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Numerical Investigation of Subharmonic Solutions to Duffing's Equation

By

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1. Introduction

The present paper is concerned with subharmonic solutions to Duffing's equation

$$\frac{d^2q}{d\tau^2}+\alpha\frac{dq}{d\tau}+\kappa^2q(1+\beta q^2)=P\cos\gamma\tau\,. \tag{1.1}$$

As far as the author is aware, the analytical or experimental investigation of subharmonic solutions to Duffing's equation (for analytical investigation, e.g. see [6], [10], [11], [12] and, for experimental investigation, e.g. see [5]) has been limited till recently to the equation in which the nonlinear term is small, that is, $|\beta|\ll 1$. Recently, for the strongly nonlinear equation, that is, the equation in which the nonlinear term is not necessarily small, subharmonic solutions have been investigated analytically by P. A. T. Christopher [2, 3, 4] by the use of the method developed by Cesari [1], and numerically by M. E. Levenson [7, 8] by the use of a digital computer and by C. A. Ludeke and J. E. Cornett [9] by the use of an analog computer. Christopher established analytically the existence of a subharmonic solution of order one-third in some region of parameters, but the region of parameters obtained by him does not seem to be large enough for practical use. Numerical investigations by Levenson, Ludeke and Cornett are all based on step-by-step numerical integration of ordinary differential equations and they do

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not provide the mathematical guarantee for the existence of a subharmonic solution.

In his papers [13, 16], for nonlinear periodic differential systems, the author established a mathematical theory of Galerkin's procedure and gave a practical method of getting an error bound to a periodic approximate solution obtained by Galerkin's procedure. In his method, in the course of calculation of an error bound, the existence of an exact periodic solution can be assured automatically and moreover the stability of a periodic solution can be decided easily. In the present paper, making use of the above method, we have computed approximations to subharmonic solutions of order one-third for various values of parameters and calculated error bounds to the approximations. Naturally the existence of the corresponding subharmonic solutions has been assured and, in addition, the stability of these subharmonic solutions has been decided. In the present paper, harmonic solutions related with subharmonic solutions has been also computed.

By the transformation

$$\kappa\tau = t\,, \quad \frac{\kappa^2}{P}q = x\,, \quad \frac{\alpha}{\kappa} = \sigma\,, \quad \frac{\beta P^2}{\kappa^4} = \varepsilon\,, \quad \frac{\gamma}{\kappa} = \omega\,,$$

equation (1. 1) can be reduced to the equation

$$\frac{d^2x}{dt^2} + \sigma\frac{dx}{dt} + x(1+\varepsilon x^2) = \cos\omega t\,, \tag{1. 2}$$

which, replacing ωt by t, one can rewrite as follows:

$$\frac{d^2x}{dt^2} + \frac{\sigma}{\omega}\frac{dx}{dt} + \frac{1}{\Omega}x(1+\varepsilon x^2) = \frac{1}{\Omega}\cos t\,, \tag{1. 3}$$

where

$$\Omega = \omega^2\,. \tag{1. 4}$$

To a subharmonic solution of order one-third to (1. 1), corresponds a solution to (1. 3) of the form

$$x(t) = c_1 + \sum_{n=1}^{\infty}\left(c_{2n}\sin\frac{n}{3}t + c_{2n+1}\cos\frac{n}{3}t\right). \tag{1. 5}$$

Hence replacing t by $3t$ in (1.3) and (1.5), one can reduce the problem to the one to find a solution of the form

$$x(t) = c_1+\sum_{n=1}^{\infty}(c_{2n}\sin nt+c_{2n+1}\cos nt) \tag{1.6}$$

to the equation

$$\frac{d^2x}{dt^2}+\frac{3\sigma}{\omega}\frac{dx}{dt}+\frac{9}{\Omega}x(1+\varepsilon x^2) = \frac{9}{\Omega}\cos 3t\,. \tag{1.7}$$

In the present paper, *we assume that*

$$\varepsilon>0\,. \tag{1.8}$$

For equation (1.7) with $\sigma=0$ (that is, the *equation with damping absent*), from the symmetricity of the equation, we have sought solutions of the form

$$x(t) = \sum_{n=1}^{\infty}a_n\cos(2n-1)t\,. \tag{1.9}$$

For equation (1.7) with $\sigma\neq 0$ (that is, the *equation with damping present*), we have sought solutions of the general form (1.6) for small $\sigma>0$.

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2. Galerkin's Procedure

2.1 Galerkin's procedure. Consider a real periodic differential system

$$\frac{d\boldsymbol{x}}{dt} = \boldsymbol{X}(\boldsymbol{x}, t)\,, \tag{2.1}$$

where $\boldsymbol{x}$ and $\boldsymbol{X}(\boldsymbol{x}, t)$ are vectors of the same dimension and $\boldsymbol{X}(\boldsymbol{x}, t)$ is periodic in t of period 2π. To get an approximation to a 2π-periodic solution to (2.1), we consider a trigonometric polynomial

$$\boldsymbol{x}_m(t) = \boldsymbol{c}_1+\sum_{n=1}^{m}(\boldsymbol{c}_{2n}\sin nt+\boldsymbol{c}_{2n+1}\cos nt) \tag{2.2}$$

with unknown coefficients $\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \cdots, \boldsymbol{c}_{2m}, \boldsymbol{c}_{2m+1}$, and we determine these unknown coefficients by the equation

$$(2.3)\qquad \frac{d\boldsymbol{x}_m(t)}{dt} = \frac{1}{2\pi}\int_0^{2\pi} \boldsymbol{X}[\boldsymbol{x}_m(s), s]ds + \frac{1}{\pi}\sum_{n=1}^{m}\Big\{\sin nt\cdot\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{x}_m(s), s]\sin ns\cdot ds + \cos nt\cdot\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{x}_m(s), s]\cos ns\cdot ds\Big\}.$$

Equation (2. 3) is evidently equivalent to the equation

$$(2.4)\qquad \begin{cases} \boldsymbol{F}_1(\boldsymbol{c}) \triangleq \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{x}_m(t), t]dt = 0, \\ \boldsymbol{F}_{2n}(\boldsymbol{c}) \triangleq \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{x}_m(t), t]\sin nt\cdot dt + n\boldsymbol{c}_{2n+1} = 0, \\ \boldsymbol{F}_{2n+1}(\boldsymbol{c}) \triangleq \dfrac{1}{\pi}\displaystyle\int_0^{2\pi}\boldsymbol{X}[\boldsymbol{x}_m(t), t]\cos nt\cdot dt - n\boldsymbol{c}_{2n} = 0 \\ \qquad\qquad (n=1, 2, \cdots, m), \end{cases}$$

where $\boldsymbol{c} = \text{col}\,(\boldsymbol{c}_1, \boldsymbol{c}_2, \boldsymbol{c}_3, \cdots, \boldsymbol{c}_{2m}, \boldsymbol{c}_{2m+1})$. A trigonometric polynomial with coefficients satisfying (2. 4) will be called a *Galerkin approximation* of order m to a 2π-periodic solution to (2. 1) and the equation (2. 4) will be called a *determining equation* for Galerkin approximations of order m. A method of getting an approximation to a 2π-periodic solution by computing a Galerkin approximation is called a *Galerkin's procedure.*

Galerkin's procedure can be justified mathematically by the following theorem.

Theorem 1. *Suppose that $\boldsymbol{X}(\boldsymbol{x}, t)$ and its Jacobian matrix $\Psi(\boldsymbol{x}, t)$ with respect to $\boldsymbol{x}$ are continuously differentiable with respect to $\boldsymbol{x}$ and t in the region $D\times L$, where D is a closed bounded region of the $\boldsymbol{x}$-space and L is the real line. If differential system* (2. 1) *possesses an isolated 2π-periodic solution $\boldsymbol{x}=\hat{\boldsymbol{x}}(t)$ lying inside D, then for sufficiently large m_0, there is a Galerkin approximation $\boldsymbol{x}=\bar{\boldsymbol{x}}_m(t)$ to any order $m\geqq m_0$ such that*

$$\bar{\boldsymbol{x}}_m(t)\to\hat{\boldsymbol{x}}(t), \quad \dot{\bar{\boldsymbol{x}}}_m(t)\to\dot{\hat{\boldsymbol{x}}}(t) \qquad (\cdot = d/dt)$$

uniformly as $m\to\infty$.

For the proof of the theorem, see [13].

By an *isolated* 2π-periodic solution, is meant a 2π-periodic solution such that the multipliers of solutions of the relative first variation equation are all different from unity.

2. 2 Determining equation for Duffing's equation with damping present. Clearly equaton (1. 7) is of the form

$$\ddot{x} = X(x, \dot{x}, t) \qquad (\cdot = d/dt)\,, \tag{2.5}$$

where $X(x, y, t)$ is periodic in t of period 2π. Equation (2. 5) is clearly equivalent to the first order system

$$\begin{cases} \dot{x} = y\,, \\ \dot{y} = X(x, y, t)\,. \end{cases} \tag{2.6}$$

For (2. 6), a Galerkin approximation of order m is of the form

$$\begin{cases} x_m(t) = c_1 + \sum_{n=1}^{m} (c_{2n} \sin nt + c_{2n+1} \cos nt)\,, \\ y_m(t) = \sum_{n=1}^{m} (-nc_{2n+1} \sin nt + nc_{2n} \cos nt)\,. \end{cases} \tag{2.7}$$

Hence, for (2. 6), the determining equation for Galerkin approximations of order m can be reduced to the equation [16]

$$\begin{cases} F_1(\boldsymbol{c}) \triangleq \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} X[x_m(t), y_m(t), t]\,dt = 0\,, \\ F_{2n}(\boldsymbol{c}) \triangleq \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} X[x_m(t), y_m(t), t] \sin nt\,dt + n^2 c_{2n} = 0\,, \\ F_{2n+1}(\boldsymbol{c}) \triangleq \dfrac{1}{\pi} \displaystyle\int_0^{2\pi} X[x_m(t), y_m(t), t] \cos nt\,dt + n^2 c_{2n+1} = 0 \\ \qquad\qquad (n = 1, 2, \cdots, m), \end{cases} \tag{2.8}$$

where $\boldsymbol{c} = \mathrm{col}\,(c_1, c_2, c_3, \cdots, c_{2m}, c_{2m+1})$. Equation (2. 8) will be called a *determining equation for Galerkin approximations for the second order equation* (2. 5).

2. 3 Determining equation for Duffing's equation and damping absent. As is seen from (1. 7), Duffing's equation with damping absent is of the form

$$\ddot{x} = X_0(x, t) = \Xi(x) + T(t) \tag{2.9}$$

and

$$\begin{cases} \Xi(-x) = -\Xi(x)\,; \\ T(-t) = T(t)\,, \quad T(t+\pi) = -T(t)\,. \end{cases} \tag{2.10}$$

Corresponding to (1.9), we consider a trigonometric polynomial of the form

$$x_{m'}(t) = \sum_{n=1}^{m'} a_n \cos(2n-1)t\,. \tag{2.11}$$

Equality (2.11) implies that

$$\begin{cases} x_{m'}(-t) = x_{m'}(t)\,, \\ x_{m'}(t+\pi) = -x_{m'}(t)\,. \end{cases} \tag{2.12}$$

Then from (2.10) readily follows that

$$\begin{cases} \Xi[x_{m'}(-t)] = \Xi[x_{m'}(t)]\,, \\ \Xi[x_{m'}(t+\pi)] = -\Xi[x_{m'}(t)]\,. \end{cases} \tag{2.13}$$

Now comparing (2.11) with (2.7) we see that

$$\begin{cases} m = 2m'-1\,, \\ c_{2n} = 0 \qquad (n=1, 2, \cdots, m)\,, \\ c_{2n+1} = \begin{cases} 0 & (n=0, 2, 4, \cdots, m-1)\,, \\ a_{(n+1)/2} & (n=1, 3, 5, \cdots, m)\,. \end{cases} \end{cases} \tag{2.14}$$

Then by (2.10) and (2.13), we readily see that the determining equation (2.8) can be reduced to the equation

$$F_n(\boldsymbol{a}) \triangleq \frac{1}{\pi}\int_0^{2\pi} X_0[x_{m'}(t), t] \cos(2n-1)t \cdot dt + (2n-1)^2 a_n = 0 \qquad (n=1, 2, \cdots, m'), \tag{2.15}$$

where $\boldsymbol{a} = \mathrm{col}\,(a_1, a_2, \cdots, a_{m'})$. Equation (2.15) will be called a *determining equation for Galerkin approximations of the form* (2.11) *for the second order equation* (2.9).

2.4 Numerical solution of determining equations by Newton's method. In order to get Galerkin approximations to 2π-periodic solutions to equation (1.7), it is necessary to solve numerically determining equations of the form (2.8) or (2.15). In the present paper, we have solved determining equations (2. 8) and (2. 15) numerically by Newton's method.

In order to practise Newton's iterative process on a computer, as is seen from (2. 8) and (2. 15), it is necessary to evaluate Fourier coefficients of known functions on a computer. For this purpose, we have used the following formula [16]:

$$(2.16)\qquad \frac{1}{\pi}\int_0^{2\pi} f(t)\begin{Bmatrix}\sin pt\\ \cos pt\end{Bmatrix}dt \doteqdot \frac{1}{N}\sum_{i=1}^{2N} f(t_i)\begin{Bmatrix}\sin pt_i\\ \cos pt_i\end{Bmatrix}$$
$$(p=0, 1, 2, \cdots, \nu),$$

where

$$(2.17)\qquad t_i = \frac{2i-1}{2N}\pi \qquad (i=1, 2, \cdots, 2N)$$

and

$$(2.18)\qquad N \geqq \nu+1\,.$$

In the present paper, for (2. 7) and (2. 8), we have chosen m and N so that

$$(2.19)\qquad m = 15\,,\quad N = 2^5 = 32\,,$$

and, for (2. 11) and (2. 15), we have chosen m' and N so that

$$(2.20)\qquad m' = 15\,,\quad N = 2^6 = 64\,.$$

When we use the formula (2. 16) for evaluation of Fourier coefficients appearing in Newton's iterative process, it is necessary to evaluate trigonometric polynomials of the form (2. 7) or (2. 11) for $t=t_i$ $(i=1, 2, \cdots, 2N)$. We have evaluated these trigonometric polynomials by the use of following recurrence formulas.

Recurrence formula 1 [16]. Let

$$\phi(t) = c_1 + \sum_{n=1}^{m}(c_{2n}\sin nt + c_{2n+1}\cos nt)$$

and

$$\bar{c}_\mu = c_\mu + 2\bar{c}_{\mu+2}\cos t - \bar{c}_{\mu+4}$$
$$\begin{pmatrix}\mu = 2m+1, 2m, 2m-1, \cdots, 3, 2, 1\,; \\ \bar{c}_{2m+5} = \bar{c}_{2m+4} = \bar{c}_{2m+3} = \bar{c}_{2m+2} = 0\end{pmatrix},$$

then

$$\phi(t) = \bar{c}_1 + \bar{c}_2\sin t - \bar{c}_3\cos t\,.$$

Recurrence formula 2. Let

$$\phi(t) = \sum_{n=1}^{m'} a_n \cos(2n-1)t$$

and

$$\bar{a}_n = a_n + 2\bar{a}_{n+1}\cos 2t - \bar{a}_{n+2} \quad \begin{pmatrix} n = m', m'-1, \cdots, 2, 1; \\ \bar{a}_{m'+2} = \bar{a}_{m'+1} = 0 \end{pmatrix},$$

then

$$\phi(t) = (\bar{a}_1 - \bar{a}_2)\cos t.$$

Recurrence formula 2 can be proved analogously to formula 1.

2.5 Starting approximations for Newton's iterative process. In order to solve determining equations by Newton's iterative process, it is necessary to find the starting approximate solutions to determining equations.

1° Duffing's equation with damping absent. We consider equation (1.7) with $\sigma = 0$. In this case, Galerkin approximations under question is of the form (2.11) and the determining equation is of the form (2.15). To find a starting approximate solution for Newton's iterative process, corresponding to (2.11), we consider a Galerkin approximation of the form

$$\bar{x}(t) = a_1 \cos t + a_2 \cos 3t. \tag{2.21}$$

By (2.15), we then have the determining equation as follows.

$$\begin{cases} a_1[(9-\Omega) + \frac{27}{4}\varepsilon(a_1^2 + a_1 a_2 + 2a_2^2)] = 0, \\ (1-\Omega)a_2 + \frac{1}{4}\varepsilon(a_1^3 + 6a_1^2 a_2 + 3a_2^3) - 1 = 0. \end{cases} \tag{2.22}$$

From the first equation of (2.22), we have

$$a_1 = 0 \quad \text{or} \quad (9-\Omega) + \frac{27}{4}\varepsilon(a_1^2 + a_1 a_2 + 2a_2^2) = 0.$$

Combining these equations with the second equation of (2.22), we have the following two cases.

Case I.

$$\begin{cases} a_1 = 0 , \\ \Omega = 1+\frac{3}{4}\varepsilon a_2^2-\frac{1}{a_2} . \end{cases} \tag{2.23}$$

Case II.

$$\begin{cases} \Omega = 9+\frac{27}{4}\varepsilon(a_1^2+a_1a_2+2a_2^2) , \\ 51a_2^3+27a_1a_2^2+21a_1^2a_2-a_1^3+\frac{4}{\varepsilon}(8a_2+1) = 0 . \end{cases} \tag{2.24}$$

In Case II, from the first equation of (2.24), we readily see that real solutions of (2.22) can exist only for

$$\Omega>9 . \tag{2.25}$$

Now the derivative of the left member of the second equation of (2.24) with respect to a_2 is always positive, therefore the second equation of (2.24) can have only one real solution a_2 for any given value of a_1. Such being the case, for $\varepsilon=1/8$, $1/2$, 1 and $a_1=-5(1)5$, we have computed a_2 satisfying the second equation of (2.24) by Newton's method and then we have computed the corresponding values of Ω using the first equation of (2.24). Making use of the results obtained, we have drawn the graphs of (Ω, a_1) and (Ω, a_2) and, from these graphs, we have found the approximate solutions of (2.22) for

$$\Omega = 3.05^2,\ 3.1^2,\ 3.2^2,\ 4^2 . \tag{2.26}$$

Finally, starting from these approximate solutions, we have computed the solutions of (2.22) by Newton's method for values of Ω specified in (2.26).

In Case I, drawing the graph of (Ω, a_2) by the use of the second equation of (2.23), we have found the approximate values of a_2 satisfying the second equation of (2.23) for values of Ω specified in (2.26). Next, starting from these approximate values of a_2, by Newton's method, we have computed the values of a_2 satisfying the second equation of (2.23) for values of Ω specified in (2.26).

However, in Case I, we have computed only the values of a_2 lying near those in Case II.

Figures 1.1, 1.2 and 1.3 show the graphs of (Ω, a_1) and (Ω, a_2) in Case II by solid lines and the graphs of (Ω, a_2) in Case I lying near those in Case II by broken lines for $\varepsilon=1/8$, $1/2$ and 1 respectively. Tables 1.1, 1.2 and 1.3 show the solutions of (2.22) obtained in the above way for $\varepsilon=1/8$, $1/2$ and 1 respectively.

Let $(\bar{a}_1, \bar{a}_2)$ be any one of the values listed in Tables 1.1, 1.2 and 1.3 such that $\bar{a}_1 \neq 0$. Then, for the determining equation (2.15), we can take

$$a_1 = \bar{a}_1\,, \quad a_2 = \bar{a}_2\,, \quad a_3 = a_4 = \cdots = a_{15} = 0$$

for the starting approximate solutions from which Newton's iterative process should be started. Practically, starting from these values, by Newton's iterative process, we have got the solutions to (2.15) shown in the first two columns of Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4. Clearly these give Fourier coefficients of Galerkin approximations of the form (2.11) to subharmonic solutions of order one-third.

From the values $(\bar{a}_1, \bar{a}_2)$ listed in Tables 1.1, 1.2 and 1.3 such that $\bar{a}_1=0$, we get in the same way solutions of (2.15) which however slightly differ in the last digits from the solutions shown in the last columns of Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4. These solutions to (2.15) clearly give Galerkin approximations to 2π-periodic solution to equation (1.3). However, as is seen from Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, these 2π-periodic solutions to (1.3) are supposed to be again of the form (1.9). Hence Galerkin approximations to these solutions can be computed in the same way as Galerkin approximations to subharmonic solutions (that is, 2π-periodic solutions to (1.7)) replacing $9/\Omega$ and $\cos 3t$ in (1.7) by $1/\Omega$ and $\cos t$ respectively. The values obtained in this way are shown in the last columns of Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4. Clearly these give Fourier coefficients of Galerkin approximations of the form (2.11) to $2\pi/3$-periodic solutions to (1.7), that is, 2π-periodic solutions to (1.3) which are nothing else harmonic solutions to the given Duffing's equation.

2° **Duffing's equation with damping present.** For equation (1.7) with small $|\sigma|>0$, by **2.2**, Galerkin approximations are of the form (2.7) and the determining equation is of the form (2.8).

Now Galerkin approximations to 2π-periodic solutions to equation (1.7) or (1.3) with small $|\sigma|>0$ may be supposed to be close to those to (1.7) or (1.3) with $\sigma=0$, that is, the Galerkin approximations obtained in 1°. Hence, for equation (1.7), one can start Newton's iterative process for the determining equation (2.8) from the value

$$\text{(2.27)} \qquad \begin{cases} c_{2n}=0 & (n=1,2,3,\cdots,m)\,, \\ c_{2n+1}=\begin{cases} 0 & (n=0,2,4,\cdots,m-1)\,, \\ a_{(n+1)/2} & (n=1,3,5,\cdots,m)\,, \end{cases} \end{cases}$$

where $m=15$ and a_p $(p=1,2,\cdots,8)$ are the Fourier coefficients of Galerkin approximations listed in the first two columns of Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4. Galerkin approximations obtained in this way for $\sigma=2^{-10}=0.0009765625$ are shown in Tables 3.1.1, 3.1.2, $\cdots$, 3.2.2. Clearly these are Galerking approximations to subharmonic solutions to Duffing's equation with small damping present.

For equation (1.3), one can start Newton's iterative process for the determining equation (2.8) analogously using the Fourier coefficients of Galerkin approximations listed in the last columns of Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4. Galerkin approximations obtained in this way for $\sigma=2^{-4}=0.0625$ are shown in Table 4. Clearly these are Galerkin approximations to harmonic solutions to Duffing's equation with small damping present.

3. Error Estimation of Galerkin Approximations and the Stability of Corresponding Periodic Solutions

3.1 Basic theorem. Let the symbol $\|\cdots\|$ denote the Euclidean norm of vectors or the corresponding norm of matrices. Then the theorem on which our method of error estimation is based reads as follows.

Theorem 2. *In differential system* (2.1), *suppose that* $X(x, t)$ *is* *continuously differentiable with respect to* $\boldsymbol{x}$ *in the region* $D\times L$, *where* D *is a given region of the* $\boldsymbol{x}$*-space and* L *is the real line.*

Assume that (2.1) *possesses a periodic approximate solution* $\boldsymbol{x}=\bar{\boldsymbol{x}}(t)$ *lying inside* D *such that the multipliers of solutions of the linear homogeneous system*

$$\frac{d\boldsymbol{y}}{dt}=\Psi[\bar{\boldsymbol{x}}(t), t]\boldsymbol{y} \tag{3.1}$$

are all different from unity, where $\Psi(\boldsymbol{x}, t)$ *is the Jacobian matrix of* $\boldsymbol{X}(\boldsymbol{x}, t)$ *with respect to* $\boldsymbol{x}$.

Let $\Phi(t)$ *be a fundamental matrix of* (3.1) *satisfying the initial condition* $\Phi(0)=E$ (E *the unit matrix*) *and* $H(t, s)=(H_{kl}(t, s))$ *be a piecewise continuous matrix such that*

$$H(t, s)=\begin{cases}\Phi(t)[E-\Phi(2\pi)]^{-1}\Phi^{-1}(s) & \text{for } 0\leqq s\leqq t\leqq 2\pi\,,\\ \Phi(t)[E-\Phi(2\pi)]^{-1}\Phi(2\pi)\Phi^{-1}(s) & \text{for } 0\leqq t< s\leqq 2\pi\,.\end{cases} \tag{3.2}$$

Let M *be a positive number such that*

$$\left[2\pi\cdot\max_{0\leqq t\leqq 2\pi}\int_0^{2\pi}\sum_{k,l}H_{kl}^2(t, s)ds\right]^{1/2}\leqq M\,, \tag{3.3}$$

and r *be a non-negative number such that*

$$\left\|\frac{d\bar{\boldsymbol{x}}(t)}{dt}-\boldsymbol{X}[\bar{\boldsymbol{x}}(t), t]\right\|\leqq r\,. \tag{3.4}$$

If there exist positive constants δ *and* $k<1$ *such that*

$$\begin{cases}\text{(i)} & D_\delta\triangleq\{\boldsymbol{x}\,|\,\|\boldsymbol{x}-\boldsymbol{x}(t)\|\leqq\delta \text{ for some } t\}\subset D\,,\\ \text{(ii)} & \|\Psi(\boldsymbol{x}, t)-\Psi[\bar{\boldsymbol{x}}(t), t]\|\leqq k/M \text{ for all}\\ & \qquad (\boldsymbol{x}, t) \text{ satisfying } \|\boldsymbol{x}-\boldsymbol{x}(t)\|\leqq\delta\,,\\ \text{(iii)} & Mr/(1-k)\leqq\delta\,,\end{cases} \tag{3.5}$$

then the given differential system (2.1) *possesses one and only one periodic solution* $\boldsymbol{x}=\hat{\boldsymbol{x}}(t)$ *in* D_δ *and this is an isolated periodic solution. Moreover, for* $\boldsymbol{x}=\hat{\boldsymbol{x}}(t)$, *it holds that*

$$\|\bar{\boldsymbol{x}}(t)-\hat{\boldsymbol{x}}(t)\|\leqq Mr/(1-k)\,. \tag{3.6}$$

For the proof of the theorem, see [13].

When a Galerkin approximation $\bar{\boldsymbol{x}}_m(t)$ has been obtained, as will be shown later, for $\bar{x}(t)=\bar{x}_m(t)$ one can easily find the numbers M and r satisfying (3. 3) and (3. 4) respectively. Then, as will be illustrated with an example, one can easily check the existence of the constants δ and k satisfying the condition (3. 5). If there exist such constants δ and k, then by the above theorem one can know the existence of an exact periodic solution of (2. 1) and, in addition, by (3. 6) one can find an error bound to the Galerkin approximation $\bar{x}_m(t)$.

As is seen from (3. 2) and (3. 3), in order to find the number M satisfying (3. 3), one has to compute a fundamental matrix $\Phi(t)$ of (3. 1) satisfying the initial condition $\Phi(0)=E$. If $\bar{x}(t)$ is close to the exact solution $\hat{x}(t)$, then the eigenvalues of $\Phi(2\pi)$ is close to the multipliers of solutions of the first variation equation of (2. 1) with respect to the exact periodic solution. Hence one may decide the stability of the exact periodic solution by inspecting the absolute values of eigenvalues of the matrix $\Phi(2\pi)$. In Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, 3.1.1, $\cdots$, 3.2.2 and 4, eigenvalues of $\Phi(2\pi)$ are shown under the sign λ_i $(i=1, 2)$.

3. 2 The number *r*. For equation (1. 7) with $\sigma\neq 0$, Galerkin approximations are of the form (2. 7). Therefore, as is seen from (2. 6), the number r is a number such that

$$(3.7)\qquad |\ddot{x}_m(t)-X[x_m(t), \dot{x}_m(t), t]|\leqq r\,.$$

Let $\bar{x}_m(t)$ be a Galerkin approximation obtained and let

$$(3.8)\qquad \begin{aligned}&\ddot{\bar{x}}_m(t)-X[\bar{x}_m(t), \dot{\bar{x}}_m(t), t]\\&\quad= C_1+\sum_{n=1}^{\infty}(C_{2n}\sin nt+C_{2n+1}\cos nt)\,.\end{aligned}$$

Then inequality (3. 7) is valid if

$$(3.9)\qquad |C_1|+\sum_{n=1}^{m_1}[C_{2n}^2+C_{2n+1}^2]^{1/2}<r$$

with large m_1. In our computations, we have chosen m_1 so that

$$m_1 = 25$$

and, for computation of $C_1, C_2, \cdots, C_{2m_1+1}$, we have used the formula

(2.16) with $N=32$. By inequality (3.9), for r, we have taken a a number slightly greater than the quantity

$$|C_1|+\sum_{n=1}^{25}[C_{2n}^2+C_{2n+1}^2]^{1/2}.$$

For equation (1.7) with $\sigma=0$, Galerkin approximations are of the form (2.11). Hence corresponding to a Galerkin approximation $\bar{x}_m(t)$ obtained, we have the following expansion instead of (3.8):

$$\ddot{\bar{x}}_m(t)-X_0[\bar{x}_m(t),\, t] = \sum_{n=1}^{\infty} A_n \cos(2n-1)t\,. \tag{3.10}$$

Then inequality (3.7) is valid if

$$\sum_{n=1}^{m_1} |A_n| < r \tag{3.11}$$

with large m_1. In our computations, we have chosen m_1 so that

$$m_1 = 25\,,$$

and, for computation of $A_1, A_2, \cdots, A_{25}$, we have used the formula (2.16) with $N=64$. By inequality (3.11), for r, we have taken a number slightly greater than the quantity

$$\sum_{n=1}^{25}|A_n|.$$

The above method applies also to equation (1.3) without any change.

In Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, 3.1.1, $\cdots$, 3.2.2 and 4, are shown the numbers r found by the above method.

3.3 The number *M*. To find the number M corresponding to a Galerkin approximation $\bar{x}_m(t)$, we first have to compute a fundamental matrix $\Phi(t)$ of (3.1) with $\bar{x}(t)=\bar{x}_m(t)$ satisfying the initial condition $\Phi(0)=E$. In the present paper, by the method developed in [14] and [15], we have computed the desired fundamental matrix in the form

$$\Phi(t) \doteqdot \frac{1}{2}B_0+\sum_{n=1}^{30} B_n T_n\left(\frac{t}{\pi}-1\right), \tag{3.12}$$

where $T_n(t)$ $(n=1, 2, \cdots, 30)$ are Chebyshev polynomials such that

$$T_n(\cos\theta) = \cos n\theta\,.$$

By means of (3.2), we then compute

$$H(p\pi/128,\ q\pi/128)\qquad \begin{pmatrix} p = 0, 2, 4, \cdots, 256\,; \\ q = 0, 1, 2, \cdots, 256 \end{pmatrix}.$$

Making use of $H(p\pi/128, q\pi/128)$ obtained, we compute the integrals

$$\int_0^{2\pi} \sum_{k,l} H_{kl}^2(p\pi/128, s)ds \qquad (p=0, 2, 4, \cdots, 256)$$

by Simpson's rule with mesh size $\pi/128$. Then, by (3.3), a number

$$\left[2\pi \cdot \max_p \int_0^{2\pi} \sum_{k,l} H_{kl}^2(p\pi/128, s)ds\right]^{1/2} \qquad (p=0, 2, 4, \cdots, 256)$$

will give the desired number M.

The numbers M calculated in the above way are shown in Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, 3.1.1, $\cdots$, 3.2.2 and 4.

3.4 The numbers δ and k. We shall illustrate with an example how the existence of the numbers δ and k satisfying the condition (3.5) of Theorem 2 can be checked for Galerkin approximations to periodic solutions to the equation of the form (1.7) or (1.3).

Example. Equation (1.7) with $\sigma=2^{-10}$, $\varepsilon=1$, $\omega=3.1$.

By Table 3.2.1, we have two Galerkin approximations, of which the following one will be brought into consideration:

$$\begin{aligned}(3.13)\qquad \bar{x}(t) = \bar{x}_{15}(t) = {} & 0.00374\ 84022 \sin t + 0.31593\ 24676 \cos t \\ & + 0.00007\ 53137 \sin 3t - 0.11741\ 26343 \cos 3t \\ & - 0.00000\ 95962 \sin 5t - 0.00021\ 62865 \cos 5t \\ & + 0.00000\ 06531 \sin 7t + 0.00006\ 37792 \cos 7t \\ & + 0.00000\ 00195 \sin 9t - 0.00000\ 45873 \cos 9t \\ & - 0.00000\ 00014 \sin 11t - 0.00000\ 00458 \cos 11t \\ & + 0.00000\ 00050 \cos 13t \\ & - 0.00000\ 00002 \cos 15t\,.\end{aligned}$$

Equation (1.7) is evidently equivalent to the first order system

$$(3.14)\qquad \begin{cases} \dfrac{dx}{dt} = y\,, \\ \dfrac{dy}{dt} = -\dfrac{3\sigma}{\omega}y - \dfrac{9}{\omega^2}(x+\varepsilon x^3) + \dfrac{9}{\omega^2}\cos 3t\,. \end{cases}$$

The Jacobian matrix $\Psi(x, y, t)$ of the right member of (3. 14) with respect to x and y is

$$\Psi(x, y, t) = \begin{bmatrix} 0 & 1 \\ -\dfrac{9}{\omega^2}(1+3\varepsilon x^2) & -\dfrac{3\sigma}{\omega} \end{bmatrix},$$

therefore we have

$$(3.15)\qquad \|\Psi(x, y, t) - \Psi[\bar{x}(t), \bar{y}(t), t]\| = \frac{27}{\omega^2}\varepsilon\,|x^2 - \bar{x}^2(t)|\,,$$

where $\bar{y}(t) = \dot{\bar{x}}(t)$. From (3. 15), for

$$(3.16)\qquad [\,|x-\bar{x}(t)|^2 + |y-\bar{y}(t)|^2]^{1/2} \leqq \delta\,,$$

we then have

$$(3.17)\qquad \|\Psi(x, y, t) - \Psi[\bar{x}(t), \bar{y}(t), t]\| \leqq \frac{27}{\omega^2}\varepsilon\delta[\delta + 2\,|\bar{x}(t)|\,]\,,$$

therefore we see that for system (3. 14), the condition (3. 5) is satisfied by the numbers δ and k. They satisfy

$$(3.18)\qquad \begin{cases} \dfrac{27}{\omega^2}\varepsilon\delta[\delta + 2\cdot\max|\bar{x}(t)|\,] \leqq \dfrac{k}{M}\,, \\ \dfrac{Mr}{1-k} \leqq \delta\,, \end{cases}$$

where r and M are numbers specified in Theorem 2 for Galerkin approximation (3. 13). By Table 3.2.1, we see that

$$(3.19)\qquad r = 2.9\times 10^{-9}, \quad M = 32.1\,.$$

From (3. 13), we readily see that

$$\begin{aligned} |\bar{x}(t)| \leqq\; & [0.00374\ 84022^2 + 0.31593\ 24676^2]^{1/2} \\ & + [0.00007\ 53137^2 + 0.11741\ 26343^2]^{1/2} \\ & \qquad \vdots \\ & + [0.0^2 \qquad\qquad\quad + 0.00000\ 00050^2]^{1/2} \\ & + [0.0^2 \qquad\qquad\quad + 0.00000\ 00002^2]^{1/2} \\ =\; & 0.43365\ 22820. \end{aligned}$$

Hence substituting $\varepsilon=1$, $\omega=3.1$ and (3.19) into (3.18), we see that inequality (3.18) is valid if

$$(3.20)\qquad \begin{cases} \dfrac{27}{9.61}\times\delta\times(\delta+0.86730\ 45640)\leqq\dfrac{k}{32.1}\,, \\ \dfrac{32.1\times 2.9\times 10^{-9}}{1-k}\leqq\delta\,. \end{cases}$$

The second inequality of (3.20) means

$$(3.21)\qquad \frac{9.309\times 10^{-8}}{1-k}\leqq\delta\,.$$

Now we expect $k\ll 1$. Therefore taking into account inequality (3.21), we assume that

$$(3.22)\qquad \delta\leqq 1\times 10^{-7}\,.$$

Then the first inequality of (3.20) is valid if

$$\frac{27}{9.61}\times 0.86730\ 46640\times\delta\leqq\frac{k}{32.1}\,,$$

that is,

$$\delta\leqq\frac{9.61}{27\times 0.86730\ 46640\times 32.1}k$$
$$=0.01278\ 44\cdots\times k\,.$$

Hence we suppose that

$$(3.23)\qquad \delta\leqq 0.01278k\,.$$

Then combining (3.23) with (3.21), we have

$$(3.24)\qquad \frac{9.309\times 10^{-8}}{1-k}\leqq\delta\leqq 0.01278k\,.$$

Now from

$$\frac{9.309\times 10^{-8}}{1-k}\leqq 0.01278k\,,$$

we have

$$9.309\times 10^{-8}\leqq 0.01278k(1-k)\,,$$

that is,

$$\frac{9.309\times 10^{-8}}{0.01278}=7.284037\cdots\times 10^{-6}\leqq k(1-k)\,.$$

Since we expect small k, we suppose that

$$(3.25) \qquad k = 8\times 10^{-6}.$$

For this value of k,

$$\begin{cases} \dfrac{9.309\times 10^{-8}}{1-k} = 9.3090744\cdots\times 10^{-8}, \\ 0.01278k = 10.224\cdots\times 10^{-8}. \end{cases}$$

Hence taking into account the assumption (3.22), from (3.24) and (3.25), we see that inequality (3.20) is valid for

$$(3.26) \qquad k = 8\times 10^{-6} \quad \text{and} \quad 9.30908\times 10^{-8}\leqq\delta\leqq 1\times 10^{-7},$$

in other words, the condition (3.5) of Theorem 2 is satisfied by the numbers δ and k specified in (3.26).

By Theorem 2, we thus see that equation (1.7) with $\sigma=2^{-10}$, $\varepsilon=1$, $\omega=3.1$ possesses a unique periodic solution $\hat{x}(t)$ in the region

$$(3.27) \qquad [x-\bar{x}|(t)|^2+|\dot{x}-\dot{\bar{x}}(t)|^2]^{1/2}\leqq\delta_0 = 1\times 10^{-7}$$

and moreover

$$(3.28) \qquad [|\bar{x}(t)-\hat{x}(t)|^2+|\dot{\bar{x}}(t)-\dot{\hat{x}}(t)|^2]^{1/2}\leqq E = 9.31\times 10^{-8}.$$

Remark. All Galerkin approximations listed in Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, 3.1.1, $\cdots$, 3.2.2 and 4 satisfy the condition of Theorem 2. Therefore *corresponding to each Galerkin approximation listed in these tables, an exact periodic solution exists.* The error bounds E to Galerkin approximations obtained by the application of Theorem 2 are shown in Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, 3.1.1, $\cdots$, 3.2.2 and 4. In Tables 3.1.1, $\cdots$, 3.2.2 and 4, the numbers δ_0 which, as in (3.27), fix the region of the existence of exact periodic solutions, are also shown.

4. Conclusions

By Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, we see that for $\varepsilon=1/8, 1/2$, 1 and $\omega=3.05$, 3.1, 3.2, 4, *Duffing's equation with damping absent possesses a harmonic solution with the neutral stability and two kinds* *of subharmonic solutions, of which one has the neutral stability and the other is unstable.* For Duffing's equation with damping absent, subharmonic solutions with the neutral stability will be called *subharmonic solutions of the first kind* and unstable subharmonic solutions will be called *subharmonic solutions of the second kind.* By Tables 3.1.1, 3.1.2, $\cdots$, 3.2.2 and 4, we further see that for $\varepsilon=1/8$, 1 and $\omega=3.1$, 4, *Duffing's equation with small positive damping present possesses a stable harmonic solution and two kinds of subharmonic solutions, of which one close to subharmonic solutions of the first kind of the equation with damping absent is stable and the other close to subharmonic solutions of the second kind is unstable.*

From Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, 3.1.1, $\cdots$, 3.2.2 and 4, we further observe some properties of periodic solutions to Duffing's equation. They will be stated in the following sections.

4.1 Symmetricity of exact periodic solutions.

1° Periodic solutions to Duffing's equation with damping absent. In Tables 2.1.1, 2.1.2, $\cdots$, 2.3.4, every Galerkin approximation to a subharmonic solution, that is, a 2π-periodic solution to (1.7) with $\sigma=0$ and one to a harmonic solution, that is, a 2π-periodic solution to (1.3) with $\sigma=0$ are both of the form

$$\bar{x}(t) = \sum_{n=1}^{m} a_n \cos(2n-1)t \,. \tag{4.1}$$

Let $\hat{x}(t)$ be a corresponding exact periodic solution, then by Theorem 2 we have

$$[\,|\hat{x}(t)-\bar{x}(t)|^2+|\hat{x}'(t)-\bar{x}'(t)|^2]^{1/2} \leqq E \leqq \delta_0\,, \tag{4.2}$$

where $'$ denotes the differentiation with respect to the argument. Now from (4.1) we have

$$\bar{x}(-t) = -\bar{x}(t+\pi) = \bar{x}(t)\,, \tag{4.3}$$

therefore from (4.2) we have

$$\begin{aligned}
&\left[\,|\hat{x}(-t)-\bar{x}(t)|^2+\left|\frac{d}{dt}\hat{x}(-t)-\bar{x}'(t)\right|^2\right]^{1/2}\\
&=[\,|\hat{x}(-t)-\bar{x}(-t)|^2+|-\hat{x}'(-t)+\bar{x}'(-t)|^2]^{1/2}\\
&=[\,|\hat{x}(-t)-\bar{x}(-t)|^2+|\hat{x}'(-t)-\bar{x}'(-t)|^2]^{1/2}\\
&\leqq E \leqq \delta_0
\end{aligned}$$

and

$$\left[|-\hat{x}(t+\pi)-\bar{x}(t)|^2+\left|-\frac{d}{dt}\hat{x}(t+\pi)-\bar{x}'(t)\right|^2\right]^{1/2}$$
$$=[|-\hat{x}(t+\pi)+\bar{x}(t+\pi)|^2+|-\hat{x}'(t+\pi)+\bar{x}'(t+\pi)|^2]^{1/2}$$
$$\leqq E\leqq \delta_0\,.$$

However from the symmetricity of equations (1. 7) and (1. 3), $\hat{x}(-t)$ and $-\hat{x}(t+\pi)$ are also periodic solutions to (1. 7) or (1. 3) correspondingly. Then, since a periodic solution to (1. 7) or (1. 3) satisfying inequality (4. 2) is unique by Theorem 2, we see that

$$\hat{x}(-t) = -\hat{x}(t+\pi) = \hat{x}(t)\,,$$

which means that *the Fourier series of* $\hat{x}(t)$ *is of the form*

$$\hat{x}(t) = \sum_{n=1}^{\infty} \hat{a}_n \cos(2n-1)t\,. \tag{4. 4}$$

2° **Periodic solutions to Duffing's equation with damping present.** In Tables 3.1.1, 3.1.2, 3.2.1 and 3.2.2, every Galerkin approximation $\bar{x}(t)$ to a subharmonic solution, that is, a 2π-periodic solution to (1. 7) with $\sigma \neq 0$ satisfies the inequality

$$[|\bar{x}(t+\pi)+\bar{x}(t)|^2+|\bar{x}'(t+\pi)+\bar{x}'(t)|^2]^{1/2}\leqq\sqrt{52}\times10^{-10}\,. \tag{4. 5}$$

Let $\hat{x}(t)$ be a corresponding exact periodic solution, then by Theorem 2 we have

$$[|\hat{x}(t)-\bar{x}(t)|^2+|\hat{x}'(t)-\bar{x}'(t)|^2]^{1/2}\leqq E\,. \tag{4. 6}$$

Then from (4. 6) and (4. 5) we readily get

$$\begin{aligned}(4.\ 7)\qquad &[|\hat{x}(t+\pi)+\bar{x}(t)|^2+|\hat{x}'(t+\pi)+\bar{x}'(t)|^2]^{1/2}\\ &\leqq[|\hat{x}(t+\pi)-\bar{x}(t+\pi)|^2+|\hat{x}'(t+\pi)-\bar{x}'(t+\pi)|^2]^{1/2}\\ &+[|\bar{x}(t+\pi)+\bar{x}(t)|^2+|\bar{x}'(t+\pi)+\bar{x}'(t)|^2]^{1/2}\\ &\leqq E+\sqrt{52}\times10^{-10}\,.\end{aligned}$$

However, as is seen from Tables 3.1.1, 3.1.2, 3.2.1, and 3.2.2,

$$E+\sqrt{52}\times10^{-10}<E+8\times10^{-10}<\delta_0\,,$$

which by (4. 7) implies

$$[|\hat{x}(t+\pi)+\bar{x}(t)|^2+|\hat{x}'(t+\pi)+\bar{x}'(t)|^2]^{1/2}<\delta_0\,. \tag{4. 8}$$

Now from the symmetricity of equation (1. 7), $-\hat{x}(t+\pi)$ is also a periodic solution of (1. 7). Then, since a periodic solution of (1. 7) satisfying the inequality

$$[|\hat{x}(t)-\bar{x}(t)|^2+|\hat{x}'(t)-\bar{x}'(t)|^2]^{1/2}\leqq\delta_0$$

is unique by Theorem 2, we see that

$$\hat{x}(t+\pi) = -\hat{x}(t)\,,$$

which means that *the Fourier series of* $\hat{x}(t)$ *is of the form*

$$\hat{x}(t) = \sum_{n=1}^{\infty}[\hat{c}_{2n}\sin(2n-1)t+\hat{c}_{2n+1}\cos(2n-1)t]\,. \tag{4.9}$$

For all harmonic solutions $\hat{x}(t)$ corresponding to Galerkin approximations listed in Table 4, it can be proved in the way similar to 1° that $\hat{x}(t)$ *can be expanded in Fourier series of the form* (4. 9).

4. 2 Remarkable character of periodic solutions to Duffing's equation with damping absent. Tables 2.1.1, 21.2, ⋯, 2.3.4 show that in the Fourier series of the subharmonic solutions, the first two coefficients a_1 and a_2 dominate remaining ones strongly and, in the Fourier series of the harmonic solutions, the first coefficient a_2 dominates remaining ones strongly. Comparing Tables 2.1.1, 2.1.2, ⋯, 2.3.4 with Tables 1.1, 1.2, 1.3, we further see that the above dominant Fourier coefficients of the subharmonic solutions and the harmonic solutions are all very close to the values of a_1 and a_2 listed in Tables 1.1, 1.2, 1.3, that is, the Fourier coefficients of Galerkin approximations of the form

$$\bar{x}(t) = a_1\cos t+a_2\cos 3t\,. \tag{4.10}$$

This implies that *even for non-small* $\varepsilon>0$, *one can know the qualitative character of periodic solutions to Duffing's equation with damping absent by investigating the character of the Galerkin approximations of the form* (4. 10). Then we may suppose that *Figures* 1.1, 1.2 *and* 1.3 *are valid also for periodic solutions to Duffing's equation with damping absent.*

4.3 A remark to periodic solutions to Duffing's equation with damping present. In **4.1**, we have observed that periodic solutions to Duffing's equation with small damping present are of the form

$$x(t) = \sum_{n=1}^{\infty} b_n \sin(2n-1)t + \sum_{n=1}^{\infty} c_n \cos(2n-1)t.$$

Comparing Tables 3.1.1, 3.1.2, 3.2.1, 3.2.2 and 4 with Tables 2.1.2, 2.1.4, 2.3.2, 2.3.4, we observe that

$$\sum_{n=1}^{\infty} c_n \cos(2n-1)t \fallingdotseq \sum_{n=1}^{\infty} a_n \cos(2n-1)t,$$

where $\sum_{n=1}^{\infty} a_n \cos(2n-1)t$ is a corresponding periodic solution to Duffing's equation with damping absent. In Tables 3.1.1 and 3.1.2, we further observe that for $\varepsilon=1/8$,

$$\sum_{n=1}^{\infty} b_n \sin(2n-1)t$$

is not a small quantity of the order $\sigma=2^{-10}=0.00097\ 65625$.

Fig. 1.1 ($\varepsilon=1/8$).

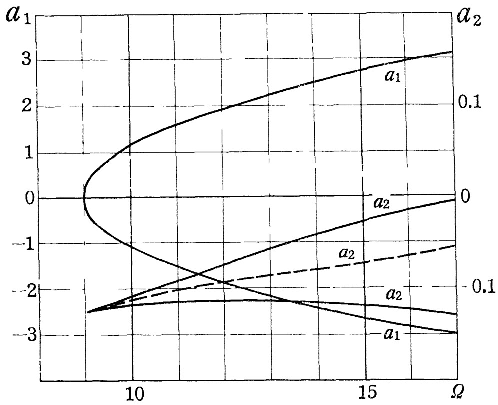

Fig. 1.2 ($\varepsilon=1/2$).

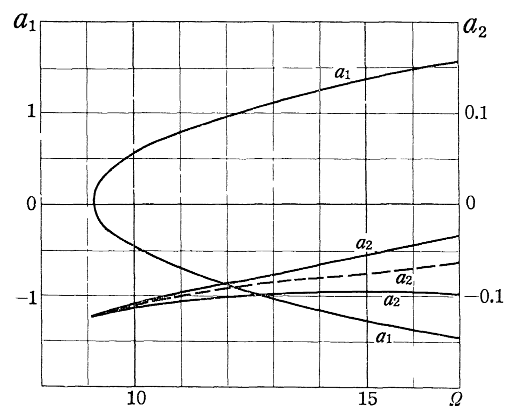

Fig. 1.3 ($\varepsilon=1$)

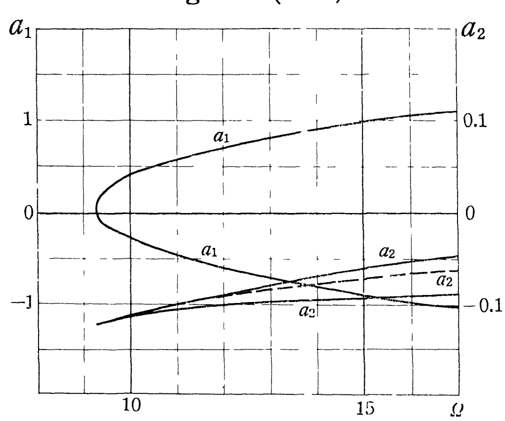

Table 1.1 ($\varepsilon=1/8$)

ω	Ω	a_1	a_2
3.05	9.3025	0.63741 8236	−0.12059 7225
		0.0	−0.12046 5389
		−0.51585 2082	−0.12171 4132
3.1	9.61	0.89420 1639	−0.11557 8278
		0.0	−0.11616 1085
		−0.77575 5977	−0.11942 2055
3.2	10.24	1.25675 9007	−0.10488 5156
		0.0	−0.10823 7974
		−1.14426 3122	−0.11640 0863
4.0	16.0	2.88938 9313	−0.01832 4240
		0.0	−0.06666 8519
		−2.81281 9167	−0.12545 0320

Table 1.2 ($\varepsilon=1/2$).

ω	Ω	a_1	a_2
3.05	9.3025	0.31345 1389	−0.12113 7420
		0.0	−0.12052 4726
		−0.19244 6982	−0.12103 7994
3.1	9.61	0.45446 1607	−0.11695 5127
		0.0	−0.11621 2376
		−0.33651 0725	−0.11793 1971
3.2	10.24	0.64311 9357	−0.10831 4549
		0.0	−0.10827 6627
		−0.53100 9109	−0.11289 2860
4.0	16.0	1.46165 9437	−0.04550 7215
		0.0	−0.06667 4077
		−1.38515 4020	−0.09826 4008

Table 1.3 ($\varepsilon=1$)

ω	Ω	a_1	a_2
3.05	9.3025	0.19879 5839	−0.12123 5672
		0.0	−0.12060 4116
		−0.07853 7428	−0.12075 3860
3.1	9.61	0.31608 8307	−0.11741 1722
		0.0	−0.11628 0975
		−0.19879 4931	−0.11732 0538
3.2	10.24	0.45813 5606	−0.10945 9489
		0.0	−0.10832 8293
		−0.34653 8367	−0.11164 0431
4.0	16.0	1.04268 2507	−0.05360 9552
		0.0	−0.06668 1491
		−0.96626 3409	−0.09015 7062

Table 2.1.1

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/8$, $\omega=3.05$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.63738 94700	−0.51581 55306	0.0
2	−0.12059 77217	−0.12171 46120	−0.12046 53891
3	−0.00015 03858	−0.00015 14040	0.0
4	0.00001 74607	−0.00001 45577	0.0
5	−0.00000 06320	−0.00000 07096	−0.00000 06604
6	−0.00000 00038	−0.00000 00033	0.0
7	0.00000 00002	−0.00000 00002	⋮
8	0.0	0.0	⋮
9	⋮	⋮	⋮
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	8.63×10^{-9}	1.31×10^{-8}	5×10^{-10}
M	101.5	123.7	9.9
E	8.760×10^{-7}	1.6207×10^{-6}	5.0×10^{-9}
λ_i	0.99321 02880 ±0.11633 28121i	1.09964 4524, 0.90938 47852	−0.47244 71969 ±0.88135 89769i
$\|\lambda_i\|$	0.99999 99996		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.1.2

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/8$, $\omega=3.1$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.89416 37367	−0.77570 72093	0.0
2	−0.11558 04311	−0.11942 42151	−0.11616 10854
3	−0.00029 52928	−0.00030 39846	0.0
4	0.00002 15640	−0.00002 06942	0.0
5	−0.00000 04842	−0.00000 07033	−0.00000 05730
6	−0.00000 00063	−0.00000 00064	0.0
7	0.00000 00002	−0.00000 00002	⋮
8	0.0	0.0	⋮
9	⋮	⋮	⋮
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	9.6×10^{-9}	1.86×10^{-8}	5×10^{-10}
M	76.1	86.0	10.2
E	7.306×10^{-7}	1.5998×10^{-6}	5.2×10^{-9}
λ_i	0.98267 41279 ±0.18534 17320i	1.17720 8198, 0.84946 74103	−0.44269 32662 ±0.89667 31133i
$\|\lambda_i\|$	0.99999 99997		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.1.3

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/8$, $\omega=3.2$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	1.25671 76442	−1.14419 82013	0.0
2	−0.10489 29084	−0.11640 92077	−0.10823 79741
3	−0.00052 42595	−0.00057 89532	0.0
4	0.00002 26175	−0.00002 82413	0.0
5	−0.00000 02233	−0.00000 07476	−0.00000 04347
6	−0.00000 00080	−0.00000 00117	0.0
7	0.00000 00001	−0.00000 00003	⋮
8	0.0	0.0	⋮
9	⋮	⋮	⋮
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	1.68×10^{-8}	1.71×10^{-8}	2×10^{-10}
M	61.8	66.2	10.8
E	1.0384×10^{-6}	1.1321×10^{-6}	2.2×10^{-9}
λ_i	0.96021 45254 ±0.27926 34323i	1.29745 2642, 0.77074 10404	−0.38467 39051 ±0.92305 25373i
$\lvert\lambda_i\rvert$	0.99999 99997		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.1.4

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/8$, $\omega=4$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	2.88939 64903	−2.81264 00834	0.0
2	−0.01834 34265	−0.12557 78220	−0.06666 85187
3	−0.00034 06838	−0.00232 01890	0.0
4	−0.00000 20357	−0.00007 13067	0.0
5	0.00000 00117	−0.00000 19406	−0.00000 00648
6	0.00000 00002	−0.00000 00528	0.0
7	0.0	−0.00000 00014	⋮
8	⋮	0.0	⋮
9	⋮	⋮	⋮
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	1.15×10^{-8}	2.37×10^{-8}	5×10^{-10}
M	70.9	71.3	16.4
E	8.155×10^{-7}	1.6902×10^{-6}	8.3×10^{-9}
λ_i	0.88108 83555 ±0.47295 16979i	1.62112 2770, 0.61685 64262	−0.00065 44034 ±0.99999 97858i
$\lvert\lambda_i\rvert$	0.99999 99993		0.99999 99999
Stability	neutral	unstable	neutral

Table 2.2.1

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/2$, $\omega=3.05$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.31340 53558	−0.19237 42156	0.0
2	−0.12113 75427	−0.12103 80304	−0.12052 47278
3	−0.00011 05583	−0.00011 03701	0.0
4	0.00003 47781	−0.00002 13758	0.0
5	−0.00000 26472	−0.00000 27122	−0.00000 26459
6	−0.00000 00131	−0.00000 00086	0.0
7	0.00000 00015	−0.00000 00010	0.0
8	−0.00000 00001	−0.00000 00001	−0.00000 00001
9	0.0	0.0	0.0
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	2.19×10^{-8}	2.05×10^{-8}	1.2×10^{-9}
M	60.6	97.0	9.9
E	1.3275×10^{-6}	1.9889×10^{-6}	1.19×10^{-8}
λ_i	0.98851 25646 $\pm$0.15113 87076i	1.09880 6886, 0.91007 80240	−0.47982 71512 $\pm$0.87736 30406i
$\lvert\lambda_i\rvert$	0.99999 99997		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.2.2

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/2$, $\omega=3.1$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.45439 19264	−0.33641 48660	0.0
2	−0.11695 64970	−0.11793 31876	−0.11621 23773
3	−0.00026 33429	−0.00026 41626	0.0
4	0.00004 53681	−0.00003 46282	0.0
5	−0.00000 22006	−0.00000 25224	−0.00000 22951
6	−0.00000 00253	−0.00000 00203	0.0
7	0.00000 00017	−0.00000 00015	⋮
8	0.0	−0.00000 00001	
9	⋮	0.0	
⋮		⋮	
15	0.0	0.0	0.0
r	2.34×10^{-8}	2.08×10^{-8}	1.4×10^{-9}
M	40.3	52.7	10.1
E	9.431×10^{-7}	1.0963×10^{-6}	1.42×10^{-8}
λ_i	0.96636 03021 $\pm$0.25719 20796i	1.2162 9642, 0.82216 80007	−0.44956 48933 $\pm$0.89324 76738i
$\lvert\lambda_i\rvert$	0.99999 99997		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.2.3

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/2$, $\omega=3.2$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.64303 26531	−0.53088 90496	0.0
2	−0.10832 12797	−0.11289 93867	−0.10827 66276
3	−0.00051 46997	−0.00053 17972	0.0
4	0.00005 10049	−0.00004 80110	0.0
5	−0.00000 13945	−0.00000 23212	−0.00000 17408
6	−0.00000 00368	−0.00000 00375	0.0
7	0.00000 00014	−0.00000 00019	⋮
8	0.0	−0.00000 00001	⋮
9	⋮	0.0	⋮
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	1.21×10^{-8}	1.59×10^{-8}	1.0×10^{-9}
M	31.5	36.4	10.7
E	3.812×10^{-7}	5.788×10^{-7}	1.08×10^{-8}
λ_i	0.91968 76243 $\pm$0.39265 08284i	1.40503 2563, 0.71172 72764	−0.39062 79056 $\pm$0.92054 86621i
$\lvert\lambda_i\rvert$	0.99999 99997		0.99999 99999
Stability	neutral	unstable	neutral

Table 2.2.4

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1/2$, $\omega=4$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	1.46165 38180	−1.38496 30370	0.0
2	−0.04555 49212	−0.09836 10693	−0.06667 40766
3	−0.00084 49916	−0.00180 57494	0.0
4	0.00000 59508	−0.00007 68957	0.0
5	0.00000 02492	−0.00000 26025	−0.00000 02591
6	−0.00000 00018	−0.00000 00865	0.0
7	−0.00000 00001	−0.00000 00030	⋮
8	0.0	−0.00000 00001	⋮
9	⋮	0.0	⋮
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	7.0×10^{-9}	1.49×10^{-8}	2×10^{-10}
M	35.6	35.9	16.3
E	2.493×10^{-7}	5.350×10^{-7}	3.3×10^{-9}
λ_i	0.76357 40830 $\pm$0.64572 02327i	1.96414 8870, 0.50912 63771	−0.00261 64669 $\pm$0.99999 65770i
$\lvert\lambda_i\rvert$	0.99999 99995		0.99999 99999
Stability	neutral	unstable	neutral

Table 2.3.1

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1$, $\omega=3.05$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.19874 95570	−0.07843 84541	0.0
2	−0.12123 55889	−0.12075 36311	−0.12060 41229
3	−0.00005 65814	−0.00005 70524	0.0
4	0.00004 42107	−0.00001 73127	0.0
5	−0.00000 53693	−0.00000 53370	−0.00000 53029
6	−0.00000 00176	−0.00000 00078	0.0
7	0.00000 00039	−0.00000 00015	0.0
8	−0.00000 00002	−0.00000 00003	−0.00000 00002
9	0.0	0.0	0.0
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	2.46×10^{-8}	2.47×10^{-8}	1.8×10^{-9}
M	63.9	158.8	9.8
E	1.5723×10^{-6}	3.9256×10^{-6}	1.77×10^{-8}
λ_i	0.98984 12174 $\pm$0.14217 72262i	1.05869 0072, 0.94456 34995	−0.48959 37021 $\pm$0.87195 06906i
$\lvert\lambda_i\rvert$	0.99999 99997		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.3.2

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1$, $\omega=3.1$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.31600 21647	−0.19866 15617	0.0
2	−0.11741 22723	−0.11732 07872	−0.11628 09810
3	−0.00021 64496	−0.00021 57463	0.0
4	0.00006 37913	−0.00004 03006	0.0
5	−0.00000 45871	−0.00000 48456	−0.00000 45989
6	−0.00000 00459	−0.00000 00309	0.0
7	0.00000 00050	−0.00000 00033	0.0
8	−0.00000 00002	−0.00000 00002	−0.00000 00002
9	0.0	0.0	0.0
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	2.60×10^{-8}	1.35×10^{-8}	3×10^{-10}
M	32.0	49.3	10.1
E	8.321×10^{-7}	6.656×10^{-7}	3.1×10^{-9}
λ_i	0.95776 42812 $\pm$0.28755 44838i	1.20610 4705, 0.82911 54117	−0.45866 69814 $\pm$0.88860 82377i
$\lvert\lambda_i\rvert$	0.99999 99998		1.00000 0000
Stability	neutral	unstable	neutral

Table 2.3.3

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1$, $\omega=3.2$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	0.45801 73216	−0.34637 31608	0.0
2	−0.10946 46843	−0.11164 50565	−0.10832 82962
3	−0.00048 31870	−0.00048 78993	0.0
4	0.00007 49046	−0.00006 06922	0.0
5	−0.00000 31416	−0.00000 42355	−0.00000 34869
6	−0.00000 00747	−0.00000 00644	0.0
7	0.00000 00045	−0.00000 00045	0.0
8	−0.00000 00001	−0.00000 00002	−0.00000 00001
9	0.0	0.0	0.0
⋮	⋮	⋮	⋮
15	0.0	0.0	0.0
r	2.48×10^{-8}	9.9×10^{-9}	1.1×10^{-9}
M	23.2	28.8	10.7
E	5.754×10^{-7}	2.852×10^{-7}	1.18×10^{-8}
λ_i	0.88939 86110 ±0.45713 24868i	1.44677 1135, 0.69119 43257	−0.39852 58925 ±0.91715 70819i
$\|\lambda_i\|$	0.99999 99998		0.99999 99999
Stability	neutral	unstable	neutral

Table 2.3.4

Periodic solutions to (1.7) with $\sigma=0$ ($\varepsilon=1$, $\omega=4$)

	Subharmonic solutions		Harmonic solution
n	a_n	a_n	a_n
1	1.04265 32218	−0.96604 99115	0.0
2	−0.05366 54105	−0.09024 34248	−0.06668 14915
3	−0.00099 23715	−0.00164 48577	0.0
4	0.00001 80707	−0.00008 58323	0.0
5	0.00000 04148	−0.00000 33351	−0.00000 05184
6	−0.00000 00123	−0.00000 01232	0.0
7	−0.00000 00002	−0.00000 00049	⋮
8	0.0	−0.00000 00002	
9	⋮	0.0	
⋮		⋮	
15	0.0	0.0	0.0
r	1.55×10^{-8}	1.48×10^{-8}	5×10^{-10}
M	25.3	25.7	16.3
E	3.922×10^{-7}	3.804×10^{-7}	8.2×10^{-9}
λ_i	0.66814 27831, ±0.74403 30775i	2.21194 1080, 0.45209 16072	−0.00522 98781 ±0.99998 63240i
$\|\lambda_i\|$	0.99999 99995		0.99999 99999
Stability	neutral	unstable	neutral

Table 3.1.1

Periodic solutions to (1.7) with $\sigma=2^{-10}$ ($\varepsilon=1/8$, $\omega=3.1$)

1° $\bar{x}_{15}(t)=0.03028\ 80058 \sin t \ +0.89331\ 82959 \cos t$
$+0.00030\ 91923 \sin 3t \ -0.11559\ 53736 \cos 3t$
$-0.00002\ 37202 \sin 5t \ -0.00029\ 44095 \cos 5t$
$+0.00000\ 05627 \sin 7t \ +0.00002\ 15577 \cos 7t$
$+0.00000\ 00139 \sin 9t \ -0.00000\ 04849 \cos 9t$
$-0.00000\ 00004 \sin 11t - 0.00000\ 00063 \cos 11t$
$+0.00000\ 00002 \cos 13t,$

$r=1.9\times10^{-9}$, $M=76.6$, $E=1.456\times10^{-7}$, $\delta_0=1.47\times10^{-7}$,
λ_1, $\lambda_2=0.97988\ 03088\pm0.18415\ 80459i$,
$|\lambda_1|$, $|\lambda_2|=0.99703\ 54082$,
Stability: stable.

2° $\bar{x}_{15}(t)=0.03063\ 35257 \sin t \ -0.77548\ 34440 \cos t$
$+0.00024\ 57725 \sin 3t \ -0.11941\ 61668 \cos 3t$
$+0.00001\ 99608 \sin 5t \ -0.00030\ 33481 \cos 5t$
$+0.00000\ 09372 \sin 7t \ -0.00002\ 06791 \cos 7t$
$+0.00000\ 00123 \sin 9t \ -0.00000\ 07027 \cos 9t$
$+0.00000\ 00005 \sin 11t - 0.00000\ 00064 \cos 11t$
$-0.00000\ 00002 \cos 13t,$

$r=2.0\times10^{-9}$, $M=86.5$, $E=1.731\times10^{-7}$, $\delta_0=1.8\times10^{-7}$,
$\lambda_1=1.17326\ 4292$, $\lambda_2=0.84727\ 67891$,
Stability: unstable.

Table 3.1.2

Periodic solutions to (1.7) with $\sigma=2^{-10}$ ($\varepsilon=1/8$, $\omega=4$)

1° $\bar{x}_{15}(t)= \ +0.00000\ 00001$
$+0.06210\ 04049 \sin t \ +2.88864\ 67924 \cos t$
$-0.00000\ 00001 \sin 2t$
$+0.00362\ 47077 \sin 3t \ -0.01846\ 37771 \cos 3t$
$+0.00005\ 19900 \sin 5t \ -0.00034\ 55740 \cos 5t$
$-0.00000\ 00777 \sin 7t \ -0.00000\ 21181 \cos 7t$
$-0.00000\ 00082 \sin 9t \ +0.00000\ 00113 \cos 9t$
$-0.00000\ 00001 \sin 11t + 0.00000\ 00002 \cos 11t,$

$r=4.1\times10^{-9}$, $M=71.0$, $E=2.912\times10^{-7}$, $\delta_0=3\times10^{-7}$,
λ_1, $\lambda_2=0.87932\ 24423\pm0.47138\ 16650i$,
$|\lambda_1|=|\lambda_2|=0.99770\ 16747$,
Stability: stable.

2° $\bar{x}_{15}(t)= \ +0.00000\ 00002$
$+0.06299\ 89812 \sin t \ -2.81201\ 85119 \cos t$
$-0.00000\ 00001 \cos 2t$
$+0.00349\ 53718 \sin 3t \ -0.12546\ 59671 \cos 3t$
$+0.00016\ 47728 \sin 5t \ -0.00231\ 31766 \cos 5t$
$+0.00000\ 63287 \sin 7t \ -0.00007\ 09561 \cos 7t$
$+0.00000\ 02221 \sin 9t \ -0.00000\ 19254 \text{ cas } 9t$
$+0.00000\ 00074 \sin 11t - 0.00000\ 00522 \cos 11t$
$+0.00000\ 00002 \sin 13t - 0.00000\ 00014 \cos 13t,$

$r=4.5\times10^{-9}$, $M=71.4$, $E=3.214\times10^{-7}$, $\delta_0=3.3\times10^{-7}$,
$\lambda_1=1.61657\ 6060$, $\lambda_2=0.61575\ 11731$,
Stability: unstable.

Table 3.2.1

Periodic solutions to (1.7) with $\sigma=2^{-10}$ ($\varepsilon=1, \omega=3.1$)

1° $\bar{x}_{15}(t)=0.00374\ 84022 \sin t \ +0.31593\ 24676 \cos t$
$+0.00007\ 53137 \sin 3t \ -0.11741\ 26343 \cos 3t$
$-0.00000\ 95962 \sin 5t \ -0.00021\ 62865 \cos 5t$
$+0.00000\ 06531 \sin 7t \ +0.00006\ 37792 \cos 7t$
$+0.00000\ 00195 \sin 9t \ -0.00000\ 45873 \cos 9t$
$-0.00000\ 00014 \sin 11t -0.00000\ 00458 \cos 11t$
$+0.00000\ 00050 \cos 13t$
$-0.00000\ 00002 \cos 15t,$

$r=2.9\times10^{-9}$, $M=32.1$, $E=9.31\times10^{-8}$, $\delta_0=1\times10^{-7}$,
$\lambda_1, \lambda_2=0.95498\ 22496\pm0.28651\ 09291i$,
$|\lambda_1|=|\lambda_2|=0.99703\ 54104$,
Stability : stable.

2° $\bar{x}_{15}(t)=0.00385\ 11023 \sin t \ -0.19869\ 89516 \cos t$
$+0.00005\ 49550 \sin 3t \ -0.11732\ 12521 \cos 3t$
$+0.00000\ 38874 \sin 5t \ -0.00021\ 57613 \cos 5t$
$+0.00000\ 08258 \sin 7t \ -0.00004\ 03077 \cos 7t$
$+0.00000\ 00114 \sin 9t \ -0.00000\ 48456 \cos 9t$
$+0.00000\ 00008 \sin 11t -0.00000\ 00309 \cos 11t$
$+0.00000\ 00001 \sin 13t -0.00000\ 00033 \cos 13t$
$-0.00000\ 00002 \cos 15t,$

$r=2.2\times10^{-9}$, $M=49.3$, $E=1.085\times10^{-7}$, $\delta_0=1.1\times10^{-7}$,
$\lambda_1=1.20252\ 8159$, $\lambda_2=0.82665\ 80649$,
Stability : unstable.

Table 3.2.2

Periodic solutions to (1.7) with $\sigma=2^{-10}$ ($\varepsilon=1$, $\omega=4$)

1° $\bar{x}_{15}(t)=0.00765\ 38598 \sin t \ +1.04261\ 49858 \cos t$
$+0.00048\ 30974 \sin 3t \ -0.05367\ 09094 \cos 3t$
$-0.00000\ 71715 \sin 5t \ -0.00099\ 24868 \cos 5t$
$-0.00000\ 05051 \sin 7t \ +0.00001\ 80779 \cos 7t$
$+0.00000\ 00098 \sin 9t \ +0.00000\ 04150 \cos 9t$
$+0.00000\ 00005 \sin 11t -0.00000\ 00123 \cos 11t$
$-0.00000\ 00002 \cos 13t,$

$r=1.6\times10^{-9}$, $M=25.3$, $E=4.05\times10^{-8}$, $\delta_0=4.2\times10^{-8}$,
$\lambda_1, \lambda_2=0.66669\ 89022\pm0.74224\ 06603i$,
$|\lambda_1|=|\lambda_2|=0.99770\ 16708$,
Stability : stable.

2° $\bar{x}_{15}(t)= \quad +0.00000\ 00001$
$+0.00796\ 97109 \sin t \ -0.96602\ 78183 \cos t$
$+0.00043\ 69056 \sin 3t \ -0.09023\ 89741 \cos 3t$
$+0.00003\ 25368 \sin 5t \ -0.00164\ 44793 \cos 5t$
$+0.00000\ 17935 \sin 7t \ -0.00008\ 58069 \cos 7t$
$+0.00000\ 00868 \sin 9t \ -0.00000\ 33336 \cos 9t$
$+0.00000\ 00042 \sin 11t -0.00000\ 01231 \cos 11t$
$+0.00000\ 00002 \sin 13t -0.00000\ 00049 \cos 13t$
$-0.00000\ 00002 \cos 15t,$

$r=3.9\times10^{-9}$, $M=25.7$, $E=1.003\times10^{-7}$, $\delta_0=1.08\times10^{-7}$,
$\lambda_1=2.20663\ 9824$, $\lambda_2=0.45109\ 70035$,
Stability : unstable.

Table 4

Periodic solutions to (1.3) with $\sigma=2^{-4}$.

1° $\varepsilon=1/8$, $\omega=3.1$.

$\bar{x}_{15}(t)=0.00261\ 30219\sin t\ -0.11610\ 22676\cos t$
$+0.00000\ 00425\sin 3t\ -0.00000\ 05709\cos 3t$,

$r=2\times10^{-10}$, $M=10.2$, $E=2.1\times10^{-9}$, $\delta_0=2.1\times10^{-9}$,

$\lambda_1,\ \lambda_2=-0.41468\ 98606\pm0.84205\ 13087i$,

$|\lambda_1|=|\lambda_2|=0.93862\ 56373$,

Stability: stable

2° $\varepsilon=1/8$, $\omega=4.0$.

$\bar{x}_{15}(t)=0.00111\ 08642\sin t\ -0.06665\ 00033\cos t$
$+0.00000\ 00036\sin 3t\ -0.00000\ 00646\cos 3t$,

$r=1\times10^{-10}$, $M=16.4$, $E=1.7\times10^{-9}$, $\delta_0=1.9\times10^{-9}$,

$\lambda_1,\ \lambda_2=0.00010\ 72410\pm0.95209\ 79207i$,

$|\lambda_1|=|\lambda_2|=0.95209\ 79267$,

Stability: stable.

3° $\varepsilon=1$, $\omega=3.1$.

$\bar{x}_{15}(t)=0.00261\ 84132\sin t\ -0.11622\ 19199\cos t$
$+0.00000\ 03415\sin 3t\ -0.00000\ 45826\cos 3t$
$-0.00000\ 00002\cos 5t$,

$r=2\times10^{-10}$, $M=10.1$, $E=2.1\times10^{-9}$, $\delta_0=2.7\times10^{-9}$,

$\lambda_1,\ \lambda_2=-0.42969\ 03383\pm0.83449\ 64351i$,

$|\lambda_1|=|\lambda_2|=0.93862\ 56373$,

Stability: stable.

4° $\varepsilon=1$, $\omega=4$.

$\bar{x}_{15}(t)=0.00111\ 12964\sin t\ -0.06666\ 29616\cos t$
$+0.00000\ 00286\sin 3t\ -0.00000\ 05174\cos 3t$,

$r=1\times10^{-10}$, $M=16.3$, $E=1.7\times10^{-9}$, $\delta_0=2\times10^{-9}$,

$\lambda_1,\ \lambda_2=-0.00424\ 99789\pm0.95208\ 84411i$,

$|\lambda_1|=|\lambda_2|=0.95209\ 79267$,

Stability: stable.

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