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# On the Structure of $(n-1)$-connected $2n$-dimensional $\pi$-manifolds

Dedicated to Professor Atuo Komatu
on his 60th birthday

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## Introduction

In this paper, we will try to decompose $(n-2)$-connected $2n$-dimensional closed $\pi$-manifolds into a connected sum of certain familiar manifolds. Our main theorems are given in the section 5. I show there that under some conditions such $\pi$-manifolds are decomposed as a connected sum of a homotopy $2n$-sphere, some copies of the product of the original $n$-spheres, the total spaces of some $(n-1)$-sphere bundles over $(n+1)$-spheres, and the boundary of a handlebody. And I give a sufficiency condition so that the handlebody may vanish.

Throughout this paper, all manifolds are $C^\infty$ and compact connected.

I would like to express my thanks to Professor N. Shimada for his kind advices.

## 1. Notes for $(n-1)$-connected Case

**Lemma 1.1.** *Let* $M^{2n}$ *be an* $(n-1)$*-connected* $2n$*-dimensional closed* $\pi$*-manifold* $(n\geqslant 3)$. *We assume that* $\operatorname{Arf} M=0$ *if* $n=4k+3$. *Then, there exists such basis* $\{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_p\}$ *for* $H_nM$ *with*

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*intersection numbers* $\lambda_i \cdot \lambda_j = \mu_i \cdot \mu_j = 0$, $\lambda_i \cdot \mu_j = \delta_{ij}$ *that the imbedded n-spheres* $S_i^n$, $S'^n_j$ *representing* $\lambda_i$, $\mu_j$ *respectively have trivial normal bundles.*

**Proof.** If $n$ is even, the assertion is well known from Lemma 9 and Lemma 7 of [7]. Let $n$ be odd. There exists a symplectic basis $\{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_p\}$ such that $\lambda_i \cdot \lambda_j = \mu_i \cdot \mu_j = 0$, $\lambda_i \cdot \mu_j = \delta_{ij}$ ([7]). The Arf invariant of $M$ is defined as $\operatorname{Arf} M = \varepsilon(\lambda_1) \cdot \varepsilon(\mu_1) + \varepsilon(\lambda_2) \cdot \varepsilon(\mu_2) + \cdots + \varepsilon(\lambda_p) \cdot \varepsilon(\mu_p) \pmod 2$, where $\varepsilon$ is a certain function from $H_n M$ to $Z_2$ and satisfies the relation $\varepsilon(\lambda + \mu) = \varepsilon(\lambda) + \varepsilon(\mu) + \lambda \cdot \mu \pmod 2$ [5].

Now, if a pair $(\lambda_i, \mu_i)$ satisfies $\varepsilon(\lambda_i) = 0$, $\varepsilon(\mu_i) = 1$, replace these by $\lambda_i' = \lambda_i$, $\mu_i' = \lambda_i + \mu_i$. Then, we know $\varepsilon(\lambda_i') = 0$, $\varepsilon(\mu_i') = \varepsilon(\lambda_i) + \varepsilon(\mu_i) + \lambda_i \cdot \mu_i = 0 \pmod 2$. The case when $\varepsilon(\lambda_i) = 1$, $\varepsilon(\mu_i) = 0$ is also similar. If two pairs $(\lambda_i, \mu_i)$, $(\lambda_j, \mu_j)$ satisfy $\varepsilon(\lambda_i) = \varepsilon(\mu_i) = \varepsilon(\lambda_j) = \varepsilon(\mu_j) = 1$, replace these pairs by $\lambda_i' = \lambda_i + \lambda_j$, $\mu_i' = \lambda_i + \lambda_j + \mu_i$ and $\lambda_j' = \mu_i - \mu_j$, $\mu_j' = \lambda_j + \mu_i - \mu_j$. Then, similarly we know $\varepsilon(\lambda_i') = \varepsilon(\mu_i') = \varepsilon(\lambda_j') = \varepsilon(\mu_j') = 0$. Since we assumed that $\operatorname{Arf} M = 0$, thus we have a new basis $\{\lambda_1', \cdots, \lambda_p', \mu_1', \cdots, \mu_p'\}$ which satisfies $\varepsilon(\lambda_i') = \varepsilon(\mu_j') = 0$. We note, if we represent $\lambda_i$, $\mu_j$ by imbedded spheres, we can identify $\varepsilon(\lambda_i)$, $\varepsilon(\mu_j)$ as the characteristic elements ($\in Z_2$) of the normal bundles of those spheres. This completes the proof.

From this lemma, we can easily show that an $(n-1)$-connected $2n$-dimensional closed $\pi$-manifold $M^{2n}$ ($n \geqslant 3$) is, under the assumption that $\operatorname{Arf} M = 0$ when $n = 4K+3$, diffeomorphic to $S^n \times S^n \# \cdots \# S^n \times S^n \# \tilde{S}^{2n}$, that is, a connected sum of $p$ copies of $S^n \times S^n$ and a homotopy sphere $\tilde{S}^{2n}$, where $S^n$ is the $n$-dimensional ordinary sphere and $2p$ is the rank of $H_n M$.

**Notes.** The above shows that any two differentiable $\pi$-structures on an $(n-1)$-connected $2n$-dimensional closed manifold are equivalent modulo $\theta_{2n}$. On the other hand, R.K. Lashof has shown in [6] that if two given $(n-1)$-connected $2n$-dimensional closed differentiable manifolds have a homotopy equivalence which induces the stable equivalence of those tangent bundles, then they are diffeomorphic modulo $\theta_{2n}$. This shows that any two differentiable structures $\mathfrak{D}_1$, $\mathfrak{D}_2$ on an $(n-1)$-connected $2n$-dimensional closed manifold $M^{2n}$ are equivalent modulo $\theta_{2n}$ if $n\equiv 0,3,4,5,6,7 \pmod 8$.[1] It is clear if $n\equiv 3,5,6,7 \pmod 8$, since $M^{2n}$ is almost parallelizable. If $n\equiv 0,4 \pmod 8$, the obstruction to construct a stable equivalence of the tangent bundles $\tau_1=\tau(\mathfrak{D}_1)$, $\tau_2=\tau(\mathfrak{D}_2)$ is given by the differences of the Pontryagin classes, $P_k(\tau_2)-P_k(\tau_1)$ and $P_{2k}(\tau_2)-P_{2k}(\tau_1)$ $(n=4k)$. But these obstructions vanish from the topological invariance of rational Pontryagin classes [8].

## 2. Surgeries

In this section we study that if we kill elements of the $(n-1)$-th homology group of a given $2n$-dimensional manifold by surgeries, then how it affects the $n$-th homology group of the modified manifold.

Let $M^{2n}$ be a $2n$-dimensional closed manifold and let $\varphi$: $S^{n-1}\times D^{n+1}\to M^{2n}$ be an imbedding. We donote by $\lambda$ the homology class of $\varphi(S^{n-1}\times 0)$. Let $M'^{2n}=\chi(M,\varphi)$ be the modified manifold [7] and let $\varphi': D^n\times S^n\to M'^{2n}$ be the dual of $\varphi$. We denote by $\lambda'$ the homology class of $\varphi'(0\times S^n)$. Let $M_0=M-\mathrm{Int}\,\varphi(S^{n-1}\times D^{n+1})$. This is also equal to $M'-\mathrm{Int}\,\varphi'(D^n\times S^n)$.

**Lemma 2.1.** *If the order of $\lambda$ is infinite, then $\lambda'$ must be zero or a torsion element.*

(1) *If $\lambda'$ is zero, the homomorphisms*

$$H_nM \xrightarrow{i_*} H_nM_0 \xrightarrow{i'_*} H_nM'$$

*are respectively isomorphisms, where, $i$, $i'$ denote inclusion maps.*

(2) *If $\lambda'$ is a torsion element, the homomorphisms*

$$FH_nM \xrightarrow{i_*} FH_nM_0 \xrightarrow{i'_*} FH_nM'$$

$$TH_nM \xrightarrow{i_*} TH_nM_0/(\varphi(*\times S^n)) \xrightarrow{i'_*} TH_nM'/(\lambda')$$

1) Mr. H. Sato informed me that he proved all the case.

*are respectively isomorphisms, where* $F(\ )$ *and* $T(\ )$ *denote the free part and the torsion part of the group respectively.*

**Proof.** By excision, there are isomorphisms

$$H_i(M, M_0) \cong H_i(S^{n-1}\times D^{n+1},\ S^{n-1}\times S^n) \cong \begin{cases} Z & \text{for } i=n+1,\ 2n \\ 0 & \text{otherwise} \end{cases}$$

$$H_j(M', M_0) \cong H_j(D^n\times S^n,\ S^{n-1}\times S^n) \cong \begin{cases} Z & \text{for } j=n,\ 2n \\ 0 & \text{otherwise.} \end{cases}$$

So, considering the homology exact sequences of $(M, M_0)$ and $(M', M_0)$, we have the following commutative diagram

$$\begin{array}{ccccccccccc} & & Z & \overset{\lambda'}{\searrow} & & & & & & & \\ & & \downarrow{\scriptstyle \varepsilon'} & & & & & & & & \\ 0 & \longrightarrow & M_nM_0 & \xrightarrow{i'_*} & H_nM' & \xrightarrow{\cdot\lambda'} & Z & \xrightarrow{\varepsilon} & H_{n-1}M_0 & \longrightarrow H_{n-1}M' & \longrightarrow 0 \\ & & \downarrow{\scriptstyle i_*} & & & & & & & & \\ & & H_nM & & & & & & & & \\ & & \downarrow & & & & & & & & \\ & & 0 & & & & & & & & \end{array}$$

such that the horizontal and vertical sequences are exact (cf. [5, Lemma 5.6]).

Here $\lambda': Z\to H_nM'$ denotes the homomorphism which caries 1 into $\lambda'$, and $\cdot\lambda': H_nM'\to Z$ denotes the homomorphism which caries each element of $H_nM'$ into the intersection number with $\lambda'$. We note that $\varepsilon'(1)$ is the homology class of $\varphi(*\times S^n)$ and $\varepsilon(1)$ is the homology class $i_*^{-1}(\lambda)$ where $i_*: H_{n-1}M \longrightarrow H_{n-1}M$ is an isomorphism.

Since we assumed that the order of $\lambda$ is infinite, also the order of $i_*^{-1}\lambda$ is infinite. So, $\operatorname{Ker}\varepsilon$ is equal to zero. Therefore, $i'_*: H_nM_0\to H_nM'$ is an isomorphism. On the otherhand, since any intersection number with $\lambda'$ is zero, $\lambda'$ must be zero or a torsion element by Poincaré duality. (1) If $\lambda'=0$, $\varepsilon'(1)$ must be zero, so that $i_*: H_nM_0\to H_nM$ is an isomorphism. (2) Let $\lambda'$ be a torsion element. At the short,exact sequence $Z \xrightarrow{\varepsilon'} H_nM_0 \xrightarrow{i_*} H_nM \longrightarrow 0$, $\varepsilon'(1)$ is a torsion element. So, $\varepsilon'(Z)=\operatorname{Ker} i_*$ is a subgroup of $TH_nM_0$. It is easy to see that $i_*(TH_nM_0)=TH_nM$. Therefore we have an exact sequence

$0\longrightarrow \mathcal{E}'(Z)\longrightarrow TH_nM_0 \xrightarrow{i_*} TH_nM\longrightarrow 0$. Since there is an isomorphism $i'_*: TH_nM_0\to TH_nM'$, we have the half of the desired relation of (2). On the other hand, it is easy to see that

$$H_nM=i_*(FH_nM_0)+i_*(TH_nM_0) = i_*(FH_nM_0)+TH_nM\,.$$

Since $i_*: FH_nM_0\to i_*(FH_nM_0)$ maps isomorphically, we may adopt $i_*(FH_nM_0)$ as a free part of $H_nM$. We denote this by $FH_nM$. We also adopt $i'_*(FH_nM_0)$ as a free part of $H_nM'$ and denote this by $FH_nM'$. Thus we have the desired isomorphisms of the rest of (2). This completes the proof.

**Lemma 2.2.** *If $\lambda$ is a torsion element, then the order of $\lambda'$ is infinite, and*

$$\text{rank } H_nM' = \text{rank } H_nM+2\,.$$

**Proof.** Let the order of $\lambda$ be $P$. Since $i_*: H_{n-1}M_0\to H_{n-1}M$ is an isomorphism and $\mathcal{E}(1)=i_*^{-1}(\lambda)$, we have the following short exact sequence from the above diagram.

$$0 \longrightarrow H_nM_0 \longrightarrow H_nM' \xrightarrow{\cdot\lambda'} (P) = \text{Ker}\,\mathcal{E} \longrightarrow 0\,,$$

where $(P)$ is the subgroup of $Z$ generated by $P$.
Since this sequence splits, there is an isomorphism

$$H_nM' \cong H_nM_0+(P)\,.$$

We note that $\lambda'$ is not a torsion element and so $\mathcal{E}'(Z)\cong\lambda'(Z)\cong Z$ at the above diagram.
Thus we have,

$$\begin{aligned}\text{rank } H_nM' &= \text{rank } H_nM_0+1\\ \text{rank } H_nM &= \text{rank } (H_nM_0/\mathcal{E}'(Z)) = \text{rank } H_nM_0-1\end{aligned}$$

This completes the proof.

**Proposition 2.3.** *Let $M^{2n}$ be an $(n-2)$-connected $2n$-dimensional closed $\pi$-manifold and suppose that $H_{n-1}M$ has no torsion subgroup. Then we can kill $H_{n-1}M$ so that the surgeries do not affect $H_nM$, that is, the produced $(n-1)$-connected $2n$-dimensional $\pi$-manifold has the same $n$-th homology group as $M^{2n}$.*

**Proof.** Let $H_{n-1}M \cong Z+\cdots+Z$ with generators $\lambda_1, \cdots, \lambda_r$. Let $\varphi_1: S^{n-1}\times D^{n+1}\to M^{2n}$ be an imbedding such that $\varphi_1(S^{n-1}\times 0)$ represents $\lambda_1$, and let $M'^{2n}=\chi(M, \varphi_1)$. Since $H_{n-1}M' \cong H_{n-1}M/(\lambda_1)$, $H_{n-1}M'$ has no torsion. Therefore, by the universal coefficient theorem $H_nM' \cong H^nM \cong \mathrm{Hom}\,(H_nM', Z)+\mathrm{Ext}\,(H_{n-1}M', Z)$, where the torsion part vanishes.

This means that $\lambda_1'=0$. Thus we have the isomorphisms $H_nM \xleftarrow{i_*} H_nM_0 \xrightarrow{i'_*} H_nM'$ from Lemma 2.1. Repeating this, we have the proposition.

### 3. Splitting Theorems

Using the results of sections 1 and 2, we can decompose $(n-2)$-connected $2n$-dimensional $\pi$-manifolds.

**Theorem 3.1.** *Let $M^{2n}$ be a $(n-2)$-connected $2n$-dimensional closed $\pi$-manifold $(n\geqslant 3)$ such that $H_{n-1}M$ has no torsion. Then there exists the following decomposition;*

$$M^{2n} = S^n\times S^n \# \cdots \# S^n\times S^n \# M_1^{2n},$$

*where $S^n$ is the ordinal $n$-sphere and $M_1^{2n}$ is a $(n-2)$-connected $2n$-dimensional closed $\pi$-manifold such that*

$$H_iM_1 \cong \begin{cases} H_iM & \text{if } \ i=n-1, n+1 \\ 0 & \text{if } \ i=n. \end{cases}$$

(*We assume that the* Arf *invariant is zero if* $n=4k+3$.)

**Proof.** Let $H_{n-1}M \cong Z+\cdots+Z$ with generators $\kappa_1, \kappa_2, \cdots, \kappa_r$. Let $\varphi_i: S^{n-1}\times D^{n+1}\to M^{2n}$ $i=1, 2, \cdots, r$ be imbeddings such that each $\varphi_i(S^{n-1}\times 0)$ represents $\kappa_i$, and let $M_0=M-\bigcup_{i=1}^{r}\mathrm{Int}\,\varphi_i(S^{n-1}\times D^{n+1})$, $M^{2n}$ be the $(n-1)$-connected $2n$-dimensional $\pi$-manifold obtained by those spherical modifications $\chi(\varphi_1), \cdots, \chi(\varphi_r)$. From Proposition 2.3, we have the isomorphisms

$$H_nM \xleftarrow{i_*} H_nM_0 \xrightarrow{i'_*} H_nM'.$$

where $i, i'$ are inclusion map.

On the other hand, by Lemma 1.1 there exists such basis $\{\lambda_1, \cdots, \lambda_p, \mu_1, \cdots, \mu_p\}$ for $H_nM'$ with intersection numbers $\lambda_i\cdot\lambda_j=\mu_i\cdot\mu_j=0$, $\mu_i\cdot\mu_j=\delta_{ij}$ that every imbedded $n$-sphere which represents these homology classes has trivial normal bundle. Let $\lambda'_i=i_*^{-1}(\lambda_i)$, $\mu'_j=i_*^{-1}(\mu_j)$ for $i, j=1, 2, \cdots, p$. Then $\{\lambda'_1, \cdots, \lambda'_p, \mu'_1, \cdots, \mu'_p\}$ is a basis for $H_nM_0$ with intersection numbers $\lambda'_i\cdot\lambda'_j=\mu'_i\cdot\mu'_j=0$, $\lambda'_i\cdot\mu'_j=\delta_{ij}$. Since $M_0$ is also $(n-2)$-connected, by Hurewicz's Theorem any element of $H_nM_0$ is spherical. So we can represent $\lambda'_i$, $\mu'_j$ by imbeded spheres $S^n_i$, $S'^n_j$. Using Whitney's method we may assume that $S^n_i$ and $S'^n_i$ meet transversely at only one point and any other pair of spheres does not intersect. We note that $S^n_i, S'^n_j$ also represent $\lambda_i$, $\mu_j$ respectively and that whether the normal bundles of $S^n_i$ and $S'^n_j$ are trivial or not depends only on the homology classes $\lambda_i$, $\mu_j$ respectively [5, Lemma 8.3]. So the normal bundles of $S^n_i$ and $S'^n_j$ in $M_0$ are trivial. Therefore $S^n_i, S'^n_j$ in $M^{2n}$ make a basis for $H_nM$ with trivial normal bundles.

The tubular neighbourhood of $S_i\vee S'_i$ in $M^{2n}$ makes a plumbing manifold $S^n_i\times D^n \veebar S'^n_i\times D^n$ with the boundary $S^{2n-1}$ for each $i$, and $S^n_i\times D^n\veebar S'^n_i\times D^n$ is diffeomorphic to $S^n\times S^n-\operatorname{Int} D^{2n}$. Let $N=M-\bigcup_{i=1}^{p}\operatorname{Int}(S^n_i\times D^n\veebar S'^n_i\times D^n)$ and attach $p$ copies of $D^{2n}$ to $N$. Then we have a closed manifold $M^{2n}_1$ which is almost parallelizable. Thus we can decompose $M^{2n}$ as $M^{2n}=S^n\times S^n\#\cdots\# S^n\times S^n\# M^{2n}_1$.

$M^{2n}_1$ is simply connected by van Kampen Theorem and has such homology groups as asserted using the Mayer-Vietoris sequence. We note that $M^{2n}_1$ is a $\pi$-manifold since the Index of $M^{2n}_1$ is zero.

This completes the proof.

**Remark.** In theorem 3.1, if $M^{2n}$ is $(n-1)$-connected then $M^{2n}_1$ is a homotopy sphere. This induces the form asserted in section 1.

**Theorem 3.2.** *$M^{2n}_1$ is decomposed as the form $M^{2n}_1=\tilde{S}^{2n}\#\partial W^{2n+1}$, where $\tilde{S}^{2n}$ is a homotopy 2n-sphere and $W^{2n+1}$ is a handlebody* $D^{2n+1}\bigcup_{\{\varphi_i\}}\{\bigcup_{i=1}^{r} D^{n+1}_i\times D^n_i\}$, $r=\operatorname{rank} H_{n-1}M$.

**Proof.** If we kill the generators of $H_{n-1}M_1$, then by proposition

2.3 we have a homotopy sphere $\tilde{S}^{2n}$. So, from the manifold $M_1^{2n}\#(-\tilde{S}^{2n})$ we obtain the standard sphere $S^{2n}$ by the surgery. Thus we may assume that the surgery deforms $M_1^{2n}$ to the standard sphere $S^{2n}$. This means that $M_1^{2n}$ can be obtained from $S^{2n}$ by the converse construction, that is, by surgery on a disjoint set of imbeddings $\varphi_i: S^n\times D^n\to S^{2n}$ $i=1, 2, \cdots, r$. Thus $M_1^{2n}$ is clearly the boundary of a handlebody $W^{2n+1}=D^{2n+1}\underset{\{\varphi_i\}}{\cup}\{\overset{r}{\underset{i=1}{\cup}}D_i^{n+1}\times D_i^n\}$. This completes the proof.

## 4. Linking Elements

Let $W^{2n+1}=D^{2n+1}\underset{\{\varphi_i\}}{\cup}\{\overset{r}{\underset{i=1}{\cup}}D_i^{n+1}\times D_i^n\}$ be a handlebody with attaching maps $\varphi_i: S^n\times D^n\to S^{2n}$ $i=1, 2, \cdots, r$, and let $M^{2n}=\partial W^{2n+1}$. When we restrict the imbeddings $\varphi_i$ to $\partial D_i^{n+1}\times 0=S_i^n\times 0$, we have an $n$-link $\varphi_1(S_1^n\times 0)\cup\varphi_2(S_2^n\times 0)\cup\cdots\cup\varphi_r(S_r^n\times 0)$ in $S^{2n}=\partial D^{2n+1}$. Let $S_1^n\cup S_2^n\cup\cdots\cup S_r^n$ be an $n$-link in $S^{2n}$ and let $X_i=S^{2n}-\{\underset{j\neq i}{\cup}S_j^n\}$. Then there is an isomorphism $\pi_n(X_i)\cong\pi_n(\underset{j\neq i}{\vee}S_j^{n-1})\cong Z_2+Z_2+\cdots+Z_2$ $(n\geqslant 4)$ and the $n$-sphere $S_i^n\subset X_i$ defines an element $\lambda^i$ of $\pi_n(\underset{j\ i}{\vee}S_j^{n-1})$ which is called the linking element of $S_i^n$ [3. p. 243]. $\lambda^i: i=1, 2, \cdots, r$ determine the isotopy class of the $n$-link $S_1^n\cup S_2^n\cup\cdots\cup S_r^n$.

In this section we study a sufficiency condition so that all the linking elements for the $n$-link $\varphi_1(S_1^n\times 0)\cup\varphi_2(S_2^n\times 0)\cup\cdots\cup\varphi_r(S_r^n\times 0)$ may be zero.

**Lemma 4.1.**

$$H_iM^{2n}\cong\begin{cases}\overbrace{Z+\cdots+Z}^{r} & \textit{if}\quad i=n-1, n+1\\ Z & \textit{if}\quad i=0, 2n\\ 0 & \textit{otherwise}\end{cases}$$

*and the generators are given as follows;*

$\varphi_i(x_i\times S_i^{n-1})$ $(x_i\in\partial D_i^{n+1},\ i=1, 2, \cdots, r)$ *generates* $H_{n-1}M^{2n}$

$(j_*)^{-1}(\psi_i(D_i^{n+1}\times y_i))$ $(y_i\in\partial D_i^n,\ i=1, 2, \cdots, r)$ *generates* $H_{n+1}M^{2n}$.

**Proof.** Let $S_0=S^{2n}-\overset{r}{\underset{i=1}{\cup}}\operatorname{Int}\varphi_i(S_i^n\times D_i^n)=M^{2n}-\overset{r}{\underset{i=1}{\cup}}\operatorname{Int}\psi_i(D_i^{n+1}\times S_i^{n-1})$.

From the homology exact sequences of $(M^{2n}, S_0)$ and $(S^{2n}, S_0)$, we have isomorphisms

$$Z \cong H_n(S^{2n}, S_0) \underset{\cong}{\overset{\partial_*}{\longrightarrow}} H_{n-1}(S_0) \underset{\cong}{\overset{i_*}{\longrightarrow}} H_{n-1}M$$

$$H_{n-1}M \underset{\cong}{\overset{j_*}{\longrightarrow}} H_{n+1}(M, S_0) \cong Z\,.$$

This implies the lemma.

**Proposition 4.2.** *If* $n \geqslant 4$ *and* $Sq^2 : H^{n-1}(M : Z_2) \to H^{n+1}(M ; Z_2)$ *is trivial, then the linking elements for the n-link* $\varphi_1(S_1^n \times 0) \cup \varphi_2(S_2^n \times 0) \cup \cdots \cup \varphi_r(S_r^n \times 0)$ *are all zero.*

**Proof.** Let $Y = S^{2n} - \bigcup_{i=1}^{r} \mathrm{Int}\,\varphi(S_i^n \times D_i^n)$ and let $S_i^{n-1} = D_i^{n-1} \cup D'^{n-1}_i$. We note that $M^{2n} = \partial W^{2n+1} = \{S^{2n} - \bigcup_{i=1}^{r} \mathrm{Int}\,\varphi_i(S_i^n \times D_i^n)\} \underset{\{\varphi_i\}}{\cup} \{\bigcup_{i=1}^{r} D_i^{n+1} \times S_i^{n-1}\}$. $Y \underset{\{\varphi'_i\}}{\cup} \{\bigcup_{i=1}^{r} D_i^{n+1} \times y_i\}$ is a deformation retract of $M^{2n} \bigcup_{i=1}^{r} \mathrm{Int}\,\{D_i^{n+1} \times D'^{n-1}_i\}$, where $\varphi'_i = \varphi_i \,|\, S_i^n \times y_i$ and $y_i \in D_i^{n-1} \subset S_i^{n-1}$. Then we have the following commutative diagram.

$$\begin{array}{ccc}
H^{n-1}(M ; Z_2) & \xrightarrow{\quad Sq^2 \quad} & H^{n+1}(M ; Z_2) \\
{\scriptstyle i^*} \big\downarrow \cong & & {\scriptstyle i^*} \big\downarrow \cong \\
H^{n-1}(Y \underset{\{\varphi'_i\}}{\cup} \{\bigcup_i D_i^{n+1} \times y_i\} ; Z_2) & \xrightarrow{\quad Sq^2 \quad} & H^{n+1}(Y \underset{\{\varphi'_i\}}{\cup} \{\bigcup_i D_i^{n+1} \times y_i\} ; Z_2) \\
{\scriptstyle F^*} \big\downarrow \cong & & {\scriptstyle F^*} \big\downarrow \cong \\
H^{n-1}(S_1^{n-1} \vee \cdots \vee S_r^{n-1} \underset{\{\omega_i\}}{\cup} \{\bigcup_i D_i^{n+1}\} ; Z_2) & \xrightarrow{Sq^2} & H^{n+1}(S_1^{n-1} \vee \cdots \vee S_r^{n-1} \underset{\{\omega_i\}}{\cup} \{\bigcup_i D_i^{n+1}\} ; Z_2)
\end{array}$$

The first part of the diagram is clear and the vertical maps induced by the inclusion map are isomorphisms.

The second part is given as follows. Let $X = S^{2n} - \bigcup_{i=1}^{r} \varphi_i(S_i^n \times 0)$ and let $S_i^{n-1}$ $i = 1, 2, \cdots, r$ be $r$ copies of $(n-1)$-spheres. Define a continuous map $f : S_1^{n-1} \vee S_2^{n-1} \vee \cdots \vee S_r^{n-1} \to X$ so that each $f(S_i^{n-1})$ is homotopic in $X$ to $\varphi_i(x_i \times S_i^{n-1})$ which has linking number $+1$ with $\varphi_i(S_i^n \times 0)$ in $S^{2n}$. Then, using Alexander duality, the map $f$ induces the isomorphisms of their homology groups up to dimension $2n-2$, so the isomorphisms of their homotopy groups up to dimension $2n-3$ $(n \geqslant 3)$. Since $Y$ is a deformation retract of $X$ we may assume that $f$ is a map into $Y$. Thus we have an isomorphism $f_*: \pi_n(\bigvee_{i=1}^{r} S_i^{n-1}) \to \pi_n(Y)$. Choose a map $\omega_i: S_i^n \to \bigvee_{i=1}^{n} S_i^{n-1}$ such that $f \circ \omega_i$ is homotopic to $\varphi_i'$. We may assume $\varphi_i' = f \circ \omega_i$. Define a continuous map $F: \bigvee_{i=1}^{r} S_i^{n-1} \bigcup_{\{\omega_i\}}^{r} \{\bigcup_{i=1} D_i^{n+1}\} \to Y \bigcup_{\{\varphi'_i\}}^{r} \{\bigcup_{i=1} D_i^{n+1} \times y_i\}$ as $F | \bigvee_i S_i^{n-1} = f$, $F | \bigcup_i D_i^{n+1} = \text{identity}$. Then, by Lemma 4.1, it is easy to see that $F^*$ carries the generators of the one onto the generators of the other. Therefore $F^*$ is an isomorphism and we have the second diagram.

Thus, if $Sq^2: H^{n-1}(M: Z_2) \to H^{n+1}(M; Z_2)$ is trivial, $\omega_i$ must be homotopic zero for each $i$. So, $\varphi_i'$ is homotopic zero in $X \subset S^{2n} - \bigcup_{j \neq i} \varphi_j(S_j^n \times 0)$. Since $\varphi_i'$ is homotopic to $\varphi_i | S_i^n \times 0$ in $S^{2n} - \bigcup_{j \neq i} \varphi_j(S_j^n \times 0)$, this implies that $\lambda^i = 0$ for all $i$. This completes the proof.

**Remark.** If $\varphi_i': S_i^n \times y_i \to S^{2n} - \varphi_i(S^n \times 0)$ is homotopic zero for all $i$, the converse of Proposition 4.2 is also valid.

## 5. Structure Theorems

Let $M^{2n}$ be a $(n-2)$-connected $2n$-dimensional closed $\pi$-manifold and assume that $H_{n-1}M$ is free with rank $r$ and $H_nM \cong 0$. Then, from Theorem 3.2, $M^{2n}$ is decomposed as $M^{2n} = \tilde{S}^{2n} \# \partial W^{2n+1}$. $W^{2n+1}$ is a handlebody $D^{2n+1} \bigcup_{\{\varphi_i\}} \{\bigcup_{i=1}^{r} D_i^{n+1} \times D_i^n\}$ with attaching maps $\varphi_i: S_i^n \times D_i^n \to S^{2n}$ $i = 1, 2, \cdots, r$. We study more precise structure of $M^{2n}$.

Let $\lambda^i$ be the linking element of $\varphi_i(S_i^n \times 0)$ for the $n$-link $\varphi_1(S_1^n \times 0) \cup \varphi_2(S_2^n \times 0) \cup \cdots \cup \varphi_r(S_r^n \times 0)$. $\lambda^i$ $i = 1, 2, \cdots, r$ are all zero if $r = 1$ or, by Proposition 4.2, if $Sq^2: H^{n-1}(M; Z_2) \to H^{n+1}(M; Z_2)$ is trivial. But $\lambda^i$ $i = 1, 2, \cdots, r$ are not always all zero. Now, we assume that $\lambda^1 = \lambda^2 = \cdots = \lambda^q = 0$ and $\lambda^i \neq 0$ for $i > q$.

Since $\lambda^1 = 0$, $\varphi_1 | S^n \times 0$ is homotopic to zero in $X_1 = S^{2n} - \bigcup_{j \neq 1} \varphi_j(S_j^n \times 0)$, which is 2-connected if $n \geqq 4$. So, by Haefliger [2], we know that $\varphi_1(S_1^n \times 0)$ is isotopic in $X_1$ to a $n$-sphere which bounds an imbedded $(n+1)$-disk $D^{n+1}$, and also that $D^{n+1}$ is contained in the interior of an imbedded $(n+1)$-disk $C^{n+1}$. Since there exists an isotopy $f_t$ of the identity of $S^{2n}$ such that $f_1 \circ (\varphi_1 | S_1^n \times 0)$ equals the restriction of the imbedding of $D^{n+1}$ to the boundary and other $\varphi_j | S_j^n \times 0$ $j > 1$ are fixed [9], we may assume that $\varphi_1(S_1^n\times 0)$ is the boundary of $D^{n+1}$. $C^{n+1}$ does not intersect other $n$-spheres and the normal bundle is a product. So, there exists an imbedded $2n$-disk $D^{2n}$ which contains $D^{n+1}$ in its interior and other $n$-spheres in its complement. Then extend $D^{2n}$ by an isotopy of $S^{2n}$ onto the hemisphere of $S^{2n}$. Thus we may assume that $\varphi_1$ maps $S_1^n\times D_1^n$ into the interior of the upper hemisphere and $\varphi_j$ $j>1$ maps $S_j^n\times D_j^n$ into the interior of the lower hemisphers. This implies that $W^{2n+1}$ is decomposed to a sum of handlebodies, $W^{2n+1}=W_1^{2n+1}\# W'^{2n+1}$, where $W_1^{2n+1}=D^{2n+1}\cup_{\varphi_1} D_1^{n+1}\times D_1^n$ and $W'^{2n+1}=D^{2n+1}\cup_{\{\varphi_j\}}\{\cup_{j>1} D_j^{n+1}\times D_j^n\}$. Clearly $W_1^{2n+1}$ is diffeomorphic to the total space of a $D^n$-bundle over the $(n+1)$-sphere. Repeating this for $\lambda^i$ $i=2,\cdots,q$, we have

**Theorem 5.1.** *Let $M^{2n}$ be an $(n-2)$-connected $2n$-dimensional closed $\pi$-manifold $(n\geqslant 4)$ such that $H_{n-1}M$ is free of rank $r$ and $H_nM\cong 0$. Let $M^{2n}=\tilde{S}^{2n}\#\partial W^{2n+1}$ and let the linking elements $\lambda^i$ $i=1,2,\cdots,r$ defined by the attaching maps $\varphi_i$ $i=1,2,\cdots,r$ are zero for $1\leqslant i\leqslant q$.*

*Then $M^{2n}$ is decomposed as*

$$M^{2n}=\tilde{S}^{2n}\#B_1\#B_2\#\cdots\#B_q\#\partial W'^{2n+1}$$

*where $\tilde{S}^{2n}$ is a homotopy $2n$-sphere, $B_i$ is the total space of an $(n-1)$-sphere bundle over the $(n+1)$-sphere and $W'^{2n+1}$ is a handlebody $D^{2n+1}\cup_{\{\varphi_j\,;\,j>q\}}\{\cup_{j>q} D_j^{n+1}\times D_j^n\}$ with non-zero linking elements.*

**Corollary 5.2.** *If $r=1$ or if $Sq^2: H^{n-1}(M;Z_2)\to H^{n+1}(M;Z_2)$ is trivial[2)] then $W'^{2n+1}$ vanish, that is, $q=r$. And the characteristic elements of $B_i$ $i=1,2,\cdots,r$ are in the image of the natural homomorphism $i_*:\pi_nSO_{n-1}\to\pi_nSO_n$.*

**Proof.** We consider only on the characteristic elements. Let $\mu_i$ be the characteristic element of $B_i$. Using the Mayer-Vietoris sequence, $Sq^2$: $H^{n-1}(B_i:Z_2)\to H^{n+1}(B_i;Z_2)$ is also trivial for $i=1,2,\cdots,r$. This shows that in the cell decomposition of $B_i$ the attaching map

2) More precise structure of $M^{2n}$ in this case has been given by Tamura [13].

of the $(n+1)$-cell to the $(n-1)$-sphere must be homotopic zero. So, $B_i$ admits a cross section. This implies that $\mu_i$ is in the image of $i_*$.

Combining Theorem 3.1 and Theorem 5.1, we have

**Theorem 5.3.** *Let $M^{2n}$ be an $(n-2)$-connected $2n$-dimensional closed $\pi$-manifold $(n\geqslant 4)$ such that $H_{n-1}M$ is free of rank $r$. Let the linking elements $\lambda^i$ $i=1,2,\cdots,r$ defined as above are zero for $i=1,2,\cdots,q$. Then $M^{2n}$ is decomposed as $M^{2n}=\tilde{S}^{2n}\# S^n\times S^n\#\cdots\# S^n\times S^n\# B_1\# B_2\#\cdots\# B_q\#\partial W'^{2n+1}$, where $\tilde{S}^{2n}$ is a homotopy $2n$-sphere, $B_i$ is the total space of an $(n-1)$-sphere bundle over the $(n+1)$-sphere, and $W'^{2n+1}$ is a handlebody $D^{2n+1}\bigcup_{\{\varphi_j\,;\,j>q\}}\{\bigcup_{j>q} D_j^{n+1}\times D_j^n\}$ with non-zero linking elements.* (*We also assume that the* Arf *invariant is zero if $n=4k+3$.*)

**Corollary 5.4.** *If $r=1$ or if $Sq^2: H^{n-1}(M;Z_2)\to H^{n+1}(M;Z_2)$ is trivial then $W'^{2n+1}$ vanish, that is, $q=r$. And the characteristic elements of $B_i$ $i=1,2,\cdots,r$ are in the image of the natural homomorphism $i_*: \pi_n SO_{n-1}\to\pi_n SO_n$.*

## 6. Notes on Parallelizable Manifolds

It is also interesting how many parallelizable manifolds are and what style they have. By [11], an $m$-dimensional closed $\pi$-manifold $M^m$ is parallelizable if and only if

(1) $m$ is even and the Euler characteristic of $M$ is zero, or

(2) $m$ is odd, $m\neq 1,3,7$, and the semi-characteristic of $M$ is zero mod. 2, or

(3) $m=1,3,7$.

From this we have the following results:

**Proposition 6.1.** *Let $M^{2n}$ be an $(n-1)$-connected $2n$ dimensional closed parallelizable manifold $(n\geqslant 3)$. Then $n$ must be odd and $M^{2n}$ has the form as $M^{2n}=S^n\times S^n\#\tilde{S}^{2n}$, under the assumption that the* Arf *invariant is zero if $n=4k+3$.*

**Proposition 6.2.** *Let $M^{2n}$ be an $(n-2)$-connected $2n$ dimensional closed parallelizable manifold $(n\geqslant 4)$ such that $H_{n-1}M$ has no torsion.*

*Let* $r = \operatorname{rank} H_{n-1}M$ *and* $2p = \operatorname{rank} H_nM$. *Then* $p = r + (-1)^{n-1}$ *and* $M^{2n}$ *is a connected sum of* $(r-1)$ *or* $(r+1)$ *copies of* $S^n \times S^n$ *according as* $n$ = *even or odd and such manifolds* $M_1^{2n}$ *as obtained in Theorem* 5.1. *We also assume that the* Arf *invariant is zero if* $n = 4k+3$.

On the other hand, as an example for the odd dimensional case we have the following

**Proposition 6.3.** *In the set of diffeomorphism classes of simply connected* 5-*dimensional closed* π-*manifolds, exactly the half consists of parallelizable manifolds and the other half consists of non-parallelizable manifolds.*

**Proof.** Smale [10] has classified simply connected closed 5-manifolds with vanishing 2nd Stiefel-Whitney classes up to diffeomorphism. This is exactly the classification of simply connected 5-dimensional closed π-manifolds. From his results, we can easily obtain the proposition.

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