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Finiteness of the Number of Discrete Eigenvalues of the Schrödinger Operator for a Three Particle System

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§ 1. Introduction

In this paper we shall study discrete eigenvalues of the Schrödinger operator for a three particle system with infinitely heavy nucleus. The operator

$$H = -\Delta_1 - \Delta_2 - \frac{Z}{r_1} - \frac{Z}{r_2} + \frac{1}{|r_1 - r_2|} \qquad (Z>0) \tag{1.1}$$

is a most interesting case. Žislin [11] and Jörgens [6] has shown that the essential spectrum of the operator (1.1) consists of $\left[-\frac{Z^2}{4}, \infty\right)$. In fact $-\frac{Z^2}{4}$ is the least eigenvalue of the operators $-\Delta_i - \frac{Z}{r_i}$ $(i=1, 2)$ (see (2.7) and (2.8)). In case $Z=2$ in (1.1) (Schrödinger operator for helium atom), Kato [7] has shown that there exist an infinite number of discrete eigenvalues in $\left(-\infty, -\frac{Z^2}{4}\right)$. Moreover, Žislin [11] and the author [9] have given the same results as Kato's for $Z>1$ (positive ions composed of one nucleus and two electrons). In case $0<Z\leq 1$, no such knowledge as for discrete eigenvalues seems to have been obtained. However, for $0<Z<1$ (in this case the operator (1.1) has no physical meaning), we can assert by Theorem 1 in § 2 that there exists at most a

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finite number of discrete eigenvalues in $\left(-\infty, -\frac{Z^2}{4}\right)$. For $Z=1$ (hydrogen negative ion), the problem is unsolved.

Theorem 2 is an extension of the well-kown fact that the operator in $L^2(R^3)$

(1. 2) $$L = -\Delta + q(r)$$

has at most a finite number of discrete eigenvalues in $(-\infty, 0)$, where $q(r) \geq -\frac{1}{4} \cdot \frac{1}{r^2}$ for $r>R$ and tends to zero as $r \to \infty$.

§ 2. Statement of the Theorems

Let Ω be a domain in the m-dimensional Euclidean space R^m. We write for $f, g \in L^2(\Omega)$, $\int_\Omega f(x)\overline{g(x)}dx = (f, g)_\Omega$ and $||f||_\Omega = (f, f)_\Omega^{1/2}$. For simplicity we write $\boldsymbol{r}_i = (x_{3i-2}, x_{3i-1}, x_{3i})$, $r_i = |\boldsymbol{r}_i| = (\sum_{\nu=0}^{2} x_{3i-\nu}^2)^{1/2}$, $d\boldsymbol{r}_i = dx_{3i-2}dx_{3i-1}dx_{3i}$, $\Delta_i f = \sum_{\nu=0}^{2} \frac{\partial^2 f}{\partial x_{3i-\nu}^2}$ and $|\nabla_i f| = \left(\sum_{\nu=0}^{2} \left|\frac{\partial f}{\partial x_{3i-\nu}}\right|^2\right)^{1/2}$ $(i=1, 2)$. Let $C_0^\infty(R^m)$ be the space of all C^∞ functions with compact support, $\mathcal{D}_{L^2}^n(R^m)$ be the completion of $C_0^\infty(R^m)$ with the norm $||f||_{\mathcal{D}_{L^2}^n(R^m)} = (\sum_{|\alpha| \leq n} ||D^\alpha f||_{R^m}^2)^{1/2}$, where $D^\alpha f = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_m^{\alpha_m}} f$ and $|\alpha| = \alpha_1 + \cdots + \alpha_m$, and $Q_\alpha(R^m) = \left\{ f\,;\ \sup_{x \in R^m} \int_{|x-y| \leq 1} \frac{|f(y)|^2}{|x-y|^{m-4+\alpha}} dy < +\infty \right\}$. Now let us consider the Schrödinger operator of the form

(2. 1) $$(H\psi)(x) = -\Delta_1\psi(x) - \Delta_2\psi(x) + q_1(\boldsymbol{r}_1)\psi(x) + q_2(\boldsymbol{r}_2)\psi(x) + P(\boldsymbol{r}_1, \boldsymbol{r}_2)\psi(x)\,.$$

For each term of this operator, we assume that

(2. 2) $q_i(\boldsymbol{r}_i) \in L_{\text{loc}}^2(R^3)$ $(i = 1, 2)$ and $P(\boldsymbol{r}_1, \boldsymbol{r}_2) \in Q_\alpha(R^6)$ (for some $\alpha > 0$) are real-valued functions,

(2. 3) $q_i(\boldsymbol{r}_i)$ $(i = 1, 2)$ converge uniformly to zero as $r_i \to \infty$,

(2. 4) $P(\boldsymbol{r}_1, \boldsymbol{r}_2) \geq 0$ in R^6,

(2. 5) $P(\boldsymbol{r}_1, \boldsymbol{r}_2)$ converges uniformly to zero as $r_1 \to \infty$ whenever $\boldsymbol{r}_2$ is fixed, and as $r_2 \to \infty$ whenever $\boldsymbol{r}_1$ is fixed,

(2. 6) there exist some constants k, k' $(1<k<k'<+\infty)$, β $(0<\beta\leq 1)$, $\varepsilon>0$, $R>0$, $c>0$ such that

$$P(r_1, r_2)+q_1(r_1)\begin{cases} \geq \dfrac{c}{r_1^{2\beta-\varepsilon}} & \text{for} \quad k \leq \dfrac{r_1^\beta}{r_2} \leq k' \quad \text{and} \quad r_1 \geq R\,, \\ \geq 0 & \text{for} \quad k' < \dfrac{r_1^\beta}{r_2} \quad \text{and} \quad r_1 \geq R\,, \end{cases}$$

$$P(r_1, r_2)+q_2(r_2)\begin{cases} \geq \dfrac{c}{r_2^{2\beta-\varepsilon}} & \text{for} \quad k \leq \dfrac{r_2^\beta}{r_1} \leq k' \quad \text{and} \quad r_2 \geq R\,, \\ \geq 0 & \text{for} \quad k' < \dfrac{r_2^\beta}{r_1} \quad \text{and} \quad r_2 \geq R\,. \end{cases}$$

Then we have

Theorem 1. *The Schrödinger operator H of the form* (2.1) *has the following properties*:

(*i*) *If we assume* (2.2), (2.3) *and if the domain of H is $\mathcal{D}_{L^2}^2(R^6)$, H is a lower semi-bounded selfadjoint operator in $L^2(R^6)$.*

(*ii*) *Under the conditions* (2.2)–(2.5) *the essential spectrum $\sigma_e(H)$ of H is $[\mu, \infty)$, where*

$$\mu = \min_{i=1,2} \inf_{\substack{\varphi \in \mathcal{D}_{L^2}^2(R^3) \\ \|\varphi\|_{R^3}=1}} (H_i\varphi, \varphi)_{R^3} \leq 0\,, \tag{2.7}$$

$$H_i = -\Delta_i + q_i(r_i)\,. \tag{2.8}$$

(*iii*) *If we assume* (2.2)–(2.6) *and $\mu<0$, there exists at most a finite number of discrete eigenvalues in $(-\infty, \mu)$.*

Remark 1. The condition (2.6) is satisfied by $q_i(r_i)$ $(i=1, 2)$ and $P(r_1, r_2)$ having the following properties; for some $\gamma(0<\gamma<2)$

$$q_i(r_i) \geq -\frac{a}{r_i^\gamma} \quad \text{for} \quad r_i \geq R \qquad (i=1, 2)\,, \tag{2.9}$$

$$P(r_1, r_2) \geq \frac{b}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^\gamma} \quad \text{for} \quad |\boldsymbol{r}_1 - \boldsymbol{r}_2| \geq R\,, \tag{2.10}$$

$$b > a\,, \tag{2.11}$$

where $|\boldsymbol{r}_1 - \boldsymbol{r}_2| = (\sum_{\nu=1}^{3} (x_\nu - x_{3+\nu})^2)^{1/2}$.

In fact for $k>1$ large enough to satisfy $\dfrac{b}{(1+k^{-1})^\gamma} - a > 0$, we have $|\boldsymbol{r}_1 - \boldsymbol{r}_2| \geq r_1 - r_2 \geq (1-k^{-1})r_1 \geq R$ for $r_2 \leq k^{-1} r_1$ and sufficiently large r_1, and by (2.9) and (2.10) $P(r_1, r_2) + q_1(r_1) \geq \left\{\dfrac{b}{(1+k^{-1})^\gamma} - a\right\} \cdot \dfrac{1}{r_1^\gamma}$ for

$r_2 \leq k^{-1} r_1$ and sufficiently large r_1. Therefore (2.6) is satisfied for $\beta = 1$. On the other hand, assume (2.2)-(2.5) and

$$q_i(r_i) \leq -\frac{a}{r_i^\gamma} \quad \text{for} \quad r_i \geq R \quad (i=1,2)\ (0<\gamma\leq 2), \tag{2.12}$$

$$P(\boldsymbol{r}_1, \boldsymbol{r}_2) \leq \frac{b}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^\gamma} \quad \text{for} \quad |\boldsymbol{r}_1 - \boldsymbol{r}_2| \geq R, \tag{2.13}$$

$$a - b \begin{cases} >0 & \text{for} \quad 0<\gamma<2, \\ >\dfrac{1}{4} & \text{for} \quad \gamma = 2 \end{cases} \tag{2.14}$$

together with some conditions on $P(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for $|\boldsymbol{r}_1 - \boldsymbol{r}_2| < R$. Then the existence of an infinite number of discrete eigenvalues has been shown by the author [9].

Remark 2. In case $\mu = 0$, we can see that H is a non-negative operator in $L^2(R^6)$ (see the proof of Lemma 6), and has no discrete eigenvalues.

If $q_i(r_i)$ $(i=1, 2)$ tend to zero more rapidly than (2.9), we have only to assume (2.4) in place of (2.6) and (2.4) as for $P(\boldsymbol{r}_1, \boldsymbol{r}_2)$. Namely, we have

Theorem 2. *If we assume* (2.2)-(2.5) *and the condition*

$$q_i(r_i) \geq -\frac{1}{4}\frac{1}{r_i^2} \quad \textit{for} \quad r_i \geq R \quad (i=1,2), \tag{2.15}$$

the Schrödinger operator H of the form (2.1) *has at most a finite number of discrete eigenvalues in $(-\infty, \mu)$, where μ is given by* (2.7) *and* (2.8).

Remark 3. If only one of $q_i(r_i)$ $(i=1, 2)$ satisfies (2.15), we can not, in general, assert that H has at most a finite number of discrete eigenvalues in $(-\infty, \mu)$.

In fact let

$$q_1(r_1) = \begin{cases} -V_0 & \text{for} \quad 0 \leq r_1 < R, \quad (V_0 > 0), \\ 0 & \text{for} \quad r_1 \geq R, \end{cases} \tag{2.16}$$

$$q_2(r_2) \leq -\left(\frac{1}{4} + \varepsilon\right)\frac{1}{r_2^2} \quad \text{for} \quad r_2 \geq R \quad (\varepsilon > 0), \tag{2.17}$$

(2.18) $$P(r_1, r_2) = 0 .$$

Since H_2 has an infinite number of discrete eigenvalues in $(-\infty, 0)$ (see, e.g. Glazman [4] or Uchiyama [9]), we write its eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots < 0$. If V_0 is sufficiently large, the least discrete eigenvalue μ of H_1 is smaller than λ_1. Then by (2.7) and (2.8) $\sigma_e(H) = [\mu, \infty)$ and H has an infinite number of discrete eigenvalues $\{\mu + \lambda_k\}_{k=1,2,\cdots}$ in $(-\infty, \mu)$.

§ 3. Proof of the Theorems

As for Theorem 1 we have only to prove (iii), since it is known that (i) holds by Ikebe-Kato [5] and Jorgens [6], and (ii) by Jorgens [6] and Žislin [11]. Hereafter we assume (2.2)–(2.6) till the completion of the proof of Theorem 1.

Let $g(t)$ be a function having the following properties: $g(t) \in C^\infty(0, \infty)$, $g(t) \equiv 1$ for $t \geq k'$, $g(t) \equiv 0$ for $0 < t < k$ and $0 \leq g(t) \leq 1$ for $0 < t < +\infty$. By the condition (2.3) and $\mu < 0$, we can choose $R > 1$ large enough to satisfy the following inequalities:

$$(3.1) \qquad \begin{aligned} q_1(r_1) &> \frac{\mu}{2} \qquad \text{for} \quad r_1 > \frac{R^\beta}{k'} , \\ q_2(r_2) &> \frac{\mu}{2} \qquad \text{for} \quad r_2 > \frac{R^\beta}{k'} , \end{aligned}$$

$$(3.2) \qquad t^4 g(t) g''(t) + cR^\varepsilon \geq 0 \qquad \text{for} \quad k \leq t \leq k' .$$

We define domains $\{\Omega_1\}_{i=1,\cdots 4}$ in R^6 as follows:

$$\Omega_1 = \{r_1 < R \quad \text{and} \quad r_2 < R\}, \quad \Omega_2 = \left\{r_1 \geq R \quad \text{and} \quad r_2 \leq \frac{r_1^\beta}{k}\right\},$$

$$\Omega_3 = \left\{r_2 \geq R \quad \text{and} \quad r_1 \leq \frac{r_2^\beta}{k}\right\} \quad \text{and} \quad \Omega_4 = R^6 - \bigcup_{i=1}^{3} \Omega_i .$$

Then by $R > 1$ and $0 < \beta \leq 1$, we have $\Omega_i \cap \Omega_j = \phi (i \neq j)$ and $\bigcup_{i=1}^{4} \Omega_i = R^6$. For convenience, let us introduce the following notation for $\psi \in \mathcal{D}_{L^2}^2(R^6)$:

$$(3.3) \qquad L[\psi] \equiv (H\psi, \psi)_{R^6} = \sum_{i=1}^{4} \{ ||| \nabla_1 \psi | ||_{\Omega_i}^2 + ||| \nabla_2 \psi | ||_{\Omega_i}^2 + (q_1 \psi, \psi)_{\Omega_i} + (q_2 \psi, \psi)_{\Omega_i} + (P\psi, \psi)_{\Omega_i} \} \equiv \sum_{i=1}^{4} L_i[\psi] .$$

Now we shall show the following lemma.

Lemma 1. *For any* $\psi\in\mathcal{D}^2_{L^2}(R^6)$, $L_2[\psi]\geq\mu\|\psi\|^2_{\Omega_2}$ *and* $L_3[\psi]\geq\mu\|\psi\|^2_{\Omega_3}$.

Proof. Let $\psi(x)\in C_0^\infty(R^6)$. By Green's theorem, we have

$$(3.4)\quad \begin{aligned}\int_{\Omega_2}\left|\nabla_2\left(g\left(\frac{r_1^\beta}{r_2}\right)\psi(x)\right)\right|^2dx &= \sum_{j=4}^6\int_{\Omega_2}\left|\frac{\partial g}{\partial x_j}\psi+\frac{\partial\psi}{\partial x_j}g\right|^2dx\\ &= \int_{\Omega_2}\{|\nabla_2 g|^2|\psi|^2+g^2|\nabla_2\psi|^2\}dx+\frac{1}{2}\sum_{j=4}^6\int_{\Omega_2}\frac{\partial g^2}{\partial x_j}\frac{\partial|\psi|^2}{\partial x_j}dx\\ &= \int_{\Omega_2}\{|\nabla_2 g|^2|\psi|^2+g^2|\nabla_2\psi|^2\}dx\\ &\quad+\frac{1}{2}\int_{r_1\geq R}dr_1\int_{(r_1/k)\geq r_2\geq 0}\sum_{j=4}^6\frac{\partial g^2}{\partial x_j}\frac{\partial|\psi|^2}{\partial x_j}dr_2\\ &= \int_{\Omega_2}g^2|\nabla_2\psi|^2dx+\int_{\Omega_2}\left\{|\nabla_2 g|^2-\frac{1}{2}\Delta_2(g^2)\right\}|\psi|^2dx\\ &= \int_{\Omega_2}g\left(\frac{r_1^\beta}{r_2}\right)^2|\nabla_2\psi|^2dx-\int_{\Omega_2}g\left(\frac{r_1^\beta}{r_2}\right)g''\left(\frac{r_1^\beta}{r_2}\right)\frac{r_1^{2\beta}}{r_2^4}|\psi|^2dx\,.\end{aligned}$$

On the other hand, since $g\left(\frac{r_1^\beta}{r_2}\right)r_2\psi(x)\in\mathcal{D}^2_{L^2}(R^3)$ whenever r_1 is fixed $(r_1\geq R)$, we have by (2.7) and (2.8)

$$(3.5)\quad \int_{R^3}|\nabla_2(g\psi)|^2dr_2+\int_{R^3}q_2(r_2)|g\psi|^2dr_2\geq\mu\int_{R^3}|g\psi|^2dr_2\,.$$

Integrating (3.5) on the subdomain $\{r_1\,;\,r_1\geq R\}$ in R^3 with respect to r_1, we have

$$(3.6)\quad \int_{\Omega_2}|\nabla_2(g\psi)|^2dx+\int_{\Omega_2}q_2(r_2)|g\psi|^2dx\geq\mu\int_{\Omega_2}|g\psi|^2dx\,.$$

Then by (3.4) and (3.6), we have

$$(3.7)\quad \int_{\Omega_2}\{g^2|\nabla_2\psi|^2+q_2g^2|\psi|^2\}\,dx\geq\int_{\Omega_2}\left\{\mu g^2+gg''\cdot\frac{r_1^{2\beta}}{r_2^4}\right\}|\psi|^2dx\,.$$

Therefore by (3.7) and $0\leq g(t)\leq 1$ for $k\leq t<+\infty$,

$$(3.8)\quad \begin{aligned}L_2[\psi] &= \int_{\Omega_2}\{|\nabla_1\psi|^2+(1-g^2)|\nabla_2\psi|^2\}dx\\ &\quad+\int_{\Omega_2}\{g^2|\nabla_2\psi|^2+q_2g^2|\psi|^2\}dx\end{aligned}$$

$$+\int_{\Omega_2}\{q_2(1-g^2)+p+q_1\}|\psi|^2dx$$
$$\geq\int_{\Omega_2}\left\{\mu g^2+gg''\frac{r_1^{2\beta}}{r_2^4}+q_2(1-g^2)+p+q_1\right\}|\psi|^2dx\,.$$

Let $\frac{r_1^\beta}{r_2}=t$. We have by (2.6), (3.1) and (3.2)

$$(3.9)\qquad (1-g(t)^2)(-\mu+q_2(r_2)) + \frac{1}{r_1^{2\beta}}\{g(t)g''(t)t^4+r_1^{2\beta}(P(r_1,r_2)+q_1(r_1))\}\geq 0$$

for $t\geq k$ and $r_1\geq R$. In fact $g(t)\equiv 1$ and $g''(t)\equiv 0$ for $t\geq k'$, and $-\mu+q_2(r_2)>-\frac{\mu}{2}>0$ for $k\leq t<k'$ i.e. $r_2>\frac{r_1^\beta}{k'}>\frac{R^\beta}{k'}$. Then by (3.8) and (3.9) we have for any $\psi(x)\in C_0^\infty(R^6)$

$$(3.10)\qquad L_2[\psi]\geq\mu\|\psi\|_{\Omega_2}^2\,.$$

Making use of Lemma 2 below due to Ikebe-Kato [5] and Jörgens [6], (3.10) holds for any $\psi\in\mathcal{D}_{L^2}^2(R^6)$. In a similar fashion we have $L_3[\psi]\geq\mu\|\psi\|_{\Omega_3}^2$ for any $\psi\in\mathcal{D}_{L^2}^2(R^6)$. q.e.d.

Lemma 2. *We have:*

(*i*) *Under the conditions* (2.2) *and* (2.3)

$$(3.11)\qquad q_i\in Q_\alpha(R^6)\quad\textit{for}\quad 1>\alpha>0\qquad(i=1,2)\,.$$

(*ii*) *If* $q(x)\in Q_\alpha(R^m)$ *for some* $\alpha>0$, *then for any* $\eta>0$ *there exists some constant* $c(\eta)>0$ *such that for any* $\varphi\in\mathcal{D}_{L^2}^1(R^m)$

$$(3.12)\qquad \int_{R^m}|q||\varphi|^2dx\leq\eta\||\nabla\varphi|\|_{R^m}^2+c(\eta)\|\varphi\|_{R^m}^2,\quad\textit{where}\quad |\nabla\varphi|=\left(\sum_{j=1}^m\left|\frac{\partial\varphi}{\partial x_j}\right|^2\right)^{1/2}.$$

Next we have

Lemma 3. *For any* $\psi\in\mathcal{D}_{L^2}^2(R^6)$, $L_4[\psi]\geq\mu\|\psi\|_{\Omega_4}^2$.

Proof. By (2.4) and (3.1), we have

$$L_4[\psi]=\int_{\Omega_4}\{|\nabla_1\psi|^2+|\nabla_2\psi|^2+P|\psi|^2\}\,dx+\int_{\Omega_4}(q_1+q_2)|\psi|^2dx$$
$$\geq\mu\|\psi\|_{\Omega_4}^2\,.$$

q.e.d.

Last of all we shall show the following lemma. The method of proof is similar to that of Glazman [4], who has used it for the Schrödinger operators for two particle systems.

Lemma 4. *There exists some finite dimensional subspace* $\mathfrak{M}$ *in* $L^2(R^6)$ *such that for any* $\psi\in\mathcal{D}^2_{L^2}(R^6)\cap\mathfrak{M}^\perp$

$$L_1[\psi]\geq\mu\|\psi\|^2_{\Omega_1}, \tag{3.13}$$

where $\mathfrak{M}^\perp$ *denotes the orthogonal complement subspace of* $\mathfrak{M}$ *in* $L^2(R^6)$.

Before proving Lemma 4, we introduce the function space $\mathcal{E}^n_{L^2}(\Omega)=\{f: D^\alpha f\in L^2(\Omega)$ for any $\alpha(|\alpha|\leq n)\}$, where derivatives are taken in the distribution sense, and bring out the next lemma (see e.g. Mizohata [8]).

Lemma 5. *There exists an "extension operator"* Φ *which maps* $\mathcal{E}^1_{L^2}(\Omega_1)$ *to* $\mathcal{D}^1_{L^2}(R^6)$ *and some constant* $\tilde{c}=\tilde{c}(\Omega_1,\Phi)>0$ *such that for any* $\varphi\in\mathcal{E}^1_{L^2}(\Omega_1)$

$$(\Phi\varphi)(x)=\varphi(x) \qquad \textit{for} \quad x\in\Omega_1, \tag{3.14}$$

and

$$\begin{cases}\|\Phi\varphi\|^2_{\mathcal{D}^1_{L^2}(R^6)}\leq\tilde{c}\|\varphi\|^2_{\mathcal{E}^1_{L^2}(\Omega_1)}, \\ \|\Phi\varphi\|^2_{R^6} \qquad \leq\tilde{c}\|\varphi\|^2_{\Omega_1},\end{cases} \tag{3.15}$$

where $\|\varphi\|^2_{\mathcal{E}^1_{L^2}(\Omega_1)}=\{\|\varphi\|^2_{\Omega_1}+\||\nabla\varphi|\|^2_{\Omega_1}\}$.

Proof of Lemma 4. By Lemma 2 and Lemma 5, we have for any $\psi\in\mathcal{D}^2_{L^2}(R^6)$

$$\begin{aligned}\left|\int_{\Omega_1}q_i|\psi|^2dx\right|&\leq\int_{R^6}|q_i||\Phi\psi|^2dx\leq\eta\||\nabla(\Phi\psi)|\|^2_{R^6}+c(\eta)\|\Phi\psi\|^2_{R^6}\\ &\leq\tilde{c}\eta\||\nabla\psi|\|^2_{\Omega_1}+(\eta+c(\eta))\tilde{c}\|\psi\|^2_{\Omega_1} \qquad (i=1,2).\end{aligned} \tag{3.16}$$

Then by (2.4) we have for any $\psi\in\mathcal{D}^2_{L^2}(R^6)$

$$L_1[\psi]\geq(1-2\tilde{c}\eta)\||\nabla\psi|\|^2_{\Omega_1}-2(\eta+c(\eta))\tilde{c}\|\psi\|^2_{\Omega_1}. \tag{3.17}$$

Now let $A=-\Delta$ and its definition domain $D(A)=\left\{f: f\in\mathcal{E}^2_{L^2}(\Omega_1)\right.$ and $\left.\left.\frac{\partial f}{\partial n}\right|_{\partial\Omega_1}=0\right\}$, where $\frac{\partial f}{\partial n}$ means the derivative along the normal to the boundary $\partial\Omega_1$ of Ω_1. It is known that A is a non-negative selfadjoint operator in $L^2(\Omega_1)$ and the eigenvalue problem for A can be solved by the variational method concerning the form $|||\nabla\varphi|||^2_{\Omega_1}$ in the admissible function space $\mathcal{E}^1_{L^2}(\Omega_1)$ (see e.g. Courant-Hilbert [2], [3]). We choose $\eta>0$ to satisfy $2\tilde{c}\eta<1$. Since the spectrum of A consists only of discrete eigenvalues, the number of eigenvalues smaller than $\dfrac{2(\eta+c(\eta))\tilde{c}+\mu}{1-2\tilde{c}\eta}$ is finite. Let this number be p, where multiple eigenvalues are counted repeatedly, and $\{\varphi_n\}_{n=1,\cdots,p}\subset D(A)$ be orthonormal eigenfunctions in $L^2(\Omega_1)$ belonging to these eigenvalues. We define $\tilde{\varphi}_n(x)\in L^2(R^6)$ by $\tilde{\varphi}_n(x)=\varphi_n(x)$ for $x\in\Omega_1$ and $\tilde{\varphi}_n(x)=0$ for $x\notin\Omega_1$. Let $\mathfrak{M}$ be the subspace of $L^2(R^6)$ spanned by $\{\tilde{\varphi}_n\}_{n=1,\cdots,p}$. Then the dimension of $\mathfrak{M}$ is finite, and by (3.17) we have for any $\psi\in\mathcal{D}^2_{L^2}(R^6)\cap\mathfrak{M}^\perp$,

$$L_1[\psi]\geq\mu||\psi||^2_{\Omega_1}. \tag{3.18}$$

q.e.d.

Proof of (iii) of Theorem 1. Let $E(\lambda)$ be the right-continuous resolution of the identity associated with H. If the dimension of the subspace $E(\mu-0)L^2(R^6)$ is larger than that of $\mathfrak{M}$, we choose some constant $\delta>0$ and some function $\psi\in E(\mu-\delta)L^2(R^6)\subset\mathcal{D}^2_{L^2}(R^6)$ to satisfy $\psi\in\mathfrak{M}^\perp$ and $||\psi||_{R^6}\neq0$. Then by $\psi\in E(\mu-\delta)L^2(R^6)$ we have $L[\psi]\leq(\mu-\delta)||\psi||^2_{R^6}$. On the other hand, by Lemma 1, Lemma 3 and Lemma 4 we have $L[\psi]\geq\mu||\psi||^2_{R^6}$. These two inequalities are incompatible. Therefore there exists at most a finite number of discrete eigenvalues in $(-\infty,\mu)$. q.e.d.

Remark 4. As for the operator of the form (1.1), there exists some $Z_0(1>Z_0>0)$ such that for any $Z(0<Z<Z_0)$ the operator (1.1) has no discrete eigenvalues.

Indeed, instead of (3.16) we have

$$(3.16)' \quad \left|\int_{\Omega_1} q_i|\psi|^2dx\right|\leq Z\tilde{c}\eta|||\nabla\psi|||^2_{\Omega_1}+(\eta+c(\eta))\tilde{c}Z||\psi||^2_{\Omega_1} \qquad (i=1,2).$$

Then if we take into consideration $P(r_1,r_2)>\dfrac{1}{2R}$ on Ω_1, we have for any $\psi\in\mathcal{D}^2_{L^2}(R^6)$

$$(3.17)' \quad L_1[\psi]\geq(1-2\tilde{c}\eta Z)|||\nabla\psi|||^2_{\Omega_1}+\Big(\frac{1}{2R}-2Z(\eta+c(\eta)\tilde{c}\Big)||\psi||^2_{\Omega_1}$$

in place of (3.17). Take Z sufficiently small. We have for any $\psi\in\mathcal{D}_{L^2}^2(R^6)$

$$L_1[\psi]\mu\geq\|\psi\|_{\Omega_1}^2. \tag{3.18'}$$

By Lemma 1, Lemma 3 and (3.18)′, there exists no discrete eigenvalues in $(-\infty, \mu)$.

Proof of Theorem 2. Under the conditions (2.2), (2.3) and (2.15), the selfadjoint operators $H_i(i=1,2)$ in $L^2(R^3)$ (whose domains $D(H_i)$ are $\mathcal{D}_{L^2}^2(R^3)$) has at most a finite number of discrete eigenvalues in $(-\infty, 0)$ and $\sigma_e(H_i)=[0, \infty)$ (see, e.g. Birman [1]). Let the discrete eigenvalues of H_i be $\lambda_{i,1}\leq\lambda_{i,2}\leq\cdots\leq\lambda_{i,n_i}<0$, if they exist, and the orthonormal eigenfunctions belonging to these eigenvalues be $\{\varphi_{i,k}(r_i)\}_{k=1,\cdots,n_i;\, i=1,2}\subset\mathcal{D}_{L^2}^2(R^3)$. Let $\mathfrak{N}$ be the finite dimensional subspace in $L^2(R^6)$ spanned by $\{\varphi_{1,k}(r_1)\varphi_{2,l}(r_2)\}_{k=1,\cdots n_1,\,;\, l=1,\cdots,n_2}$, and T be the operator of the form

$$T=-\Delta_1-\Delta_2+q_1(r_1)+q_2(r_2). \tag{3.19}$$

If $D(T)=\mathcal{D}_{L^2}^2(R^3)$, T is a selfadjoint operator in $L^2(R^6)$ by Theorem 1 (i). Then we have

Lemma 6. *If there exists some $f\in\mathcal{D}_{L^2}^2(R^6)\cap\mathfrak{N}^{\perp}$ such that $Tf=\kappa f$ and $\|f\|_{R^6}=1$, then $\kappa\geq\mu$, where*

$$\mu=\begin{cases}\min(\lambda_{1,1}, \lambda_{2,1}), & \textit{if } \lambda_{1,1} \textit{ or } \lambda_{2,1} \textit{ exists,}\\ 0, & \textit{if neither } \lambda_{1,1} \textit{ nor } \lambda_{2,1} \textit{ exists.}\end{cases} \tag{3.20}$$

Proof. Let $\mu<0$ and $\kappa<\mu$. Put $f_{1,k}(r_2)=\int_{R^3}f(r_1, r_2)\overline{\varphi_{1,k}(r_1)}dr_1$, then we have $f_{1,k}(r_2)\in\mathcal{D}_{L^2}^2(R^3)$ and

$$\begin{aligned}(\kappa-\lambda_{1,k})f_{1,k}(r_2) &= \int_{R^3}Tf(r_1, r_2)\overline{\varphi_{1,k}(r_1)}dr_1\\ &-\int_{R^3}H_1f(r_1, r_2)\overline{\varphi_{1,k}(r_1)}dr_1 = H_2f_{1,k}(r_2).\end{aligned} \tag{3.21}$$

By (3.21) we have $f_{1,k}(r_2)=0$ or $f_{1,k}(r_2)$ is an eigenfunction for H_2 belonging to the eigenvalue $\kappa-\lambda_{1,k}<0$. In the latter case, $f_{1,k}(r_2)$ is represented by $f_{1,k}(r_2)=\sum_{l=1}^{n_2}c_{k,l}\varphi_{2,l}(r_2)$. By $f\in\mathfrak{N}^{\perp}$ we have $c_{k,l}=((f_{1,k}(r_2), \varphi_{2,l}(r_2))_{R^3}=(f,\varphi_{1,k}\cdot\varphi_{2,l})_{R^6}=0$. Thus we have $f_{1,k}(r_2)=0$ for any $k(1 \leq k \leq n_1)$. Since $\mathcal{D}^2_{L^2}(R^m) = \mathcal{E}^2_{L^2}(R^m)$, by Fubini's theorem $f(r_1, r_2) \in \mathcal{D}^2_{L^2}(R^3)$ as a function of r_1, whenever a.e. $r_2 \in R^3$ is fixed. Then by $\int_{R^3} f(r_1, r_2)\overline{\varphi_{1,k}(r_1)}dr_1 = 0$ ($k=1, \cdots, n_1$) and $\sigma_e(H_1) = [0, \infty)$, we have for a.e. $r_2 \in R^3$ $\int_{R^3} H_1 f \cdot \bar{f} dr_1 \geq 0$. By integrating this inequality over R^3 with respect to r_2, we have

$$(3.22) \qquad (H_1 f, f)_{R^6} \geq 0 .$$

In a similar fashion, we have

$$(3.23) \qquad (H_2 f, f)_{R^6} \geq 0 .$$

Then by (3.22) and (3.23)

$$(3.24) \qquad \kappa = (Tf, f)_{R^6} = (H_1 f, f)_{R^6} + (H_2 f, f)_{R^6} \geq 0 ,$$

which contradicts $\kappa < \mu < 0$. In case $\mu = 0$, $H_i (i=1, 2)$ are non-negative operator in R^3, and so we have (3.24). q.e.d.

Now we continue the proof of Theorem 2. By Lemma 6, T has only discrete eigenvalues $\{\lambda_{1,k} + \lambda_{2,l}\}$ in $(-\infty, \mu)$ and eigenfunctions $\{\varphi_{1,k}(r_1)\varphi_{2,l}(r_2)\}$, if $\lambda_{1,k} + \lambda_{2,l} < \mu$. By (ii) of Theorem 1 we have $\sigma_e(H) = \sigma_e(T) = [\mu, \infty)$, where μ is given by (3.20). Then for any $f \in \mathcal{D}^2_{L^2}(R^6) \cap \mathfrak{N}^{\perp}$ we have

$$(3.25) \qquad (Tf, f)_{R^6} \geq \mu \|f\|^2_{R^6} .$$

By (3.25) and (2.4), we have for any $f \in \mathcal{D}^2_{L^2}(R^6) \cap \mathfrak{N}^{\perp}$

$$(3.26) \qquad (Hf, f)_{R^6} \geq (Tf, f)_{R^6} \geq \mu \|f\|^2_{R^6} .$$

Since $\mathfrak{N}$ is a finite dimensional subspace in $L^2(R^6)$, we have the assertion of Theorem 2 by the same method as applied to the proof of (iii) of Theorem 1. q.e.d.

Remark 5. Let

$$(3.27) \qquad H = \sum_{\nu=1}^{2} \left\{ \sum_{j=0}^{2} \left(\frac{1}{i} \frac{\partial}{\partial x_{3\nu-j}} + b_{3\nu-j}(r_\nu) \right)^2 + q_\nu(r_\nu) \right\} + P(r_1, r_2)$$
$$= H_1 + H_2 + P(r_1, r_2) ,$$

where $H_\nu = \sum_{j=0}^{2} \left(\frac{1}{i} \frac{\partial}{\partial x_{3\nu-j}} + b_{3\nu-j}(r_\nu) \right)^2 + q_\nu(r_\nu)$. If we assume (2.2)–(2.5) and

(3.28) $|b_{3\nu-j}(r_\nu)| \leq \dfrac{\text{const}}{r_\nu^{1+\varepsilon}}$ for $r_\nu \geq R$ $(\nu = 1, 2\,;\ j = 1, 2, 3)$,

(3.29) $b_{3\nu-j}(r_\nu) \in \mathscr{B}^1(R^3)$ are real-valued functions $(\nu = 1, 2\,;\ j = 1, 2, 3)$,

(3.30) $q_\nu(r_\nu) \geq -\dfrac{\text{const}}{r_\nu^{2+\varepsilon}}$ for $r_\nu \geq R$ $(\nu = 1, 2)$,

where $f(x) \in \mathscr{B}^1(R^3)$ means that $f(x)$ has continuous derivatives of first order in R^3 and $\sup_{x\in R^3} |f(x)| + \sup_{x\in R^3} \sum_{k=1}^{3} \left|\dfrac{\partial f}{\partial x_k}(x)\right| < +\infty$, then we have the same results as Theorem 2.

In fact if $D(H) = \mathscr{D}^2_{L^2}(R^6)$, H is a lower semi-bounded selfadjoint operator in $L^2(R^6)$ and $\sigma_e(H) = [\mu, \infty)$, where

(3.31) $$\mu = \min_{\nu=1,2} \inf_{\substack{\|\varphi\|_{R^3}=1 \\ \varphi \in \mathscr{D}^{L^2}_2(R^3)}} (H_\nu \varphi, \varphi)$$

(see, Jörgens [6]). On the other hand the operators $H_\nu(\nu=1, 2)$ in $L^2(R^3)$ have at most a finite number of discrete eigenvalues in $(-\infty, 0)$ and $\sigma_e(H_\nu) = [0, \infty)$ (see, e.g. Uchiyama [10]). Then we have for the operator $H_1 + H_2$ the same results as Lemma 6 and we can prove the assertion in a similar fashion to the proof of Theorem 2.

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