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Triple Cohomology of Algebras and Two Term Extensions*

Dedicated to Professor Atuo Komatu for his 60th birthday

By

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Introduction

It has been recognized that the notion of triple [12] (or monad [8]) provides a unified simplicial method for defining homology and cohomology in categorical setting (Godement [15], Huber [19], Eilenberg-Moore [12], Dold-MacLane-Oberst [8], Beck [5]).

It has been shown that many known, classical or special, (co)-homology theories of groups, modules and algebras (Eilenberg-MacLane [10], Cartan-Eilenberg [6], Hochschild [17], Harrison [16], Shukla [27], etc.) are triple cohomologies (Barr-Beck [1], Barr [2], [3], Iwai [20]).

In the former announcement [26], we treated triple cohomologies viewing them as derived functors (in a functor category) in the sense of relative homological algebra [11], [25]. Since then such interpretations have appeared (Dold-MacLane-Oberst [8], Dubuc [9]). Therefore we will not discuss this subject here.

We will treat the calculation or interpretation of triple cohomology of an algebra with coefficients in a module [5]. The 0th

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and 1st cohomology groups H^0 and H^1 (the dimension indices in triple cohomology being one less than usual) were discussed by J. M. Beck in his dissertation [5]. The purpose of the present paper is to interpret the second cohomology, $H^2(A, M)$, of an algebra A with coefficients in an A-module M as the set of equivalence classes after Yoneda [29] of two term extensions of A by M (see §3, Lichtenbaum-Schlessinger [23] or Gerstenhaber [13] for two term extensions).

Our interpretation appears to be more direct than those through classical obstruction theory for algebra extensions (MacLane [24], [10], Hochschild [18], Shukla [27], Barr [4]) and suggests a close relationship between H^n and n term extensions for $n>2$ (see §4). In fact, such an interpretation of H^n has been obtained by A. Iwai, one of the present authors, and is to appear in his subsequent paper [21].

In the sequel we choose and deal with a specific category, the category of Lie algebras over a commutative ground ring. The argument is functorial, at least in substance, so is applicable to other categories of algebraic systems with tripleable underlying object functors [5].

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§1. Preliminaries

Recall the notion of cotriple [12]. A *cotriple* $\boldsymbol{G}=(G, \varepsilon, \delta)$ in a category $\mathfrak{A}$ consists of a (covariant) functor $G: \mathfrak{A} \to \mathfrak{A}$ and natural transformations $\varepsilon: G \to I_{\mathfrak{A}}$ ($I_{\mathfrak{A}}$ denotes the identity functor) and $\delta: G \to G^2 = G \circ G$ satisfying

$$\varepsilon G \circ \delta = G\varepsilon \circ \delta = 1_G \quad (1_G \text{ denotes the identity natural transformation}),$$
$$G\delta \circ \delta = \delta G \circ \delta,$$

where εG means the natural transformation defined by

$$\varepsilon G(A) = \varepsilon(G(A)) : G^2(A) \to G(A), \qquad \text{for} \quad A \in \mathrm{ob}\mathfrak{A},$$

and similarly for $G\varepsilon$, $G\delta$, and δG. *Triples* are defined dually.

A cotriple comes usually from an *adjoint pair of functors* (S, U),

$U:\mathcal{A}\to\mathcal{C}$ and $S:\mathcal{C}\to\mathcal{A}$ in the following way. Let S be the *left adjoint* of U (notation $S\dashv U$). This means that there exists a natural equivalence λ between the Hom set functors:

$$(1.1)\qquad \lambda(C, A):\ \mathrm{Hom}_{\mathcal{A}}(S(C), A)\approxeq \mathrm{Hom}_{\mathcal{C}}(C, U(A))$$

for $C\in\mathrm{ob}\,\mathcal{C}$, $A\in\mathrm{ob}\,\mathcal{A}$. Define natural transformations ε and η by

$$(1.2)\qquad \begin{aligned} \varepsilon(A) &= \lambda^{-1}(1_{U(A)}):\ SU(A)\to A, &&\text{for}\quad A\in\mathrm{ob}\,\mathcal{A},\\ \eta(C) &= \lambda(1_{S(C)}):\ C\to US(C), &&\text{for}\quad C\in\mathrm{ob}\,\mathcal{C}, \end{aligned}$$

with abbreviation $\lambda=\lambda(U(A), A)$ resp. $\lambda=\lambda(C, S(C))$. Then we have

$$(1.3)\qquad \lambda(\rho) = U(\rho)\circ\eta(C), \qquad \text{for}\quad \rho\in\mathrm{Hom}_{\mathcal{A}}(S(C), A),$$

with $\lambda=\lambda(C, A)$ and

$$\begin{aligned} \varepsilon S\circ S\eta &= 1_S:\ S\to SUS\to S,\\ U\varepsilon\circ\eta U &= 1_U:\ U\to USU\to U. \end{aligned}$$

It follows that the adjoint pair (S, U) yields a cotriple $(SU, \varepsilon, S\eta U)$ in $\mathcal{A}$ (dually a triple $(US, \eta, U\varepsilon S)$ in $\mathcal{C}$).

Conversely, given a cotriple (G, ε, δ) in $\mathcal{A}$. It is known ([12], [22]) that there exist a category $\mathcal{C}$ and an adjoint pair of functors (S, U), $U:\mathcal{A}\to\mathcal{C}$, $S:\mathcal{C}\to\mathcal{A}$ with $S\dashv U$, inducing the cotriple (G, ε, δ) as above.

From now on we assume that all categories considered are pointed (i.e. have zero objects) with kernels and all functors are also pointed (i.e. $T(0)=0$), unless otherwise stated.

Given an adjoint pair of functors $U:\mathcal{A}\to\mathcal{C}$ and $S:\mathcal{C}\to\mathcal{A}$ with $S\dashv U$ which induces a cotriple $\boldsymbol{G}=(G, \varepsilon, \delta)$ in $\mathcal{A}$. There is naturally defined a simplicial object $G_*=\{G_n;\ n\geq 0\}$ in the category of endofunctors: $\mathcal{A}\to\mathcal{A}$, with $G_n=G^{n+1}=G\circ\cdots\circ G$(($n+1$)-fold iterated composition of G) and

$$\begin{aligned} \varepsilon_n^i &= G^i\varepsilon G^{n-i}:\ G_n\to G_{n-1}, && 0\leq i\leq n \text{ (face operator)},\\ \delta_n^i &= G^i\delta G^{n-i}:\ G_n\to G_{n+1}, && 0\leq i\leq n \text{ (degeneracy operator)}. \end{aligned}$$

Here we have the usual commutation rule:

$$\varepsilon^i\delta^j = \begin{cases} \delta^{j-1}\varepsilon^i & (i<j)\\ \text{identity} & (i=j \quad\text{or}\quad j+1)\\ \delta^j\varepsilon^{i-1} & (i>j+1) \end{cases}$$

$$\varepsilon^i\varepsilon^j = \varepsilon^{j-1}\varepsilon^i \qquad (i<j)$$
$$\delta^i\delta^j = \delta^{j-1}\delta^i \qquad (i\leq j)$$

with abbreviations $\varepsilon^i=\varepsilon^i_n$ and $\delta^j=\delta^j_n$. For any object A in $\mathfrak{A}$ the corresponding simplicial object $G_*(A)=\{G_n(A)\,;\,n\geq 0\}$ in $\mathfrak{A}$ is called the *standard simplicial complex over* A *associated with the cotriple* $\mathbf{G}$. And the augmented simplicial complex $G^+_*(A)=\{G_n(A)\,;\,n\geq -1\}$ with natural augmentation $\varepsilon(A): G_0(A)\to G_{-1}(A)=A$, i.e. the sequence

$$\cdots\cdots G_n(A) \overset{\varepsilon^0}{\underset{\varepsilon^n}{\vdots}} G_{n-1}(A) \cdots\cdots \rightrightarrows G_1(A) \overset{\varepsilon^0}{\underset{\varepsilon^1}{\rightrightarrows}} G_0(A) \xrightarrow{\varepsilon} A\,,$$

together with degeneracy operators, is called the *standard simplicial resolution of* A.

The following lemma is a slight modification of Theorem in [20].

Lemma 1.4. *Suppose that* $U(A)$ *is an abelian group object in* $\mathcal{C}$ *for every object* A *in* $\mathfrak{A}$. *Then the augmented chain complex* $UG^+_*(A)$ *with differential* $d_n(A)=\sum_{i=0}^{n}(-1)^iU\varepsilon^i(A)$ $(n>0)$ *and augmentation* $U\varepsilon(A)$ *is acyclic. Precisely, there exists a contracting homotopy* $s_n(A)$: $UG_n(A)\to UG_{n+1}(A)$ $(n\geq -1)$ *such that* $U\varepsilon\circ s_{-1}=1$ *and*

$$d_{n+1}s_n+s_{n-1}d_n = 1 \qquad (n\geq 0)\,.$$

Proof. Define morphisms $t_n(A)$: $UG_m(A)\to UG_m(A)$ and $u_n(A)$: $UG_m(A)\to UG_{m+1}(A)$, for $0\leq n\leq m$, as follows:

$$\begin{aligned}
&t_0 = 1\,,\quad u_{-1}=0\,,\quad u_0=0\\
(1.5)\qquad &t_n = (1-U\delta^0\varepsilon^1)(1-U\delta^1\varepsilon^2)\cdots(1-U\delta^{n-1}\varepsilon^n) \qquad (n\geq 1)\\
&u_n = t_0\circ U\delta^0-t_1\circ U\delta^1+\cdots+(-1)^{n-1}t_{n-1}\circ U\delta^{n-1} \qquad (n\geq 1)\,.
\end{aligned}$$

Then we have

$$(1.6)\qquad \begin{aligned}
&U\varepsilon^i\circ t_n = \begin{cases} t_{n-1}\circ d_n & (i=0)\\ 0 & (0<i\leq n)\end{cases}\\
&1-t_n = d_{n+1}u_n+u_{n-1}d_n \qquad (n\geq 0)\,.
\end{aligned}$$

Now define morphisms $\eta_n(A)=\eta UG_n(A)$: $UG_n(A)\to UG_{n+1}(A)$, which satisfy

$$U\varepsilon^i\circ\eta_n = \begin{cases} 1 & (i=0) \\ \eta_{n-1}\circ U\varepsilon^{i-1} & (0<i\leq n+1) \end{cases} \tag{1.7}$$
$$U\varepsilon\circ\eta_{-1} = 1\,.$$

Using these we get the required contracting homotopy:

$$s_n = \eta_n t_n + u_n \qquad (n\geq 0) \tag{1.8}$$
$$s_{-1} = \eta_{-1} = \eta U\,.$$

Note that all these t_n, u_n, η_n and s_n are natural transformations of functors $\mathcal{A}\to\mathcal{C}$.

For later use, we give the following variant of the above lemma which will be similarly proved.

Lemma 1.9. *Suppose that there is given another category* $\mathcal{C}'$ *and a functor* $U'\colon\mathcal{C}\to\mathcal{C}'$ *such that* $U'U(A)$ (*but not* $U(A)$) *is an abelian group object in* $\mathcal{C}'$ *for every object* A *in* $\mathcal{A}$. *Then the augmented chain complex* $U'UG_*^{+}(A)$ *with differential* $d_n(A)=\sum_{i=0}^{n}(-1)^i U'U\varepsilon^i(A)$ $(n>0)$ *and augmentation* $U'U\varepsilon(A)$ *is acyclic.*

For the purpose of defining cotriple cohomology, we first fix an object A in $\mathcal{A}$ and consider the *comma category* $(\mathcal{A}, A)$ [5]. By definition an object in $(\mathcal{A}, A)$ is a pair (B, γ) of an object B in $\mathcal{A}$ and a morphism $\gamma: B\to A$ in $\mathcal{A}$, and a morphism $(B, \gamma)\to(B', \gamma')$ is such a commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\varphi} & B' \\ & {\scriptstyle\gamma}\searrow \quad \swarrow{\scriptstyle\gamma'} & \\ & A & \end{array}$$

An object (B, γ) in $(\mathcal{A}, A)$ will be offen denoted by $B\to A$ or simply by B. Note that the comma category $(\mathcal{A}, A)$ is not pointed for $A\neq 0$, but has kernels and the terminal object $A=(A, 1_A)$.

An adjoint pair $\mathcal{C}\overset{S}{\to}\mathcal{A}\overset{U}{\to}\mathcal{C}$ induces canonically an adjoint pair $(\mathcal{C}, U(A))\overset{S}{\to}(\mathcal{A}, A)\overset{U}{\to}(\mathcal{C}, U(A))$ which will be denoted by the same symbols S and U as before, and similarly for the induced cotriple $\boldsymbol{G}=(G, \varepsilon, \delta)$, $G=SU$, $\delta=S\eta U$. Note that $G^n(A)$ is regarded canonically as an object in $(\mathcal{A}, A)$ with the unique morphism $G^n(A)\to A$ expressed by a composition of face operators.

Now denote by Ab the category of abelian groups and let $T:(\mathfrak{A}, A)\to Ab$ be a contravariant functor. Then we have a cochain complex $TG_*(A)=\{TG_n(A); n\geq 0\}$ with differential $d^*(A)=\sum(-1)^i \times T\varepsilon^i(A)$. Its derived groups $H^n(TG_*(A))$ are, by definition, the *cotriple cohomology groups* of A with initial group $H^0(TG_*(A))=T(A)$. For example take an abelian group object $Y\to A$ in $(\mathfrak{A}, A)$ and define a functor $T:(\mathfrak{A}, A)\to Ab$ by $T(X)=\mathrm{Hom}_{(\mathfrak{A},A)}(X, Y)$. Then we have the cohomology groups $H_G^n(A, Y)=H^n(TG_*(A))$ with $H_G^0(A, Y)=\mathrm{Hom}_{(\mathfrak{A},A)}(A, Y)$.

§2. Lie Algebras and Cotriples

Let K be a commutative ring with unit which we fix as ground ring. By a *K-Lie algebra* we mean a K-module Γ with a K-bilinear product $[x, y]\in\Gamma$ such that

$$[x, x] = 0\,,$$
$$[x, [y, z]]+[y, [z, x]]+[z, [x, y]] = 0$$

for $x, y, z\in\Gamma$. By a *Γ-module* for a K-Lie algebra Γ we mean a K-module M with left operation of Γ on M such that $[x, y]m=xym-yxm$ for $x, y\in\Gamma$, $m\in M$.

Let $\mathcal{L}$ be category of all K-Lie algebras with obvious morphisms, and ${}_\Gamma\mathfrak{M}$ be the category of all Γ-modules. Then every abelian group object in the comma category $(\mathcal{L}, \Gamma)$ is known to be of the form of the split extension $\Gamma*M$ of Γ by a Γ-module M. This may be called also the *idealization* of a Γ-module M and is defined as a direct product $\Gamma\times M$ of K-modules with bracket product

$$[(x, m), (y, n)] = ([x, y], xn-ym) \tag{2.1}$$

for $x, y\in\Gamma$ and $m, n\in M$. With the injection $m\to(0, m)$ and the projection $(x, m)\to x$, and regarding M as an abelian Lie algebra, we have a split exact sequence

$$0\to M\to\Gamma*M\to\Gamma\to 0$$

in $\mathcal{L}$ (also in $(\mathcal{L}, \Gamma)$).

Denote by $\mathrm{Ab}(\mathcal{L}, \Gamma)$ the full subcategory of $(\mathcal{L}, \Gamma)$ formed of all abelian group objects. Then we have an equivalence of categories

$$\mathrm{Ab}(\mathcal{L}, \Gamma) \underset{\Theta}{\overset{\mathrm{Ker}}{\rightleftarrows}} {}_{\Gamma}\mathfrak{M}$$

where Ker denotes the kernel functor and $\Theta(M) = \Gamma * M$ [5].

For a Γ-module M and an object (L, γ) in $(\mathcal{L}, \Gamma)$, a *K-derivation* (or simply *derivation*) $f : L \to M$ is defined as a K-linear map such that

$$f[x, y] = \gamma(x)f(y) - \gamma(y)f(x). \tag{2.2}$$

The set of all such K-derivations $f : L \to M$ forms an abelian group denoted by $\mathrm{Der}_M(L, \gamma)$ (or simply by $\mathrm{Der}_M(L)$), and it defines the derivation functor $\mathrm{Der}_M : (\mathcal{L}, \Gamma) \to \mathrm{Ab}$. As is well known, there is a canonical isomorphism

$$\mathrm{Der}_M(L) \approx \mathrm{Hom}_{(\mathcal{L}, \Gamma)}(L, \Theta(M)). \tag{2.3}$$

Now we consider the cotriple cohomology $H^n_G(\Gamma, \Theta(M)) = H^n(\mathrm{Der}_M G_*(\Gamma))$ with respect to a cotriple G in $(\mathcal{L}, \Gamma)$ as defined at the end of §1. For brevity we occasionally denote $H^n_G(\Gamma, \Theta(M))$ by $H^n_G(\Gamma, M)$ (or $H^n(\Gamma, M)$) and call this the *n-th cohomology group of* Γ *with coefficients in M.*

To calculate explicitly $H^n_G(\Gamma, M)$, we shall choose some typical cotriples G in $\mathcal{L}$. Take $\mathcal{L}$ for $\mathcal{A}$ in the preceding section. Let $\mathcal{C}$ be either one of the following pointed categories: 1) the category ${}_K\mathfrak{M}$ of K-modules, 2) the category $\mathcal{S}$ of pointed sets (i.e. sets with base points and base points preserving maps) and 3) the category $\mathcal{S}^\times$ of pointed sets with multiplications. More explanation is needed for the last category. An object in $\mathcal{S}^\times$ is a pointed set (X, x_0) with multiplication (may be non-associative) $X \times X \xrightarrow{\cdot} X$ such that $x_0 \cdot x = x \cdot x_0 = x_0$ for any $x \in X$ and a morphism $f : (X, x_0) \to (Y, y_0)$ is a multiplication preserving set map.

According to each of the above cases 1), 2) and 3), $\mathcal{C}$ will be also denoted by $\mathcal{C}_i$ $(i = 1, 2, 3)$. Let $U_i : \mathcal{L} \to \mathcal{C}_i$ be the underlying object functor. It is clear that the U_i are faithful and known to be tripleable in the sense of Beck [5] for all the above cases of $\mathcal{C}$. The left adjoint $S_i : \mathcal{C}_i \to \mathcal{L}$ of U_i will be given as follows in respective cases.

In case $\mathcal{C}_1 = {}_K\mathfrak{M}$, S_1 is given by the functor L described in [6], that is, for a K-module M, $S_1(M)=L(M)$ is the quotient K-Lie algebra of the free non-associative algebra $A(M)=M+M\otimes M+(M\otimes M)\otimes M+M\otimes(M\otimes M)+\cdots$ by the two-sided ideal generated by elements of the form $m\otimes m$ and $m_1\otimes(m_2\otimes m_3)+m_2\otimes(m_3\otimes m_1)+m_3\otimes(m_1\otimes m_2)$ for $m, m_i\in M$.

In case $\mathcal{C}_2=\mathcal{S}$, S_2 is given by $S_2(X, x_0)=LF(X, x_0)$ where $F(X, x_0)=K(X)/K(x_0)$ is the free K-module generated by the set X with identification $x_0=0$.

In case $\mathcal{C}_3=\mathcal{S}^\times$, the functor S_3 is given as follows: Let (X, x_0) be an object in $\mathcal{S}^\times$. Construct first the free K-module $F(X, x_0)$ as above and introduce in it a unique K-bilinear multiplication induced from the multiplication of X. Then the K-Lie algebra $S_3(X, x_0)$ is defined as the quotient of the non-associative K-algebra $F(X, x_0)$ by the two-sided ideal generated by elements of the form $x\cdot x$ and $x\cdot(y\cdot z)+y\cdot(z\cdot x)+z\cdot(x\cdot y)$.

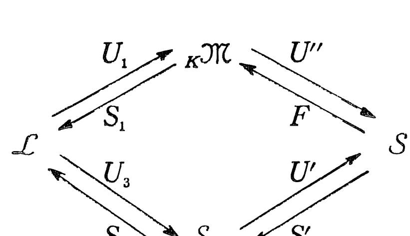
The corresponding cotriples $\boldsymbol{G}$ and the cohomologies $H^n_G(\Gamma, M)$ will be denoted by $\boldsymbol{G}_i$ and $H^n_i(\Gamma, M)$ in respective cases $i=1, 2, 3$.

We remark that $H^*_1(\Gamma, M)$ is the Hochschild cohomology [17] and $H^*_2(\Gamma, M)$ is related to the cohomologies of Dixmier [7] and Shukla [28]. It is known in [5] and [1] that

$$H^n_i(\Gamma, M)\approx\begin{cases}\mathrm{Der}_M(\Gamma) & (n=0)\\ \mathrm{Ex}^1_i(\Gamma, M) & (n=1)\end{cases}$$

for $i=1, 2, 3$, where $\mathrm{Ex}^1_i(\Gamma, M)$ denotes the set of all isomorphism classes of singular U_i-split extensions of Γ by M in $\mathcal{L}$. The bijective correspondence $H^1\approx\mathrm{Ex}^1$ becomes an isomorphism of K-modules if we introduce a suitable Baer sum in $\mathrm{Ex}^1_i(\Gamma, M)$.

To conclude this section, we are situated in the following commutative diagram



$$U_2=U''U_1=U'U_3,$$
$$S_2=S_1F=S_3S',$$

where U' and U'' are the forgetful functors and S' is the left adjoint of U'.

§ 3. Two Term Extensions and Main Theorem

Let M be a Γ-module which may be regarded as an abelian K-Lie algebra. By a U-split exact sequence in $\mathcal{L}$ we means a sequence in $\mathcal{L}$ of which transformation by U is split exact (in $\mathcal{C}$).

Definition. By a *two term extension of* Γ *by* M *with respect to the underlying object functor* U_i we mean a U_i-split exact sequence in $\mathcal{L}$ $(i=1,2,3)$:

$$(e): \quad 0 \to M \overset{\varphi_2}{\to} X_1 \overset{\varphi_1}{\to} X_0 \overset{\varphi_0}{\to} \Gamma \to 0$$

where X_0 operates on X_1 in the following way:

$$\varphi_2(\varphi_0(x)m) = x\cdot\varphi_2(m)\,, \tag{3.1}$$

$$\varphi_1(x\cdot u) = [x, \varphi_1(u)]\,, \tag{3.2}$$

$$\varphi_1(u)\cdot v = [u, v] \tag{3.3}$$

for $x\in X_0$, $m\in M$, $u, v\in X_1$. Moreover, in the case $i=3$ (i.e. $\mathcal{C}=\mathcal{S}^{\times}$), we put an extra condition: there exists a set map $\beta : X_0\to X_1$ such that

$$\begin{aligned} \varphi_1\beta &= 1-\sigma\varphi_0\,, \\ \beta[x, y] &= x\beta(y)-\sigma\varphi_0 y\cdot\beta(x) \qquad \text{for} \quad x, y\in X_0\,, \end{aligned} \tag{3.4}$$

where $\sigma : \Gamma\to X_0$ is a morphism in $\mathcal{S}^{\times}$ satisfying $\varphi_0\sigma=1$ (the existence of such a map σ is ensured by the U-splitness of the sequence (e)).

The totality of all such two term extensions of Γ by M with respect to U_i form a category $\mathcal{E}_i$ of which a morphism $(e)\to(e')$ is given by a commutative diagram:

$$\begin{array}{rccccccccc} (e): & 0 \to & M & \overset{\varphi_2}{\to} & X_1 & \overset{\varphi_1}{\to} & X_0 & \overset{\varphi_0}{\to} & \Gamma & \to 0 \\ & & \| & & \downarrow{\scriptstyle\psi_1} & & \downarrow{\scriptstyle\psi_0} & & \| & \\ (e'): & 0 \to & M & \overset{\varphi_2'}{\to} & Y_1 & \overset{\varphi_1'}{\to} & Y_0 & \overset{\varphi_0'}{\to} & \Gamma & \to 0 \end{array}$$

where ψ_0 and ψ_1 are morphisms in $\mathcal{L}$ compatible with the operations of X_0 and Y_0 on X_1 and Y_1 $(i=1,2,3)$. To extensions (e) and (e')

are called *equivalent* (notation $(e)\sim(e')$) if they are connected by a sequence of morphisms of both directions: e.g.,

$$(e) = (e_0) \leftarrow (e_1) \rightarrow (e_2) \rightarrow \cdots \leftarrow (e_n) = (e')\,.$$

The set of all equivalence classes of two term extensions of Γ by M with respect to the underlying object functor U_i is denoted by $\mathrm{Ex}_i^2(\Gamma, M)$ $(i=1, 2, 3)$.

Now we can state our main theorem.

Theorem 3.5. *There is a bijective correspondence*

$$H_i^2(\Gamma, M) \approx \mathrm{Ex}_i^2(\Gamma, M)$$

for each $i=1, 2, 3$.

The proof of this theorem will be given in §5.

We remark that, for an object L in $\mathcal{L}$, the underlying K-module $U_1(L)$ and the underlying set $U_2(L)$ are clearly abelian group objects in ${}_K\mathfrak{M}$ and $\mathcal{S}$ respectively, but $U_3(L)$ is not so in $\mathcal{S}^\times$, which is the reason why we need the exceptional condition (3.4) in the definition of two term extensions with respect to U_3.

§4. Standard Two Term Extensions

Take any one of the cotriples $\boldsymbol{G}_i$ in $(\mathcal{L}, \Gamma)$, $i=1, 2, 3$, as before. The standard simplicial complex $G_*(\Gamma)$ over Γ induces the underlying chain complex of K-modules, denoted by the same notation $G_*(\Gamma)$, with differential $d_n=\sum_{i=0}^{n}(-1)^i\varepsilon^i(\Gamma)$ $(n\geq 1)$ and the augmented complex $G_*^+(\Gamma)$ is acyclic by Lemma 1.4 in cases $\mathcal{C}={}_K\mathfrak{M}$ or $\mathcal{S}$, by Lemma 1.9 in case $\mathcal{C}=\mathcal{S}^\times$.

Let M be a Γ-module. $G^n(\Gamma)$ operates naturally on M via the canonical morphism $G^n(\Gamma)\rightarrow\Gamma$ (see §1), so that M is considered as a $G^n(\Gamma)$-module for every $n\geq 0$.

A *derivation 2-cocycle* $f\in\mathrm{Der}_M(G^3(\Gamma))$ is a K-linear map $f: G^3(\Gamma)\rightarrow M$ and $f[x, y]=xf(y)-yf(x)$ for $x, y\in G^3(\Gamma)$ and $d^*f=fd_3=0: G^4(\Gamma)\rightarrow M$. Two such cocycles f and f' are *D-cohomologous* (notation $f\underset{D}{\sim}f'$) if there exists a derivation $\omega: G^2(\Gamma)\rightarrow M$ and $\omega d_2=f-f'$.

Given a derivation 2-cocycle $f\in\mathrm{Der}_M(G^3(\Gamma))$. We shall construct a K-Lie algebra E_f as follows. Put $\tilde{G}_1(\Gamma)=\mathrm{Ker}(\varepsilon^1 : G^2(\Gamma)\to G(\Gamma))$. Let $\tilde{G}_1(\Gamma)\times M$ be the direct product of K-Lie algebras (M being an abelian Lie algebra). Let I be the ideal of $\tilde{G}_1(\Gamma)\times M$ generated by elements of the form $(-t_1d_2y, f(t_2y))$ for $y\in G^3(\Gamma)$ (see (1.5)). Define E_f to be the quotient K-Lie algebra $\tilde{G}_1(\Gamma)\times M/I$. Then E_f has a set-presentation $N(\Gamma)\times M$, where $N(\Gamma)=\mathrm{Ker}(\varepsilon : G(\Gamma)\to\Gamma)$. To see this, we define set maps

$$\pi:\ \tilde{G}_1(\Gamma)\times M\to N(\Gamma)\times M$$

and

$$\kappa:\ N(\Gamma)\times M\to\tilde{G}_1(\Gamma)\times M$$

by

$$\pi(x, m) = (d_1x, m+f(\bar{x}))$$

and

$$\kappa(n, m) = (\bar{n}, m)$$

respectively, where $\bar{x}=\eta_n(x)$ for $x\in UG_n(\Gamma)$ (see (1.7)).

Direct calculations show that

$$\pi(I) = (0, 0)\,,$$
$$\pi\circ\kappa = \text{identity}\,,$$

and

$$\kappa\circ\pi \equiv \text{identity mod } I\,.$$

It follows that π induces a canonical one to one correspondence $E_f\approx N(\Gamma)\times M$ as set.

Denote by (n, m) an element of E_f for $n\in N(\Gamma)$ and $m\in M$ in this presentation. Then the K-Lie algebra structure of E_f is given explicitly by

(4.1) $$(n_1, m_1)+(n_2, m_2) = (n_1+n_2,\ m_1+m_2+f(\overline{\bar{n}_1+\bar{n}_2}))\,,$$

(4.2) $$k(n, m) = (kn, km+f(\overline{k\bar{n}}))\,, \qquad \text{for} \quad k\in K\,,$$

(4.3) $$[(n_1, m_1), (n_2, m_2)] = ([n_1, n_2], f(\overline{[\bar{n}_1, \bar{n}_2]}))\,,$$

where we used the notation

(4.4) $$\bar{x} = \eta_n(x)\in UG_{n+1}(\Gamma)$$

for $x\in UG_n(\Gamma)$ (see §1, (1.7)).

Remark. If $\mathcal{C}={}_K\mathcal{M}$, then $E_f=N(\Gamma)\oplus M$ (direct sum) as K-

module. If $\mathcal{C}=\mathcal{S}^{\times}$, then (4.3) reduces to

$$[(n_1, m_1), (n_2, m_2)] = ([n_1, n_2], 0)\,.$$

Define morphisms $\iota: M\to E_f$ and $\varphi: E_f\to G(\Gamma)$ by $\iota(m)=(0, m)$ and $\varphi(n, m)=n$ respectively. Then we have a U-split exact sequence in $\mathcal{L}$:

$$(4.5)\qquad (e_f):\ 0\to M\xrightarrow{\iota} E_f\xrightarrow{\varphi} G(\Gamma)\xrightarrow{\varepsilon}\Gamma\to 0$$

which will be called a *standard two term extension* of Γ by M (with respect to the underlying object functor $U:\mathcal{L}\to\mathcal{C}$). By (4.3) M is contained in the centre of E_f. $G(\Gamma)$ operates on E_f by

$$(4.6)\qquad x(n, m) = ([x, n],\ xm+f([\overline{x}, \overline{n}]))\,,$$

so that we have

$$(4.7)\qquad \iota(xm) = x\iota(m)\,,$$

$$(4.8)\qquad \varphi(xu) = [x, \varphi(u)]$$

and

$$(4.9)\qquad \varphi(u)v = [u, v]$$

for $x\in G(\Gamma)$, $m\in M$ and $u, v\in E_f$.

Further we have the following commutative diagram of K-modules:

$$(4.10)\qquad \begin{array}{ccccccccc} G^4(\Gamma) & \xrightarrow{d_3} & G^3(\Gamma) & \xrightarrow{d_2} & G^2(\Gamma) & \xrightarrow{d_1} & G(\Gamma) & \xrightarrow{\varepsilon} & \Gamma\to 0 \\ \downarrow & & \downarrow f & & \downarrow \alpha & & \| & & \| \\ 0 & \to & M & \xrightarrow{\iota} & E_f & \xrightarrow{\varphi} & G(\Gamma) & \xrightarrow{\varepsilon} & \Gamma\to 0 \end{array}$$

where α is a canonical K-linear map $G^2(\Gamma)\to E_f$ given as follows. Using the canonical decomposition (see Lemmas 1.4, 1.9)

$$x = d_2s_1x+s_0d_1x\,,\qquad \text{for}\quad x\in G^2(\Gamma)\,,$$

we define α by

$$(4.11)\qquad \alpha(x) = (d_1x,\ fs_1x)\,.$$

Then we can verify that α is K-linear.

Proposition 4.12. *The K-linear map* α *satisfies*

$$\alpha[x, y] = \varepsilon^0 x \cdot \alpha(y) - \varepsilon^1 y \cdot \alpha(x) \qquad \textit{for} \quad x, y \in G^2(\Gamma) .$$

Proof. The right-hand side will be computed by using (4.10), (4.1~3) as follows:

$$\varepsilon^0 x\alpha(y) - \varepsilon^1 y \cdot \alpha(x) = (d_1[x, y],\ m_1 + m_2) \tag{4.13}$$

where

$$m_1 = \varepsilon^0 x \cdot f s_1 y - \varepsilon^1 y \cdot f s_1 x ,$$
$$m_2 = f(\overline{[\overline{\varepsilon^0 x}, \overline{d_1 y}] - [\overline{\varepsilon^1 y}, \overline{d_1 x}]} + \overline{[\overline{\varepsilon^0 x}, \overline{d_1 y}] - [\overline{\varepsilon^1 y}, \overline{d_1 x}]}) .$$

Since f is a derivation, we have

$$m_1 = f([\bar{x}, \overline{t_1 y}] + [\overline{t_1 x}, \overline{\delta^0 \varepsilon^1 y}] + [\delta^0 x, \delta^0 y]) ,$$

where $t_1 = 1 - \delta^0 \varepsilon^1 : G^2(\Gamma) \to G^2(\Gamma)$ (Cf. (1.5)). Therefore $m_1 + m_2$ is written as $f(\xi)$ for a certain element $\xi \in G^3(\Gamma)$, and a direct calculation shows that $d_2 \xi = d_2 s_1[x, y]$. Since f is a cocycle and $G_*^{\ddagger}(\Gamma)$ is acyclic as above, we obtain

$$m_1 + m_2 = f(\xi) = f s_1[x, y] . \tag{4.14}$$

The proposition follows from (4.13) and (4.14).

We have thus assigned canocically the standard two term extension (e_f) to each derivation 2-cocycle $f : G^3(\Gamma) \to M$.

We show that the extension (e_f) belongs to $\mathcal{E}_i$ (see §3). This is clear in case of $\mathcal{C} = {}_K\mathcal{M}$ or $\mathcal{S}$ by (4.6~8). In case of $\mathcal{C} = \mathcal{S}^\times$, we define a set map $\beta : G(\Gamma) \to E_f$ by

$$\beta(x) = \alpha\eta_0(x) = \alpha(\bar{x}) .$$

Then, we have, by (1.7)

$$\varphi\beta = \varphi\alpha\eta_0 = d_1\eta_0 = 1 - \eta\varepsilon ,$$

and by Prop. 4.12

$$\beta[x, y] = \alpha[\overline{x, y}] = \alpha[\bar{x}, \bar{y}] = \varepsilon^0 \bar{x} \cdot \alpha(\bar{y}) - \varepsilon^1 \bar{y} \cdot \alpha(\bar{x})$$
$$= x\beta(y) - \eta\varepsilon y \cdot \beta(x) ,$$

so that the condition (3.4) is satisfied.

Proposition 4.15. *If derivation 2-cocycles f and f' are D-cohomologous ($f \underset{D}{\sim} f'$), then we have $(e_f) \sim (e_{f'})$.*

Proof. Suppose that $f'=f+\omega d_2$ for a derivation $\omega: G^2(\Gamma)\to M$. Define $\psi: E_f\to E_{f'}$ by

$$\psi(n, m) = (n, m-\omega(\bar{n}))\in E_{f'}.$$

Then ψ is a (bijective) morphism (in $\mathcal{L}$) and commutes with the operations of $G(\Gamma)$ on E_f and $E_{f'}$. We have thus a morphism $(e_f)\to(e_{f'})$ in the category $\mathcal{E}_i$ of two term extensions of Γ by M (see §3).

Remark. In a parallel way as above, we can define what to be called the standard n term extension (e_f) of Γ by M for derivation n-cocycle $f: G^{n+1}(\Gamma)\to M$ $(n\geq 2)$ as follows.

Define first the *Moore subcomplex* $\tilde{G}_*(\Gamma)$ of the chain complex $G_*(\Gamma)$ of K-modules by

(4.16)
$$\tilde{G}_0(\Gamma) = G(\Gamma), \quad \tilde{G}_k(\Gamma) = \bigcap_{i=1}^{k} \mathrm{Ker}\,(\varepsilon^i: G^{k+1}(\Gamma)\to G^k(\Gamma)) \quad \text{for} \quad k>0.$$

Then $\tilde{G}_k(\Gamma)$ is an ideal of $G^{k+1}(\Gamma)$ and we have a commutative diagram of K-modules:

(4.17)
$$\begin{array}{ccccccccccc}
\to G^{n+1}(\Gamma) & \overset{d_n}{\to} & G^n(\Gamma) & \to \cdots \to & G^2(\Gamma) & \overset{d_1}{\to} & G(\Gamma) & \overset{\varepsilon}{\to} & \Gamma & \to & 0 \\
\downarrow t_n & & \downarrow t_{n-1} & & \downarrow t_1 & & \| t_0 & & \| & & \\
\to \tilde{G}_n(\Gamma) & \overset{\varepsilon_0}{\to} & \tilde{G}_{n-1}(\Gamma) & \to \cdots \to & \tilde{G}_1(\Gamma) & \overset{\varepsilon_0}{\to} & \tilde{G}_0(\Gamma) & \overset{\varepsilon}{\to} & \Gamma & \to & 0
\end{array}$$

where

(4.18)
$$t_n = (1-\delta^0\varepsilon^1)(1-\delta^1\varepsilon^2)\cdots(1-\delta^{n-1}\varepsilon^n), \qquad n\geq 0$$

are retractions and define a chain equivalence $G_*(\Gamma)\sim\tilde{G}_*(\Gamma)$.

Now let $\tilde{G}_{n-1}(\Gamma)\times M$ be the direct product of K-Lie algebras (M being an abelian Lie algebra). Let I be the ideal of $\tilde{G}_{n-1}(\Gamma)\times M$ generated by elements of the form $(-t_{n-1}d_n(y), f(t_n y))$ for $y\in G^{n+1}(\Gamma)$. Define $E_f=\tilde{G}_{n-1}(\Gamma)\times M/I$ to be the quotient K-Lie algebra. Then E_f has a canonical set presentation $N_{n-2}(\Gamma)\times M$, where $N_{n-2}(\Gamma)=\mathrm{Ker}\,(\varepsilon^0: \tilde{G}_{n-2}(\Gamma)\to\tilde{G}_{n-3}(\Gamma))$ for $n\geq 3$ and $N_0(\Gamma)=N(\Gamma)=\mathrm{Ker}\,(\varepsilon: G(\Gamma)\to\Gamma)$ as before. Using this set presentation, we can give the K-Lie algebra structure of E_f explicitly by the same form of formulas as in (4.1), (4.2) and (4.3) (in fact, we have only to replace $N(\Gamma)$ by $N_{n-2}(\Gamma)$).

Then we have a U-split exact sequence (e_f) in $\mathcal{L}$:

$$0 \to M \overset{\iota}{\to} E_f \overset{\varphi}{\to} \tilde{G}_{n-2}(\Gamma) \overset{\varepsilon^0}{\to} \cdots \overset{\varepsilon^0}{\to} \tilde{G}_0(\Gamma) \overset{\varepsilon}{\to} \Gamma \to 0 \tag{4.19}$$

which we call a *standard n term extension* of Γ by M with respect to the underlying object functor U.

Similarly as in (4.10), we have the following commutative diagram

$$\begin{array}{ccccccccccccc} G^{n+2}(\Gamma) & \to & G^{n+1}(\Gamma) & \overset{d_n}{\to} & G^n(\Gamma) & \overset{d_{n-1}}{\longrightarrow} & G^{n-1}(\Gamma) & \to & \cdots & \overset{d_1}{\to} & G(\Gamma) & \overset{\varepsilon}{\to} & \Gamma \\ \downarrow & & \downarrow f & & \downarrow \alpha & & \downarrow t_{n-2} & & & & \| t_0 & & \| \\ 0 & \to & M & \overset{\iota}{\to} & E_f & \overset{\varphi}{\to} & \tilde{G}_{n-2}(\Gamma) & \to & \cdots & \overset{\varepsilon^0}{\to} & \tilde{G}_0(\Gamma) & \overset{\varepsilon}{\to} & \Gamma \end{array} \tag{4.20}$$

where α is the canonical K-map $G^n(\Gamma) \to E_f$ explicitly given by

$$\alpha(x) = (t_{n-2}d_{n-1}x,\ fs_{n-1}x)\,. \tag{4.21}$$

§5. Proof of Theorem 3.5

In the last section we have defined a map $\Phi: H_i^2(\Gamma, M) \to \mathrm{Ex}_i^2(\Gamma, M)$ $(i=1, 2, 3)$ (see Proposition 4.15). We shall prove the following two propositions of which the first asserts the ontoness of Φ and the second one asserts Φ to be 1-1.

Proposition 5.1. *Given a two term extension* (e) *in* $\mathcal{E}_i$ $(i=1,2,3)$. *Then there exists a derivation 2-cocycle* $f: G^3(\Gamma) \to M$ *such that* $(e_f) \to (e)$ *in* $\mathcal{E}_i$.

Proposition 5.2. *If* $(e_f) \to (e) \leftarrow (e_{f'})$ *in* $\mathcal{E}_i$ $(i=1,2,3)$, *then* f *and* f' *are D-cohomologous* (*i.e.* f *and* f' *determine the same cohomology class* $\in H_i^2(\Gamma, M)$).

Proof of Proposition 5.1. Given a two term extension $(e) \in \mathcal{E}_i$:

$$(e):\ 0 \to M \overset{\varphi_2}{\to} X_1 \overset{\varphi_1}{\to} X_0 \overset{\varphi_0}{\to} \Gamma \to 0$$

which is U_i-split exact sequence in $\mathcal{L}$ and X_0 operates on X_1 as in (3.1), (3.2) and (3.3) with additional condition (3.4) in case of $\mathcal{C}_3 = \mathcal{S}^\times$.

There exists a morphism $\sigma: U(\Gamma) \to U(X_0)$ in $\mathcal{C}$ with $\varphi_0\sigma = 1$, so that σ determines a unique morphism $\tau = \lambda^{-1}(\sigma): G(\Gamma) \to X_0$ in $\mathcal{L}$ by

(1.1). Then we have

$$\varphi_0\tau = \varepsilon .$$

Define the idealization $X_0 * X_1$ of X_0-module X_1. That is, $X_0 * X_1$ is the direct sum of X_0 and X_1 as K-module and the bracket product is given by

$$[(x, u), (y, v)] = ([x, y], [u, v]+x\cdot v-y\cdot u) \quad \text{for} \quad x, y\in X_0,\ u, v\in X_1 . \tag{5.3}$$

We have a morphism $\beta: U(X_0)\to U(X_1)$ in $\mathcal{C}$ such that

$$\varphi_1\beta = 1-\sigma\varphi_0$$

in case of $\mathcal{C}={}_K\mathcal{M}$ or $\mathcal{S}$. In case of $\mathcal{C}=\mathcal{S}^{\times}$, we must take such a β as in (3.4).

Now define a morphism $\rho': UG(\Gamma)\to U(X_0 * X_1)$ in $\mathcal{C}$ by

$$\rho'(x) = (\tau\eta\varepsilon x, \beta\tau x) = (\sigma\varphi_0\tau x, \beta\tau x) \qquad \text{for} \quad x\in G(\Gamma) . \tag{5.4}$$

This determines a unique morphism $\rho: G^2(\Gamma)\to X_0 * X_1$ in $\mathcal{L}$.

Lemma 5.5. *The morphism $\rho: G^2(\Gamma)\to X_0 * X_1$ is expressed by*

$$\rho(x) = (\tau\varepsilon^1 x, g(x)) ,$$

where $g: G^2(\Gamma)\to X_1$ is K-linear and satisfies:

$$\varphi_1 g = \tau d_1$$

and

$$g[x, y] = \tau\varepsilon^0 x\cdot g(y)-\tau\varepsilon^1 x\cdot g(x)$$

for $x, y\in G^2(\Gamma)$.

Proof. Define two morphisms $\theta^0, \theta^1 : X_0 * X_1 \to X_0$ in $\mathcal{L}$ by

$$\theta^0(x, u) = x+\varphi_1(u)$$

and

$$\theta^1(x, u) = x .$$

Then using (1.3), we have

$$\theta^0\rho(\bar{x}) = \theta^0\rho'(x) = \tau\eta\varepsilon(x)+\varphi_1\beta\tau(x) = \tau x = \tau\varepsilon^0(\bar{x})$$

and

$$\theta^1\rho(\bar{x}) = \theta^1\rho'(x) = \tau\eta\varepsilon(x) = \tau\varepsilon^1(\bar{x}) .$$

Using again (1.3), we conclude that

$$\theta^0\rho = \tau\varepsilon^0$$

and

$$\theta^1\rho = \tau\varepsilon^1 ,$$

as morphisms $G^2(\Gamma)\rightarrow X_0$ in $\mathcal{L}$. The lemma follows from these properties of ρ.

Returning to the proof of Prop. 5.1, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
 & G^3(\Gamma) & \xrightarrow{d_2} & G^2(\Gamma) & \xrightarrow{d_1} & G(\Gamma) & \xrightarrow{\varepsilon} & \Gamma & \rightarrow 0 \\
 & \downarrow f & & \downarrow g & & \downarrow \tau & & \| & \\
0 \rightarrow & M & \xrightarrow[\varphi_2]{} & X_1 & \xrightarrow[\varphi_1]{} & X_0 & \xrightarrow[\varphi_0]{} & \Gamma & \rightarrow 0
\end{array}$$

where $f=\varphi_2^{-1}gd_2$ is a derivation 2-cocycle by Lemma 5.5. We have now a morphism $(e_f)\rightarrow(e)$ in $\mathcal{E}_i$:

$$\begin{array}{cccccccccc}
(e_f): & 0 \rightarrow & M & \xrightarrow{\iota} & E_f & \xrightarrow{\varphi} & G(\Gamma) & \xrightarrow{\varepsilon} & \Gamma & \rightarrow 0 \\
\downarrow & & \| & & \downarrow \psi & & \downarrow \tau & & \| & \\
(e): & 0 \rightarrow & M & \xrightarrow[\varphi_2]{} & X_1 & \xrightarrow[\varphi_1]{} & X_0 & \xrightarrow[\varphi_0]{} & \Gamma & \rightarrow 0
\end{array}$$

where $\psi: E_f\rightarrow X_1$ is a morphism in $\mathcal{L}$ defined by

$$\psi(n, m) = \varphi_2(m)+g(\bar{n}) \qquad \text{q. e. d.}$$

Proof of Proposition 5.2. Suppose that there is given a commutative diagram.

$$\begin{array}{cccccccccc}
(e_f): & 0 \rightarrow & M & \xrightarrow{\iota} & E_f & \xrightarrow{\varphi} & G(\Gamma) & \xrightarrow{\varepsilon} & \Gamma & \rightarrow 0 \\
\downarrow & & \| & & \downarrow \psi & & \downarrow \tau & & \| & \\
(e): & 0 \rightarrow & M & \xrightarrow{\varphi_2} & X_1 & \xrightarrow{\varphi_1} & X_0 & \xrightarrow{\varphi_0} & \Gamma & \rightarrow 0 \\
\uparrow & & \| & & \uparrow \psi' & & \uparrow \tau' & & \| & \\
(e_{f'}): & 0 \rightarrow & M & \xrightarrow[\iota']{} & E_{f'} & \xrightarrow[\varphi']{} & G(\Gamma) & \xrightarrow[\varepsilon']{} & \Gamma & \rightarrow 0
\end{array}$$

Define a morphism $\rho': U(\Gamma)\rightarrow U(X_0*X_1)$ by

$$\rho'(x) = (\tau(\bar{x}), \beta\tau'(\bar{x})) .$$

Then ρ' determines a unique morphism $\rho: G(\Gamma)\to X_0 * X_1$ in $\mathcal{L}$.

Lemma 5.6. *The morphism ρ is expressible in the form*

$$\rho(x) = (\tau(x), \omega(x)),$$

where $\omega: G(\Gamma)\to X_1$ is K-linear and satisfies:

$$\varphi_1\omega = \tau' - \tau$$

and

$$\omega[x, y] = \tau'(x)\cdot\omega(y) - \tau(y)\cdot\omega(x), \qquad for \quad x, y\in G(\Gamma).$$

The proof is similar as in that of Lemma 5.5 and hence ommited.

Now consider a K-linear map

$$\omega_1 = \psi'\alpha' - \psi\alpha - \omega d_1: \quad G^2(\Gamma) \to X_1, \tag{5.7}$$

where $\alpha': G^2(\Gamma)\to E_{f'}$ and $\alpha: G^2(\Gamma)\to E_f$ are defined in (4.11). Then

$$\varphi_1\omega_1 = \tau'\varphi'\alpha' - \tau\varphi\alpha - (\tau' - \tau)d_1 = 0.$$

Therefore ω_1 is regarded as a map: $G^2(\Gamma)\to M$. A straightforward calculation shows that

$$\begin{aligned}\omega_1[x, y] &= \tau'\varepsilon^0 x\cdot\omega_1(y) - \tau\varepsilon^1 y\cdot\omega_1(x)\\ &= \varepsilon\varepsilon^0 x\cdot\omega_1(y) - \varepsilon\varepsilon^1 y\cdot\omega_1(x).\end{aligned}$$

This means that $\omega_1: G^2(\Gamma)\to M$ is a derivation. And clearly we have

$$\varphi_2\omega_1 d_2 = \psi'\alpha' d_2 - \psi'\alpha d_2 = \psi'\iota' f' - \psi\iota f = \varphi_2(f' - f),$$

that is, $\omega_1 d_2 = f' - f$. q.e.d.

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