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On the Cauchy Problem for a Simple Degenerate Diffusion System

By

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§1. Introduction

We consider the following Cauchy problem for the system of equations:

$$
(1.1)\qquad
\begin{aligned}
\frac{\partial u_1}{\partial t} &= \frac{\partial^2 u_1}{\partial x^2} - d_1 u_1 u_4 - d_2 u_1 u_3 \\
\frac{\partial u_2}{\partial t} &= \frac{\partial^2 u_2}{\partial x^2} - d_3 u_2 u_4 + d_2 u_1 u_3 \\
\frac{\partial u_3}{\partial t} &= \qquad\quad d_3 u_2 u_4 - d_2 u_1 u_3 \\
\frac{\partial u_4}{\partial t} &= \qquad -d_1 u_1 u_4 - d_3 u_2 u_4
\end{aligned}
$$

in $R^T = (-\infty < x < +\infty,\ 0 < t \leqslant^{V} T)$ with the initial data

$$
(1.2)\qquad
\begin{aligned}
u_1(x, 0) &= u_{10}(x) \\
u_2(x, 0) &= u_{20}(x) \\
u_3(x, 0) &= u_{30}(x) \\
u_4(x, 0) &= u_{40}(x),
\end{aligned}
$$

where the initial data are all non-negative and the coefficients d_1, d_2 and d_3 are positive constants.

Such a system is considered one of the problems of diffusion accompanied by an immobilizing reaction of second order (2), (4).

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In this paper, we intend to prove the uniqueness and the existence of the solution of the problem (1. 1) and (1. 2) by using a suitable simple difference scheme.

Using the following vector and matrix notations,

$$U = {}^t(u_1, u_2, u_3, u_4),$$

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A(U) = \begin{pmatrix} -d_2u_3 & 0 & 0 & -d_1u_1 \\ d_2u_3 & -d_3u_4 & 0 & 0 \\ -d_2u_3 & d_3u_4 & 0 & 0 \\ 0 & -d_3u_4 & 0 & -d_1u_1 \end{pmatrix},$$

the system (1. 1) and (1. 2) is reduced to

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} + A(U)U, \tag{1.3}$$

$$U(x, 0) = U_0(x). \tag{1.4}$$

We introduce the following difference scheme to (1. 1) and (1. 2),

$$\begin{aligned} u_1^{n+1,j} &= P(u_1^{n,j}) - k(d_1u_1^{n+1,j}u_4^{n,j} + d_2u_1^{n+1,j}u_3^{n,j}) \\ u_2^{n+1,j} &= P(u_2^{n,j}) - k(d_3u_2^{n+1,j}u_4^{n,j} - d_2u_1^{n+1,j}u_3^{n,j}) \\ u_3^{n+1,j} &= u_3^{n,j} + k(d_3u_2^{n,j}u_4^{n+1,j} - d_2u_1^{n,j}u_3^{n+1,j}) \\ u_4^{n+1,j} &= u_4^{n,j} - k(d_1u_1^{n,j}u_4^{n+1,j} + d_3u_2^{n,j}u_4^{n+1,j}) \end{aligned} \tag{1.5}$$

in R_h^T=(the rectangular lattices with mesh sizes (h, k) in R^T) with the initial data

$$\begin{aligned} u_1^{0,j} &= u_{10}(jh) \\ u_2^{0,j} &= u_{20}(jh) \\ u_3^{0,j} &= u_{30}(jh) \\ u_4^{0,j} &= u_{40}(jh) \end{aligned} \quad \text{or} \quad U^{0,j} = U_0(jh), \tag{1.6}$$

where $U^{n,j} = U(jh, nk)$ for integers j and n $(0 \leqslant nk \leqslant T)$, $P(u_i^{n,j}) = \lambda u_i^{n,j-1} + (1-2\lambda)u_i^{n,j} + \lambda u_i^{n,j+1}$ for $i=1, 2$ and $\lambda = k/h^2 = \text{const.}$

§ 2. Existence Theorem

The methods used in this section are essentially the same as those used by Fritz John [6].

First we have the following lemma.

Lemma 1 (Stability).

A sufficient condition for stability of the difference scheme (1. 5) and (1. 6) is given by the following inequality

$$(2.1) \qquad 0<\lambda\leqslant\frac{1}{2}\,.$$

Proof. We can write (1. 5) as follows:

$$(2.2)\qquad \begin{aligned} u_1^{n+1,j} &= \frac{P(u_1^{n,j})}{1+k(d_1u_4^{n,j}+d_2u_3^{n,j})}\\ u_2^{n+1,j} &= \frac{P(u_2^{n,j})+kd_2u_1^{n+1,j}u_3^{n,j}}{1+kd_3u_4^{n,j}}\\ u_3^{n+1,j} &= \frac{u_3^{n,j}+kd_3u_2^{n,j}u_4^{n+1,j}}{1+kd_2u_1^{n,j}}\\ u_4^{n+1,j} &= \frac{u_4^{n,j}}{1+k(d_1u_1^{n,j}+d_3u_2^{n,j})}\,. \end{aligned}$$

If $U^{0,j}$ is non-negative, it follows from (2. 2) that $U^{n,j}$ is non-negative for any n. Therefore we have from (1. 5) that

$$(2.3)\qquad \begin{aligned} &u_1^{n+1,j}\leqslant P(u_1^{n,j})\,,\\ &u_1^{n+1,j}+u_2^{n+1,j}\leqslant P(u_1^{n,j}+u_2^{n,j})\,,\\ &u_3^{n+1,j}+u_4^{n+1,j}\leqslant u_3^{n,j}+u_4^{n,j}\,,\\ &u_4^{n+1,j}\leqslant u_4^{n,j}\,. \end{aligned}$$

Thus we have the following estimates for any n,

$$(2.4)\qquad \begin{aligned} &|u_1^n|\leqslant|u_1^0|\,,\\ &|u_2^n|\leqslant|u_1^0|+|u_2^0|\,,\\ &|u_3^n|\leqslant|u_3^0|+|u_4^0|\,,\\ &|u_4^n|\leqslant|u_4^0|\,, \end{aligned}$$

where $\quad |u_i^n|=\sup_j|u_i^{n,j}| \qquad \text{for} \quad i=1,2,3,4\,.$

Lemma 1 is proved.

Secondly we can show that the x-difference quotient $U_x^{n,j} =(U^{n,j+1}-U^{n,j})/h$ satisfying (1.5) and (1.6) is bounded uniformly in h. In fact we have the following equations from (1.5) and (1.6),

$$
(2.5)\quad
\begin{aligned}
(u_1^{n+1,j})_x &= P((u_1^{n,j})_x) - k[d_1\{u_4^{n,j+1}(u_1^{n+1,j})_x \\
&\quad + u_1^{n+1,j}(u_4^{n,j})_x\} + d_2\{u_3^{n,j+1}(u_1^{n+1,j})_x \\
&\quad + u_1^{n+1,j}(u_3^{n,j})_x\}],\\
(u_2^{n+1,j})_x &= P((u_2^{n,j})_x) - k[d_3\{u_4^{n,j+1}(u_2^{n+1,j})_x \\
&\quad + u_2^{n+1,j}(u_4^{n,j})_x\} - d_2\{u_3^{n,j+1}(u_1^{n+1,j})_x \\
&\quad + u_1^{n+1,j}(u_3^{n,j})_x\}],\\
(u_3^{n+1,j})_x &= (u_3^{n,j})_x + k[d_3\{u_2^{n,j+1}(u_4^{n+1,j})_x \\
&\quad + u_4^{n+1,j}(u_2^{n,j})_x\} - d_2\{u_1^{n,j+1}(u_3^{n+1,j})_x \\
&\quad + u_3^{n+1,j}(u_1^{n,j})_x\}],\\
(u_4^{n+1,j})_x &= (u_4^{n,j})_x - k[d_1\{u_1^{n,j+1}(u_4^{n+1,j})_x \\
&\quad + u_4^{n+1,j}(u_1^{n,j})_x\} + d_3\{u_2^{n,j+1}(u_4^{n+1,j})_x \\
&\quad + u_4^{n+1,j}(u_2^{n,j})_x\}],
\end{aligned}
$$

and also the initial conditions

$$
(2.6)\qquad U_x^{0,j} = U_0((j+1)h) - U_0(jh))/h\,.
$$

Then it follows from (2.5) and Lemma 1 that

$$
(2.7)\quad
\begin{aligned}
&|(u_1^{n+1})_x| \leqslant |(u_1^n)_x| + kDM(|(u_4^n)_x| + |(u_3^n)_x|)\,,\\
&|(u_2^{n+1})_x| \leqslant |(u_2^n)_x| + kDM(|(u_4^n)_x| + |(u_3^n)_x| + |(u_1^{n+1})_x|)\,,\\
&|(u_3^{n+1})_x| \leqslant |(u_3^n)_x| + kDM(|(u_4^{n+1})_x| + |(u_2^n)_x| + |(u_1^n)_x|)\,,\\
&|(u_4^{n+1})_x| \leqslant |(u_4^n)_x| + kDM(|(u_1^n)_x| + |(u_2^n)_x|)\,,
\end{aligned}
$$

where $M = \max_i \{\sup_{R_h^T} |u_i^{n,j}|\}$ and $D = \max(d_1, d_2, d_3)$. Therefore, we obtain

$$
(2.8)\qquad |U_x^n| \leqslant e^{3DMT} |U_x^0|,
$$

where $\qquad |U_x^n| = \sum_{i=1}^{4} |(u_i^n)_x|.$

Consequently, if $U^{0,j}$ is of class $\mathscr{B}^1$, $U_x^{n,j}$ is bounded uniformly in h. Similarly, we have the following equations for $U_t^{n,j}$;

$$
\begin{aligned}
(u_1^{n+1,j})_t &= P((u_1^{n,j})_t) - k[d_1\{u_4^{n+1,j}(u_1^{n+1,j})_t \\
&\quad + u_1^{n+1,j}(u_4^{n,j})_t\} + d_2\{u_3^{n+1,j}(u_1^{n+1,j})_t \\
&\quad + u_1^{n+1,j}(u_3^{n,j})_t\}], \\
(u_2^{n+1,j})_t &= P((u_2^{n,j})_t) - k[d_3\{u_4^{n+1,j}(u_2^{n+1,j})_t \\
&\quad + u_2^{n+1,j}(u_4^{n,j})_t\} - d_2\{u_3^{n+1,j}(u_1^{n+1,j})_t \\
&\quad + u_1^{n+1,j}(u_3^{n,j})_t\}], \\
(u_3^{n+1,j})_t &= (u_3^{n,j})_t + k[d_3\{u_2^{n+1,j}(u_4^{n+1,j})_t \\
&\quad + u_4^{n+1,j}(u_2^{n,j})_t\} - d_2\{u_1^{n+1,j}(u_2^{n+1,j})_t \\
&\quad + u_3^{n+1,j}(u_1^{n,j})_t\}], \\
(u_4^{n+1,j})_t &= (u_4^{n,j})_t - k[d_1\{u_1^{n+1,j}(u_4^{n+1,j})_t \\
&\quad + u_4^{n+1,j}(u_1^{n,j})_t\} + d_3\{u_2^{n+1,j}(u_4^{n+1,j})_t \\
&\quad + u_4^{n+1,j}(u_2^{n,j})_t\}],
\end{aligned}
\tag{2.9}
$$

and the initial conditions

$$U_t^{0,j} = (U^{1,j} - U_0(jh))/k\,, \tag{2.10}$$

where $U_t^{n,j} = (U^{n+1,j} - U^{n,j})/k$. Hence we have the following estimate;

$$|U_t^n| \leqslant e^{3DMT}(|(u_{10})_{xx}| + |(u_{20})_{xx}| + 8DM^2)\,. \tag{2.11}$$

Therefore, if u_{10}, u_{20}, u_{30} and u_{40} are of class $\mathscr{B}^2$, $\mathscr{B}^2$, $\mathscr{B}^1$ and $\mathscr{B}^1$ respectively, $U^{n,j}$, $U_x^{n,j}$ and $U_t^{n,j}$ are bounded uniformly in h from (2.8) and (2.11). Proceeding in this manner we see that, if u_{10}, u_{20}, u_{30} and $u_{40}$are of class $\mathscr{B}^4$, $\mathscr{B}^4$, $\mathscr{B}^3$ and $\mathscr{B}^3$ respectively, all quantities

$$U^{n,j},\ U_x^{n,j},\ U_{xx}^{n,j},\ U_{xxx}^{n,j},\ U_t^{n,j},\ U_{xt}^{n,j},\ U_{xxt}^{n,j},\ U_{tt}^{n,j} \tag{2.12}$$

are bounded uniformly in h.

Now we consider a sequence of lattices $R_{h_m}^T$ with mesh sizes $(h_m = h_0/2^m,\ k_m = \lambda h_m^2)$ for a positive fixed number h_0 and non-negative integers m. Let define $R_{h_\infty}^T$ be the sum of all the sets $R_{h_m}^T$, then $R_{h_\infty}^T$ is a denumerable set and everywhere dense in R^T.

Since $U^{n,j}$ is bounded uniformly in h, it follows that there exists a sequence I of non-negative integers m such that

$$\lim_{\substack{m\subset I\\ m\to\infty}} U_m^{n,j} = U(x,t) \qquad \text{in} \quad R_{h_\infty}^T\,. \tag{2.13}$$

Here $U_m^{n,j} = U(jh_m, hk_m)$ is defined in $R_{h_m}^T$. If $U_m^{n,j}$ and $U_m^{n',j'}$ are both defined in $R^T_{h_m}$ for large m, then from the boundness of $U^{n,j}_x$ and $U^{n,j}_t$,

$$U_m^{n',j'}-U_m^{n,j} = h_m\sum_{\alpha=1}^{|j'-j|}(U_m^{n',j+\alpha-1})_x+k_m\sum_{\beta=1}^{|n'-n|}(U_m^{n+\beta-1,j})_t \tag{2.14}$$
$$= 0(x-x')+0(t-t') \qquad \text{in} \quad R^T_{h_m},$$

where $(x, t)=(jh_m, nk_m)$ and $(x', t')=(j'h_m, n'k_m)$. For $m\subset I$ and $m\to\infty$, we have

$$U(x', t')-U(x, t)=0(x'-x)+0(t'-t) \qquad \text{in} \quad R^T_{h_\infty}. \tag{2.15}$$

Consequently the limit function $U(x, t)$ is of class $\mathcal{E}^0_t(\mathcal{B}^0)$ in $R^T_{h_\infty}$ and we can continue $U(x, t)$ into the whole set R^T. Then $U(x, t)$ is of class $\mathcal{E}^0_t(\mathcal{B}^0)$ in R^T.

Since all quantities of (2.12) are uniformly bounded, we conclude in the same discussion that $(U^{n,j})_x$, $(U^{n,j})_{xx}$ and $(U^{n,j})_t$ converge respectively towards the corresponding derivatives of $U(x, t)$ and that the limits are of class $\mathcal{E}^0_t(\mathcal{B}^0)$ in R^T.

Finally in the limit as $m\subset I$ and $m\to\infty$ for $h=h_m$ and $k=k_m$ in the equations (1.5) and (1.6), we find that $U(x, t)$ satisfies (1.1) and (1.2) in $R^T_{h_\infty}$ and in R^T. Thus we obtain that, if u_{10}, u_{20}, u_{30} and u_{40} are of class $\mathcal{B}^4$, $\mathcal{B}^4$, $\mathcal{B}^3$ and $\mathcal{B}^3$ respectively, the limit function $U(x, t)$ is a solution of (1.1) and (1.2) such that

$$\begin{aligned} &u_1(x, t) \text{ and } u_2(x, t) \text{ are of class } \mathcal{E}^0_t(\mathcal{B}^2)\cap\mathcal{E}^1_t(\mathcal{B}^0), \\ &u_2(x, t) \text{ and } u_2(x, t) \text{ are of class } \mathcal{E}^0_t(\mathcal{B}^1)\cap\mathcal{E}^2_t(\mathcal{B}^0) \\ &\text{in } R^T. \end{aligned} \tag{2.16}$$

Further, if we use the properties of the fundamental solution of heat equation, we can modify that, if u_{10}, u_{20}, u_{30} and u_{40} are of class $\mathcal{B}^2$, $\mathcal{B}^2$, $\mathcal{B}^1$ and $\mathcal{B}^1$ respectively, the same results (2.16) are obtained.

Thus we have the following Theorem;

Theorem 1.

If u_{10}, u_{20}, u_{30} and u_{40} are of class $\mathcal{B}^2$, $\mathcal{B}^2$, $\mathcal{B}^1$ and $\mathcal{B}^1$ respectively and they are all non-negative, then there exists the non-negative solution, which satisfies (2.16), of the problem (1.1) and (1.2).

§ 3. Uniqueness Theorem

We have the following lemma.

Lemma 2 (Convergence).

If $U(x, t)$ is a solution of (1. 1) and (1. 2), for which $u_1(x, t)$ and $u_2(x, t)$ are of class $\mathcal{E}_t^0(\mathcal{B}^2) \cap \mathcal{E}_t^1(\mathcal{B}^0)$, and $u_3(x, t)$ and $u_4(x, t)$ are of class $\mathcal{E}_t^1(\mathcal{B}^0)$ in R^T, and if $U^{n,j}$ is the solution of (2. 1) and (2. 2) under the assumption of Lemma 1, then there exists a $\delta(\varepsilon)$ for any ε, such that for $(x=jh, t=nk)$ in R_h^T and $0<h, k\leqslant\delta$

$$||U^{n,j}-U(x, t)||<\varepsilon$$

where $$||u|| = \sup_{R_h^T} |u(x, t)| \qquad \text{and} \quad ||U||=\sum_{i=1}^{4} ||u_i|| .$$

Proof. We have

$$\begin{aligned} P(u_1(x, t)) &= \lambda u_1(x+h, t)+(1-2\lambda)u_1(x, t)+\lambda u_1(x-h, t) \\ &= u_1(x, t)+\frac{\lambda h^2}{2}\left\{\frac{\partial^2}{\partial x^2}u_1(x+\theta h, t)+\frac{\partial^2}{\partial x^2}u_1(x-\theta' h, t)\right\} \\ &\qquad\qquad (0<\theta, \theta'<1) \\ &= u_1(x, t)+k\left\{\frac{\partial}{\partial t}u_1(x, t)+d_1u_1(x, t)u_4(x, t)\right. \\ &\quad \left. +d_2u_1(x, t)u_3(x, t)\right\}+0(k) \\ &= u_1(x, t+k)+k\{d_1u_1(x, t)u_4(x, t)+d_2u_1(x, t) \\ &\quad u_3(x, t)\}+0(k) \\ &= u_1(x, t+k)+k\{d_1u_1(x, t+k)u_4(x, t) \\ &\quad +d_2u_1(x, t+k)u_3(x, t)\}+0(k) . \end{aligned}$$

Thus, it follows that

$$\begin{aligned} u_1(x, t+k) &= P(u_1(x, t))-k\{d_1u_1(x, t+k)u_4(x, t) \\ &\quad +d_2u_1(x, t+k)u_3(x, t)\}+0(k) . \end{aligned} \tag{3. 1}$$

By similar calculation we have

$$\begin{aligned} u_2(x, t+k) &= P(u_2(x, t))-k\{d_3u_2(x, t+k)u_4(x, t) \\ &\quad -d_2u_1(x, t+k)u_3(x, t)\}+0(k) , \end{aligned} \tag{3. 2}$$

$$\begin{aligned} u_3(x, t+k) &= u_3(x, t)+k\{d_3u_2(x, t)u_4(x, t+k) \\ &\quad -d_2u_1(x, t)u_3(x, t+k)\}+0(k) , \end{aligned} \tag{3. 3}$$

$$u_4(x, t+k) = u_4(x, t) - k\{d_1 u_4(x, t+k) u_1(x, t) + d_3 u_2(x, t) u_4(x, t+k)\} + 0(k) . \tag{3.4}$$

Putting $s_i(x, t) = u_i^{n,j} - u_i(x, t)$ for $i=1, 2, 3, 4$, we get the following difference scheme from (3.1), (3.2), (3.3) and (3.4),

$$\begin{aligned}
s_1(x, t+k) &= \frac{P(s_1(x, t)) - kf_1 + 0(k)}{1 + k(d_1 u_4^{n,j} + d_2 u_3^{n,j})} \\
s_2(x, t+k) &= \frac{P(s_2(x, t)) - kf_2 + 0(k)}{1 + kd_3 u_4^{n,j}} \\
s_3(x, t+k) &= \frac{s_3(x, t) - kf_3 + 0(k)}{1 + kd_2 u_1^{n,j}} \\
s_4(x, t+k) &= \frac{s_4(x, t) - kf_4 + 0(k)}{1 + k(d_1 u_1^{n,j} + d_3 u_2^{n,j})}
\end{aligned} \tag{3.5}$$

in R_h^T with the initial data

$$S(x, 0) = 0 \tag{3.6}$$

where $S = {}^t(s_1, s_2, s_3, s_4)$ and

$$\begin{aligned}
f_1 &= \{d_1 s_4(x, t) + d_2 s_3(x, t)\} u_1(x, t+k) , \\
f_2 &= d_3 u_2(x, t+k) s_4(x, t) - d_2 u_1(x, t+k) s_3(x, t) \\
&\quad - d_2 u_3^{n,j} s_1(x, t+k) , \\
f_3 &= -d_3 u_4(x, t+k) s_2(x, t) + d_2 u_3(x, t+k) s_1(x, t) \\
&\quad - d_3 u_2^{n,j} s_4(x, t+k) , \\
f_4 &= \{d_1 s_1(x, t) + d_3 s_2(x, t)\} u_4(x, t+k) .
\end{aligned} \tag{3.7}$$

Here we consider the following difference scheme for $Y = {}^t(y_1, y_2, y_3, y_4)$,

$$\begin{aligned}
y_1(x, t+k) &= P(y_1(x, t)) + k(F_1 + G_1) \\
y_2(x, t+k) &= P(y_2(x, t)) + k(F_2 + G_2) \\
y_3(x, t+k) &= y_3(x, t) + k(F_3 + G_3) \\
y_4(x, t+k) &= y_4(x, t) + k(F_4 + G_4)
\end{aligned} \tag{3.8}$$

in R_h^T with the initial data

$$Y(x, 0) = 0 , \tag{3.9}$$

where $F_i = \sup_{R_h^T} |f_i(x, t, k)|$. G_i is the upper bound of g_i and $g_i = g_i(x, t, k)$ tends to 0 as $k \to 0$ for $i=1, 2, 3, 4$.

We have the following equations from (3.8) and (3.9),

$$y_i(x, t) = t(F_i+G_i) \qquad \text{for} \quad i=1,2,3,4. \tag{3.10}$$

Therefore we get the following inequalities from (3.5) and (3.9)

$$|s_i(x, t)| \leqslant y_i(x, t) \qquad \text{for} \quad i=1,2,3,4 \tag{3.11}$$

and then we have in $0\leqslant t\leqslant T$

$$||s_i|| \leqslant T(F_i+G_i) \qquad \text{for} \quad i=1,2,3,4. \tag{3.12}$$

Thus, from (3.12) we have

$$||S|| \leqslant \sum_{i=1}^{4} T(F_i+G_i). \tag{3.13}$$

On the other hand, we find from (3.7) that

$$\sum_{i=1}^{4} F_i \leqslant 3DM||S||. \tag{3.14}$$

Then we get

$$(1-3DMT)||S|| \leqslant T\sum_{i=1}^{4} G_i. \tag{3.15}$$

If we select $T=T_1$ such that $T_1<1/3DM$, there exists a $\delta(\varepsilon)$ for any ε such that $0<h, k\leqslant\delta$ and

$$||S|| < \varepsilon \qquad \text{in} \quad 0\leqslant t\leqslant T_1. \tag{3.16}$$

If we consider $t=T_1$ as a new initial time, we get $t=T_2$ such that $T_2<T_1+1/3DM$ and

$$||S|| < \varepsilon \qquad \text{in} \quad T_1\leqslant t\leqslant T_2. \tag{3.17}$$

Thus, if we continue the same argument, we have the same result as (3.16) in R_h^T and Lemma 2 is proved.

Here if $U_1(x, t)$ and $U_2(x, t)$ are both arbitrary functions satisfying the assumption of Lemma 2, then there exists a $\delta(\varepsilon)$ for any ε such that $0<h, k\leqslant\delta$ implies

$$\begin{aligned} ||U_1(x, t)-U_2(x, t)|| &\leqslant ||U_1(x, t)-U^{n,j}||+||U_2(x, t)-U^{n,j}|| \\ &\leqslant 2\varepsilon \qquad \text{in} \quad R_h^T, \end{aligned} \tag{3.18}$$

where $U^{n,j}$ is the solution of (1.5) and (1.6).

Thus, we have proved the following theorem;

Theorem 2.

As for a genuine solution $U(x, t)$ of (1.1) and (1.2) satisfying the assumption of Lemma 2, the solution is unique.

Remark. Similarly we can deal with the following system of an immobilizing reaction of higher order,

$$\frac{\partial U}{\partial t} = D\frac{\partial^2 U}{\partial x^2} + A(U^P)U \tag{3.19}$$

$$U(x, 0) = U_0(x) \tag{3.20}$$

where for non-negative integer P,

$$A(U^P) = \begin{pmatrix} -d_2 u_1^{p_1} u_3^{p_3} & 0 & 0 & -d_1 u_1^{p_5} u_4^{p_6} \\ d_2 u_1^{p_1} u_3^{p_3} & -d_3 u_2^{p_2} u_4^{p_4} & 0 & 0 \\ -d_2 u_1^{p_1} u_3^{p_3} & d_3 u_2^{p_2} u_4^{p_4} & 0 & 0 \\ 0 & -d_3 u_2^{p_2} u_4^{p_4} & 0 & -d_1 u_1^{p_5} u_4^{p_6} \end{pmatrix}$$

and prove the existence and the uniqueness of the solution.

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