) \equiv \alpha(x)\cdot f(x)$, $(B_0 f)(x) \equiv \beta_0(x) f(x)$ $(B_j f)(x) \equiv \beta_j(x)\frac{\partial}{\partial x_j} f(x)$ $(1 \leqslant j \leqslant n)$ then we have formally $H_1 = H_0 + AB$ for $B = \sum_{j=1}^{n} B_j$.

From now on we will take $\alpha(x) = (1 + |x|)^{-a}$, then the operator A belongs to $\mathfrak{B}$ and satisfies conditions (a) of (K. 2). Operators B_j with domains $\{f(x) | f(x) \in L^2, (B_j f)(x) \in L^2\}$ have closed extensions, which are also denoted by B_j. For the proof of the first part of (a), we have only to show that $\mathfrak{D}(H_0)$ is contained in $\mathfrak{D}(AB)$ and that for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for any $u \in \mathfrak{D}(H_0)$

$$||ABu|| \leqslant \varepsilon ||H_0 u|| + c_\varepsilon ||u||$$

(Chapt. V of [1]).

Assume that $D(B_j) \supset D(H_0^{\mu_j})$ for μ_j with $0 < \mu_j < 1$, then we have for $u \in D(H_0)$ and $k > 1$

$$\begin{aligned} ||B_j u|| &= ||B_j R_0(-1)^{\mu_j} (H_0 + 1)^{\mu_j} R_0(-k)(H_0 + k)u|| \\ &\leqslant ||B_j R_0(-1)^{\mu_j}|| \cdot ||(H_0 + 1)^{\mu_j} R_0(-k)|| \cdot \{||H_0 u|| + k||u||\}\,. \end{aligned}$$

By the closed graph theorem, $B_j R_0(-1)^{\mu_j}$ is bounded. Noticing that for $k > 1$

$$||(H_0 + 1)^{\mu_j} R_0(-k)|| \leqslant k^{\mu_j - 1}\,,$$

we have the estimate that there exists $C_\varepsilon > 0$

$$||B_j u|| \leqslant \varepsilon ||H_0 u|| + C_\varepsilon ||u||$$

for $\varepsilon > 0$ and $u \in \mathfrak{D}(H_0)$. Since A is bounded, we will obtain the desired statements if $\mathfrak{D}(B_j) \supset \mathfrak{D}(H_0^{\mu_j})$.

Now we will show the validity of this inclusion for $1 \leqslant j \leqslant n$. Since $p_1 > n$ (condition I), we can choose μ_j such that $p_1 > \frac{n}{2\mu_j - 1}$ and

$<\mu_j<1$. If $u\in D(H_0^{\mu_j})$, the Fourier transform of $u(x)$, $\hat{u}(\xi)$ satisfies $(1+|\xi|^2)^{\mu_j}\hat{u}(\xi)\in L^2(R^n)$. Since $p_1>n$, we have $\xi_j(1+|\xi|^2)^{-\mu_j}\in L^{p_1}(R^n)$. As $\xi_j\hat{u}=\xi_j(1+|\xi|^2)^{-\mu_j}(1+|\xi|^2)^{\mu_j}\hat{u}$, $\xi_j\hat{u}\in L_q(R^n)$ for $\frac{1}{q}=\frac{1}{p_1}+\frac{1}{2}$. Therefore $\frac{\partial}{\partial x_j}u\in L^{q'}$ with $\frac{1}{q'}=1-\frac{1}{q}=\frac{1}{2}-\frac{1}{p_1}$. As $\beta_j\in L^{p_1}$ and $\frac{1}{p_1}+\frac{1}{q'}=\frac{1}{2}$, we have $B_ju=\beta_j\frac{\partial}{\partial x_j}u\in L^2(R^n)$.

Next we treat the case $j=0$. Since $p_2>\frac{n}{2}$, we can choose μ_0 satisfying $p_2>\frac{n}{2\mu_0}$ and $0<\mu_0<1$. Then $(1+|\xi|^2)^{-\mu_0}\in L^{p_2}(R^n)$. As $\hat{u}=(1+|\xi|^2)^{-\mu_0}(1+|\xi|^2)^{\mu_0}\hat{u}$, $\hat{u}\in L^q(R^n)$ for $\frac{1}{q}=\frac{1}{p_2}+\frac{1}{2}$. Therefore $u\in L^{q'}(R^n)$ with $\frac{1}{q'}=1-\frac{1}{q}=\frac{1}{2}-\frac{1}{p_2}$. And finally we have $B_0u=\beta_0u\in L^2(R^n)$. Thus the first part of (a) has been proved.

Now we start to check conditions (K. 1) to (K. 5). For $r>0$ we define

$$R_n(r, z) \equiv c_n\sqrt{z}^{(n/2)-1}r^{1-(n/2)}H^{(1)}_{(n/2)-1}(\sqrt{z}\,r)$$

with

$$c_n = i2^{-(n/2)-1}\pi^{1-(n/2)}, \qquad \operatorname{Im}\sqrt{z}\geqslant 0$$

and the ν-th Hankel function of first kind $H_\nu^{(1)}(\zeta)$. Then it holds that for a nonreal or negative number z,

$$(H_0-z)^{-1}f(x) = \int_{R^n}R_n(|x-y|, z)f(y)\,dy\,.$$

The following asymptotic representations are well known:

(H. 1) $H_\nu^{(1)}(\zeta)=-\pi^{-1}2^{-\nu}\Gamma(\nu)\zeta^{-\nu}+0(\zeta^{-\nu})$ as $|\zeta|\to 0$ with $\operatorname{Im}\zeta\geqslant 0$,

(H. 2) $H_\nu^{(1)}(\zeta)=\sqrt{2}(\pi\zeta)^{-(1/2)}e^{i(\zeta-(2\nu+1)(\pi/4))}+0(\zeta^{-(3/2)})$ as $|\zeta|\to\infty$ with $\operatorname{Im}\zeta\geqslant 0$.

Taking $\phi(\zeta)\in C^\infty$ such that $\phi(\zeta)=1$ for $|\zeta|\leqslant 1$ and $\phi(\zeta)=0$ for $|\zeta|\geqslant 2$, we have

$$\begin{aligned}R_n(r, z) &= \phi(\sqrt{z}\,r)R_n(r, z)+(1-\phi(\sqrt{z}\,r))R_n(r, z)\\ &= \frac{S_n(r, z)}{r^{n-2}}+\frac{S_n(r, z)}{r^{(n-1)/2}}\,.\end{aligned}$$

Let Π be the complex plane which has a cut along the positive real axis from 0 to ∞, including both edges of the cut. Then $S_n^{(k)}(r, z)$

can be regarded to be continuous in $(r, z)\in(0, \infty)\times(\Pi-\{0\})$. From (H. 1) and (H. 2) it holds that:

$$|S_n^{(1)}(r, z)| \leqslant \text{const}, \quad S_n^{(1)}(r, z) = 0 \quad \text{for} \quad |\sqrt{z}\,r| \geqslant 2,$$

and that:

$$|S_n^{(2)}(r, z)| \leqslant \text{const}\,|z|^{(n-2)/4}, \quad S_n^{(2)}(r, z) = 0 \quad \text{for} \quad |\sqrt{z}\,r| \leqslant 1.$$

By the identity $\frac{d}{d\zeta}(\zeta^{-\nu}H_\nu^{(1)}(\zeta)) = -\zeta^{-\nu}H_{\nu+1}^{(1)}(\zeta)$, we deduce that for $r=(\sum_{j=1}^{n} x_j^2)^{1/2}$,

$$\begin{aligned}\frac{\partial}{\partial x_j}R_n(r, z) &= -\frac{c_n}{c_{n+2}}\cdot\frac{x_j}{r}\cdot R_{n+2}(r, z)\\ &= \frac{S_{n,j}^{(1)}(x, z)}{r^{n-1}} + \frac{S_{n,j}^{(2)}(r, z)}{r^{(n-1)/2}}\end{aligned}$$

where $S_{n,j}^{(k)}(x\,;z)\equiv -c_n/c_{n+2}\cdot\frac{x_j}{r}\cdot S_{n+2}^{(k)}(r, z)$ $(k=1, 2, 1\leqslant j\leqslant n)$. These functions satisfy the estimates of the same type as $S_{n+2}^{(k)}$. Define $\Pi_N^+\equiv\{z:\ N^{-1}\leqslant|z|\leqslant N,\ \text{Im}\, z\geqslant 0\}$ for $N>1$, and similarly Π_N^- for $\text{Im}\, z\leqslant 0$. Noticing that $S_n^{(1)}(r, z)=r^{n-2}\phi(\sqrt{z}\,r)R_n(r, z)$ and that $S_n^{(2)}(r, z)=r^{(n-1)/2}(1-\phi\sqrt{z}\,r))R_n(r, z)$, using the identity $\frac{d}{d\zeta}(\zeta^\nu H_\nu^{(1)}(\zeta)) =\zeta^\nu H_{\nu-1}^{(1)}(\zeta)$, we have the following estimates:

$$\left|\frac{\partial}{\partial z}S_n^{(1)}(r, z)\right| \leqslant c_N, \quad \text{and} \quad \left|\frac{\partial}{\partial z}S_n^{(2)}(r, z)\right| \leqslant c_N r$$

for $z\in\Pi_N^\pm$ where c_N is some constant depending only on N. Therefore we have

$$\left|\frac{\partial}{\partial z}S_{n,j}^{(1)}(x, z)\right| \leqslant c'_N \quad \text{and} \quad \left|\frac{\partial}{\partial z}S_{n,j}^{(2)}(x, z)\right| \leqslant c'_N r$$

for $z\in\Pi_N^\pm$ and $1\leqslant j\leqslant n$.

Letting $S_{n,0}^{(k)}(x, z)\equiv S_n^{(k)}(|x|, z)$, we define integral kernels:

$$Q_j^{(k)}(x, y\,; z) = \frac{\beta_j(x)S_{n,j}^{(k)}(x-y, z)\alpha(y)}{|x-y|^{\lambda_{j,k}}} \qquad (k=1, 2,\ 0\leqslant j\leqslant n)$$

where $\lambda_{0,1}=n-2$, $\lambda_{j,1}=n-1$ $(1\leqslant j\leqslant n)$ and $\lambda_{j,2}=\frac{n-1}{2}$ $(0\leqslant j\leqslant n)$ and integral operators:

$$(Q_j^{(k)}(z)f)(x) = \int_{R^n} Q_j^{(k)}(x, y; z)f(y)\,dy.$$

Formally we have $Q(z) = BR_0(z)A = \sum_{j=0}^{n}\sum_{k=1}^{2} Q_j^{(k)}(z)$. We need:

Lemma. *For a given λ satisfying $n>\lambda>0$, assume that there exist p, q such that $\frac{1}{p}, \frac{1}{q} < \frac{1}{2}$ and $\frac{1}{p}+\frac{1}{q}=1-\frac{\lambda}{n}$. Consider an integral operator: $(K_\lambda f)(x) = \int_{R^n} \frac{\beta(x)\gamma(x, y)\alpha(y)}{|x-y|^\lambda} f(y)\,dy$, where $\gamma(x, y)$ is continuous and bounded on $R^{2n}-\{x=y\}$.*

If $\alpha(x)\in L^p(R^n)$ and $\beta(x)\in L^q(R^n)$, then K_λ is a bounded operator in $L^2(R^n)$ satisfying

$$||K_\lambda||_{\mathfrak{B}} \leqslant C(p, q)||\gamma||_\infty \cdot ||\beta||_q \cdot ||\alpha||_p$$

where $||\gamma||_\infty = \sup_{x \neq y} |\gamma(x, y)|$, $||u||_p = \left(\int_{R_n} |u(x)|^p dx\right)^{1/p}$.

Moreover if we assume further that $\alpha(x)\in L^{p'}(R^n)$ and $\beta(x)\in L^{q'}(R^n)$ where p' and q' satisfy $\frac{1}{2}-\frac{\lambda}{n} \leqslant \frac{1}{p'} < \frac{1}{2}$ and $\frac{1}{p'}+\frac{1}{q'} < 1-\frac{\lambda}{n}$, then K_λ is completely continuous (due to Kuroda, see lemma 5.3 of [4]).

Proof. Sobolev's inequality shows that $\left|\int_{R^n}\int_{R^n} \frac{g(x)f(y)}{|x-y|^\lambda} dx\,dy\right| \leqslant C(P, Q)||g||_Q||f||_P$ for $P>1, Q>1, \frac{1}{P}+\frac{1}{Q}>1$ and $\lambda = n\left(2-\frac{1}{P}-\frac{1}{Q}\right)$ ([6]). Let $\frac{1}{P}=\frac{1}{2}+\frac{1}{p}$ and $\frac{1}{Q}=\frac{1}{2}+\frac{1}{q}$. Substituting the above inequility for $g=\beta v$ and $f=\alpha u$ where $u, v\in L^2(R^n)$, we have readily the first part of the lemma.

Let $\chi_N(x)$ be the characteristic function of $D_N = \{x \mid |x| \leqslant N\}$. Define $(K_\lambda^{(N,M)}f)(x) \equiv \int_{R^n} \frac{\chi_M(x)\beta(x)\gamma(x, y)\alpha(y)\chi_N(y)}{|x-y|^\lambda} f(y)\,dy$.
By our assumption on p' and q', Kondrašev's Theorem ([7]) asserts that an integral kernel, $\frac{\gamma(x, y)}{|x-y|^\lambda}$, determins a completely continuous integral operator from $L^{P'}(D_N)$ to $L^{Q'}(D_N)$ for $N<\infty$ with $\frac{1}{P'}=\frac{1}{p'}+\frac{1}{2}$ and $\frac{1}{Q'}=\frac{1}{2}-\frac{1}{q'}$. Since we can regard $\alpha\cdot\chi_N$ as a bounded operator from $L^2(R^n)$ to $L^{P'}(D_N)$ and $\beta\cdot\chi_N$ as that from $L^{Q'}(D_N)$ to

$L^2(R^n)$, we have that $K_\lambda^{(N,N)}$ is completely continuous from $L^2(R^n)$ to $(L^2(R^n)$. But from the first part of the lemma, we have

$$\begin{aligned}||K_\lambda - K_\lambda^{(N,N)}|| &\leqslant ||K_\lambda - K_\lambda^{(\infty,N)}|| + ||K_\lambda^{(\infty,N)} - K_\lambda^{(N,N)}|| \\ &\leqslant C(p, q)\{||(1-\chi_N)\alpha||_p ||\beta||_q + ||\chi_N \alpha||_p ||(1-\chi_N)\beta||_q\} \\ &\to 0 \qquad (\text{as } N \text{ tends to } \infty).\end{aligned}$$

Therefore K_λ is completely continuous. Thus the lemma has been proved.

In the above lemma, if we take $\alpha(x)=(1+|x|)^{-a}$ where $a>\frac{n}{2}$, and $\beta(x)\in L^q$ where q satisfies $\frac{1}{2}-\frac{\lambda}{n}<\frac{1}{q}<\frac{1}{2}$ for $\frac{1}{2}-\frac{\lambda}{n}>0$ and $\frac{1}{q}<1-\frac{\lambda}{n}$ for $\frac{1}{2}-\frac{\lambda}{n}\leqslant 0$, then $\alpha(x)\in L^p$ for any p with $\frac{1}{p}\leqslant\frac{1}{2}$. Since we can choose p and p' appropriately, the above lemma asserts that K_λ is completely continuous in $L^2(R^n)$. More precisely, if $\beta(x) \in L^{q_j}$ where q_j satisfy that:

$$\begin{aligned}&\text{for} \quad \lambda_1 = n-1, \quad q_1>n, \\ &\text{for} \quad \lambda_2 = n-2, \quad \text{if} \quad n=3, \quad \text{then} \quad 6>q_2>2, \\ &\qquad\qquad\qquad\qquad\quad \text{if} \quad n>3, \quad \text{then} \quad q_2>\frac{n}{2}, \\ &\text{for} \quad \lambda_3 = \frac{n-1}{2}, \quad 2n>q_3>2,\end{aligned}$$

then K_{λ_j} are completely continuous. Therefore $Q_j^{(k)}(z)$ are completely continuous if $\beta_0(x)\in L^{q_2}\cap L^{q_3}$ and if $\beta_j(x)\in L^{q_1}\cap L^{q_2}$ $(1\leqslant j\leqslant n)$. These being assumed in condition (I), we will obtain (K. 4) if the operator $Q(z)$, treated above, coincides with $BR_0(z)A$. We will show this fact. From Sobolev's inequality it holds that if $f\in L^2$, then

$$\int_{R^n}\frac{\partial}{\partial x_j}R_n(|x-y|, z)\alpha(y)f(y)\,dy\in L^p \qquad \text{for} \quad 1\geqslant\frac{1}{p}\geqslant\frac{1}{2}.$$

This implies that $\mathfrak{D}(B)\supset\mathfrak{R}(R_0(z)A)$ ((b) of (K. 2)) and that our integral operator $Q(z)$ is equal to $BR_0(z)A$.

Naturally we can define $Q(\lambda\pm i0)$ for $\lambda>0$ as the integral operator with the kernel $\sum_{j,k} Q_j^{(k)}(x, y\,;\lambda\pm i0)=\lim_{\varepsilon\downarrow 0}\sum_{j,k} Q_j^{(k)}(x, y\,;\lambda\pm i\varepsilon)$ for $x\neq y$. Also $Q(0)$ can be defined by putting $R_n(r, 0)=(n-2)^{-1}(2\pi)^{-(n/2)}$

$\times \Gamma\left(\frac{n}{2}\right) r^{2-n}$. We denote by $Q^{(N)}(z)$ the operator which is obtained by the replacing $\beta_j(x)$ by $\chi_N(x)\beta_j(x)$ and $\alpha(x)$ by $\chi_N(x)\alpha(x)$ in $Q(z)$. As in the proof of the lemma, we can show $\lim_{N\to\infty} Q^{(N)}(z)=Q(z)$ in $\mathfrak{B}$ uniformly in z belonging to a bounded set of either the upper or the lower half-plane including the real axis. The asymptotic formula (H. 1) asserts that if $\operatorname{Im} z_1$, $\operatorname{Im} z_2 \geqslant 0$ (or $\leqslant 0$) and if $|z_1|$, $|z_2|$ and $|r| \leqslant N$, we have $R_n(r, z_1)-R_n(r, z_2)=\frac{o(1)}{r^{n-2}}$ uniformly as $|z_1-z_2|$ tends to zero. This assures that $Q^{(N)}(z)$ is a $\mathfrak{B}$-valued continuous function on $\operatorname{Im} z\geqslant 0$ (or $\operatorname{Im} z \leqslant 0$). Therefore $Q(z)$ is a $\mathfrak{B}$-valued continuous function, which implies (K. 5).

As for (K. 3), we must consider integral operators $A^{(k)}(z)$ with kernels: $A^{(k)}(x, y\,; z)=\dfrac{\overline{\alpha(x)}\, S_n^{(k)}(x-y, z)\,\alpha(y)}{|x-y|^{\lambda_{0,k}}}$ $(k=1, 2)$. Since $A^{(k)}(z)$ behave similarly to $Q_0^{(k)}(z)$, we omit to describe the check of this condition.

Finally we will show that (K. 6) holds under conditions (I) and (II). Let I be a closed interval on the real axis which does not contain zero. Since $\frac{\partial}{\partial z} S_{n,j}^{(1)}(x, z)$ is bounded in $z \in \Pi_N^{\pm}$, $Q_j^{(1)}(z)$ $(0\leqslant j\leqslant n)$ is Lipschitz continuous on $I^{\pm}$. Next we notice that for $z_1, z_2 \in \Pi_N^{\pm}$

$$
\begin{aligned}
\frac{|S_{n,j}^{(2)}(x, z_1)-S_{n,j}^{(2)}(x, z_2)|^{\theta}}{|r|^{\theta}} &= \frac{1}{r^{\theta}}\left|\int_{z_2}^{z_2}\frac{\partial}{\partial z} S_{n,j}^{(2)}(x, z)\, dz\right|^{\theta} \\
&\leqslant \text{const}\, |z_1-z_2|^{\theta}
\end{aligned}
$$

since $\left|\frac{\partial}{\partial z} S_{n,j}^{(2)}\right| \leqslant C_N |r|$. Putting $\theta=\frac{1}{2}+\varepsilon$ and $\lambda_4=\frac{n}{2}-1-\varepsilon$ $(\varepsilon>0)$, the integral kernel of $Q_j^{(2)}(z_1)-Q_j^{(2)}(z_2)$ is estimated by const $|z_1-z_2|^{(1/2)+\varepsilon}\dfrac{\beta_j(x)\,\alpha(y)}{|x-y|^{\lambda_4}}$ for $z_1, z_2 \in \Pi_N^{\pm}$. By the lemma if $\beta_j \in L^{q_4}$ with $\frac{1+\varepsilon}{n}<\frac{1}{q_4}<\frac{1}{2}$, the above kernel defines a bounded operator $\in \mathfrak{B}$. Since we can choose ε arbitrarily small, condition (II) asserts that $Q_j^{(2)}(z)$ is Hölder continuous on $\Pi_N^{\pm}$ with the exponent greater than $1/2$.

The Hölder continuity of $\left\{\frac{1}{2\pi i} S(\lambda)\right\}^{1/2}$ has been shown by Kuroda in case $n=3$ ([2]). He used the spherical coordinate representation of $S(\lambda)$. His method is also valid for $n>3$ if we take the n-dimensional spherical harmonics instead of the 3-dimensional.

Since $\lim_{\substack{\lambda<0 \\ \lambda\to 0}} Q(\lambda)=Q(0)$ holds, applying Theorem 1 of [5] to our case, we know that negative eigenvalues do not accumulate at zero. Another conclusions of our theorem follow from the result of Kuroda.

§ 3. Remarks

1. Outline of the proof of Theorem K.

By (K. 1) we have the second resolvent equation:

$$R_1(z) = R_0(z) - R_1(z)\,(H_1 - H_0)\,R_0(z)\,.$$

Since A is bounded ((a) of (K. 2)), it holds that

$$R_1(z)A = R_0(z)A - R_1(z)\,(H_1 - H_0)\,R_0(z)A\,.$$

Noticing that $\mathfrak{D}(B)\cap\mathfrak{D}\supset\mathfrak{R}(R_0(z)A)$ ((b) of (K. 2)), we have

$$\begin{aligned}(H_1 - H_0)\,R_0(z)A &= ABR_0(z)A \quad ((\text{c}) \text{ of } (\text{K. }2)) \\ &= AQ(z) \qquad ((\text{K. }3))\,.\end{aligned}$$

Therefore the following identity is obtained.

$$R_1(z)A(1+Q(z)) = R_0(z)A\,. \tag{3. 1}$$

Define $G_0(z)\equiv 1+Q(z)$ for $\operatorname{Im} z\neq 0$. If $u\in\mathfrak{H}$ satisfies that $u+Q(z)u=0$ then $v=R_0(z)Au$ satisfies $H_1v=zv$. Since $\operatorname{Im} z\neq 0$, self-adjointness of H_1 implies that $v=0$. Hence $u=0$. Moreover $Q(z)$ is completely continuous ((K. 4)). Therefore there exists a bounded inverse of $(1+Q(z))$. We write $G_1(z)\equiv(1+Q(z))^{-1}$ for $\operatorname{Im} z\neq 0$. Then we have from (3. 1) identities

$$\begin{cases} R_0(z)\,AG_1(z) = R_1(z)\,A\,, \\ R_1(z)\,AG_0(z) = R_0(z)\,A\,. \end{cases} \tag{3. 2}$$

Let $S_j(z)\equiv A^*\{R_j(z)-R_j(\bar z)\}A$ for $j=0, 1$ and $\operatorname{Im} z\neq 0$. We have

$$\begin{aligned}A^*\{R_j(z)-R_j(\bar z)\}A &= (z-\bar z)A^*R_j(\bar z)\,R_j(z)A \\ &= (z-\bar z)\,(R_j(z)\,A)^*R_j(z)A\end{aligned}$$

$$= (z-\bar{z})(R_k(z)AG_j(z))^* R_k(z)AG_j(z) \quad \text{(by (3.2))}$$
$$= G_j(z)^* A^* \{R_k(z) - R_k(\bar{z})\} AG_j(z) .$$

This relation is rewritten as

$$S_j(z) = G_j(z)^* S_k(z) G_j(z) \tag{3.3}$$

where $(j, k) = (0, 1)$ or $(1, 0)$.

On the other hand it is well known that for α and β with $-\infty \leqslant \alpha < \beta \leqslant \infty$,

$$\int_\alpha^\beta dE_j(\lambda) = \text{s-}\lim_{\varepsilon \downarrow 0} \frac{1}{2\pi i} \int_\alpha^\beta (R_j(\lambda+i\varepsilon) - R_j(\lambda - i\varepsilon))\, d\lambda .$$

From this identity we obtain that for $u, v \in \mathfrak{H}$,

$$\int_\alpha^\beta d(E_j(\lambda)Au, Av)\, d\lambda = \frac{1}{2\pi i} \lim_{\varepsilon \downarrow 0} \int_\alpha^\beta (S_j(\lambda+i\varepsilon)u, v)\, d\lambda . \tag{3.4}$$

For $j=0$ condition (K. 3) assures that

$$\int_\alpha^\beta d(E_0(\lambda)Au, Av) = \frac{1}{2\pi i} \int_\alpha^\beta (S(\lambda)u, v)\, d\lambda .$$

By the continuity of $S(\lambda)$, we can conclude that for any measurable set ω of the real line

$$A^* E_0(\omega) A = \frac{1}{2\pi i} \int_\omega S(\lambda)\, d\lambda$$

(the integration can be performed in the operator topology). This identity implies that $\Re(A)$ is contained in $\mathfrak{H}_{0,ac}$. But $\Re(A)$ is dense in $\mathfrak{H}$ and $\mathfrak{H}_{0,ac}$ is closed. We know $\mathfrak{H}_{0,ac} = \mathfrak{H}$. This means conclusion 1° of Theorem K.

By (K. 5) there exists

$$G_0(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} G_0(\lambda \pm i\varepsilon) = 1 + Q(\lambda \pm i0) .$$

Since $G_0(z) - 1 = Q(z)$ is completely continuous for $\operatorname{Im} z \neq 0$ ((K. 4)), $G_0(\lambda \pm i0) - 1$ is also completely continuous. And $G_1(z) = G_0(z)^{-1}$ for $\operatorname{Im} z \neq 0$. Therefore lemma 6.2 of [3] (or lemma 5.2 of [2]) assures that there exists a bounded inverse of $G_0(\lambda \pm i0)$, $G_1(\lambda \pm i0)$, for almost every λ of the real axis. We denote by e the subset of the real line which consists of points λ such that $G_0(\lambda \pm i0)$ can not be invertible. Since $G_0(\lambda \pm i0)$ is continuous in λ of the real axis, the complement of e, e' is open. Moreover $G_1(z)$ is continuous on e'^+ and e'^- respectively, where $e'^+(e'^-)=\{z : \operatorname{Re} z \in e',\ \operatorname{Im} z \geqslant 0\ (\operatorname{Im} z \leqslant 0)\}$. Therefore we have for $\lambda \notin e$

$$G_1(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} G_1(\lambda \pm i\varepsilon) \qquad \text{in } \mathfrak{B}\,.$$

Define $S_1(\lambda \pm i0) \equiv \pm G_1(\lambda \pm i0)^* S(\lambda) G_1(\lambda \pm i0)$ for $\lambda \notin e$. By (3.3) it is obtained that for $\lambda \notin e$

$$S_1(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} S_1(\lambda \pm i\varepsilon) \qquad \text{in } \mathfrak{B}\,.$$

If $(\alpha, \beta) \cap e = \phi$, the equality (3.4) implies that for $u, v \in \mathfrak{H}$

$$\int_\alpha^\beta d(E_1(\lambda) Au, Av) = \frac{\pm 1}{2\pi i} \int_\alpha^\beta (S_1(\lambda \pm i0) u, v)\, d\lambda\,.$$

From this identity we conclude that for any closed set ω such that $\omega \cap e = \phi$, it holds that

$$A^* E_1(\omega) A = \frac{\pm 1}{2\pi i} \int_\omega S_1(\lambda \pm i0)\, d\lambda\,.$$

Using this relation we can conclude that $\sigma_{ac}(H_1) = \sigma(H_0)$, and that $\sigma_s(H_1) \subset e$. This means conclusion 2 of Theorem K. We can also deduce conclusion 3° of Theorem K as in the proof of Theorems 5.1 to 5.4 of [3] (or Theorem 4.1 of [2]). Finally conclusion 4° of Theorem K follows from Theorem 7.1 of [3] (Professor Kuroda kindly informed the author that the condition of Hölder continuity of $Q(z)$ was missing in [3]).

2. Eigenfunction expansions.

Under conditions (K.1) to (K.5), we have more concrete knowledge concerning the spectral representation of $H_{1,ac}$ by Kuroda's criterion. Consider the following integral equation:

$$\begin{aligned}\varphi^\pm(x, \xi) = e^{ix\xi} - \int_{R^n} \varphi^\pm(y, \xi) \\ \times \left\{2i \sum_{j=1}^n \left(b_j(y) \frac{\partial}{\partial y_j} + \frac{\partial b_j(y)}{\partial y_j} + b_j(y)^2\right) + q(y)\right\} \\ \times R_n(|x-y| : |\xi|^2 \mp i0)\, dy\end{aligned}$$

If $|\xi|^2 \notin \sigma_s(H_1)$, we can find the unique (x, ξ)-measurable solution of $\varphi_1^{\pm}(x, \xi)$ with $\int_{R^n}(1+|x|)^{-2a}|\varphi_1^{\pm}(x, \xi)|^2 dx < \infty$. Define $(F^{\pm}f)(\xi) = (2\pi)^{-(n/2)}\int_{R^n} f(x)\varphi^{\pm}(x, \xi)dx$ for f with $(1+|x|)^a f \in L^2(R^n)$. Kuroda's abstract theorem (shown in [2]) asserts that F can be extended to an isometric operator from $\mathfrak{H}_{1,ac}$ onto $L^2(R^n)$, satisfying $(F^{\pm}P_1E_1(\lambda)f)(\xi) = \chi_\lambda(\xi)(F^{\pm}P_1 f)(\xi)$ where $\chi_\lambda(\xi)$ is the characteristic function of $\{\xi \mid |\xi|^2 \leqslant \lambda\}$. Moreover if it holds that $(1+|x|)^a H_1 f \in L^2$ for $f \in \mathcal{S}$, then we have for $|\xi|^2 \notin \sigma_s(H_1)$, $\left[\sum_{j=1}^{n}\left(\frac{\partial}{i\partial x_j} + b_j(x)\right)^2 + q(x)\right]\varphi^{\pm}(x, \xi) = |\xi|^2\varphi^{\pm}(x, \xi)$ in the sense of distribution. Some additional information of regularity of $\varphi^{\pm}(x, \xi)$ will be obtained under appropriate conditions on b_j and q.

3. After this work was completed, the author heard the work of Ikebe-Tayoshi ([8]) from Professor Ikebe. They treated a 3-dimensional Schrödinger operator : $L = -\Delta + \sum_{j,k=1}^{3}\alpha_{j,k}(x)\frac{\partial^2}{\partial x_j \partial x_k} + \sum_{k=1}^{3}\beta_k\frac{\partial}{\partial x_k} + \gamma$. They have established the unitary equivalence of L and $-\Delta$ chiefly using the condition of α_{jk}, β_k and $\gamma \in L^1(R^3)$. Their method is based on the fact that if $R_1^2(z) - R_0^2(z)$ is an operator of the trace class, then the part (a) of our Theorem follows from Kato's criterion (Chapt. X of [1]).

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