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A Necessary Condition for the Well-posedness of the Cauchy Problem for the First Order Hyperbolic System with Multiple Characteristics

By

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1. Introduction

The Cauchy problem for a hyperbolic equation (either single or system) with distinct characteristics is well posed. Moreover it is well posed for any perturbation by lower order operators, namely it is a so called *strongly hyperbolic* equation (cf. K. Kasahara-M. Yamaguti [1]).

A higher order single equation with constant coefficients is strongly hyperbolic if and only if the characteristics are all real and distinct. In [2] the author showed that if a higher order single equation with *variable* coefficients is strongly hyperbolic, then the characteristics are necessarily real and simple. This is done, however, for only the case when the multiplicity of the characteristics is constant with respect to the variables (x, t, ξ).

Now consider the following first order hyperbolic system:

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{k} A_j \frac{\partial u}{\partial x_j} + Bu + f \,. \tag{1.1}$$

It is well known that if $\{A_j\}$ are all constant matrices, then it is necessary for this system to be strongly hyperbolic that the matrix $A\cdot\xi = \sum_{j=1}^{k} A_j\xi_j$ is diagonalisable. But if $\{A_j\}$ are matrices of functions, it seems that there is no prosperous result concerning the

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necessary condition for (1.1) to be strongly hyperbolic except the work of G. Strang [3].

In this paper we consider the non-symmetric hyperbolic system with variable coefficients and give a partial answer to the following problem: *when the non-symmetric hyperbolic system has multiple characteristics, what is a necessary condition for it to be strongly hyperbolic?*

Our result is quite parallel with the case when the coefficients are constant. There is, however, some essential restriction, namely we also assume that the multiplicity of characteristics is independent of the variables (x, t, ξ), and if otherwise we have no good result.

Let us now consider

$$M[u] = \frac{\partial u}{\partial t} - \sum_{j=1}^{k} A_j(x, t)\frac{\partial u}{\partial x_j} - B(x, t)u = f\,, \tag{1.2}$$

where $\{A_j\}$ and B are $m\times m$ matrices with entries in L. Schwartz's $\mathcal{B}_{x,t}$, and u intimates an unknown m-vector $u={}^t(u_1, u_2, \cdots, u_m)$. Assume that M is hyperbolic, namely all the roots of

$$\det(\lambda I - \sum_{j=1}^{k} A_j(x, t)\xi_j) = 0 \tag{1.3}$$

are real for any real $\xi \neq 0$. Now we formulate the condition on M:

Condition (M). Assume that the characteristic equation of M can be expressed like the following:

$$\det(\lambda I - \sum A_j(x, t)\xi_j) = (\lambda-\lambda_1)^{p_1}\cdots(\lambda-\lambda_s)^{p_s}\,,$$

where $\{p_j\}_{j=1,2,\cdots,s}$ are integers independent of (x, t, ξ) and p_1 is actually greater than unity. Here $\lambda_1, \cdots, \lambda_s$ are distinct.

We consider the Cauchy problem for (1.2) in L^2 sense.

Definition. The forward Cauchy problem for $M[u]=f$ is said to be uniformly L^2-well posed in $[0, T]$, if for every $t_0\in[0, T]$ the following (i) and (ii) are fulfilled.

(i) For any $\varphi(x)={}^t(\varphi_1(x), \cdots, \varphi_m(x))\in(L^2)^m$ and for $f(x, t)={}^t(f_1(x, t), \cdots, f_m(x, t))\in(\mathcal{E}_t^0(L^2))^m$, the Cauchy problem

$$\begin{aligned} &M[u] = f \\ &u(x, t_0) = \varphi(x)\,, \qquad t_0\leqq t\leqq T\,, \end{aligned}$$

has a unique solution

$$u(x, t, t_0) \in \mathcal{E}_t^0(L^2) \cap \mathcal{E}_t^1(\mathcal{D}_{L^2}^{1'}), \qquad t_0 \leqq t \leqq T.$$

(ii) The solution $u(x, t, t_0)$ satisfies the energy inequality

$$\|u(\cdot, t, t_0)\|_{L^2} \leqq C_T\Big(\|u(\cdot, t_0, t_0)\|_{L^2} + \int_{t_0}^t \|f(s)\|_{L^2} ds\Big),$$

where $\|u(t)\|_{L^2}^2 = \sum_{j=1}^m \|u_j(t)\|_{L^2}^2$.

Now our result is the following

Theorem. *Suppose that the first order hyperbolic system $M[u]=f$ satisfies the condition (M). If the Cauchy problem is uniformly L^2-well posed in $[0, T]$, then the matrix $A \cdot \xi = \sum_{j=1}^k A_j(x, t)\xi_j$ is necessarily diagonalisable for any (x, t, ξ).*

To prove this, the pseudo-differential operators of type P introduced in [2] is useful. Then we prepare in §2 some lemmas concerning them.

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2. Some Lemmas on Pseudo-differential Operators

We introduced in [2] the pseudo-differential operators H_γ associated with Puiseux series. Namely the pseudo-differential operator H_γ of type P is defined by the symbol

$$\sigma(H_\gamma) = \sum_{j=0}^\infty h_j(x, \xi)\hat{\gamma}(\xi)|\xi|^{-j/p},$$

where $\hat{\gamma}(\xi)$ is a C^∞-function vanishing for $|\xi| \leqq R$ and takes the value 1 for $|\xi| \geqq R+1$, and p is a positive integer (see §2 in [2]). In this paragraph, we give as lemmas some properties of these pseudo-differential operators. In details we refer readers to our previous paper [2]. First we define the orders of the operator H_γ.

Definition 2.1. A pseudo-differential operator H_γ is said to be of order $-r$ if there exists a positive constant C_r independent of u and we have

$$||H_\gamma\Lambda^r u||_{L^2}\leqq C_r||u||_{L^2}$$

for any u in L^2.

Lemma 2.1. *Let H_γ and K_γ be two pseudo-differential operators of type P, then $(H_\gamma K_\gamma - H_\gamma\circ K_\gamma)\Lambda$*[1] *is of order zero.*

Lemma 2.2. *Suppose that $\inf_{x,|\xi|=1}|h_0(x,\xi)|=\delta>0$. Then there exists for a sufficiently large number R*[2] *a positive constant δ'*[3] *such that*

$$||H_\gamma u||_{L^2}\geqq\delta'||u||_{L^2}$$

holds for any L^2 function $\hat{u}(\xi)$ with the support out of the sphere $|\xi|=R$.

For the proof, see Lemma 2.4 in [2] and see also Lemma 2.1 in [6].

Next we take an arbitrary sequence $\{\hat{u}_n(\xi)\}$ of L^2 whose supports lie between two concentric spheres $|\xi|=c_1n$ and $|\xi|=c_2n$ $(c_1<c_2,$ $n=1,2,\cdots)$. On this sequence we have the following

Lemma 2.3. *In the case $p\geqq 2$, suppose that $\mathrm{Re}\,h_0(x,\xi)=0$ and $\inf_{x,|\xi|=1}\mathrm{Re}\,h_1(x,\xi)=\delta>0$, then for sufficiently large n we have*

$$\mathrm{Re}\,(H_\gamma\Lambda u_n, u_n)\geqq Cn^{1-(1/p)}||u_n||^2_{L^2},$$

where C is independent of n.

Finally we give a property of so-called hypoelliptic pseudo-differential operators in [2]. Let $\phi(D)$ be a hypoelliptic pseudo-differential operator and denote the hypoelliptic pseudo-differential operator $\overline{\mathscr{F}}[\hat{\gamma}(\xi)\hat{\phi}(\xi)]$ by $\phi_1(D)$. Now we have

Lemma 2.4. *Take a function $b(x)$ in $\mathscr{B}$. We can express the commutator $[b(x),\phi_1(D)]=b(x)\phi_1(D)-\phi_1(D)b(x)$ as*

$$[b(x),\phi_1(D)]=\sum_{|\nu|=1}^{s-1}(-1)^{|\nu|+1}\frac{b^{(\nu)}(x)}{\nu!}(x^\nu\phi_1)(D)+B_0,$$

where B_0 is a bounded operator in L^2 and for any positive integer s we have

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1) The operator $H_\gamma\circ K_\gamma$ is defined by the symbol $\sigma(H_\gamma)\sigma(K_\gamma)$.

2) This R is the same that defines the function $\hat{\gamma}(\xi)$.

3) This δ' is determined by R as

$$\delta'\geqq\left(\frac{\delta}{2}-c_1R^{-1}-c_2\sum_{j=1}^{\infty}M_{H_j}R^{-j/p}\right),\quad M_{H_j}=\sum_{|\nu|\leqq 2k}\sup\left|\left(\frac{\partial}{\partial\xi}\right)^\nu h_j(x,\xi)\right|$$

$$\|B_0\| \leqq C|b(x)|_{\mathscr{B}^s}(\|D_\xi^s\hat{\phi}_1(\xi)\|_{L^1}+\|D_\xi^{s+2k}\hat{\phi}_1(\xi)\|_{L^1}),$$

where $\|B_0\|$ *is the operator norm of* B_0 *and* C *is independent of* s.

Now we apply this lemma on the hypoelliptic pseudo-differential operator defined by $\overline{\mathscr{F}}[\hat{\alpha}_n(\xi)]$. Here $\hat{\alpha}_n(\xi)=\hat{\alpha}(\xi/n)$, and $\hat{\alpha}(\xi)$ is a C^∞-function which is identically equal to 1 in the neighbourhood of $\xi_0 \neq 0$ and vanishes in $\left\{\xi;\ |\xi-\xi_0| \geqq \frac{|\xi_0|}{4}\right\}$, (also see §2 in [2]). We get the following

Lemma 2.5. *Let* $b(x)$ *be a function in* $\mathscr{B}$. *Then we have the following representation*:

$$[b(x), \alpha_n(D)] = \sum_{|\nu|=1}^{s-1}(-1)^{|\nu|+1}\frac{b^{(\nu)}(x)}{\nu!}(x^\nu\alpha_n)(D)+B_0,$$

where $\|B_0\|=O(n^{-s})$, $n\to\infty$.

3. Pseudo-canonicalisation of a Matrix

In this paragraph we prove a proposition concerning matrices which plays a fundamental role in the proof of our Theorem. This Proposition 3.1 is essentially the same as Proposition 3.1 in [5] (also see Chapter 3, §1 in [7]).

Let $A(x, \xi)$ be a $m\times m$ matrix all entries of which are functions of $\mathscr{B}_{x,\xi}$, $(x, \xi)\in R_x^{k+1}\times R_\xi^k$. Moreover let all entries be analytic in ξ.

Proposition 3.1. *Assume that the matrix* $A(x, \xi)$ *satisfies the following two conditions*:

(1) *Let the characteristic polynomial of* $A(x, \xi)$ *has the form*

$$\det(\lambda I-A(x, \xi)) = (\lambda-\lambda_1)^{p_1}\cdots(\lambda-\lambda_s)^{p_s},$$

where $\{p_j\}_{j=1,2,\dots,s}$ *are integers independent of* (x, ξ) *and* $p_1>1$. *Finally* $\lambda_1, \lambda_2, \dots, \lambda_s$ *are all distinct.*

(2) *There exists a point* (x_1, ξ_1) *such that* $A(x_1, \zeta_1)$ *is not diagonalisable. In particular we assume that the dimension of the eigenspace corresponding to* $\lambda_1(x_1, \xi_1)$ *is smaller than* p_1.

Then there exists an open set $\mathcal{O}_0\subset R_x^{k+1}\times R_\xi^k$ *such that*

(*a*) *We can define an eigenvector* $z_0(x, \xi)$ *and the corresponding root* *vector* $z_1(x, \xi)$ *of* $A(x, \xi)$ *associated with* $\lambda_1(x, \xi)$ *on* $\overline{\mathcal{O}}_0$[4] *so that they have the same regularity as* $A(x, \xi)$:

$$\begin{aligned} Az_0 &= \lambda_1 z_0 \\ Az_1 &= \lambda_1 z_1 + z_0, \quad (x, \xi) \in \overline{\mathcal{O}}_0 . \end{aligned}$$

(*b*) *There exists a* $m \times m$ *matrix* $S(x, \xi)$ *which has the same regularity as* $A(x, \xi)$ *and satisfies the following* (*b.* 1) *and* (*b.* 2):

(*b.* 1) $S(x, \xi)$ *is uniformly regular on* $\overline{\mathcal{O}}_0$, *namely for some positive* σ *we get*

$$|\det S(x, \xi)| \geqq \sigma \quad on \quad \overline{\mathcal{O}}_0 ,$$

$$(b.2) \qquad S(x, \xi)A(x, \xi)S^{-1}(x, \xi) = \begin{bmatrix} \lambda_1, 0, \ 0, \cdots, 0 \\ 1, \lambda_1, 0, \cdots, 0 \\ * \end{bmatrix}.$$

Proof. 1°. Let us denote by M_s a minor of $(\lambda_1 - A)$ of order s. First we shall prove that *there exist an open set* $\mathcal{O}_0 \subset R_x^{k+1} \times R_\xi^k$ *and an integer* $s_0 > m - p_1$ *attached to* $\mathcal{O}_0$ *such that*

1) *if* $s > s_0$, *then all* $M_s \equiv 0$ *on* $\overline{\mathcal{O}}_0$,

2) *if* $s \leqq s_0$, *there exists at least one minor* M_s *which does not vanish on* $\overline{\mathcal{O}}_0$.

There exists, by the condition (2), an integer $s_1 > m - p_1$ such that $M_{s_1}(x_1, \xi_1) \neq 0$. Since $\{\lambda_j(x, \xi)\}$ have, by the implicit function theorem, the same regularity as $A(x, \xi)$, it is also the case about $M_{s_1}(x, \xi)$. Then taking a positive constant δ, we can find a neighbourhood $\mathcal{O}_1$ of (x_1, ξ_1) such that

$$\inf_{\overline{\mathcal{O}}_1} |M_{s_1}(x, \xi)| \geqq \delta .$$

Next, if for some $s' > s_1$ there exists a minor $M_{s'}$ which does not vanish at some point $(x_2, \xi_2) \in \mathcal{O}_1$ then we necessarily find one minor M_{s_1+1} such that

$$M_{s_1+1}(x_2, \xi_2) \neq 0 , \quad (x_2, \xi_2) \in \mathcal{O}_1 .$$

Now we can also find a neighbourhood $\mathcal{O}_2 (\subset \mathcal{O}_1)$ of (x_2, ξ_2) such that

4) $\overline{\mathcal{O}}_0$ means the closure of $\mathcal{O}_0$.

$$\inf_{\bar{\mathcal{O}}_2} |M_{s_1+1}(x, \xi)| \geqq \delta .$$

After the finite times of the same way, we can get the above mentioned open set $\mathcal{O}_0$ and an integer s_0.

2°. Now let us consider to define an eigenvector $z_0(x, \xi)$ and an associated root vector $z_1(x, \xi)$ on $\bar{\mathcal{O}}_0$. Denote $(\lambda_1 - A(x, \xi))$ by A_1. Then we can find, by the same reasoning in 1°, an open set $\mathcal{O}_0$ and an integer $s_0>0$ associated with A_1^2 such that

1) if $s>s_0$, all minor M_s of A_1^2 are identically zero on $\bar{\mathcal{O}}_0$.

2) if $s\leqq s_0$, there exists at least one minor of A_1^2 which does not vanish on $\bar{\mathcal{O}}_0$.

By this fact, we can get non-trivial solutions of

$$A_1^2 z = 0$$

on $\bar{\mathcal{O}}_0$ (Cramer's formula). Evidently we can find[5] $z_1(x, \xi)$ in $\{z(x, \xi) : A_1^2 z=0 \text{ on } \bar{\mathcal{O}}_0,\ z\not\equiv 0\}$ and finally $z_0(x, \xi)$ is given by

$$z_0 = A_1 z_1 .$$

3°. *Construction of* $S(x, \xi)$. Since $z_0(x, \xi)$ and $z_1(x, \xi)$ are linearly independent on $\bar{\mathcal{O}}_0$, there exists an integer j and k such that

$$\det \begin{bmatrix} z_{0j} & z_{0k} \\ z_{1j} & z_{1k} \end{bmatrix} \neq 0$$

on $\bar{\mathcal{O}}_0$.[6] Now if we set

$$\tilde{S}(x, \xi) = \begin{pmatrix} z_{0j}, z_{0k}, z_{01}, \cdots, z_{0j-1}, z_{0j+1}, \cdots, z_{0k-1}, z_{0k+1}, \cdots, z_{0m} \\ z_{1j}, z_{1k}, z_{11}, \cdots, z_{1j-1}, z_{1j+1}, \cdots, z_{1k-1}, z_{1k+1}, \cdots, z_{1m} \\ 0\ ,\ 0\ ,\ 1\ ,\ 0\ ,\ \cdots\cdots\cdots\cdots\cdots\cdots\ ,\ 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0\ ,\ 0\ ,\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\ ,\ 0\ ,\ 1 \end{pmatrix},$$

this matrix is uniformly regular on $\bar{\mathcal{O}}_0$. Changing the orders of rows in $\tilde{S}(x, \xi)$ we get $S(x, \xi)$ as

5) Since the rank r of A_1^2 is smaller than m, we can take arbitrary functions for $(m-r)$ components of z_0 and z_1. Of course we take functions with the same regularity as $A(x, \xi)$ for them.

6) $z_0 = {}^t(z_{01}, z_{02}, \cdots, z_{0m})$, $z_1 = {}^t(z_{11}, z_{12}, \cdots, z_{1m})$.

$$S(x, \xi) = \begin{bmatrix} z_{01}, z_{02}, \cdots, z_{0m} \\ z_{11}, z_{12}, \cdots, z_{1m} \\ * \end{bmatrix}.$$

Of course $S(x, \xi)$ is uniformly regular on $\overline{\mathcal{O}}_0$ and the last assertion (b. 2) is evident.

4. The Proof of Theorem

1° We shall prove our theorem by a contradiction. Therefore we assume that the matrix $A(x, t, \xi)=\sum_{j=1}^{k} A_j(x, t)\xi_j$ is not diagonalisable at $(0, 0, \xi_0)$, $\xi_0 \neq 0$. More explicitly we assume that the dimension of the eigenspace corresponding to $\lambda_1(0, 0, \xi_0)$ is actually smaller than p_1 (=the multiplicity of the eigenvalue λ_1). Under this circumstances, we can prove that the Cauchy problem for

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{k} A_j(x, t)\frac{\partial u}{\partial x_j} \tag{4.1}$$

is not uniformly L^2-well posed in $[0, T]$.

2°. Under the assumption we can find, by Proposition 3. 1, an open set $\mathcal{O}_0 \subset R_x^k \times R_\xi^k \times [0, T]$ and get a $m\times m$ matrix $n_0(x, t, \xi)$ satisfying the following

1) $n_0(x, t, \xi)$ is uniformly regular on $\overline{\mathcal{O}}_0$,

2) $$n_0(x, t, \xi)A(x, t, \xi)n_0(x, t, \xi)^{-1} = \begin{bmatrix} i\lambda_1, 0, \cdots\cdots\cdots, 0 \\ i, i\lambda_1, 0, \cdots, 0 \\ * \end{bmatrix},$$

$$(x, t, \xi)\in\overline{\mathcal{O}}_0.$$

Now we localize our equation (4. 1) in this $\mathcal{O}_0$. Take a C_0^∞-function $\beta(x)$ which satisfies (i) $0\leqq\beta(x)\leqq 1$, (ii) supp $[\beta]$ is contained in the x-section of $\mathcal{O}_0$. Multiply (4. 1) by $\beta(x)$ and apply the pseudo-differential operator $\alpha_n(D)$, then we get

$$\frac{\partial}{\partial t}\alpha_n(\beta u) = \sum_{j=1}^{k} A_j(x, t)\frac{\partial}{\partial x_j}\alpha_n(\beta u) + \\ +\sum_{j=1}^{k}\left[\alpha_n, A_j\frac{\partial}{\partial x_j}\right](\beta u)+\sum_{j=1}^{k}\alpha_n\left(\left(A_j(x, t)\frac{\partial\beta}{\partial x_j}\right)u\right).$$

We take here the pseudo-differential operator $\alpha_n(D)$[7)]

$$\text{supp}\,[\hat{\alpha}(\xi)] \subset \{\text{the } \xi\text{-section of } \mathcal{O}_0\}\,.$$

Setting

$$v_n = \alpha_n(\beta u)\,,$$

$$f_n = \sum_{j=1}^{k}\Big(\Big[\alpha_n, A_j\frac{\partial}{\partial x_j}\Big](\beta u) + \alpha_n\Big(A_j\frac{\partial\beta}{\partial x_j}\Big)u\Big)$$

our equation can be written as

$$\frac{\partial v_n}{\partial t} = \sum_{j=1}^{k} A_j(x, t)\frac{\partial v_n}{\partial x_j} + f_n\,.$$

Now we express this equation by a singular integral operator $\mathcal{H}_0$:

$$\frac{d}{dt}v_n = \mathcal{H}_0\Lambda v_n + f_n\,, \tag{4.2}$$

where $\mathcal{H}_0$ is defined by a symbol $\sigma(\mathcal{H}_0) = i\sum_{j=1}^{k} A_j(x, t)\frac{\xi_j}{|\xi|}$.

3°. Let us now consider to modify the principal part $\mathcal{H}_0\Lambda$ of our equation. This modification is essential for the proof of our theorem.

Extending the definition of $n_0(x, t, \xi)$ in the whole space $R^k_x \times R^k_\xi \times [0\ T]$, we get a singular integral operator $\mathcal{N}_0(x, t, D)$[8)]:

$$\sigma(\mathcal{N}_0)(x, t, D) = n_0(x, t, \xi')\,, \quad \xi' = \xi/|\xi|\,.$$

In the same way we also get a singular integral operator $\mathcal{D}_0$ by a symbol $n_0(x, t, \xi')A(x, t, \xi')n_0^{-1}(x, t, \xi')$. Now we get

$$\sigma(\mathcal{N}_0)\sigma(\mathcal{H}_0) = \sigma(\mathcal{D}_0)\sigma(\mathcal{N}_0)\,,$$

that is

$$\mathcal{N}_0\mathcal{H}_0\Lambda = \mathcal{D}_0\Lambda\mathcal{N}_0 + B_1$$

where B_1 is a bounded operator in L^2.

We shall prove that there exists a perturbation of $\mathcal{D}_0\Lambda$ with a pseudo-differential operator which breaks the well-posedness of the Cauchy problem. This fact proves the statement of our theorem.

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7) A pseudo-differential operator $\alpha_n(D)$ is generally defined in §2.

8) We refer readers to §3 in [2].

For the perturbation operator, take a pseudo-differential operator $\mathcal{D}_{02}\gamma$ defined by the following $m\times m$ matrix

$$\sigma(\mathcal{D}_{02}\gamma)=\begin{bmatrix}0, & b\hat{\gamma}(\xi)|\xi|^{-1}, & 0, & \cdots, & 0\\ & & 0 & & \end{bmatrix},$$

where b is a real constant, we shall define its size later (see Lemma 4.2).

Denoting the pseudo-differential operator $(\mathcal{D}_0+\mathcal{D}_{02})\gamma$ by $\mathcal{D}_{01}\gamma$, we decompose $\mathcal{D}_0\Lambda\mathcal{N}_0$ into

$$\mathcal{D}_0\Lambda\mathcal{N}_0=\mathcal{D}_{01}\gamma\Lambda\mathcal{N}_0+(\mathcal{D}_0(1-\gamma)\Lambda-\mathcal{D}_{02}\gamma\Lambda)\mathcal{N}_0\,.$$

Then $(\mathcal{D}_0(1-\gamma)-\mathcal{D}_{02}\gamma)\Lambda\mathcal{N}_0\equiv B_2$ is a pseudo-differential operator of order zero. In the sequel we get

$$\frac{d}{dt}w_n=\mathcal{D}_{01}\gamma\Lambda w_n+(B_1+B_2)v_n+\mathcal{N}_0'v_n+\mathcal{N}_0f_n\,, \tag{4.3}$$

where $w_n=\mathcal{N}_0v_n$ and $\mathcal{N}_0'$ is defined by $\sigma(\mathcal{N}_0')=-\frac{d}{dt}\sigma(\mathcal{N}_0)$. Using Lemma 2.4–2.5, we get

Lemma 4.1. (1)[9] *There exists a positive constant c_1 independent of n and we have*

$$\left\|\left[\alpha_n, A_j(x,t)\frac{\partial}{\partial x_j}\right]\right\|\leqq c_1\,,\qquad 0\leqq t\leqq T.$$

(2) *There exists a positive constant c_2 independent of n such that*

$$\|B_1v_n+B_2v_n+\mathcal{N}_0'v_n+\mathcal{N}_0f_n\|\leqq c_2\|u(t)\|_{L^2}\,.$$

Now let us prove that the principal part $\mathcal{D}_{01}\gamma\Lambda$ of (4.3) violates the Cauchy problem for (4.1). To show this, we first consider to diagonalize the 2×2 block of $\mathcal{D}_{01}\gamma\Lambda$ at the top-left corner. Calculate the perturbed characteristic roots of $\sigma(\mathcal{D}_{01})$:

$$\begin{aligned}\det(\mu I-\sigma(\mathcal{D}_{01}))&=((\mu-i\lambda_1)^2-ib|\xi|^{-1})(\mu-i\lambda_1)^{p_1-2}\cdots(\mu-i\lambda_s)^{p_s}\\&=(\mu-\mu_1)(\mu-\mu_2)(\mu-i\lambda_1)^{p_1-2}\cdots(\mu-i\lambda_s)^{p_s}\,,\end{aligned}$$

namely we get the perturbed characteristic roots

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9) This is stated here only for the use to prove (2).

$$\mu_1 = i\lambda_1(x, t, \xi') + \sqrt{ib}\,|\xi|^{-1/2},$$
$$\mu_2 = i\lambda_1(x, t, \xi') - \sqrt{ib}\,|\xi|^{-1/2}.$$

Here we notice that

Lemma 4.2. (1) *Let* $\delta > 0$ *be given, taking* b *suitably* μ_1 *satisfies*[10]

$$\begin{aligned}\operatorname{Re}\mu_1(x, t, \xi) &= \operatorname{Re}\sqrt{ib}\,|\xi|^{-1/2}\\ &\geqq \delta|\xi|^{-1/2}.\end{aligned}$$

(2) *The perturbed characteristic roots* μ_1 *and* μ_2 *define pseudo-differential operators of type P.*

In the same way for $n_0(x, t, \xi')$, we get a $m\times m$ matrix $n_1(x, t, \xi)$ such that

1) regular for $0<|\xi|<+\infty$.

2) $n_1(x, t, \xi)\sigma(\mathcal{D}_{01})n_1^{-1}(x, t, \xi) = \begin{bmatrix}\mu_1, & 0, & \cdots, & 0\\ 0, & \mu_2, & \cdots, & 0\\ & & * & \end{bmatrix}.$

Extending the definition of $n_1(x, t, \xi)$ in the whole space, we get a pseudo-differential operator $\mathcal{N}_1\gamma$ of type P: we define $\mathcal{N}_1\gamma$ by $\sigma(\mathcal{N}_1)\hat{\gamma}(\xi) = |\xi|^{-1}n_1(x, t, \xi)$.[11] Now defining the pseudo-differential operator $\mathcal{D}_\gamma$ by

$$\sigma(\mathcal{D}_\gamma) = \begin{bmatrix}\mu_1\hat{\gamma}, & 0, & \cdots, & 0\\ 0, & \mu_2\hat{\gamma}, & \cdots, & 0\\ & & * & \end{bmatrix},$$

we get

$$\mathcal{N}_1\gamma\mathcal{D}_{01}\gamma\Lambda = \mathcal{D}_\gamma\Lambda\mathcal{N}_1\gamma + B_3$$

where B_3 is a bounded operator in L^2. Thus our equation is expressed as

$$\frac{d}{dt}W_n(t) = \mathcal{D}_\gamma\Lambda W_n(t) + G_n(t), \tag{4.4}$$

where

10) By the hyperbolicity of the operator M, $\lambda_1(x, t, \xi)$ is real.

11) Here this $n_1(x, t, \xi)$ is the extended one. The multiplication by $|\xi|^{-1}$ is necessary for $N_1\gamma$ to be of *type P*.

$$W_n(t) = \mathcal{N}_1\gamma w_n(t)\,,$$
$$G_n = \mathcal{N}_1\gamma((B_1+B_2)v_n + \mathcal{N}_0' v_n + \mathcal{N}_0 f_n) + \mathcal{N}_1'\gamma w_n + B_3 w_n\,.$$ [12]

For this G_n we have

Lemma 4.3. *We see by Lemma 4.1 that*

$$||G_n(t)||_{L^2} \leqq C||u(t)||_{L^2}\,,$$

for some constant C independent of n.

Let us define a pseudo-differential operator $\mathcal{R}_{11}\gamma$ by

$$\sigma(\mathcal{R}_{11}\gamma) = \mu_1(x, t, \xi)\hat{\gamma}(\xi)\,,$$

then the first column of the system (4.4) is

$$\frac{d}{dt} W_n^{(1)}(t) = \mathcal{R}_{11}\gamma\Lambda W_n^{(1)}(t) + G_n^{(1)}(t)\,. \tag{4.5}$$

From this equation, we get a fundamental inequality

$$\frac{d}{dt} S_n(t) \geqq c_1\sqrt{n}\; S_n(t) - c_2||u(t)||_{L^2}^2\,, \tag{4.6}$$

for $S_n(t) = ||W_n^{(1)}(t)||_{L^2}^2$ just as in §3 of [2]. This inequality shows that for the real b defined in Lemma 4.2 the perturbation of $\mathcal{D}_0\Lambda$ by $\mathcal{D}_{02}\gamma\Lambda$ violates the well-posedness of the Cauchy problem.

4°. Taking $\hat{\psi}(\xi) \in C_0^\infty(R_\xi^k)$ such that

$$\text{supp}\,[\hat{\psi}(\xi)] \subset \{\xi : \hat{\alpha}(\xi) \equiv 1\}\,,$$

we define $\psi_n(x)$ by $\psi_n(x) = e^{i(n-1)\xi_0 \cdot x}\psi(x) \in L^2$, where $\psi(x)$ is the Fourier inverse image of $\hat{\psi}(\xi)$. Let $\mathcal{M}_0$ and $\mathcal{M}_1$ be pseudo-differential operators defined by the symbols $\sigma(\mathcal{M}_0) = n_0(x, t, \xi')^{-1}$ and $\sigma(\mathcal{M}_1) = n_1(x, t, \xi)^{-1}$ respectively. Then they have the following properties:

Lemma 4.4.

1) $\mathcal{M}_0 \circ \mathcal{N}_0 = I$.

2) $\mathcal{M}_1 \circ \mathcal{N}_1\gamma = \gamma I$.

Consider now the following Cauchy problem:

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12) The operator $N_1'\gamma$ is defined by $-\frac{d}{dt}\sigma(N_1\gamma)$.

$$\frac{\partial u}{\partial t} = \sum_{j=1}^{k} A_j(x, t) \frac{\partial u}{\partial x_j}, \qquad t_0 \leqq t \leqq T,$$
$$u(x, t_0) = \mathcal{M}_0 \mathcal{M}_1 \gamma \Psi_n, \quad \Psi_n = {}^t(\psi_n(x), 0, \cdots, 0).$$

Denoting the solution of this problem by

$$u_n(x, t, t_0) \in \mathcal{E}_t^0(L^2) \cap \mathcal{E}_t^1(\mathcal{D}_{L^2}'), \qquad t_0 \leqq t \leqq T,$$

we have by the energy inequality

Lemma 4.5. *There exists a positive constant C independent of n and t_0 such that*

$$\|u_n(\cdot, t, t_0)\|_{L^2} \leqq C.$$

If we take

$$\begin{aligned} \hat{S}_n &= \| W_n^{(1)}(t) \|^2 \\ &= \| \mathcal{N}_1 \gamma \mathcal{N}_0 \alpha_n (\beta u_n(t, t_0))^{(1)} \|^2 \end{aligned}$$

for $S_n(t)$, using Lemma 4.4, we get by (4.6) an inequality

$$\frac{d}{dt} \tilde{S}_n(t) \geqq c_1 \sqrt{n}\, \tilde{S}_n(t) - c_2, \tag{4.7}$$

for sufficiently large n, where c_1 and c_2 are constants independent of n. Finally we get the following lemma. We shall prove this lemma in Appendix.

Lemma 4.6. *There exist constants c_3 and c_4 independent of n such that*

(i) $\tilde{S}_n(t_0) \geqq c_3, \quad 0 \leqq \forall t_0 \leqq T.$
(ii) $\tilde{S}_n(t) \leqq c_4, \quad 0 \leqq t_0 \leqq \forall t \leqq T.$

Integrating (4.7) by t, we get by Lemma 4.6

$$c_4 > \tilde{S}_n(t) > c_3 e^{c_1 \sqrt{n}(t-t_0)} + \frac{c_2}{c_1 \sqrt{n}} (1 - e^{c_1 \sqrt{n}(t-t_0)}) \qquad 0 \leqq t_0 \leqq t \leqq T,$$

and this is a contradiction as n tends to infinity except $t = t_0$. Thus our proof is complete.

Appendix

Let us now prove here Lemma 4. 6 in §4. First we see that $W_n(t_0, t_0)$ can be expressed as

$$\begin{aligned} W_n(t_0, t_0) &= \mathcal{N}_1\gamma\mathcal{N}_0\mathcal{M}_0\mathcal{M}_1\gamma(\alpha_n(\beta\Psi_n)) + \\ &+\mathcal{N}_1\gamma\mathcal{N}_0\Big[[\alpha_n, \mathcal{M}_0]\mathcal{M}_1\gamma(\beta\Psi_n) + \mathcal{M}_0[\alpha_n, \mathcal{M}_1\gamma](\beta\Psi_n) + \\ &+\alpha_n\mathcal{M}_0[\beta, \mathcal{M}_1\gamma]\Psi_n + \alpha_n[\beta, \mathcal{M}_0]\mathcal{M}_1\gamma\Psi_n\Big]. \end{aligned}$$

Setting

$$\begin{aligned} [\alpha_n, \mathcal{M}_0]\mathcal{M}_1\gamma(\beta\Psi_n) &\equiv X_1, \\ \mathcal{M}_0[\alpha_n, \mathcal{M}_1\gamma](\beta\Psi_n) &\equiv X_2, \\ \alpha_n\mathcal{M}_0[\beta, \mathcal{M}_1\gamma]\Psi_n &\equiv X_3, \\ \alpha_n[\beta, \mathcal{M}_0]\mathcal{M}_1\gamma\Psi_n &\equiv X_4, \end{aligned}$$

we have

$$\text{(A. 1)} \qquad \begin{aligned} W_n(t_0, t_0) &= \mathcal{N}_1\gamma\mathcal{N}_0\mathcal{M}_0\mathcal{M}_1\gamma(\alpha_n(\beta\Psi_n)) + \mathcal{N}_1\gamma\mathcal{N}_0 X, \\ X &= X_1 + X_2 + X_3 + X_4. \end{aligned}$$

Lemma A. 1. *There exists a positive constant C independent of n such that*

$$\|X\|_{L^2} \leqq \frac{C}{n}.$$

Proof. First we see by Lemma 2. 5, X_1 and X_2 are, in L^2 norm, evidently of order $O(n^{-1})$ as n tends to infinity. Next, using the decomposition of identity operator: $I = \frac{I}{I+\Lambda} + \frac{\Lambda}{I+\Lambda}$, X_3 is represented as

$$\begin{aligned} X_3 &= \alpha_n\mathcal{M}_0[\beta, \mathcal{M}_1\gamma]\Psi_n \\ &= \alpha_n\mathcal{M}_0[\beta, \mathcal{M}_1\Lambda\gamma]\frac{\Psi_n}{I+\Lambda} + \alpha_n\mathcal{M}_0[\beta, \mathcal{M}_1\gamma]\frac{\Psi_n}{I+\Lambda}. \end{aligned}$$

Since β is a C_0^∞-function, $[\beta, \mathcal{M}_1\gamma]$ and $[\beta, \mathcal{M}_1\Lambda\gamma]$ are operators of order zero (cf. Remark 1 in §2 of [2]). Then we have

$$\begin{aligned} \|X_3\| &\leqq \text{constant}\left\|\hat{\alpha}_n(\xi)\,\frac{\hat{\phi}_n(\xi)}{1+|\xi|}\right\| \\ &\leqq \frac{\text{constant}}{n}. \end{aligned}$$

In the same way, we easily get the similar estimate for X_4.

Next, we rewrite the first term of the right-handside of (A. 1):

$$\mathcal{N}_1\gamma\mathcal{N}_0\mathcal{M}_0\mathcal{M}_1\gamma = \gamma^2+(\mathcal{N}_1\gamma-\gamma\mathcal{N}_1)\mathcal{M}_1\gamma+\gamma(\mathcal{N}_1\mathcal{M}_1-\mathcal{N}_1\circ\mathcal{M}_1)\gamma+ \\ +\mathcal{N}_1\gamma(\mathcal{N}_0\mathcal{M}_0-\mathcal{N}_0\circ\mathcal{M}_0)\mathcal{M}_1\gamma ,$$

and we get

$$W_n(t_0, t_0) = \gamma^2(\alpha_n(\beta\Psi_n))+(Q_1+Q_2+Q_3)(\alpha_n(\beta\Psi_n))+\mathcal{N}_1\gamma\mathcal{N}_0 X ,$$

where

$$Q_1 = [\mathcal{N}_1, \gamma]\mathcal{M}_1\gamma$$
$$Q_2 = \gamma(\mathcal{N}_1\mathcal{M}_1-\mathcal{N}_1\circ\mathcal{M}_1)\gamma ,$$
$$Q_3 = \mathcal{N}_1\gamma(\mathcal{N}_0\mathcal{M}_0-\mathcal{N}_0\circ\mathcal{M}_0)\mathcal{M}_1\gamma .$$

Then we have the following

Lemma A. 2. *For some constant C independent of n, we have*

$$||Q_i||\leqq\frac{n}{C}, \qquad i=1, 2, 3 .$$

Proof. Using the preceding decomposition of the identity, we can express Q_1 as

$$Q_1 = \frac{Q_1}{I+\Lambda}+\frac{Q_1\Lambda}{I+\Lambda}$$
$$= (\mathcal{N}_1\gamma-\gamma\mathcal{N}_1)\mathcal{M}_1\gamma\frac{I}{I+\Lambda}+(\mathcal{N}_1\gamma-\gamma\mathcal{N}_1)\mathcal{M}_1\Lambda\gamma\frac{I}{I+\Lambda} .$$

Now we write this as

$$Q_1\equiv(Q_{11}+Q_{12})\frac{I}{I+\Lambda} .$$

Since $\hat{\gamma}(\xi)\equiv1$ for $|\xi|\geqq R+1$, Q_{11} and Q_{12} are evidently the bounded operators in L^2. We can show, just in the same way, that $||Q_2||$ is also of order n^{-1} as n tends to infinity.

Finally, noticing

$$\mathcal{M}_1\Lambda\gamma = \Lambda\mathcal{M}_1\gamma+[\mathcal{M}_1\gamma, \Lambda] ,$$

we can get

$$||Q_3\Lambda||\leqq C(||(\mathcal{N}_0\mathcal{M}_0-\mathcal{N}_0\circ\mathcal{M}_0)\Lambda||\cdot||\mathcal{M}_1\gamma||+ \\ +||(\mathcal{N}_0\mathcal{M}_0-\mathcal{N}_0\circ\mathcal{M}_0)||\cdot||[\mathcal{M}_1, \Lambda]\gamma||)$$

and by this estimate we get $||Q_3|| \leqq \frac{C}{n}$.

These two lemmas prove that there exists a positive constant K independent of n such that

$$\tilde{S}_n(t_0) = ||W_n^{(1)}(t_0, t_0)||^2$$
$$\geqq ||\alpha_n(\beta\psi_n)||^2 - \left(\frac{K}{n}\right)^2.$$

Then Lemma 4.6 is immediate by this inequality just as Lemma 4.2 in our previous paper [2].

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