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A Family of One-parameter Subgroups of $\mathcal{O}(\mathcal{S}_r)$ Arising from the Variable Change of the White Noise

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Summary

Let L_r^2 be the real Hilbert space of square summable functions on the real line and let $\mathcal{S}_r$ be the space of rapidly decreasing functions. Then we can define the probability measure μ of the *Gaussian white noise* on the conjugate space $\mathcal{S}_r^*$ of $\mathcal{S}_r$.

Let $\mathcal{O}(\mathcal{S}_r)$ be the group of rotation which act on $\mathcal{S}_r$. Then every element $\mathfrak{g}$ of $\mathcal{O}(\mathcal{S}_r)$ induces an automorphism $\mathfrak{g}^*$ on the probability space $(\mathcal{S}_r^*, \mu)$ and so every one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ induces a flow on $(\mathcal{S}_r^*, \mu)$.

T. Hida, I. Kubo, H. Nomoto and H. Yoshizawa [1] introduced a certain kind of one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ arising from the variable change by *functions.*

In this paper, we define another kind of one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ arising from the variable change by *distributions* and show that this family contains the *shift*, the *tension* and furthermore *all that commute with the shift.*

1. Introduction

Let $L^2 = L^2(-\infty, +\infty)$ be the complex Hilbert space of complex-valued square summable functions on the real line and L_r^2 be the real Hilbert space consisting of all real-valued functions in L^2. Let $\mathcal{S}$ be the complex topological vector space of rapidly decreasing functions on the real line, that is

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$$\mathcal{S} = \left\{ \begin{array}{l} \xi(x) \in C^\infty ; \\ ||\xi||_{k,p} = \sup\limits_{-\infty<x<+\infty} |x^p \xi^{(k)}_{(x)}| < +\infty \, , \\ \qquad\qquad\qquad\qquad k, p = 0, 1, 2, 3, \cdots \end{array} \right\}$$

where C^∞ is the set of all infinitely many times continuously differentiable functions. It is well-known that $\mathcal{S}$ is a σ-normed nuclear space with the family of the norms $\{||\xi||_{k,p} ; k, p=0, 1, 2, \cdots\}$. Let $\mathcal{S}_r$ be the real topological vector space consisting of all real-valued functions in $\mathcal{S}$. Then $\mathcal{S}_r$ is also a nuclear space contained in L^2_r densely, and by Minlos' theorem a continuous postive-definite functional on $\mathcal{S}_r$ defined by

$$C(\xi) = \exp\left[-\frac{1}{2}||\xi||^2\right], \qquad \xi \in \mathcal{S}_r$$

determines a probability measure μ on $\mathcal{S}_r^*$ such that

$$C(\xi) = \int_{\mathcal{S}_r^*} \exp\,[i\langle X, \xi\rangle] d\mu(X)\,,$$

where $||\xi||$ stands for the norm on L^2_r and $\langle X, \xi\rangle$ the canonical bilinear form on $\mathcal{S}_r^* \times \mathcal{S}_r$. We call this probability measure μ the *Gaussian white noise.*

Let $\mathcal{O}(\mathcal{S}_r)$ be a group of rotations on L^2_r and that map $\mathcal{S}_r$ onto $\mathcal{S}_r$ and the restriction of that to $\mathcal{S}_r$ is homeomorphism on $\mathcal{S}_r$. Then for every $\mathfrak{g}$ in $\mathcal{O}(\mathcal{S}_r)$, we can define a homeomorphism $\mathfrak{g}^*$ on $\mathcal{S}_r^*$ by

$$\langle \mathfrak{g}^* X, \xi\rangle = \langle X, \mathfrak{g}\xi\rangle\,, \qquad \xi \in \mathcal{S}_r\,,\ X \in \mathcal{S}_r^*$$

and it is well-known that $\mathfrak{g}^*$ is an automorphism on the probability space $(\mathcal{S}_r^*, \mu)$, (see for example T. Hida [2]). By the above correspodence, every one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ induces a flow on $(\mathcal{S}_r^*, \mu)$.

In this paper, we first define a subgroup of $\mathcal{O}(\mathcal{S}_r)$ which comes from the variable change by distributions, then we find a family of one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ which contains the shift and the tension, and finally we show that the above family contains all the one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ that commute with the shift.

Let $\varphi(x)$ be a complex-valued locally summable function on the real line and let $f_\varphi(x)$ be an absolutely continuous non-decreasing function defined by

$$f_\varphi(x) = \int_0^x |\varphi(y)|^2 dy\,, \qquad -\infty < x < +\infty.$$

We add to $\varphi(x)$ two assumptions as follows.

$$\text{(A. 1)} \qquad \begin{cases} f_\varphi(+\infty) \ (= \lim_{x \to +\infty} f_\varphi(x)) = +\infty\,, \\ f_\varphi(-\infty) \ (= \lim_{x \to -\infty} f_\varphi(x)) = -\infty\,, \end{cases}$$

$$\text{(A. 2)} \qquad \varphi(x) \neq 0 \qquad \text{a.e.}$$

Then f_φ maps the real line onto itself in one-to-one manner, and therefore, the inverse function $f_\varphi^{-1}(x)$ is well-defined. Now we define a unitary transformation $\mathfrak{g}[\varphi]$ on L^2 as follows.

$$\text{(1. 1)} \qquad \mathfrak{g}[\varphi]\xi(x) = \varphi(x)\xi(f_\varphi(x))\,, \qquad \xi \in L^2\,.$$

Let $U(\mathcal{S})$ be a group of unitary transformations on L^2 the restriction of which to $\mathcal{S}$ are homeomorphisms from $\mathcal{S}$ onto $\mathcal{S}$, and let $\mathcal{U}_{\mathcal{S}}$ be the set of locally square summable functions $\varphi(x)$ which satisfy (A. 1) and (A. 2) and for which $\mathfrak{g}[\varphi]$ belong to $U(\mathcal{S})$.

In Section 2, we determine the family of functions concretely. In fact we have the following theorem.

Theorem 1. *A function $\varphi(x)$ belongs to $\mathcal{U}_{\mathcal{S}}$ if and only if it satisfies the following four conditions.*

(S. 1) $\varphi(x) \in C^\infty$.

(S. 2) $\varphi(x) \neq 0\,, \qquad -\infty < x < +\infty.$

(S. 3) *For arbitrary non-negative integers k, p, there exists a positive number $r = r(k, p)$ such that*

$$\text{(1. 2)} \qquad \lim_{|x| \to \infty} \frac{|\varphi^{(k)}(x)|\,|x|^p}{|f_\varphi(x)|^r} = 0\,.$$

(S. 4) *For every non-negative integer p, there exists a positive number $\rho = \rho(p)$ such that*

$$\text{(1. 3)} \qquad \lim_{|x| \to \infty} \frac{|f_\varphi(x)|^p}{|x|^\rho |\varphi(x)|} = 0\,.$$

Let $\mathcal{F}$ be the Fourier transform on L^2 defined by

$$(1.4) \qquad (\mathcal{F}\xi)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\lambda} \xi(\lambda) d\lambda, \qquad \xi \in L^2.$$

Then it is well-known that $\mathcal{F} \in U(\mathcal{S})$. Suppose now a function $\varphi(x)$ in $\mathcal{U}_{\mathcal{S}}$ is Hermitian, i.e.,

$$\overline{\varphi(x)} = \varphi(-x),$$

and put

$$(1.5) \qquad \tilde{\mathfrak{g}}[\varphi] = \mathcal{F}^{-1} \mathfrak{g}[\varphi] \mathcal{F}.$$

Then $\tilde{\mathfrak{g}}[\varphi]$ belongs to $\mathcal{O}(\mathcal{S}_r)$.

In Section 3, we first show that $\mathcal{U}_{\mathcal{S}}$ is a group with respect to a product operation $\circledast$ defined by

$$(1.6) \qquad (\varphi \circledast \psi)(x) = \varphi(x) \psi(f_\varphi(x)), \qquad \varphi, \psi \in \mathcal{U}_{\mathcal{S}},$$

and next, $\mathfrak{g}[\varphi]$ is a unitary representation of the group $\mathcal{U}_{\mathcal{S}}$ on L^2 that is,

$$(1.7) \qquad \mathfrak{g}[\varphi]\mathfrak{g}[\psi] = \mathfrak{g}[\varphi \circledast \psi], \qquad \varphi, \psi \in \mathcal{U}_{\mathcal{S}}.$$

Using the above relation, we find two interesting families of one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ which are given in the type (1.5) and contain the *shift* and the *tension* in the terminology of [1]. The exact statements are as follows.

Let $\mathcal{I}$ be the set of all real odd slowly increasing functions, and for every function $h(x)$ in $\mathcal{I}$ define

$$\tilde{\mathfrak{G}}_0(h) = \{\tilde{\mathfrak{g}}[e^{ith(x)}]\,;\ -\infty < t < +\infty\}.$$

Then $\tilde{\mathfrak{G}}(h)$ is a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ and we have:

Theorem 5. *Let $\tilde{\mathfrak{G}}_0$ be the family of one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ defined by*

$$\tilde{\mathfrak{G}}_0 = \{\tilde{\mathfrak{G}}_0(h)\,;\ h \in \mathcal{I}\}.$$

Then we have:

1°) *$\tilde{\mathfrak{G}}_0$ contains the shift.*

2°) *For every $h(x)$, $h'(x)$ in $\mathcal{I}$, $\tilde{\mathfrak{G}}_0(h)$ and $\tilde{\mathfrak{G}}_0(h')$ commute with each other.*

Furthermore we have the following theorem.

Theorem 6. *For every function $h(x)$ in $\mathcal{T}$,*

$$\tilde{\mathfrak{G}}_\alpha(h) = \left\{\tilde{\mathfrak{g}}\left[\exp\left(\frac{\alpha}{2}t + i\{h(e^{\alpha t}x) - h(x)\}\right)\right];\ -\infty < t < +\infty\right\}$$

is a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$. The induced flow is isomorphic to the flow induced by the tension

$$\tilde{\mathfrak{G}}_\alpha(h) = \left\{\tilde{\mathfrak{g}}\left[\exp\left(\frac{\alpha}{2}t\right)\right]\right\}.$$

Finally, in Section 4, we show that $\tilde{\mathfrak{G}}_0$ contains all one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ *that commute with the shift.* In other words, $\tilde{\mathfrak{G}}_0$ is the maximal Abelian subgroup of $\mathcal{O}(\mathcal{S}_r)$ containing the shift.

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2. The Family of Functions $\mathcal{U}_\mathcal{S}$

In this section, we determine the family of functions $\mathcal{U}_\mathcal{S}$ explicitly. To this end, we prepare several lemmas.

Lemma 1. *Let $\varphi(x)$ be a function in $\mathcal{U}_\mathcal{S}$. Then the inverse of the unitary transformation $\mathfrak{g}[\varphi]$ on L^2 is given by $\mathfrak{g}[\varphi^{-1}]$ where*

$$\varphi^{-1}(x) = [\varphi(f_\varphi^{-1}(x))]^{-1}. \tag{2.1}$$

Proof. It is not difficult to show that $\varphi^{-1}(x)$ is locally square summable and satisfies (A. 1) and (A. 2).

For every $\xi(x)$ in L^2, we have

$$\begin{aligned}(\mathfrak{g}[\varphi^{-1}]\mathfrak{g}[\varphi]\xi)(x) &= \mathfrak{g}[\varphi^{-1}](\varphi(\cdot)\xi(f_\varphi(\cdot)))(x) \\ &= \frac{\varphi(f_{\varphi^{-1}}(x))}{\varphi(f_\varphi^{-1}(x))}\xi(f_\varphi \circ f_{\varphi^{-1}})(x).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}f_{\varphi^{-1}}\circ f_\varphi(x) &= \int_0^{f_\varphi(x)} \frac{dy}{|\varphi(f_\varphi^{-1}(y))|^2} \\ &= \int_0^x \frac{|\varphi(u)|^2}{|\varphi(u)|^2}du = x,\end{aligned}$$

by the variable change $y=f_\varphi(x)$. This means that $f_\varphi^{-1}=f_{\varphi^{-1}}$, which implies

$$\mathfrak{g}[\varphi^{-1}]\mathfrak{g}[\varphi]\xi = \xi\,, \qquad \text{for every } \xi \text{ in } L^2.$$

In the same manner, we have

$$\mathfrak{g}[\varphi]\mathfrak{g}[\varphi^{-1}]\xi = \xi\,, \qquad \text{for every } \xi \text{ in } L^2.$$

This proves the lemma.

Lemma 2. *Let $\varphi(x)$ and $\xi(x)$ be k-times continuously differentiable functions where k is a positive integer. Then, if the function*

$$\eta(x) = \varphi(x)\xi(f_\varphi(x)) \tag{2.2}$$

is also k-times continuously differentiable, we have

$$\eta^{(k)}(x) = \varphi^{(k)}(x)\xi(f_\varphi(x)) + \sum_{\nu=1}^{k} P_{\nu,k-1}[\varphi]\xi^{(\nu)}(f_\varphi(x))\,, \tag{2.3}$$

where $P_{\nu,k+1}[\varphi]$; $\nu=1, 2, \cdots, k$ are given by evaluating the polynomials $P_{\nu,k-1}(z_1, \bar{z}_1, \cdots, z_k, \bar{z}_k)$, which are determined independently of the functions $\varphi(x)$ and $\xi(x)$, at $z_1=\varphi(x)$, $z_2=\varphi'(x), \cdots, z_k=\varphi^{(k-1)}(x)$.

The proof of this lemma is given by an elementary calculation. Using Lemma 1 and Lemma 2, we prove the following lemma.

Lemma 3. *If a function $\varphi(x)$ is in $\mathcal{U}_{\mathcal{S}}$, then it satisfies the conditions* (S. 1) *and* (S. 2).

Proof. Let $\varphi(x)$ be a function in $\mathcal{U}_{\mathcal{S}}$. Then by definition, the $\mathfrak{g}[\varphi]$ is reduced by $\mathcal{S}$, and therefore, for every function $\xi(x)$ in $\mathcal{S}$, the function

$$\eta(x) = (\mathfrak{g}[\varphi]\xi)(x) = \varphi(x)\xi(f_\varphi(x)) \tag{2.4}$$

is in $\mathcal{S}$.

We now show that $\varphi(x)$ is continuous. In fact for arbitrary real numbers x, h, we have

$$\begin{aligned}\eta(x+h)-\eta(x) &= \varphi(x+h)\xi(f_\varphi(x+h))-\varphi(x)\xi(f_\varphi(x)) \\ &= [\varphi(x+h)-\varphi(x)]\xi(f_\varphi(x+h)) \\ &\quad +\varphi(x)[\xi(f_\varphi(x+h))-\xi(f_\varphi(x))]\,.\end{aligned}$$

By the continuity of the functions $\eta(x)$, $\xi(x)$ and $f_\varphi(x)$, we have

$$0 = \xi(f_\varphi(x)) \lim_{h\to 0} [\varphi(x+h)-\varphi(x)]+0\,,$$

as h tends to 0. Since ξ in $\mathcal{S}$ is arbitrary, we have

$$\lim_{h\to 0} [\varphi(x+h)-\varphi(x)] = 0\,.$$

Next we show by mathematical induction that $\varphi(x)$ is arbitrary times continuously differentiable. Since $\varphi(x)$ is continuous, it is sufficient to show that $\varphi(x)$ is $(k+1)$-times continuously differentiable assuming that it is k-times continuously differentiable.

Since $\eta(x)=\mathfrak{g}[\varphi]\xi(x)$ is in $\mathcal{S}$ for every $\xi(x)$ in $\mathcal{S}$, we have (2.3) by Lemma 2. For every x we may choose a function $\xi(x)$ in $\mathcal{S}$ such that $\xi(f_\varphi(x))\neq 0$, and we have

$$\varphi^{(k)}(x) = \xi(f_\varphi(x))^{-1}[\eta^{(k)}(x)-\sum_{\nu=1}^{k} P_{\nu,k-1}[\varphi]\xi^{(\nu)}(f_\varphi(x))]\,. \tag{2.5}$$

Since $P_{\nu,k-1}[\varphi]$, $\nu=1, 2, \cdots, k$, are polynomials of at most $(k-1)$-times derivatives of $\varphi(x)$ and since, by assumption, $\varphi(x)$ is k-times continuously differentiable, the right side of (2.5) is continuously differentiable. Therefore $\varphi(x)$ is $(k+1)$-times continuously differentiable. This proves (S.1).

Since $\mathfrak{g}[\varphi]$ is homeomorphism of $\mathcal{S}$, $\mathfrak{g}[\varphi]$ is also reduced by $\mathcal{S}$, and we have $\mathfrak{g}^{-1}[\varphi]=\mathfrak{g}[\varphi^{-1}]$ by Lemma 1, where $\varphi^{-1}(x)$ is given by (2.1). Therefore $\varphi^{-1}(x)$ must be in $\mathcal{U}_{\mathcal{S}}$. Applying [S.1] to $\varphi^{-1}(x)$, we have (S.2).

Before stating Lemma 5, we prove the following lemma.

Lemma 4. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be monotone non-decreasing divergent sequences of positive numbers such that*

$$\lim_{n\to+\infty} \frac{\beta_n^p}{\alpha_n} = 0\,, \quad \textit{for every positive integer } p, \tag{2.6}$$

and let β_1 be larger than 1. Let $\{\gamma_n(x)\}$ be a sequence of functions defined as follows:

$$\gamma_n(x) = \gamma_n(x\,;\,\alpha_n, \beta_n) = \begin{cases} \dfrac{1}{\alpha_n}\gamma(x-\beta_n)\,, & x\geqslant 0\,, \\ \dfrac{1}{\alpha_n}\gamma(x+\beta_n)\,, & x<0\,, \end{cases} \qquad n=1, 2, 3, \cdots, \tag{2.7}$$

where

$$\gamma(x) = \begin{cases} \exp\left[\dfrac{1}{x^2-1}\right], & |x|<1, \\ 0, & |x| \geqslant 1. \end{cases}$$

Then $\{\gamma_n\}$ *is a bounded sequence in* $\mathcal{S}$.

Proof. It is not difficult to show that each γ_n is in $\mathcal{S}$. Therefore it is sufficient to show that

$$\sup_n ||\gamma_n||_{k,p} < +\infty, \tag{2.8}$$

for arbitrary non-negative integers k, p.

Since $\gamma_n(x)$ vanishes outside the interval $[-\beta_n-1, \beta_n+1]$, we have

$$\begin{aligned} ||\gamma_n||_{k,p} &= \sup_x |x^p \gamma_n^{(k)}(x)| \\ &\leqslant (1+\beta_n)^p \sup_x |\gamma_n^{(k)}(x)| \\ &\leqslant \frac{(1+\beta_n)^p}{\alpha_n} \sup_x |\gamma^{(k)}(x)| \\ &= \frac{(1+\beta_n)^p}{\alpha_n} ||\gamma||_{k,0}. \end{aligned}$$

On the other hand, by assumption, $\alpha_n^{-1}(1+\beta_n)^p$ is bounded in n. Therefore we have (2.8).

Lemma 5. *If a function* $\varphi(x)$ *is in* $\mathcal{U}_{\mathcal{S}}$, *then it satisfies* (S. 3) *and* (S. 4).

Proof. Let $\varphi(x)$ be a function in $\mathcal{U}_{\mathcal{S}}$. Then by Lemma 3, it satisfies (S. 1) and (S. 2).

First we show (S. 3) by mathematical induction with respect to k. Let $k=0$ and assume (S. 3) is not true. Then there exists a non-negative integer p such that

$$\overline{\lim_{|x|\to+\infty}} \frac{|\varphi(x)|\,|x|^p}{|f_\varphi(x)|^r} = +\infty,$$

for every positive number r. Noting (A. 1) and changing the variable by $y=f_\varphi(x)$, we have

$$\overline{\lim_{|y|\to+\infty}} \frac{|\varphi(f_\varphi^{-1}(y))|\,|f_\varphi^{-1}(y)|}{|y|^r} = +\infty.$$

Therefore, restricting r to non-negative integers, we can choose a sequence of number $\{\beta_n\}$ such that

(2.9) $$1<|\beta_1|\leqslant|\beta_2|\leqslant\cdots\leqslant|\beta_n|\leqslant\cdots\uparrow+\infty,$$

and that

(2.10) $$\lim_{n\to+\infty}\frac{\alpha_n}{|\beta_n|^r}=+\infty, \qquad \text{for every non-negative integer } r,$$

where

$$\alpha_n=|\varphi(f_\varphi^{-1}(\beta_n))|\,|f_\varphi^{-1}(\beta_n)|^p.$$

In fact we may choose $\beta_n=2$ and determine $\beta_n(|\beta_n|\geqslant|\beta_{n-1}|)$ by

$$|\beta_n|^{-n}|\varphi(f_\varphi^{-1}(\beta_n))|\,|f_\varphi^{-1}(\beta_n)|^p\geqslant n, \qquad n=2,3,\cdots.$$

Put

$$\gamma_n(x)=\gamma_n(x\,;\sqrt{\alpha_n},|\beta_n|), \qquad n=1,2,3,\cdots.$$

Then the sequence of functions $\{\gamma_n(x)\}$ is bounded in $\mathcal{S}$ since the real sequences $\{\sqrt{\alpha_n}\}$ and $\{|\beta_n|\}$ satisfy the hypothesis in Lemma 4. Therefore the sequence of functions $\{\mathfrak{g}[\varphi]\gamma_n\}$ must be bounded in $\mathcal{S}$ because $\mathfrak{g}[\varphi]$ is a homeomorphism of $\mathcal{S}$, while, we have

$$\begin{aligned}\|\mathfrak{g}[\varphi]\gamma_n\|_{0,p}&=\sup_x|x^p\varphi(x)\gamma_n(f_\varphi(x))|\\&=\sup_x|f_\varphi^{-1}(x)^p\varphi(f_\varphi^{-1}(x))\gamma_n(x)|\\&\geqslant|f_\varphi^{-1}(\beta_n)^p\varphi(f_\varphi^{-1}(\beta_n))\gamma_n(\beta_n)|\\&=\frac{\alpha_n}{\sqrt{\alpha_n}\,e}=\frac{\sqrt{\alpha_n}}{e}\uparrow+\infty, \qquad \text{as} \quad n\to+\infty.\end{aligned}$$

Therefore $\{\mathfrak{g}[\varphi]\gamma_n\}$ is not bounded in $\mathcal{S}$, which contradicts to the assumption. Thus (S.3) is true for $k=0$.

Assume that (S.3) is true for $k=0,1,2,\cdots,h$. Then for every $k=0,1,2,\cdots,h$, and for every $p=0,1,2,\cdots$, there exists a positive number $r=r(k,p)$ for which (1.2) is valid.

For every function $\xi(t)$ in $\mathcal{S}$, set $\eta(x)=(\mathfrak{g}[\varphi]\xi)(x)\in\mathcal{S}$ and differentiate it $(h+1)$-times. Then by Lemma 2 we have

$$\eta^{(h+1)}(x)=\varphi^{(h+1)}(x)\xi(f_\varphi(x))+\sum_{\nu=1}^{h+1}P_{\nu,h}[\varphi]\xi^{(\nu)}(f_\varphi(x)).$$

Let s be the maximal degree of the polynomials $P_{\nu,h}[\varphi]$; $\nu=1,2,\cdots,h+1$, and put

$$r(p) = \max_{0 \leqslant k \leqslant h} r(k, p), \qquad p=0, 1, 2, \cdots.$$

Since $\eta^{(h+1)}(x)$ and $\xi^{(\nu)}(x)$, $\nu=1, 2, \cdots, h+1$, are in $\mathcal{S}$ for every non-negative integer p, we have

$$\begin{aligned}(2.11)\quad \mathrm{d}(\xi) &= \sup_x |x^p \varphi^{(h+1)}(x)\xi(f_\varphi(x))| \\ &\geqslant \sup_x |x^p \eta^{(h+1)}(x)| + \sum_{\nu=1}^{h+1} \sup_x |x^p P_{\nu,h}[\varphi]\xi^{(\nu)}(f_\varphi(x))| \\ &\geqslant ||\eta||_{h+1,p} + \sum_{\nu=1}^{h+1} \sup_x \frac{|x^p P_{\nu,h}[\varphi]|}{1+|f_\varphi(x)|^{sr(p)}} (1+|f_\varphi(x)|^{sr(p)})|\xi^{(\nu)}(f_\varphi(x))| \\ &\leqslant ||\mathfrak{g}[\varphi]\xi||_{h+1,p} + \sum_{\nu=1}^{h+1} K_\nu \{||\xi||_{\nu,[sr(p)]+1} + ||\xi||_{\nu,0}\},\end{aligned}$$

where

$$K_\nu = \sup_x \frac{|x^p P_{\nu,h}[\varphi]|}{1+|f_\varphi(x)|^{sr(p)}}, \qquad \nu=1, 2, \cdots, h+1,$$

and $[z]$ means the maximal integer not greater than z.

Assume that (S. 3) is not true for $k=h+1$. Then there exists a non-negative integer p such that

$$\varlimsup_{|x|\to+\infty} \frac{|\varphi^{(h+1)}(x)|\,|x|^p}{|f_\varphi(x)|^r} = \varlimsup_{|y|\to+\infty} \frac{|\varphi^{(h+1)}(f_\varphi^{-1}(y))|\,|f_\varphi^{-1}(y)|^p}{|y|^r} = +\infty,$$

for every positive number r. Therefore, in the same manner as before, we can choose sequences of real numbers $\{\alpha_n\}$ and $\{\beta_n\}$ such that

$$\alpha_n = |\varphi^{(h+1)}(f_\varphi^{-1}(\beta_n))|\,|f_\varphi^{-1}(\beta_n)|^p, \qquad n=1, 2, 3, \cdots,$$

and that (2. 9) and (2. 10) are valid.

Put

$$\gamma_n(x) = \gamma_n(x; \sqrt{\alpha_n}, |\beta_n|), \qquad n=1, 2, 3, \cdots.$$

Then by Lemma 4, $\{\gamma_n\}$ is bounded in $\mathcal{S}$. Because of (2. 12) and the continuity of $\mathfrak{g}[\varphi]$ $\{(\gamma_n)\}$ must be bounded in $\mathcal{S}$.

On the other hand we have

$$\begin{aligned}\mathrm{d}(\gamma_n) &= \sup_x |x|^p |\varphi^{(h+1)}(x)|\,|\gamma_n(f_\varphi(x))| \\ &= \sup_x |f_\varphi^{-1}(x)|^p |\varphi^{(h+1)}(f_\varphi^{-1}(x))|\,|\gamma_n(x)| \\ &\geqslant |f_\varphi^{-1}(\beta_n)|^p |\varphi^{(h+1)}(f_\varphi^{-1}(\beta_n))|\,|\gamma_n(\beta_n)| \\ &= \frac{\alpha_n}{\sqrt{\alpha_n}\, e} = \frac{\sqrt{\alpha_n}}{e} \uparrow +\infty, \qquad n \uparrow +\infty.\end{aligned}$$

This contradicts to the assumption and therefore (S. 3) is true for $k=h+1$. Thus we have proved that (S. 3) is true for arbitrary non-negative integers k and p.

Next we show (S. 4). If $\varphi(x)$ is in $\mathcal{U}_{\mathcal{S}}$, then by Lemma 1, $\varphi^{-1}(x)=[\varphi(f_\varphi^{-1}(x))]^{-1}$ is also in $\mathcal{U}_{\mathcal{S}}$. Using (S. 3) for $\varphi^{-1}(x)$ with $k=0$ and with any non-negative integer p, we can choose a positive number $\rho=\rho(p)$ such that

$$\lim_{|x|\to+\infty}\frac{|\varphi^{-1}(x)|\,|x|^p}{|f_\varphi^{-1}(x)|^\rho}=0\,,$$

and therefore we have

$$\begin{aligned}&\lim_{|x|\to+\infty}\frac{|x|^p}{|\varphi(f_\varphi^{-1}(x))|\,|f_{\varphi^{-1}}(x)|^\rho}\\ &=\lim_{|x|\to+\infty}\frac{|x|^p}{|\varphi(f_\varphi^{-1}(x))|\,|f_\varphi^{-1}(x)|^\rho}\\ &=\lim_{|x|\to+\infty}\frac{|f_\varphi(x)|^p}{|\varphi(x)|\,|(x)|^\rho}=0\,.\end{aligned}$$

Thus we have proved the lemma.

Theorem 1. *A function* $\varphi(x)$ *is in* $\mathcal{U}_{\mathcal{S}}$ *if and only if it satisfies* (S. 1), (S. 2), (S. 3) *and* (S. 4).

Proof. The necessity of (S. 1)∼(S. 4) is derived from Lemma 3 and Lemma 5. We prove only the sufficiency. Suppose that $\varphi(x)$ satisfies (S. 1)∼(S. 4). By (S. 1) $\varphi(x)$ is locally square summable, and for every $\xi(x)$ in $\mathcal{S}$ we have

$$(\mathfrak{g}[\varphi]\xi)(x)=\varphi(x)\xi(f_\varphi(x))\in\mathcal{C}^\infty.$$

Next we show that $f_\varphi(x)$ satisfies (A. 1). We begin by proving the following condition:

(S. 2′) There exists a positive number γ such that

$$\inf_x\,(1+|x|^\gamma)|\varphi(x)|>0\,. \tag{2.12}$$

To prove (S. 2′), it is enough to show that there exists a positive number γ such that

$$\varliminf_{|x|\to+\infty}|x|^\gamma|\varphi(x)|>0\,. \tag{2.13}$$

Assume that (S. 2′) is not true. Then for every positive number r we have

$$\text{(2.14)} \qquad \lim_{|x|\to+\infty} |x|^r |\varphi(x)| = 0\,.$$

Observing the relation

$$|f_\varphi(x)| \leqslant |f_\varphi(y)|, \qquad \text{if} \quad 0<x\leqslant y \text{ or } y\leqslant x<0\,,$$

together with (2.14), we have for every positive number r,

$$\overline{\lim_{|x|\to+\infty}} \frac{|f_\varphi(x)|}{|x|^r|\varphi(x)|} \geqslant \overline{\lim_{|x|\to+\infty}} \frac{\min(f_\varphi(1),\ |f_\varphi(-1)|)}{|x|^r|\varphi(x)|} = +\infty\,.$$

This contradicts to (S. 4) and hence (S. 2′) is true.

Using (S. 3) with $k=0$ and $p=[\gamma]+1$, we can choose a positive number $r=r(0, p)$ such that

$$\overline{\lim_{|x|\to+\infty}} \frac{|\varphi(x)|\,|x|^p}{|f_\varphi(x)|^r} = 0\,.$$

Noting (2.13), we have

$$\lim_{|x|\to+\infty} |f_\varphi(x)| = +\infty\,.$$

Thus (A. 1) is proved.

Our next step is to show that $\mathfrak{g}[\varphi]$ is reduced by $\mathcal{S}$ and is continuous in the topology of $\mathcal{S}$. Lemma 2 proves that for every non-negative integer k

$$\mathfrak{g}([\varphi]\xi)^{(k)}(x) = \varphi^{(k)}(x)\xi(f_\varphi(x)) + \sum_{\nu=1}^{k} P_{\nu,k-1}[\varphi]\xi^{(\nu)}(f_\varphi(x))\,.$$

Let $S^{(k)}$ be the maximum degree of the polynomials $P_{\nu,k-1}[\varphi]$, $\nu=1, 2, \cdots, k$, and let $r(p)$, $p=0, 1, 2, \cdots$, be the maximum of the (S. 3) positive numbers $r(\nu, p)$, $\nu=0, 1, \cdots, k$, where $r(\nu, p)$ is selected in (S. 3) for $\varphi(x)$. Then we have for arbitrary non-negative integers k, p,

$$\begin{aligned}
\|\mathfrak{g}[\varphi]\xi\|_{k,p} &= \sup_x |x^p(\mathfrak{g}[\varphi]\xi)^{(k)}(x)| \\
&\leqslant \sup_x \{|x^p\varphi^{(k)}(x)\xi(f_\varphi(x))| + \sum_{\nu=1}^{k} |x^p P_{\nu,k-1}[\varphi]\xi^{(\nu)}(f_\varphi(x))|\} \\
&\leqslant \sup_x \frac{|x|^p|\varphi^{(k)}(x)|}{(1+|f_\varphi(x)|^{r(k,p)})}(1+|f_\varphi(x)|^{r(k,p)})|\xi(f_\varphi(x))| \\
&\quad + \sum_{\nu=1}^{k} \sup \frac{|x|^p|P_{\nu,k-1}[\varphi]|}{(1+|f_\varphi(x)|^{r(p)s(k)})} \sup_x (1+|f_\varphi(x)|^{r(p)s(k)})|\xi^{(\nu)}(f_\varphi(x))| \\
&\leqslant C_0\{\|\xi\|_{0,[r(k,p)]+1} + \|\xi\|_{0,0}\} + \sum_{\nu=1}^{k} C_\nu\{\|\xi\|_{\nu,[r(p)s(k)]+1} + \|\xi\|_{\nu,0}\}\,,
\end{aligned}$$

where

$$C_0 = \sup_x \frac{|x^p \varphi^{(k)}(x)|}{1+|f_\varphi(x)|^{r(k,p)}},$$

$$C_\nu = \sup_x \frac{|x|^p |P_{\nu,k-1}[\varphi]|}{1+|f_\varphi(x)|^{r(p)s(k)}}, \qquad \nu=1, 2, \cdots, k.$$

Thus we have proved for arbitrary non-negative integers k, p,

$$\|\mathfrak{g}[\varphi]\xi\|_{k,p} \leqslant C_0\{\|\xi\|_{0,[r(k,p)]+1}+\|\xi\|_{0,0}\}+\sum_{\nu=1}^{k} C_\nu\{\|\xi\|_{\nu,[r(p)s(k)]+1}+\|\xi\|_{\nu,0}\},$$

which means that $\mathfrak{g}[\varphi]$ is reduced by $\mathcal{S}$ and continuous in the topology of $\mathcal{S}$. By Lemma 1 and the above estimation, it is enough to show that (S. 3) is valid for $\varphi^{-1}(x)$. We show it by mathematical induction with respect to k.

In case $k=0$, put $r(0,p)=\rho(p)$ where $\rho(p)$ is given by (S. 4). Then we have

$$\lim_{|x|\to+\infty} \frac{|\varphi^{-1}(x)|\,|x|^p}{|f_{\varphi^{-1}}(x)|^\rho} = \lim_{|x|\to+\infty} \frac{|x|^p}{|\varphi(f_\varphi^{-1}(x)|\,|f_\varphi^{-1}(x)|^\rho}$$
$$= \lim_{|x|\to+\infty} \frac{|f_\varphi(x)|^p}{|\varphi(x)|\,|x|^\rho} = 0,$$

and therefore (S. 3) is valid.

Suppose (S. 3) is valid for $\varphi^{-1}(x)$ in the cases where $k=0, 1, \cdots, h$, that is, for every pair of non-negative integers k, p; $k=0, 1, \cdots, h$, $p=0, 1, 2, \cdots$, there exists a positive number $r(k, p)$ such that (2. 9) is valid for $\varphi^{-1}(x)$. Then we have only to show that (S. 3) is valid for $k=h+1$.

From (S. 1) and (S. 2) we have $\varphi^{-1}(x)=[\varphi(f_\varphi^{-1}(x))]^{-1}\in\mathcal{C}^\infty$ and a simple calculation proves

$$\varphi^{-1}(x)\varphi(f_{\varphi^{-1}}(x)) = 1. \tag{2.15}$$

Differentiating the both sides of (2. 17) $(h+1)$-times, Lemma 2 shows that

$$\varphi^{-1(h+1)}(x)\varphi(f_{\varphi^{-1}}(x))+\sum_{\nu=1}^{h+1} P_{\nu,h}[\varphi^{-1}]\varphi^{(\nu)}(f_{\varphi^{-1}}(x)) = 0,$$

and therefore

$$\varphi^{-1(h+1)}(x) = -\sum_{\nu=1}^{h=1} \varphi^{-1}(x)P_{\nu,h}[\varphi^{-1}]\varphi^{(\nu)}(f_{\varphi^{-1}}(x)).$$

Let s be the maximum degree of the polynomials $P_{\nu,h}[\varphi]$; $\nu=1, 2, \cdots, h+1$, and let $r'(p)$ be the maximum of the positive numbers $r'(k, p)$; $k=0, 1, \cdots, h$.

Then, considering that (S. 3) is valid for $\varphi(x)$, we have for every non-negative integer p and for $r=r(\nu, 0)$ which is found in (S. 3) for $\varphi(x)$,

$$\lim_{|x|\to+\infty} \frac{|\varphi^{-1(h+1)}(x)|\,|x|^p}{|f_{\varphi^{-1}}(x)|^q}$$
$$\leqslant \lim_{|x|\to+\infty} \sum_{\nu=1}^{h+1} \frac{|x|^p|\varphi^{-1}(x)|\,|P_{\nu,h}[\varphi^{-1}]|\,|\varphi^{(\nu)}(f_{\varphi^{-1}}(x))|}{|f_{\varphi^{-1}}(x)|^q}$$
$$\leqslant \lim_{|x|\to+\infty} \sum_{\nu=1}^{h+1} \frac{|x|^{p+r}|\varphi^{-1}(x)|\,|P_{\nu,h}[\varphi^{-1}]|}{|f_{\varphi^{-1}}(x)|^q}\cdot\frac{|\varphi^{(\nu)}(f_{\varphi}^{-1}(x))|}{|x|^r} = 0,$$

where $q=r'(p+[r])(s+1)$. Therefore (S. 3) is valid for $k=h+1$.

Thus we have proved the theorem.

In Theorem 1, we determined $\mathcal{U}_S$ by the conditions on $\varphi(x)$ and $f_\varphi(x)$. It is preferable to determine $\mathcal{U}_S$ by conditions only on $\varphi(x)$. Unfortunately we have not succeeded in this direction, but we have the following propositions.

Proposition 1. *Let $\varphi(x)$ be a function for which the following condition is valid:*

(S. 2″) *There exists a positive number $\gamma<\frac{1}{2}$ such that*

(2. 16) $$m_\varphi = \inf_x (1+|x|^\gamma)|\varphi(x)|>0.$$

Then $\varphi(x)$ is in $\mathcal{U}_S$ if it is a slowly increasing function, that is, it is infinitely many times continuously differentiable and for every non-negative integer k, there exists a positive number $\tau=\tau(k)$ such that

(2. 17) $$\lim_{|x|\to+\infty} \frac{|\varphi^{(k)}(x)|}{|x|^\tau} = 0.$$

Proof. Let $\varphi(x)$ be a slowly increasing function for which (S. 2″) is valid. We have to prove (S. 1)∼(S. 4) for $\varphi(x)$, however, (S. 1) and (S. 2) are evident from the assumptions.

For every real number x, ($|x|\geqslant 1$), we have

$$
\begin{aligned}
|f_\varphi(x)| &= \left|\int_0^x |\varphi(y)|^2 dy\right| \\
&\geqslant \left|\int_0^x (1+|y|^\gamma)^2 |\varphi(y)|^2 \frac{dy}{(1+|y|^\gamma)^2}\right| \\
&\geqslant m_\varphi^2 \left|\int_0^x \frac{dy}{(1+|y|^\gamma)^2}\right| \geqslant \frac{m_\varphi^2 |x|}{(1+|x|^\gamma)^2} \\
&\geqslant \frac{m_\varphi^2}{4} |x|^{1-2\gamma},
\end{aligned}
$$

where γ is a positive number given in (S. 2″), and therefore

(2. 18) $$|f_\varphi(x)| \geqslant \frac{m_\varphi^2}{4} |x|^{1-2\gamma}, \qquad \text{if} \quad |x| \geqslant 1.$$

On the other hand, since $\varphi(x)$ is slowly increasing, there exists a positive number $\tau = \tau(0)$ such that

(2. 19) $$M_\varphi = \sup_x \frac{|\varphi(x)|}{1+|x|^\tau} < +\infty,$$

and we have for every real number x,

(2. 20) $$
\begin{aligned}
|f_\varphi(x)| &= \left|\int_0^x \frac{|\varphi(y)|^2}{(1+|y|^\tau)^2}(1+|y|^\tau)^2 dy\right| \\
&\leqslant 4M_\varphi^2 |x|^{1+2\tau}.
\end{aligned}
$$

For arbitrary non-negative integers k, p, put

$$r = r(k, p) = \frac{p+\tau(k)}{1-2\gamma}.$$

Then we have from (2. 20) and (2. 19)

$$
\begin{aligned}
&\lim_{|x|\to+\infty} \frac{|\varphi|\ |x|^p}{|f_\varphi(x)|^r} \\
&= \lim_{|x|\to+\infty} \frac{|\varphi^{(k)}(x)|}{|x|^{\tau(k)}} \cdot \frac{|x|^{p+\tau(k)}}{|f_\varphi(x)|^r} \\
&\leqslant \lim_{|x|\to+\infty} \frac{|\varphi^{(k)}(x)|}{|x|^{\tau(k)}} \left(\frac{4}{m_\varphi^2}\right)^r = 0.
\end{aligned}
$$

Thus we have proved (S. 3), while, for every non-negative integer p, put

$$\rho(p) = \gamma + (2+\tau(0))p.$$

Then we have from (2. 18) and (2. 22)

$$\lim_{|x|\to+\infty}\frac{|f_\varphi(x)|^p}{|x|^p|\varphi(x)|}$$
$$\leqslant \lim_{|x|\to+\infty}\frac{(4M_\varphi^2)^p|x|^{(1+2+\tau(0))p}}{|x|^\gamma|\varphi(x)|}$$
$$\leqslant (4M_\varphi^2)^p[\lim_{|x|\to+\infty}|x|^\gamma|\varphi(x)|]^{-1}\lim_{|x|\to+\infty}\frac{1}{|x|^p}=0\,.$$

Thus we have proved (S. 4).

Conversely we have the following proposition.

Proposition 2. *If a function* $\varphi(x)$ *is in* $\mathcal{U}_S$, *for which there exists a positive number* $\tau=\tau(0)$ *such that* (2. 19) *is true, then it is a slowly increasing function.*

Proof. Let $\varphi(x)$ be a function which satisfies the hypothesis of the proposition. Then by Theorem 1, $\varphi(x)$ satisfies (S. 1)$\sim$(S. 4) and by the estimation (2. 19), we have (2. 20). Therefore from (S. 3) and (2. 20), we have for every non-negative integer k,

$$0=\lim_{|x|\to+\infty}\frac{|\varphi^{(k)}(x)|}{|f_\varphi(x)|^{r(k,0)}}$$
$$\geqslant \lim_{|x|\to+\infty}\frac{1}{(4M_\varphi^2)^{r(k,0)}}\cdot\frac{|\varphi^{(k)}(x)|}{|x|^{(1+2\tau)r(k,0)}}\,.$$

Thus we have proved the proposition.

Summing up Proposition 1 and Proposition 2, we have the following theorem.

Theorem 2. *Let* $\varphi(x)$ *be a function for which* (S. 2″) *is true and assume that there exists a positive number* $\tau=\tau(0)$ *such that* (2. 19) *is true. Then* $\varphi(x)$ *is in* $\mathcal{U}_S$ *if and only if it is a slowly increasing function.*

Corollary. *Let* $\varphi(x)$ *be a function such that*

$$|\varphi(x)|\equiv 1\,. \tag{2. 23}$$

Then it is in $\mathcal{U}_S$ *if and only if it is a slowly increasing function.*

3. One-parameter Subgroups of $\mathcal{O}(\mathcal{S}_r)$

In this section, we first show that $\mathcal{U}_S$ is a group with the product operation $\circledast$ defined by (1. 6). Then we proceed to discuss two interesting families of one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ which contain the shift and the tension.

Lemma 6. *The correspondence between* $\varphi(x)$ *in* $\mathcal{U}_{\mathcal{S}}$ *and* $\mathfrak{g}[\varphi]$ *in* $U(\mathcal{S}_r)$ *is one-to-one.*

Proof. It is evident that the constant function $1=1(x)$ is in $\mathcal{U}_{\mathcal{S}}$ and $\mathfrak{g}[1]=\mathrm{I}$. Therefore we have only to show that if for a function $\varphi(x)$ in $\mathcal{U}_{\mathcal{S}}$, $\mathfrak{g}[\varphi]=\mathrm{I}$, then $\varphi(x)\equiv 1$.

Let $\mathfrak{g}[\varphi]=\mathrm{I}$ and let $\xi_N(x)$ be a function in $\mathcal{S}$ such that $\xi_N(x)=1$ where $-N\leqslant x\leqslant N$, $N=1, 2, 3, \cdots$. Then we have

$$\mathfrak{g}[\varphi]\xi_N(x) = \varphi(x)\xi_N(f_\varphi(x)) = \xi_N(x), \qquad N=1, 2, 3, \cdots,$$

and therefore

$$\varphi(x) = 1, \quad \max[-N, f_\varphi^{-1}(-N)]\leqslant x\leqslant \min[N, f_\varphi^{-1}(N)].$$

Thus, by letting $N\uparrow +\infty$, we have

$$\varphi(x)\equiv 1, \qquad -\infty<x<+\infty.$$

Theorem 3. $\mathcal{U}_{\mathcal{S}}$ *is a group with respect to the product operation* $\circledast$.

Proof. For every $\varphi(x)$, $\psi(x)$ in $\mathcal{U}_{\mathcal{S}}$ and every function $\xi(x)$ in $\mathcal{S}$, we have

$$\begin{aligned}\mathfrak{g}[\varphi]\mathfrak{g}[\psi]\xi(x) &= (\mathfrak{g}[\varphi]\psi(\cdot)\xi(f_\psi(\cdot)))(x)\\ &= \varphi(x)\psi(f_\varphi(x))\xi(f_\psi\circ f_\varphi(x))\\ &=(\varphi\circledast\psi)(x)\xi(f_\psi\circ f_\varphi(x)).\end{aligned}$$

On the other hand we have

$$\begin{aligned}f_\psi\circ f_\varphi(x) &= \int_0^{f_\varphi(x)}|\psi(y)|^2dy\\ &= \int_0^x|\varphi(y)\psi(f_\varphi(y))|^2dy,\end{aligned}$$

and therefore

(3.1) $$f_\psi\circ f_\varphi(x) = f_{\varphi\circledast\psi}(x).$$

Thus we have

(3.2) $$\mathfrak{g}[\varphi]\mathfrak{g}[\psi] = \mathfrak{g}[\varphi\circledast\psi].$$

It is evident that the function $(\varphi\circledast\psi)(x)$ satisfies (A. 1) and (A. 2), and that, from (3. 2), $\mathfrak{g}[\varphi\circledast\psi]=\mathfrak{g}[\varphi]\mathfrak{g}[\psi]\in\mathcal{U}(\mathcal{S})$ holds, therefore $(\varphi\circledast\psi)(x)$ is in $\mathcal{U}_{\mathcal{S}}$ for every φ, ψ in $\mathcal{U}_{\mathcal{S}}$.

Since $\mathcal{U}(\mathcal{S})$ is a subgroup of the unitary transformation group on L^2, and since the correspondence between $\mathfrak{g}[\varphi]$ and φ is one-to-one by Lemma 6, it is not difficult to prove the associative law.

Finally by simple calculations we can easily show that

$$(\varphi\circledast 1)(x) = \varphi(x)\,,$$

and

$$(\varphi\circledast\varphi^{-1})(x) = 1\,,$$

where $\varphi^{-1}(x)$ is defined by (2. 1). Thus we have proved the theorem.

Now we define three subgroup of $\mathcal{U}_{\mathcal{S}}$ as follows.

$$\mathcal{U}_{\mathcal{S}}^{+} = \{\varphi\in\mathcal{U}_{\mathcal{S}}\,;\ \varphi(x) \text{ real positive}\}\,.$$
$$\mathcal{U}_{\mathcal{S}}^{e} = \{\varphi\in\mathcal{U}_{\mathcal{S}}\,;\ |\varphi(x)|\equiv 1\}\,.$$
$$\mathcal{U}_{\mathcal{S}}^{h} = \{\varphi\in\mathcal{U}_{\mathcal{S}}\,;\ \varphi(x) = \overline{\varphi(-x)}\}\,.$$

Lemma 7. *For every φ in $\mathcal{U}_{\mathcal{S}}$ there exists a unique element φ^{+} in $\mathcal{U}_{\mathcal{S}}^{+}$ and a unique element φ^{e} in $\mathcal{U}_{\mathcal{S}}^{e}$ such that*

$$\varphi = \varphi^{e}\circledast\varphi^{+}. \tag{3. 3}$$

In other words, $U_{\mathcal{S}}$ is expressed as the product

$$\mathcal{U}_{\mathcal{S}} = \mathcal{U}_{\mathcal{S}}^{e}\circledast\mathcal{U}_{\mathcal{S}}^{+}\,. \tag{3. 4}$$

We call φ^{e} and φ^{+} the argument part and the polar part of φ, respectively.

Proof. Put $\varphi^{e}(x)=\varphi(x)/|\varphi(x)|$ and $\varphi^{+}(x)=|\varphi(x)|$. Then they are the required.

Remark. The decomposition (3. 4) is not a direct product. In fact, if $\varphi=\varphi^{e}\circledast\varphi^{+}$ and $\psi=\psi^{e}\circledast\psi^{+}$ where φ^{e}, $\psi^{e}\in\mathcal{U}_{\mathcal{S}}^{e}$ and φ^{+}, $\psi^{+}\in\mathcal{U}_{\mathcal{S}}^{+}$, then we have

$$\begin{aligned}(\varphi\circledast\psi)(x) &= \varphi^{e}(x)\varphi^{+}(x)\psi^{e}(f_{\varphi^{+}}(x))\psi^{+}(f_{\varphi^{+}}(x))\\ &= (\varphi^{e}\circledast(\psi^{e}\circ f_{\varphi^{+}}))\circledast(\varphi^{+}\circledast\psi^{+})\,.\end{aligned}$$

Hence, we have

$$(\varphi\circledast\psi)^{+} = \varphi^{+}\circledast\psi^{+}. \tag{3. 5}$$

$$(\varphi \circledast \psi)^e(x) = \varphi^e(x)\psi^e(f_{\varphi^+}(x)) . \tag{3.6}$$

Lemma 8. *Let* $\varphi(x)$ *be a function in* $\mathcal{U}_{\mathcal{S}}$. *Then* $\tilde{\mathfrak{g}}[\varphi]$ *is in* $\mathcal{O}(\mathcal{S}_r)$ *if and only if* φ *is in* $\mathcal{U}^h_{\mathcal{S}}$ *where* $\tilde{\mathfrak{g}}[\varphi]$ *is defined by* (1.5).

The proof is easy.

Here we consider one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ given by type (1.5). If for a subset $\{\varphi_t ; -\infty<t<+\infty\}$ of $\mathcal{U}_{\mathcal{S}}$, $\{\tilde{\mathfrak{g}}[\varphi_t]\}$ is a one-parameter subgroup of $\mathcal{S}(\mathcal{S}_r)$, then by Lemma 8 $\varphi_t \in \mathcal{U}^h_{\mathcal{S}}$, $-\infty<t<+\infty$, and we have

$$\begin{cases} \tilde{\mathfrak{g}}[\varphi_t]\tilde{\mathfrak{g}}[\varphi_s] = \mathcal{F}^{-1}\mathfrak{g}[\varphi_t]\mathcal{F}\mathcal{F}^{-1}\mathfrak{g}[\varphi_s]\mathcal{F} \\ \quad = \mathcal{F}^{-1}\mathfrak{g}[\varphi_t]\mathfrak{g}[\varphi_s]\mathcal{F} = \tilde{\mathfrak{g}}[\varphi_{t+s}] , \qquad -\infty<t,\ s<+\infty , \\ \tilde{\mathfrak{g}}[\varphi_0] = \mathrm{I} . \end{cases} \tag{3.7}$$

From (3.7) and Lemma 6, we have

$$\begin{cases} \varphi_s \circledast \varphi_s = \varphi_{t+s} , \qquad -\infty<t,\ s<+\infty , \\ \varphi_0 = \mathrm{I} , \end{cases} \tag{3.8}$$

that is, $\{\varphi_t\}$ is a one-parameter subgroup of $\mathcal{U}^h_{\mathcal{S}}$. Conversely, it is evident that every one-parameter subgroup $\{\varphi_t\}$ of $\mathcal{U}^h_{\mathcal{S}}$ determines a one-parameter subgroup $\{\mathfrak{g}[\varphi_t]\}$ of $\mathcal{O}(\mathcal{S}_r)$. Therefore, to obtain a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$, it is enough to obtain a one-parameter subgroup of $\mathcal{U}^h_{\mathcal{S}}$ related to the given subgroup by the formula (1.5).

Let $\{\varphi_t\}$ be a one-parameter subgroup of $\mathcal{U}^h_{\mathcal{S}}$ for which (3.8) is valid. Then by Lemma 7, the relation (3.8) is decomposed into the following two relations.

$$\begin{cases} \varphi^+_t \circledast \varphi^+_s = \varphi^+_{t+s} , \qquad -\infty<t,\ s<+\infty \\ \varphi^+_0 = 1 \end{cases} \tag{3.9}$$

$$\begin{cases} \varphi^e_t(x)\varphi^e_s(f_{\varphi^+_t}(x)) = \varphi^e_{t+s}(x) , \qquad -\infty<t,\ s<+\infty , \\ \varphi^e_0(x) = 1 \end{cases} \tag{3.10}$$

where φ^e_t and φ^+_t are the argument part and the polar part of φ_t, respectively. Therefore, to obtain a one-parameter subgroup of $U^h_{\mathcal{S}}$, we first obtain a one-parameter subgroup of $\mathcal{U}^{h+}_{\mathcal{S}} = \mathcal{U}^h_{\mathcal{S}} \cap \mathcal{U}^+_{\mathcal{S}}$ and then, corresponding to it, we must solve the equation (3.10) in $\mathcal{U}^{he}_{\mathcal{S}} = \mathcal{U}^h_{\mathcal{S}} \cap \mathcal{U}^e_{\mathcal{S}}$. However, we have not yet succeeded in obtaining a general method to find a one-parameter subgroup of $\mathcal{U}^{h+}_{\mathcal{S}}$, but at present we know two such subgroups from T. Hida, I. Kubo H. Nomoto and H. Yoshizawa [1]. One of them is the *trivial*

(3.11) $$\varphi_t^+(x)\equiv 1\,,$$

and the second is the *tension*

(3.12) $$\varphi(x) = e^{(\alpha/2)t}$$

where α is an arbitrary real number not vanishing.

To solve the equation (3.10) corresponding to the trivial and the tension, we assume the continuity of $\varphi_t(x)$ in t for every fixed real number x.

Corresponding to the trivial, the equation (3.10) becomes

(3.13) $$\begin{cases} \varphi_t^e(x)\varphi_s^e(x) = \varphi_{t+s}^e(x)\,, \\ \varphi_0^e(x)\equiv 1\,. \end{cases}$$

Noting that $\varphi_t^e(x)$ is in $\mathcal{U}_S^{he}$ and $|\varphi_t^e(x)|\equiv 1$ for every (t, x), we can define a real function

(3.14) $$\mathrm{H}(t, x) = \frac{1}{i}\log\varphi_t^e(x)\,, \qquad -\infty<t<\infty\,,$$

such that $\mathrm{H}(0, x)=0$ and that $\mathrm{H}(t, x)$ is continuous in t for every fixed x. From (3.13) we have

$$\begin{cases} \mathrm{H}(t, x)+\mathrm{H}(s, t) = \mathrm{H}(t+s, x)\,, & -\infty<t, s<+\infty\,, \\ \mathrm{H}(0, x)\equiv 0 \end{cases}$$

and because of the continuity of $\mathrm{H}(t, x)$ in t,

$$\mathrm{H}(t, x) = th(x)\,, \qquad -\infty<t<+\infty\,,$$

where $h(x)$ is a real function equal to $\mathrm{H}(1, x)$. Thus we have

(3.15) $$\varphi_t^e(x) = \exp[ith(x)]\,, \qquad -\infty<t<+\infty\,.$$

Since $\varphi_t^e(x)$ is in $\mathcal{U}_S^{he}$, by Corollary of Theorem 2, it must be an Hermitian slowly increasing function. And, it is easy to show that $\varphi_t^e(x)=\exp[ith(x)]$ is an Hermitian slowly increasing function if and only if $h(x)$ is a real odd slowly increasing function. Conversely, if $h(x)$ is a real odd slowly increasing function, then it is evident that the one-parameter set defined by (3.15) is a one-parameter subgroup of $\mathcal{U}_S^{he}$.

Summing up the above, we have the following theorem.

Theorem 4. *Let* $\{\varphi_t(x)\,;\ -\infty<t<+\infty\}$ *be a one-parameter set of functions with the trivial polar part, that is,* $|\varphi_t(x)|\equiv 1$, *and assume that for every fixed* x, $\varphi_t(x)$ *is continuous in* t. *Then it is a one-parameter subgroup of* $\mathcal{U}^h_{\mathcal{S}}$ *if and only if* $\varphi_t(x)$ *is of the form*

$$\varphi_t(x) = \exp[ith(x)]\,, \qquad -\infty<t<+\infty\,, \tag{3.16}$$

where $h(x)$ *is a real odd slowly increasing function.*

We note the following: Let $\mathcal{I}$ be the set of all real odd slowly increasing functions. Then for every $h(x)$ in $\mathcal{I}$ we have a one-parameter subgroup $\{\exp[ith(x)]\,;\ -\infty<t<+\infty\}$ of $\mathcal{U}^h_{\mathcal{S}}$ and therefore a one-parameter subgroup $\mathfrak{G}_0(h)=\{\tilde{\mathfrak{g}}[\exp[ith]]\,;\ -\infty<t<+\infty\}$ of $\mathcal{O}(\mathcal{S}_r)$.

Theorem 5. Let $\tilde{\mathfrak{G}}_0$ *be the family of one-parameter subgroups of* $\mathcal{O}(\mathcal{S}_r)$ *defined by*

$$\tilde{\mathfrak{G}}_0 = \{\tilde{\mathfrak{G}}_0(h)\,;\ h\in\mathcal{I}\}\,.$$

Then we have:

1°) $\tilde{\mathfrak{G}}_0$ *contains the shift.*

2°) *For every* $h(x)$, $h'(x)$ *in* $\mathcal{I}$, $\tilde{\mathfrak{G}}_0(h)$ *and* $\tilde{\mathfrak{G}}_0(h')$ *are commutative.*

Proof. Put $h(x)=-x$, then $\tilde{\mathfrak{G}}_0(-x)=\{\tilde{\mathfrak{g}}[e^{-itx}]\}$ is the shift. In fact, for every $\xi(x)$ in L^2_r we have

$$\begin{aligned}
\mathfrak{g}[e^{-itx}]\xi(x) &= \mathcal{F}^{-1}\mathfrak{g}[e^{-it\cdot}]\mathcal{F}\xi(x)\\
&= \mathcal{F}^{-1}[e^{-it\cdot}\tilde{\xi}(\cdot)](x)\\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ix\lambda}e^{-it\lambda}\tilde{\xi}(\lambda)d\lambda\\
&= \xi(x-t)\,, \qquad -\infty<t<+\infty\,,
\end{aligned}$$

where $\tilde{\xi}(\lambda)=(\mathcal{F}\xi)(\lambda)$.

On the other hand, if h and h' are in $\mathcal{I}$, then for every real number t, s, we have

$$\begin{aligned}
&\tilde{\mathfrak{g}}[e^{ith}]\tilde{\mathfrak{g}}[e^{ish'}]\\
&\quad= \mathcal{F}^{-1}\mathfrak{g}[e^{ith}]\mathcal{F}\cdot\mathcal{F}^{-1}\mathfrak{g}[e^{ish'}]\mathcal{F}\\
&\quad= \mathcal{F}^{-1}\mathfrak{g}[e^{ith}\circledast e^{ish'}]\mathcal{F}\\
&\quad= \mathcal{F}^{-1}\mathfrak{g}[e^{ith+ish'}]\mathcal{F}
\end{aligned}$$

$$= \mathcal{F}^{-1}\mathfrak{g}[e^{ish'}]\mathfrak{g}[e^{ith}]\mathcal{F}$$
$$= \tilde{\mathfrak{g}}[e^{ish'}]\tilde{\mathfrak{g}}[e^{ith}].$$

Thus we have proved the theorem.

Remark. The family $\tilde{\mathfrak{G}}_0$ arises from the variable change by distributions. In fact, for any function $h(x)$ in $\mathcal{T}$, we have

$$\tilde{\mathfrak{g}}[e^{ith}]\xi = \mathrm{T}_h * \xi, \qquad \xi \in \mathcal{S}_r,$$

where T_h is given by the Fourier inverse transform of e^{ith} as a distribution and $*$ stands for the convolution.

Next, we solve the equation (3.10) corresponding to the tension (3.12). In this case, the equation (3.10) becomes

$$(3.17) \qquad \begin{cases} \varphi_t^e(x)\varphi_s^e(e^{\alpha t}x) = \varphi_{t+s}^e(x), & -\infty < t,\ s < +\infty, \\ \varphi_0^e(x) \equiv 1, \end{cases}$$

where $\varphi_t^e(x)$, $-\infty < t < +\infty$, is in $\mathcal{U}_{\mathcal{S}}^{he}$. Hence we have

$$(3.18) \qquad \varphi_s^e(e^{\alpha t}x) = \frac{\varphi_{t+s}^e(x)}{\varphi_t^e(x)}, \qquad -\infty < t,\ s < +\infty$$

and therefore $\varphi_t^e(x)$ is continuous in (t, x). Then we can define a real function $\mathrm{H}(t, x)$ by

$$(3.19) \qquad \mathrm{H}(t, x) = \frac{1}{i} \log \varphi_t^e(x), \qquad -\infty < t,\ x < +\infty,$$

such that $\mathrm{H}(0, x) = 0$ and that $\mathrm{H}(t, x)$ is continuous in (t, x). According to (3.17), we have

$$(3.20) \qquad \begin{cases} \mathrm{H}(t, x) + \mathrm{H}(s, e^{\alpha t}x) = \mathrm{H}(t+s, x), \\ \mathrm{H}(0, x) \equiv 0. \end{cases}$$

Since $\varphi_t^e(x)$ is Hermitian, we have

$$(3.21) \qquad \mathrm{H}(t, x) = -\mathrm{H}(t, -x), \qquad -\infty < t, x < +\infty;$$

and since $\varphi_t^e(x)$ is slowly increasing in x for every fixed t, $\mathrm{H}(t, x)$ is also slowly increasing and we have from (3.21)

$$(3.22) \quad \frac{\partial^{2k}}{\partial x^{2k}} \mathrm{H}(t, x)\Big|_{x=0} = 0, \qquad k = 0, 1, 2, \cdots, \ -\infty < t < +\infty.$$

Conversely, if we find a solution $\mathrm{H}(t, x)$ of (3.20) for positive x

which is slowly increasing in $x>0$ and satisfies conditions as follows

$$(3.23)\qquad \lim_{x\to+\infty}\frac{\partial^{2k}}{\partial x^{2k}}\mathrm{H}(t,x)=0\,,\qquad k=0,1,2,\cdots,\ -\infty<t<+\infty\,,$$

then we can extend it to the whole line by

$$(3.24)\qquad \begin{cases}\mathrm{H}(t,0)=0\,,\\ \mathrm{H}(t,x)=-\mathrm{H}(t,-x), \qquad x<0\,.\end{cases}$$

It is easy to see that

$$(3.25)\qquad \varphi_t^e(x)=\exp\left[i\mathrm{H}(t,x)\right]$$

is a solution of (3.17). Therefore, we solve the equation (3.20) only for positive x.

Set

$$(3.26)\qquad \mathrm{G}(t,y)=\mathrm{H}(t,e^{\alpha y}),\qquad -\infty<t,\ y<+\infty\,.$$

Then we have from (3.20)

$$(3.27)\qquad \begin{cases}\mathrm{G}(t,y)+\mathrm{G}(s,t+y)=G(t+s,y)\,,\\ \mathrm{G}(0,y)=0\,.\end{cases}$$

Further, setting $y=0$, we have

$$(3.28)\qquad \begin{cases}\mathrm{G}(t,0)+\mathrm{G}(s,t)=\mathrm{G}(t+s,0)\,,\\ \mathrm{G}(0,0)=0\,.\end{cases}$$

Set

$$g(t)=\mathrm{G}(t,0)\,,\qquad -\infty<t<+\infty\,,$$

then by assumption $g(t)$ is continuous and we have from (3.28)

$$(3.29)\qquad \mathrm{G}(t,y)=g(t+y)-g(y)\,,\qquad -\infty<t,\ y<+\infty\,.$$

Finally, setting $y=\frac{1}{\alpha}\log x$, we have

$$\begin{aligned}\mathrm{H}(t,x)&=\mathrm{G}\Big(t,\frac{1}{\alpha}\log x\Big)\\ &=g\Big(t+\frac{1}{\alpha}\log x\Big)-g\Big(\frac{1}{\alpha}\log x\Big)\\ &=g\Big(\frac{1}{\alpha}\log e^{\alpha t}x\Big)-g\Big(\frac{1}{\alpha}\log x\Big)\\ &=h(e^{\alpha t}x)-h(x)\,,\qquad -\infty<t<+\infty,\ x>0\,,\end{aligned}$$

where $h(x)=g\left(\frac{1}{\alpha}\log x\right)$, $x>0$.

The problem to characterize the class of the $h(x)$ is still open but we have the following proposition.

Proposition 3. *For every function $h(x)$ in $\mathcal{I}$,*

$$\varphi_t^e(x) = \exp\left[i(h(e^{\alpha t}x)-h(x))\right], \qquad -\infty<t<+\infty$$

is a solution of the equation (3.17) *in $\mathcal{U}_{\mathcal{S}}^{he}$.*

The proof is easy.

Combining the above solution and the polar part of the tension (3.12), we have the following theorem.

Theorem 6. *For every function $h(x)$ in $\mathcal{I}$,*

$$\tilde{\mathfrak{G}}_\alpha(h) = \left\{\tilde{\mathfrak{g}}\left[\exp\left(\frac{\alpha}{2}t+i\{h(e^{\alpha t}x)-h(x)\}\right)\right];\ -\infty<t<+\infty\right\}$$

is a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$. The induced flow is isomorphic to the flow induced by the tension

$$\tilde{\mathfrak{G}}_\alpha(0) = \left\{\tilde{\mathfrak{g}}\left[\exp\frac{\alpha}{2}t\right];\ -\infty<t<+\infty\right\}.$$

Proof. It suffices to prove the latter part of the theorem. In fact we have for every t

$$\tilde{\mathfrak{g}}\left[\exp\left(\frac{\alpha}{2}t+ih\{(e^{\alpha t}x)-h(x)\}\right)\right]$$
$$= \tilde{\mathfrak{g}}[e^{-ih(x)}]\tilde{\mathfrak{g}}[e^{(\alpha/2)t}]\tilde{\mathfrak{g}}^{-1}[e^{-ih(x)}],$$

and hence

$$\tilde{\mathfrak{g}}^*\left[\exp\left(\frac{\alpha}{2}t+i\{h(e^{\alpha t}x)-h(x)\}\right)\right]$$
$$= \tilde{\mathfrak{g}}^{*-1}[e^{-ih(x)}]\tilde{\mathfrak{g}}^*[e^{(\alpha/2)t}]\tilde{\mathfrak{g}}^*[e^{-ih(x)}].$$

This proves the theorem.

Example. Put $\alpha=1$ and $h(x)=x$. Then for every $\xi(x)$ in $\mathcal{S}_r$ we have

$$\left(\tilde{\mathfrak{g}}\left[\exp\left(\frac{1}{2}+i\{e^t x-x\}\right)\right]\xi\right)(x)$$
$$= e^{-(t/2)}\xi(e^{-s}x-e^{-t}+1), \qquad -\infty<t<+\infty.$$

4. One-parameter Subgroups of $\mathcal{O}(\mathcal{S}_r)$ Which Commute with the Shift

In this section, we appeal to the following well-known theorem to prove that all one-parameter subgroups of $\mathcal{O}(\mathcal{S}_r)$ that commute with the shift are found in $\tilde{\mathfrak{G}}_0$.

Theorem 7. *Let* $\mathfrak{H}$ *be a Hilbert space and let* $\{U_t\}$ *be a continuous one-parameter unitary group in* $\mathfrak{H}$ *with the resolution of the identity* $\{E(\lambda)\}$ *of simple spectrum. Then for every continuous one-parameter unitary group* $\{V_t\}$ *which commutes with* $\{U_t\}$, *there exists a real measurable function* $h(\lambda)$ *such that*

$$V_t = \int_{-\infty}^{+\infty} e^{ith(\lambda)} dE(\lambda)\,, \qquad -\infty < t < +\infty\,. \tag{4.1}$$

Let $\{S_t\}$ be the shift on L_r^2, $\{E(\lambda)\}$ be its resolution of the identity and let $\{V_t\}$ be a continuous one-parameter unitary group on L_r^2 which commutes with the shift. Then we can extend them naturally to the operators on the complex Hilbert space L^2. We again denote them by the same notations.

It is well-known that the shift is of simple spectrum and hence Theorem 7 is applicable. Therefore there exists a real measurable function $h(\lambda)$ for which (4.1) is valid.

Furthermore put

$$\tilde{E}(\lambda) = \mathcal{F} E(\lambda) \mathcal{F}^{-1}\,, \qquad -\infty < \lambda < +\infty\,, \tag{4.2}$$

$$\tilde{V}_t = \mathcal{F} V_t \mathcal{F}^{-1}\,, \qquad -\infty < t < +\infty\,. \tag{4.3}$$

Then for every measurable set Δ, we have

$$(\tilde{E}(\Delta)\xi)(x) = \chi_\Delta(x)\xi(x)\,, \qquad \xi \in L^2\,, \tag{4.4}$$

where $\chi_\Delta(x)$ is the indicator function of the set Δ.

Proposition 4. *A continuous one-parameter unitary group* $\{V_t\}$ *on* L^2 *commutes with the shift if and only if there exists a real measurable function* $h(\lambda)$ *such that for every* $\xi(x)$ *in* L^2

$$(\tilde{V}_t \xi)(x) = e^{ith(x)}\xi(x)\,, \qquad -\infty < t < +\infty\,, \tag{4.5}$$

where $\{\tilde{V}_t\}$ *is defined by* (4.3).

Proof. Let $\xi(x)$ be an arbitrary function in L^2 and put

$$\Delta_{n,\nu} = \left\{\lambda;\ \frac{\nu-1}{n} < h(\lambda) \leqslant \frac{\nu}{n}\right\},$$
$$\nu = 0,\ \pm 1,\ \pm 2,\ \cdots,$$
$$n = 1,\ 2,\ 3,\ \cdots.$$

Then we have from (4.1), (4.2) and (4.3)

$$\tilde{V}_t = \int_{-\infty}^{+\infty} e^{ith(\lambda)} d\tilde{E}(\lambda)$$
$$= \lim_{n\to+\infty} \sum_{\nu=-\infty}^{+\infty} e^{it(\nu/n)} \tilde{E}(\Delta_{n,\nu}),$$

and according to (4.4)

$$\|\tilde{V}_t\xi - e^{ith}\xi\|^2$$
$$= \lim_{n\to+\infty} \sum_{\nu} \|e^{it(\nu/n)}\tilde{E}(\Delta_{n,\nu})\xi - \tilde{E}(\Delta_{n,\nu})\xi\|^2$$
$$= \lim_{n\to+\infty} \sum_{\nu} \int_{\Delta_{n,\nu}} |e^{it(\nu/n)} - e^{ith(\lambda)}|^2 |\xi(\lambda)|^2 d\lambda$$
$$\leqslant \lim_{n\to+\infty} \sum_{\nu} \frac{1}{n^2} \int_{\Delta_{n,\nu}} |\xi(\lambda)|^2 d\lambda$$
$$= \lim_{n\to+\infty} \frac{1}{n^2} \|\xi\|^2 = 0,$$

for every $\xi(x)$ in L^2. Thus we have proved the proposition.

Theorem 8. *A one-parameter subgroup* $\{V_t\}$ *of* $\mathcal{O}(\mathcal{S}_r)$, *which is strongly continuous on* L_r^2, *commutes with the shift if and only if there exists a function* $h(x)$ *in* $\mathcal{I}$ *such that*

$$V_t = \tilde{\mathfrak{g}}[\exp(ith(x))], \qquad -\infty < t < +\infty. \tag{4.6}$$

Proof. The sufficiency is obvious and we show only the necessity.

Let $\{V_t\}$ be a one-parameter subgroup of $\mathcal{O}(\mathcal{S}_r)$ which is strongly continuous on L_r^2 and commutes with the shift. Then we can extend it naturally to a one-parameter subgroup of $U(\mathcal{S})$ which is at the same time continuous one-parameter unitary group on L^2 and we denote it again by the same notation. Furthermore define $\{\tilde{V}_t\}$ by (4.3). Then, according to Proposition 4, we have $\tilde{V}_t = \mathfrak{g}[e^{ith(x)}]$ where $h(x)$ is a real measurable function and therefore

$$V_t = \tilde{\mathfrak{g}}[e^{ith(x)}], \qquad -\infty < t < +\infty. \tag{4.7}$$

Finally, applying Theorem 4, we see that the function $h(x)$ must be in $\mathcal{I}$.

References

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