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Factorizable Representation of Current Algebra

—Non commutative extension of the Lévy-Kinchin formula and cohomology of a solvable group with values in a Hilbert Space—

By

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Abstract

A notion of factorizable representation is defined and all factorizable representations of a commutative group of functions as well as those of the current commutation relations and canonical commutation relations are explicitly given in the continuous tensor product space (i.e. the Fock space).

The formula for a state functional of a factorizable representation is a non-commutative extension of the Levy-Kinchin formula in probability theory.

In the course of analysis, the most general form of a first order cocycle of any solvable group with values in a Hilbert space is determined. Non trivial cohomologies appear by two entirely different mechanism, namely a topological one on infinite dimensional space and an algebraic one on a finite dimensional space.

The immaginary part of an inner product of such cocycle is a second order cocycle. The condition that it is a coboundary is discussed.

§1. Introduction

A factorizable representation has been discussed for commutative groups of functions and canonical commutation relations [1], [2]. Here we extend this notion to the current algebra, which is of some interest in elementary particle physics. Given a Lie group G, we consider, so to speak, a continuous direct product of copies of G, each associated with a point on a measure space X. In the commutative case, G is the additive group of reals. In the canonical commutation relations, we take G as the group corresponding to the Lie algebra of one pair of canonical variables p, q and an identity operator (times i). If a unitary representation of the big group is

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associated with a state, which splits up into product of states of the small group at every points, then the representation can be realized in the continuous tensor product space defined in [3]. We shall attempt a determination of the structure of such a representation.

It is shown that a representation is determined by specifying at each point x of X, a unitary representation $Q_x(g)$ of G, a first order cocycle $\phi_x(g)$ with values in the representation space of Q_x, and a real valued function $c_x(g)$ whose coboundary should coincide with $\mathrm{Im}\,(\phi(g_1), \phi(g_2^{-1}))$. The representing operator can be expressed in terms of Bose creation and annihilation operators as an exponential of $a^*(x)a(x)$, $a^*(x)$ and $a(x)$ terms.

All cocycles ϕ for given unitary representation are obtained for any solvable group G. It is in general a sum of coboundary of a vector in a space $\bar{D}^+$ which is somewhat larger than the representation space, and certain vectors in the adjoint representation space of G (on its Lie algebra). The problem for a general G can be reduced to the case of a semisimple G. The cocycles for a semisimple G are not analyzed.

The problem of finding c for a given Q and ϕ is partially solved.

The general theory is applied to special examples, including the abelian case, the canonical commutation relations and current algebra over the rotation group.

§2. Current Algebra

Let X be a measure space and G be a connected Lie group with a finite dimensional Lie algebra $\mathfrak{g}$.

Definition 2.1. *The current group* $\mathrm{C}(G, X)$ *of* G *over* X *is the group of all bounded* G*-valued measurable functions equipped with the pointwise group operations. The bounded function means that its range is contained in a compact subset of* G. *The measurability refers to the* σ*-fields* B_X *of measurable sets in* X *and the* σ*-field* B_G *generated by compact sets of* G.

Definition 2.2. *The current algebra* $\mathrm{C}(\mathfrak{g}, X)$ *of* $\mathfrak{g}$ *over* X *is the Lie algebra of all bounded measurable* $\mathfrak{g}$*-valued functions equipped with the pointwise Lie algebra operations.*

For each element $\boldsymbol{l}$ of $\mathrm{C}(\mathfrak{g}, X)$ the exponential $e^{\boldsymbol{l}}$ of $\boldsymbol{l}$ is the element of $\mathrm{C}(G, X)$ defined pointwise by $[e^{\boldsymbol{l}}](x)=e^{\boldsymbol{l}(x)}$, where the last exponential is the conventional exponential mapping from Lie algebra to Lie group.

Definition 2.3. *A unitary representation* $\mathrm{g}\in\mathrm{C}(G, X)\to\mathrm{U}(\mathrm{g})$ (*a unitary operator on a Hilbert space* $\mathfrak{H}$) *is called continuous if* $\mathrm{U}(e^{t\boldsymbol{l}})$ *is strongly continuous in the real parameter t for each fixed* $\boldsymbol{l}$ *in* $\mathrm{C}(\mathfrak{g}, X)$.

Lemma 2.4. *Let* $\chi(X_0)$ *be a characteristic function of a fixed measurable subset* X_0 *of* X *and let* $g^{\chi(X_0)}$ *be the function belonging to* $\mathrm{C}(G, X)$ *which is equal to g on* X_0 *and to* 1 *on* $X-X_0$. *Then* $g\in G\to\mathrm{U}(g^{\chi(X_0)})$ *is a representation of the group G, which is continuous in the ordinary sense if* U *is continuous.*

Proof. The group representation property follows from $(g_1g_2)^{\chi(X_0)}=g_1^{\chi(X_0)}g_2^{\chi(X_0)}$. As for the continuity, we use the fact that (a) because of the representation property $\mathrm{U}(\mathrm{g})=\mathrm{U}(\mathrm{g}\mathrm{g}_0^{-1})\mathrm{U}(\mathrm{g}_0)$, it suffices to prove the continuity at $g=1$, and (b) if $\mathfrak{l}_1\cdots\mathfrak{l}_n$ is a basis of $\mathfrak{g}$, $(t_1\cdots t_n)\to e^{t_1\mathfrak{l}_1}\cdots e^{t_n\mathfrak{l}_n}$ gives a homeomorphism of a neighbourhood of 0 in R^n and a neighbourhood of 1 in G. By assumption each $\mathrm{U}(e^{t_j\boldsymbol{l}_j})$ is continuous in t and uniformly bounded, where $\boldsymbol{l}_j(x)$ is equal to $\mathfrak{l}_j$ on X_0 and to 1 on $X\cdots X_0$. By using the estimate of the following general form

$$\begin{aligned}&||A(t_1)A(t_2)\Phi-A(t_1^0)A(t_2^0)\Phi||\\&\quad\leqq||A(t_1)||\,||A(t_2)\Phi-A(t_2^0)\Phi||+||[A(t_1)-A(t_1^0)]A(t_2^0)\Phi||,\end{aligned}$$

we have the strong continuity of $\mathrm{U}(e^{t_1\boldsymbol{l}_1})\cdots\mathrm{U}(e^{t_n\boldsymbol{l}_n})$ in $(t_1\cdots t_n)$.

Definition 2.5. *A functional* E *over* $\mathrm{C}(G, X)$ *is a mapping from* $\mathrm{C}(G, X)$ *to complex numbers. A triplet of a Hilbert space* $\mathfrak{H}$, *a unitary representation* U *of* $\mathrm{C}(G, X)$ *on* $\mathfrak{H}$ *and a cyclic vector* Ψ *in* $\mathfrak{H}$ (*namely a vector* Ψ *such that vectors* $\mathrm{U}(\mathrm{g})\Psi$, $g\in\mathrm{C}(G, X)$ *span* $\mathfrak{H}$) *is said to be canonically associated with* E *if*

$$\mathrm{E}(\mathrm{g}) = (\Psi, \mathrm{U}(\mathrm{g})\Psi), \qquad \mathrm{g}\in\mathrm{C}(G, X) \tag{2.1}$$

Theorem 2.6. *A triplet* $(\mathfrak{H}, \mathrm{U}, \Psi)$ *canonically associated with a functional* E *exists if and only if* E *satisfies the following positivity* *condition*: $E(g_i^{-1}g_j)$, $i=1, \cdots, n$, $j=1, \cdots, n$, *is a nonnegative matrix for any choice of* g_i, $i=1, \cdots, n$ *from* $C(G, X)$. *For given* E, *the triplet is unique up to a unitary equivalence. The unitary representation* U *is continuous if and only if* $E(g_1 e^{tl} g_2)$ *is continuous in the real parameter* t *for every fixed* $g_1, g_2 \in C(G, X)$ *and* $l \in C(\mathfrak{g}, X)$.

The necessity of the conditions for E is obvious. The existence and uniqueness of the triplet is easily proved by a standard method.

Since $U(e^{tl})$, being unitary, is uniformly bounded, the continuity of its matrix elements between a total set $U(g)\Psi$ implies the weak continuity of $U(e^{tl})$ itself and hence its strong continuity.

Definition 2.7. *A functional* E *is called an expectation functional if* $E(1)=1$, *the positivity condition is satisfied and* $E(g_1 e^{tl} g_2)$ *is continuous in* t.

An expectation functional satisfies

$$E(g)^* = E(g^{-1}) \tag{2.2}$$

$$|E(g)| \leqq 1 \tag{2.3}$$

which immediately follows from the positivity condition with $n=2$, $g_1=1$ *and* $g_2=g$.

The definitions introduced in this section are related to customary notations in the following way.

If G is taken to be the additive group of real numbers, then $C(G, X)$ is a vector space of functions. The functional E for such $C(G, X)$ has been extensively treated in the literature.

If G is the Heisenberg group, namely if $\mathfrak{g}$ is spanned by three elements $\hat{\mathfrak{q}}$, $\hat{\mathfrak{p}}$, $\mathfrak{i}$ satisfying $[\hat{\mathfrak{p}}, \hat{\mathfrak{q}}]=\mathfrak{i}$, $[\hat{\mathfrak{q}}, \mathfrak{i}]=[\hat{\mathfrak{p}}, \mathfrak{i}]=0$ and if we require $U(e^{\mathfrak{i}f})$, for any real valued bounded measurable function f, to be an identity operator times a number $\exp i\int_X f(x)\,dx$, where dx is a measure on X, then we are dealing with a representation of canonical commutation relations. The last condition may be expressed in terms of the functional E by

$$E(g e^{\mathfrak{i}f}) = E(g)\ \exp i\int f(x)\,dx\,. \tag{2.4}$$

In this case, the customary notation is

$$U(e^{\hat{\mathfrak{q}}f}) = U(f) = e^{i\phi(f)} = e^{i\int \phi(x) f(x) dx} \tag{2.5}$$

$$U(e^{\hat{\mathfrak{p}}f}) = V(f) = e^{i\pi(f)} = e^{i\int\pi(x)f(x)\mathrm{d}x}\,. \tag{2.6}$$

If $\mathfrak{g}$ is a Lie algebra of antihermitian matrices $(\mathfrak{g}_{ij})$, then a standard notation is

$$U(e^{g}) = \exp\sum_{kl}\psi^+(x)_k\psi(x)_l\,\boldsymbol{g}(x)_{kl}\,\mathrm{d}x\,. \tag{2.7}$$

For $\gamma\in\mathfrak{g}$, the expression $\psi^+(x)\gamma\psi(x)$ is the operator valued distribution defined through a generator of one parameter subgroup:

$$\int(\psi^+(x)\gamma\psi(x))f(x)\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}U(e^{t\gamma f})|_{t=0}\,. \tag{2.8}$$

If γ is antihermitian, i times this operator is selfadjoint. The current commutation relation is

$$[\psi^+(x)\gamma_1\psi(x),\ \psi^+(y)\gamma_2\psi(y)] = \delta(x-y)\psi^+(x)[\gamma^1,\ \gamma^2]\psi(x) \tag{2.9}$$

which is to be understood as

$$\left[\int\psi^+(x)\gamma_1\psi(x)f_1(x)\,\mathrm{d}x,\ \int\psi^+(y)\gamma_2\psi(y)f_2(y)\,\mathrm{d}y\right] = \int\psi^+(x)[\gamma_1,\ \gamma_2]\psi(x)f_1(x)f_2(x)\,\mathrm{d}x \tag{2.10}$$

and holds on a dense set of vectors depending on $\gamma_1 f_1$ and $\gamma_2 f_2$.

The use of ψ^+ and ψ comes from the fact that (2.9) is formally implied by

$$[\psi^+(x),\ \psi(y)]_\pm = \delta(x-y) \tag{2.11}$$

$$[\psi^+(x),\ \psi^+(y)]_\pm = [\psi(x),\ \psi(y)]_\pm = 0 \tag{2.12}$$

where either $+$ or $-$ commutation relations are assumed, $[A,\ B]_+ = AB+BA$, $[A,\ B]_- = AB-BA$, and they lead to the same equation (2.9).

Sometimes, one is interested in a locally compact space X and the group of G-valued measurable bounded functions on X with a bounded support. The following analysis can be modified to accomodate such cases.

§3. Factorizable Functional and Type I Factorization

Definition 3.1. *An expectation functional* E *of* $C(G, X)$ *is finitely factorizable if*

(3.1) $$E(g_1 g_2) = E(g_1)E(g_2)$$

whenever the support of g_1 *is disjoint from the support of* g_2. *Here the support of a function* $g(x)$ *is the set of* $x \in X$ *for which* $g(x) \neq 1$.

If $\{X_i;\ i=1, \cdots, n\}$ is a partition of the set X into a finite number of subsets in B_X, then the subgroup of functions $g(x)$ with supports in X_i may be identified with $C(G, X_i)$. $C(G, X)$ is then a direct product of (mutually commuting groups) $C(G, X_i)$, $i=1, \cdots, n$.

If we denote the restriction of the functional E to $C(G, X_i)$ by E_i, it automatically satisfies a condition for an expectation functional of $C(G, X_i)$. For $Y \in B_X$ and $g \in C(G, X)$, let $g^{\chi(Y)}$ denote the element of $C(G, X)$ which is equal to g on Y and to 1 on $X-Y$. For a finitely factorizable functional E and for a finite partition $\{X_i\}$ of X, we have

(3.2) $$E(g) = \prod_{i=1}^{n} E_i(g^{\chi(X_i)}) .$$

Theorem 3.2. *Let* $(\mathfrak{H}, U, \Psi)$ *and* $(\mathfrak{H}_i, U_i, \Psi_i)$ *be triplets canonically associated with expectation functionals* E *and* E_i, *respectively. Let*

(3.3) $$\hat{\mathfrak{H}} = \bigotimes_{i=1}^{n} \mathfrak{H}_i , \quad \hat{\Psi} = \bigotimes_{i=1}^{n} \Psi_i$$

(3.4) $$\hat{U}(\prod_{i=1}^{n} g_i) = \bigotimes_{i=1}^{n} U_i(g_i) , \quad g_i \in C(G, X_i)$$

where $\otimes$ *denotes the tensor product. If* E *is finitely factorizable, then* $(\hat{\mathfrak{H}}, \hat{U}, \hat{\Psi})$ *is unitarily equivalent to* $(\mathfrak{H}, U, \Psi)$.

This follows from (3.2) and Theorem 2.6.

For any subset X_1 of X in B_X, we consider partition of X into X_1 and its complement $X_1^c \equiv X_2$. By the unitary equivalence of $\mathfrak{H} = \mathfrak{H}_1 \otimes \mathfrak{H}_2$ with $\mathfrak{H}$, $\mathscr{B}(\mathfrak{H}_1) \otimes 1$ defines a type I factor on $\mathfrak{H}$, which we shall write $R(X_1)$. Here $\mathscr{B}(\mathfrak{H}_1)$ denotes the set of all bounded operators of $\mathfrak{H}_1$.

Theorem 3.3. *Let* E *be finitely factorizable. For any finite partition* $\{X_i,\ i=1, \cdots, n\}$, $X_i \in B_X$ *of a given* $X_0 \in B_X$, $\{R(X_i),\ i=1, \cdots, n\}$ *is a type I factorization of* $R(X_0)$ *for which* Ψ *is a product vector. The operator* $U(g)$ *belongs to* $R(X_0)$ *if the support of* g *is in* X_0 *and is factorizable operator with respect to this factorization*: $U(g) = \Pi\, U(g^{\chi(X_i)})$, $U(g^{\chi(X_i)}) \in R(X_i)$.

This follows from Theorem 3.2 if we consider the partition $\{X_0^c, X_1, \cdots, X_n\}$ as a common subpartition of $\{X_0, X_0^c\}$ and $\{X_i, X_i^c\}$, $i=1, \cdots, n$.

Definition 3.4. *An expectation functional* E *of* $C(G, X)$ *is said to be σ-factorizable if the following condition is satisfied*:

If $\{X_i\}$ *is a countable partition of* X *into measurable sets, then*

$$\mathrm{E}(g) = \prod_i \mathrm{E}(g_i) \tag{3.5}$$

where the product converges irrespective of the ordering (*the absolute convergence*).

Definition 3.5. *A subset* X_0 *of* X *is called* E-*null set if* $X_0 \in B_X$ *and* $\mathrm{E}(g)=1$ *for an arbitrary* $g \in C(G, X_0)$. *The set of all* E-*null sets is denoted by* N_X^{E}.

Lemma 3.6. *Let* E *be a σ-factorizable functional. If* $X_1 \in N_X^{\mathrm{E}}$, $X_2 \in B_X$ *and* $X_2 \subset X_1$, *then* $X_2 \in N_X^{\mathrm{E}}$. *If* $X_i \in N_X^{\mathrm{E}}$, *then* $\bigcup_i X_i \in N_X^{\mathrm{E}}$ *where the index set is countable.*

Proof. The first statement follows from $C(G, X_2) \subset C(G, X_1)$. For the second statement, let $X_0 = \bigcup_i X_i$ and $Y_i = X_i - (\bigcup_{j<i} Y_j)$. Since $Y_i \subset X_i$, $Y_i \in N_X^{\mathrm{E}}$. Since X_0 is the union of disjoint Y_i, Definition 3.4 implies $\mathrm{E}(g) = \prod_i \mathrm{E}(g_i) = 1$ for all $g \in C(G, X_0)$. Hence $X_0 \in N_X^{\mathrm{E}}$.

Definition 3.7. *The carrier Boolean algebra* B_X^{E} *of a factorizable* E *is the Boolean algebra obtained from the σ field* B_X *by identifying any two sets which differ by an addition and a subtruction of* E-*null sets.*

Definition 3.8. E *is called separable if, for any measurable partition* $\{X_\alpha\}$ *of* X, X_α *belongs to* N_X^{E} *except for a countable number of* α.

Lemma 3.9. *If* E *is σ-factorizable and separable, then* B_X^{E} *is a complete Boolean algebra satisfying a countable chain condition.*

Proof. Since B_X is a σ-field (i.e. Boolean σ-algebra), B_X^{E} is also a Boolean σ-algebra because of Lemma 3.6. Next, if $X_\mu \in B_X$ is given for every ordinals μ between μ_0 and 1 such that $X_\mu \supset X_\nu$ and $X_\mu - X_\nu \notin N_X^{\mathrm{E}}$ for $\mu > \nu$ ($A - B \equiv A \cap B^c$), then μ_0 must be countable.

This follows from separability because $Y_\mu \equiv X_{\mu+1} - X_\mu \notin N_X^{\mathrm{E}}$ is mutually disjoint. Thus B_X^{E} satisfies the countable chain condition. Finally, let $X_\mu \in B_X$ be given. We show the existence of their least upper bound in B_X^{E}. We define an ordinal μ_0, the pair of the set Y_μ and an index $\alpha(\mu)$ for every ordinal $\mu < \mu_0$ by the transfinite induction in such a way that $Y_\mu \in B_X$, $Y_\mu \supset Y_\nu$ if $\mu > \nu$, $Y_\mu - Y_\nu \notin N_X^{\mathrm{E}}$ if $\mu > \nu$, $Y_\mu = \bigcup_{\nu<\mu} X_{\alpha(\nu)}$, and $X_\alpha - \bigcup_{\mu<\mu_0} Y_\mu \in N_X^{\mathrm{E}}$ for any α.

If this is achieved, μ_0 must be countable and $Y_{\mu_0} \equiv \bigcup_{\mu<\mu_0} Y_\mu$ is the lowest upper bound of $\{X_\alpha\}$ in B_X^{E}. Let Y_ν be defined for $\nu < \mu$. (If $\mu = 1$, this assumption holds trivially.) Then μ must be countable. Hence $\bigcup_{\nu<\mu} Y_\nu \equiv Z_\mu \in B_X$. We look for X_α such that $X_\alpha - Z_\mu \notin N_X^{\mathrm{E}}$. If no such α exists, we set $\mu = \mu_0$. Otherwise choose one such index $\alpha(\mu)$ and define $Y_\mu = X_{\alpha(\mu)} \cup Z_\mu$. Then $Y_\mu \in B_X$, $Y_\mu \supset Y_\nu$ for $\mu > \nu$. Further, if $\mu > \nu$, $Y_\mu - Y_\nu \supset Y_\mu - Z_\mu = X_{\alpha(\mu)} - Z_\mu \notin N_X^{\mathrm{E}}$ and hence $Y_\mu - Y_\nu \notin N_X^{\mathrm{E}}$. Then strictly increasing transfinite sequence $\{\alpha(\nu);\ \nu < \mu\}$, $\mu = 1, 2, \cdots$ must terminate by the axiom of well ordering and we have the desired result.

Definition 3.10. *The discrete spectrum* S_X^{dE} *of* E *in* X *is the set of points* $x \in X$ *such that* $\mathrm{E}(g) \neq 1$ *for some* $g \in \mathrm{C}(G, \{x\})$.

S_X^{dE} is countable for a separable E.

Definition 3.11. *Let* $\hat{N}_X^{\mathrm{E}}$ *be the family of subsets of* X *obtained as a union of an* E*-null set and a subset (including the empty set) of* S_X^{dE}. *The carrier Boolean algebra* B_X^{cE} *of the continuous part of* E *is the* σ*-field obtained from the* σ*-field* B_X *by identifying any two sets which differ by an addition and a subtraction of sets in* $\hat{N}_X^{\mathrm{E}}$. *(Another equivalent way is to consider the lattice of* Y-S_X^{dE}, $Y \in B_X$ *modulo null sets.)*

Lemma 3.12. B_X^{cE} *is a complete, continuous Boolean algebra with countable chain condition.* B_X^{E} *is a direct product of* B_X^{cE} *and the countable atomic complete Boolean algebra* B_X^{dE} *generated by points of* S_X^{dE}.

Definition 3.13. *A representation* π *of a complete Boolean algebra* B *by factors is a mapping from* $Y \in B$ *to a factor* $\pi(Y)$ *such that* $\pi(Y^c) = \pi(Y)'$ *and* $\pi(\cup Y_i) = (\cup \pi(Y_i))''$.

Lemma 3.14. *Let* E *be a* σ*-factorizable separable functional,* $(\mathfrak{H}, \mathrm{U}, \Psi)$ *be as in Theorem* 3.2 *and* $\mathrm{R}(Y)$ *be defined as stated after Theorem* 3.2. *Then* (1) $\mathrm{R}(X_1)=\mathrm{R}(X_2)$ *if* $X_1 \equiv X_2$ *in* B_X^{E}. *For* $Y \in B_X^{\mathrm{E}}$, *define* $\mathrm{R}(Y)=\mathrm{R}(X_1)$ *where* X_1 *is any set in the equivalence class* Y. (2) $Y \in B_X^{\mathrm{E}} \to \mathrm{R}(Y)$ *is a representation of a complete Boolean algebra by type I factors.*

Proof. If $Y \in N_X^{\mathrm{E}}$, then the corresponding $\mathfrak{H}_Y$ is one dimensional and (1) follows from Theorem 3.2. As long as the finite Boolean operations are concerned, $Y \in B_X^{\mathrm{E}} \to \mathrm{R}(Y)$ is a representation of the Boolean algebra B_X^{E} by type I factors, due to Theorem 3.2. Hence the only point to be proved is whether $(\bigcup_\alpha \mathrm{R}(X_\alpha))'' = \mathrm{R}(\bigcup_\alpha X_\alpha)$ for arbitrary X_α. Because of the countable chain condition, it is enough to see this for countable number of mutually disjoint X_α.

Let $X_0 = (\bigcup_\alpha X_\alpha)^c$ and adjoin 0 into the index set. Consider $(\mathfrak{H}_\mu, \mathrm{U}_\mu, \Psi_\mu)$ corresponding to the restriction of E to $C(G, X_\mu)$ and construct the incomplete infinite direct product $\mathfrak{H} \equiv \otimes \mathfrak{H}_\mu$ containing $\Psi \equiv \otimes \Psi_\mu$. Because of Definition (3.4) (b), $\Pi(\Psi_\mu, \mathrm{U}_\mu(g_\mu)\Psi_\mu)$ is absolutely convergent, and hence $\otimes \mathrm{U}_\mu(g_\mu)\Psi_\mu \in \hat{\mathfrak{H}}$. Hence $\otimes \mathrm{U}_\mu(g_\mu) \equiv \hat{\mathrm{U}}(g)$ also exists as a unitary operator on $\hat{\mathfrak{H}}$. Moreover $(\hat{\Psi}, \hat{\mathrm{U}}(g)\hat{\Psi}) = \mathrm{E}(g)$. Hence $(\hat{\mathfrak{H}}, \hat{\mathrm{U}}, \hat{\Psi})$ is unitarily equivalent to $(\mathfrak{H}, \mathrm{U}, \Psi)$ of Theorem 3.2. Furthermore, clearly $\hat{\mathrm{R}}_\mu = \mathscr{B}(\mathfrak{H}_\mu) \otimes (\otimes_{\nu \neq \mu} 1_\nu)$ and $\hat{\mathfrak{H}} = \mathfrak{H}_\mu \otimes (\otimes_{\nu \neq \mu} \mathfrak{H}_\nu)$ correspond to $\mathrm{R}(X_\mu)$ and the related decomposition of the Hilbert space. Hence $\mathrm{R}(\bigcup_\alpha X_\alpha) = \mathrm{R}(X_0)' = (\bigcup_\alpha \mathrm{R}(X_\alpha))''$.

Definition 3.15. *The set of* $Y \in B_X^{\mathrm{cE}}$ *such that* $\mathrm{R}(Y)$ *is one dimensional is denoted by* B_X^{c1E}. *The corresponding support* S_X^{c1E} *is the maximal element of* B_X^{c1E} (*defined as a subset of* X *modulo* E*-null sets*).

Lemma 3.16. B_X^{c1E} *is a complete Boolean sublattice of* B_X^{cE}. *If* $Y \in B_X^{\mathrm{c1E}}$, $Y_1 \subset Y$, $Y_1 \in B_X^{\mathrm{cE}}$, *then* $Y_1 \in B_X^{\mathrm{c1E}}$.

Proof. The first part follows from Lemma 3.14 and the second part from Theorem 3.2.

Definition 3.17. *The carrier Boolean algebra of the continuous tensor product part of* E *is the* σ *field obtained from* B_X^{E} *by identifying any two sets which differ by a finite number of additions and* *subtractions of* E*-null sets, sets in* B_X^{dE} *and sets in* B_X^{c1E}. (*Another way is to consider the lattice of* $Y-S_X^{\mathrm{dE}}-S_X^{\mathrm{c1E}}$, $Y\in B_X$ *modulo* E*-null sets.*) *It is denoted by* B_X^{cTE}.

Lemma 3.18. *Let* E *be a* σ*-factorizable separable functional,* ($\mathfrak{H}$, U, Ψ) *be as in Theorem* 3.2 *and* R(Y) *be defined as stated after Theorem* 3.2. (1) B_X^{E} *is a direct product of* B_X^{dE}, B_X^{c1E} *and* B_X^{cTE}. (2) $\mathrm{g}\in \mathrm{C}(G, S_X^{\mathrm{c1E}})\to \mathrm{E}(\mathrm{g})$ *is a character of the group* $\mathrm{C}(G, S_X^{\mathrm{c1E}})$ *and the corresponding Hilbert space is one dimensional,* (3) $Y\in B_X^{\mathrm{cTE}}\to \mathrm{R}(Y)$ *is a faithful representation of the complete Boolean algebra* B_X^{cTE} *by type I factors.*

Proof. (1) follows from Definition 3.11, Lemma 3.12, Definition 3.15, Lemma 3.16 and Definition 3.17. (2) follows from Definition 3.15. (3) follows from Lemma 3.14 and Definition 3.15.

§4. Realization of the Unitary Representation in the Fock Space

We now analyze the structure of the unitary representation U(g) and obtain a concrete realization in terms of creation and annihilation operators in the Fock space. First we separate out the discrete part. If $x\in S_X^{\mathrm{dE}}$, then U(g), $\mathrm{g}\in\mathrm{C}(G, \{x\})$ is a cyclic continuous unitary representation of the group G and we have

$$\mathfrak{H} = \mathfrak{H}_{\mathrm{c}}\otimes\{\bigotimes_x \mathfrak{H}_x\} \tag{4.1}$$

$$\mathrm{U}(\mathrm{g}) = \mathrm{U}_{\mathrm{c}}(\mathrm{g})\otimes\{\bigotimes_x \mathrm{U}_x(\mathrm{g}(x))\} \tag{4.2}$$

where x in $\bigotimes_x$ runs over all points in S_X^{dE}. Thus the discrete part is reduced to the study of continuous unitary representation of the Lie group G, which we do not have to study any further. We shall now turn our attention to the continuous part $\mathfrak{H}_{\mathrm{c}}$ and U_{c}. To simplify the notation we omit the c, in other words, the discrete part is assumed to be absent in the following discussion.

Next we can separate out the character part. Any $\mathrm{g}\in\mathrm{C}(G, X)$ is decomposed as $\mathrm{g}=\mathrm{g}_1\mathrm{g}_2$, supp $\mathrm{g}_1\subset S_X^{\mathrm{1E}}$, supp $\mathrm{g}_2\subset X-S_X^{\mathrm{1E}}$. Then $\mathrm{U}(\mathrm{g})=\mathrm{U}(\mathrm{g}_1)\mathrm{U}(\mathrm{g}_2)$ where $\mathrm{U}(\mathrm{g}_1)$ is a character of the group $\mathrm{C}(G, S_X^{\mathrm{1E}})$. A typical example of a character of $\mathrm{C}(G, S_X^{\mathrm{1E}})$ is given by

$$\exp\int \log \chi_x(\mathrm{g}(x))\,\mathrm{d}\mu(x)$$

where χ_x is a character of G depending measurably on x and μ is a measure.

As long as the Hilbert space structure is concerned, $\mathrm{U}(\mathrm{g}_1)$ is a multiple of identity. In the present paper, we concentrate our attention to the continuous tensor product part and hereafter omit the superfix cT (which is equivalent to assuming that the S_X^{c1E} part as well as the discrete part are absent).

For continuous tensor product part, the analysis of the continuous complete Boolean algebra of type I factors [3] reveals the following structure. $\mathfrak{H}$ is an exponential of a Hilbert space $\mathfrak{L}$. There exists a faithful representation of the complete Boolean algebra B_X^{E} by projections on $\mathfrak{L}$: $Y \in B_X \to \mathrm{P}(Y)$, $\mathrm{P}(Y)=0$ if and only if $\mathrm{R}(Y)=1$, that is $Y \in N_X^{\mathrm{E}}$. The most general forms of a product vector and a bounded product operator on $\mathfrak{H}$ are known.

We start with the analysis of the projection valued measure $\mathrm{P}(Y)$ on $\mathfrak{L}$.

Lemma 4.1. *There exists a countable partition of X into mutually disjoint $X_n \in B_X$ and positive continuous measures $\mathrm{d}\mu_n$ such that*

$$\mathfrak{L} = \bigotimes_n \{L_2(X_n, \mathrm{d}\mu_n) \otimes \mathfrak{M}_n\} \tag{4.3}$$

$$\mathrm{P}(Y) = \bigotimes_n \{\mathrm{P}_n(X_n \cap Y) \otimes 1\} \tag{4.4}$$

where $\mathfrak{M}_n$ are Hilbert spaces of distinct dimensions and $\mathrm{P}_n(Z)$ is a multiplication operator of the characteristic function of Z.

Proof. First, we find a vector χ such that $(\chi, \mathrm{P}(Y)\chi)>0$ for every $Y \in B_X$, $\notin N_X^{\mathrm{E}}$. For this purpose, take arbitrary unit vector Ψ_1 and find the largest projection $\mathrm{P}(Y_1)$ such that $\mathrm{P}(Y_1)\Psi_1=0$. Next pick up a unit vector Ψ_2 from $\mathrm{P}(Y_1)\mathfrak{L}$ and find largest projection $\mathrm{P}(Y_2)$ such that $\mathrm{P}(Y_2)\Psi_1=\mathrm{P}(Y_2)\Psi_2=0$. Continuing in this way by a transfinite induction, we have a strictly increasing sequence of projections $1-\mathrm{P}(Y_\alpha)$ which exhausts $\mathfrak{L}$: $\sum_{\alpha<\alpha_0} \mathrm{P}(Y_\alpha)(1-\mathrm{P}(Y_{\alpha+1}))=1$. $(Y_0 \equiv X.)$ By the countable chain condition, α_0 is a countable ordinal. By construction, $\mathrm{P}(Y)\Psi_\alpha=0$ for all $\alpha<\alpha_0$ implies $\mathrm{P}(Y)=0$. Let $\alpha \to n(\alpha)$ be a one to one mapping of the ordinals $\alpha<\alpha_0$ to natural numbers. Then $\chi=\sum n(\alpha)^{-1}\Psi_\alpha$ has the desired property.

To find a decomposition of the required type, we construct

$(\chi_\alpha, Z_\alpha, \mathfrak{L}_\alpha)$ for every ordinals $\alpha<\alpha_0$ by a transfinite induction in such a way that (1) $\mathfrak{L}=\oplus\mathfrak{L}_\alpha$, (2) $\{P(Y)\chi_\alpha; Y\in B_X\}$ generates $\mathfrak{L}_\alpha$, (3) $P(Z_\alpha)$ is the largest projection of the form $P(Y)$, vanishing identically on $\{\bigoplus_{\beta<\alpha}\mathfrak{L}_\beta\}^\perp$, (4) $(\chi_\alpha, P(Y)\chi_\alpha)>0$ if $Y-Z_\alpha\notin N_X^E$. Note that $\mathfrak{L}_\alpha\perp\mathfrak{L}_\beta$ for $\alpha\neq\beta$ and (3) imply $P(Z_\beta)\leq P(Z_\alpha)$ for $\beta<\alpha$ and $P(Z_\alpha)\mathfrak{L}_\alpha=0$. For the construction, assume $(\chi_\beta, Z_\beta, \mathfrak{L}_\beta)$ be given for $\beta<\alpha$. Define $P(Z_\alpha)$ by (3). Use the previous construction of χ to define χ_α as a unit vector in $\{\bigoplus_{\beta<\alpha}\mathfrak{L}_\beta\}^\perp$ satisfying (4). Define $\mathfrak{L}_\alpha$ by (2). This procedure is possible unless $\{\bigoplus_{\beta<\alpha}\mathfrak{L}_\beta\}^\perp=0$. If the last equation holds, we set $\alpha=\alpha_0$ and we have (1).

To reach the form of (4.3) and (4.4), define $P(Y_\alpha)\equiv P(Z_\alpha)-\bigvee_{\beta<\alpha} P(Z_\beta)$. By the complete Boolean property of B_X^E, $Y_\alpha\in B_X$ exists and by the countable chain condition, $P(Y_\alpha)\neq 0$ only for a countable number of $\alpha=\alpha_n$, $n=1,2,\cdots$. Set $X_n=X_{\alpha_n}$. We have $\sum P(X_n)=1$, $P(X_n)P(X_m)=0$ for $n\neq m$, and $\mathfrak{L}=\oplus\hat{\mathfrak{L}}_n$, $\hat{\mathfrak{L}}_n=P(X_n)\mathfrak{L}$. Because of the property (4), $P(Y)\chi_\beta^n\neq 0$ for $Y\subset X_n$, $Y\notin N_X^E$ where we define $\chi_\beta^n\equiv P(X_n)\chi_\beta$ for $\beta<\alpha_n$. We have $\mathfrak{L}_n=\oplus\mathfrak{L}_\beta^n$ where $\mathfrak{L}_\beta^n$ is generated by $P(Y)\chi_\beta^n$. Since two measures $(\chi_1^n, P(Y)\chi_1^n)$ and $(\chi_\beta^n, P(Y)\chi_\beta^n)$ are equivalent, we have derivatives of one with respect to the other: $A(x)=(\chi_1^n, P(dx)\chi_1^n)/(\chi_\beta^n, P(dx)\chi_\beta^n)$. Then $\varphi_\beta^n\equiv\int A(x)^{1/2}P(dx)\chi_\beta^n$ satisfies $(\varphi_\beta^n, P(Y)\varphi_\beta^n)=(\varphi_1^n, P(Y)\varphi_1^n)$. Since $\chi_\beta^n=\int A(x)^{-1/2}P(dx)\varphi_\beta^n$, $\mathfrak{L}_\beta^n$ is generated by $P(Y)\varphi_\beta^n$ and $P(Y)$ on all $\mathfrak{L}_\beta^n$ with n fixed are unitarily equivalent. Writing $\mu_n(Y)=(\chi_1^n, P(Y)\chi_1^n)$ and $\mathfrak{L}_\beta^n=\mathfrak{L}_2(X_n, d\mu_n)\otimes e_\beta$, $\mathfrak{M}_n=\bigoplus_{\beta<\alpha_n}\{ce_\beta\}$, $\varphi_\beta^n=1\otimes e_\beta(\|e_\beta\|=1)$, we have (4.3) and (4.4).

If dimensions of some $\mathfrak{M}$'s coincide, we lump together those X_n for which $\mathfrak{M}_n$ has the same dimension and obtain (4.3) and (4.4) where $\mathfrak{M}$'s have distinct dimensions. Q.E.D.

Next we analyze a product unitary operator.

Lemma 4.2. *A most general unitary product operator in a continuous tensor product is given by*

$$W(c, \phi, Q) = e^{-\frac{1}{2}(\phi,\phi)+ic}T(-Q\phi, Q, \phi) \tag{4.5}$$

where Q is a unitary operator on $\mathfrak{L}$, commuting with all $P(Y)$, $Y\in B_X$, *ϕ is a vector in $\mathfrak{L}$, c is a real number and* $T(\psi, Q, \phi)$ *is defined in Theorem* 5.3 *of ref.* [3]. *It satisfies*

$$(4.6)\qquad W(c_1, \phi_1, Q_1)W(c_2, \phi_2, Q_2) = W(c_1+c_2-\mathrm{Im}\,(\phi_1, Q_2\phi_2),\ Q_2^*\phi_1+\phi_2,\ Q_1Q_2)$$

$$(4.7)\qquad W(c, \phi, Q)\exp\Psi = e^{-\frac{1}{2}(\phi,\phi)+ic+(\phi,\Psi)}\exp Q(\Psi-\phi)$$

$$(4.8)\qquad (\Omega, W(c, \phi, Q)\Omega) = e^{-\frac{1}{2}(\phi,\phi)+ic}\,.$$

Proof. The general form of bounded product operator is given by $k\mathrm{T}(\psi, Q, \phi)$ where k is a number, Q is a bounded operator on $\mathfrak{L}$ with $||Q||\leq 1$, commuting with all $\mathrm{P}(Y)$, ψ and ϕ are vectors in $\mathfrak{L}$ and $\phi+Q^*\psi$ is in the domain of $(1-Q^*Q)^{-1/2}$. The last condition is equivalent to the requirement that $\mathrm{T}(\psi, Q, \phi)^*\mathrm{T}(\psi, Q, \phi)$ be a bounded operator and is automatically satisfied if the unitarity requirement is satisfied. The latter is given by

$$1=|k|^2\mathrm{T}(\psi, Q, \phi)^*\mathrm{T}(\psi, Q, \phi)=|k|^2\mathrm{T}(\phi+Q^*\psi, Q^*Q, \phi+Q^*\psi)e^{(\psi,\psi)}$$

$$1=|k|^2\mathrm{T}(\psi, Q, \phi)\mathrm{T}(\psi, Q, \phi)^*=|k|^2\mathrm{T}(\psi+Q\phi, QQ^*, \psi+Q\phi)e^{(\phi,\phi)}\,.$$

Since $k'\mathrm{T}(\psi', Q', \phi')=1$ implies $k'=1$, $\psi'=\phi'=0$, $Q'=1$, we obtain the following necessary and sufficient condition for the unitarity:

$$QQ^*=Q^*Q=1\,,\quad \psi=-Q\phi\,,\quad |k|^2=e^{-(\phi,\phi)}$$

The rest of the lemma follows from ref. [3].
(In terms of annihilation and creation operators, $\mathrm{T}(\psi, Q, \phi)$ can be written formally as

$$\mathrm{T}(\psi, Q, \phi)=e^{(a^*\psi)}e^{(a^*\log Q\,a)}e^{(\phi,a)}.)$$

Lemma 4.3. *If* $\lim_{n\to\infty}W(c_n, \phi_n, Q_n)=W(c, \phi, Q)$ *holds in the weak operator topology, then* $\lim\exp i(c_n-c)=1$, $\lim||\phi_n-\phi||=0$, *and* $\lim Q_n=Q$ *in the strong operator topology.*

Proof. From (4.7), we have

$$(4.9)\qquad (e^{\Phi}, W(c, \phi, Q)e^{\Psi})=e^{-\frac{1}{2}(\phi,\phi)+ic+(\phi,\Psi)-(\Phi,Q\phi)+(\Phi,Q\Psi)}\,.$$

By setting $\Psi=\Phi=0$ and separating the absolute value and phase, we have

$$(4.10)\qquad \lim||\phi_n||^2=||\phi||^2$$

$$(4.11)\qquad \lim\exp i(c_n-c)=1\,.$$

By setting $\Phi=0$, taking sufficiently small Ψ so that $|(\phi_n, \Psi)|<\pi$,

$|(\phi, \Psi)| < \pi$ and using (4.10) and (4.11), we obtain

$$\lim (\phi_n, \Psi) = (\phi, \Psi). \tag{4.12}$$

Together with (4.10), this implies $\lim \|\phi_n - \phi\| = 0$. By setting $\Psi = 0$ and using small Φ, we have

$$\lim (\Phi, Q_n\phi_n) = (\Phi, Q\phi). \tag{4.13}$$

Finally, by using small Φ and Ψ, and substituting previous results, we have

$$\lim (\Phi, Q_n\Psi) = (\Phi, Q\Psi). \tag{4.14}$$

Because Q_n and Q are unitary, this implies $\lim Q_n = Q$ in the strong sense. Q.E.D.

We now concentrate our attention to the subgroup $C_p(G, X)$ of $C(G, X)$ consisting of elements $g^{\chi(Y)}$ where $g \in G$ which is considered as a constant element of $C(G, X)$ in the notation $g^{\chi(Y)}$ and $\chi(Y)$ is a characteristic function of $Y \in B_X^E$. For each fixed Y, we have a continuous unitary representation of G

$$g \in G \rightarrow U(g^{\chi(Y)}) = W(c(g, Y), \phi(g, Y), Q(g, Y)) \tag{4.15}$$

where the continuity follows from Lemma 2.4. The relations among quantities with varying Y are given by

$$Q(g, Y) = P(Y)Q(g, X) + 1 - P(Y), \tag{4.16}$$

$$\phi(g, Y) = P(Y)\phi(g, X). \tag{4.17}$$

From

$$E(g^{\chi(Y)}) = \exp\{-\frac{1}{2}(\phi(g, X), P(Y)\phi(g, X)) + ic(g, Y)\} \tag{4.18}$$

we have

$$e^{ic(g, Y)} = \prod_n e^{ic(g, Y_n)} \tag{4.19}$$

where $\{Y_n\}$ is a countable partition of Y and the product must converge irrespective of the ordering.

Theorem 4.4. (4.15) *gives a continuous unitary representation of G for a fixed Y, if and only if* (1) *$Q(g, Y)$ is a continuous unitary representation of G,* (2) *$g \rightarrow \phi(g, Y)$ is a continuous mapping satisfying*

$$(4.20) \qquad \phi(1, Y) = 0,$$

$$(4.21) \qquad Q(g_2, Y)^* \phi(g_1, Y) = \phi(g_1 g_2, Y) - \phi(g_2, Y)$$

and (3) $g \to e^{ic(g,Y)}$ *is a continuous mapping satisfying*

$$(4.22) \qquad \exp i\{c(g_1, Y) + c(g_2, Y) - c(g_1 g_2, Y) + \mathrm{Im}[\phi(g_1, Y), \phi(g_2^{-1}, Y)]\} = 1.$$

If $\mathfrak{H}$ is spanned by $\mathrm{U}(g^{\chi(Y)})\Omega$, then $\{\phi(g, Y)\}$ must span $\mathfrak{L}$. If $\{\phi(g, Y)\}$ spans $\mathfrak{L}$, then spaces $\mathfrak{M}_n$ in (4.3) are separable.

Proof. The first half is an immediate consequence of (4.6) and Lemma 4.3. ((4.20) follows from (4.21).) From (4.7) and (4.21), $\mathrm{U}(g^{\chi(Y)})\Omega$ is a multiple of

$$(4.23) \qquad \exp -Q(g, Y)\phi(g, Y) = \exp \phi(g^{-1}, Y)$$

which is in $\exp \mathfrak{L}_1$ where $\mathfrak{L}_1$ is the space spanned by all $\phi(g, Y)$. Therefore $\mathfrak{L}_1 = \mathfrak{L}$ is necessary in order that $\{\mathrm{U}(g^{\chi(Y)})\Omega\}$ spans $e^{\mathfrak{L}}$. Finally, to prove the separability of $\mathfrak{M}_n$, we note that G is separable and hence has a countable dense set $\{g_n\}$. By continuity, $\{\phi(g_n, Y)\}$ generates $\mathfrak{L}$. Let E_N be the projection on the subspace spanned by $\phi(g_n, Y)$, $n = 1, \cdots, N$, $Y \in B_X$. E_n commutes with $\mathrm{P}(Y)$ by construction. Let $\mathfrak{L}_n$ be the subspace generated by $(1 - E_{n-1})\phi(g_n, Y)$. Then $\mathfrak{L} = \oplus \mathfrak{L}_n$ and each $\mathfrak{L}_n$ is cyclic for $\{\mathrm{P}(Y)\}$. Let $\mathrm{P}(Y_n)$ be the largest $\mathrm{P}(Y)$ vanishing on $\mathfrak{L}_n$. (Y_n defined up to a set in N_X^{E}.) Define $\mathrm{P}_{mr} \equiv \sum \prod_{i=1}^{m} \mathrm{P}(Y_i^{(c)})$ where (c) indicates Y_i or Y_i^c and the summation runs over all possibilities such that the number of Y_i^c is exactly r. Define $\chi_r = \sum_{m \geq r} m^{-1} \mathrm{P}_{mr}(1 - E_{m-1})\phi(g_m, X)$. This has the property of χ_β in Lemma 4.1 except some of χ_r here can be 0. Since its total number is countable, $\dim \mathfrak{M}_n$ is finite or countably infinite. Q.E.D.

Definition 4.5. *Let $\mathfrak{H}$ be a Hilbert space, G be a group, and $Q(g)$, $g \in G$ be a continuous unitary representation of G on $\mathfrak{H}$. An $\mathfrak{H}$-valued continuous function $\phi(g)$ of $g \in G$ is called a cocycle of first order if it satisfies*

$$(4.24) \qquad \phi(g_2) - \phi(g_1 g_2) + Q(g_2)^* \phi(g_1) = 0$$

The set of first order cocycles is denoted by $Z^1(G, \mathfrak{H})$. If

$$\phi(g) = (Q(g)^* - 1)\Omega$$

for an $\Omega \in \mathfrak{H}$, *then* ϕ *is called a coboundary. They are denoted by* $B^1(G, \mathfrak{H})$.

We shall study $\phi(g, Y)$ in a separate section. The representating operator for a general element of $\mathrm{C}(G, X)$ can be obtained from that of $\mathrm{C}_p(G, X)$ if we introduce the following additional continuity assumption.

Definition 4.6. *The functional* $\mathrm{E}(\mathrm{g})$ *is called uniformly continuous if it is continuous with respect to the uniform topology on* $\mathrm{C}(G, X)$. *The uniform topology on* $\mathrm{C}(G, X)$ *is defined by neighbourhoods* $N_{\mathfrak{N}}(\mathrm{g}) = \{\mathrm{g}' ; \mathrm{g}'(x)\mathrm{g}(x)^{-1} \subset \mathfrak{N}$ *for all* $x\}$ *where* $\mathfrak{N}$ *is a neighbourhood of* 1 *in* G.

Theorem 4.7. *An expectation functional* $\mathrm{E}(\mathrm{g})$ *is uniformly continuous if and only if the representation* $\mathrm{U}(\mathrm{g})$ *canonically associated with* E *is strongly continuous with respect to the uniform topology of* $\mathrm{C}(G, X)$.

Proof. The if part is obvious. For the only if part, we note that given a compact K in G and an open neighbourhood $\mathfrak{N}$ of 1 in G, there exists a neighbourhood $\mathfrak{N}'$ of 1 such that $\{x\mathfrak{N}'x^{-1} ; x \in K\} \subset \mathfrak{N}$. (Take a compact neighbourhood $\mathfrak{N}_x$ of 1 for each $x \in K$ such that $x\mathfrak{N}_x x^{-1} \subset \mathfrak{N}$. Then consider the set K_x of all $y \in K$ satisfying $y\mathfrak{N}_x y^{-1} \subset \mathfrak{N}$. K_x is an open covering of K and hence there exists a finite number of K_x covering K. Take $\mathfrak{N}'$ to be the intersection of $\mathfrak{N}_x$ for the finite number of x.) Hence if $\mathrm{E}(\mathrm{g})$ is uniformly continuous, then $\mathrm{E}(\mathrm{g}_1\mathrm{g}\mathrm{g}_2)$ for a fixed g_1 and g_2 is continuous in g. From this $\mathrm{U}(\mathrm{g})$ is weakly continuous on a dense set. Since it is unitary, it is strongly continuous. Q.E.D.

§5. Form of Expectation Functional

Theorem 5.1. *Let* B_X^{E} *be a continuous complete Boolean algebra of equivalence classes of subsets of* X *satisfying the countable chain condition and let* $F(g, Y)$ *be a complex valued function of* $g \in G$ *and* $Y \in B_X^{\mathrm{E}}$ *satisfying the following condition*;

(1) $F(g_i g_j^{-1}, Y) - F(g_i, Y) - F(g_j^{-1}, Y) = H(g_i, g_j ; Y) \pmod{2\pi i}$

(2) $H(g_i, g_j; Y)$ is continuous in g_i and g_j for each $Y \in B_X^{\mathrm{E}}$.
(3) $H(1, g; Y) = H(g, 1; Y) = 0$
(4) $Y \to H(g_i, g_j; Y)$ is complex finite measure on the complete Boolean algebra B_X^{E} for each fixed g_i and g_j.
(5) $H(g_i, g_j; Y)$ is a positive semidefinite matrix for any fixed g_i, $i=1, \cdots, N$, and Y.
(6) $\exp F(g, Y)$ is continuous in g.
(7) If $\{Y_i\}$ is a countable partition of Y, $\exp F(g, Y) = \prod_i \exp F(g, Y_i)$.
(8) $\mathrm{Re}\, F(g, Y) = \mathrm{Re}\, F(g^{-1}, Y)$.

Then there exists a continuous tensor product $\mathfrak{H} = e^{\mathfrak{L}}$, $\Omega = e^0$ and a continuous unitary representation $g^{\chi} \to \mathrm{U}(g^{\chi})$ of the subgroup $\mathrm{C}_p(G, X)$ such that

$$\mathrm{E}(g^{\chi(Y)}) = (\Omega, \mathrm{U}(g^{\chi(Y)})\Omega) = e^{F(g, Y)}. \tag{5.1}$$

Conversely, any separable σ-factorizable functional over $\mathrm{C}(G, X)$ with no discrete spectrum is of this form when restricted to $\mathrm{C}_p(G, X)$.

Proof. We first prove the converse part. Let $\mathrm{U}(g^{\chi(Y)})$ be given by (4.15) where Q, ϕ and c satisfy (1), (2), (3) of Theorem 4.4 and are σ-additive in Y, mod. $2\pi i$ for c. We compute $\mathrm{E}(g^{\chi(Y)}) \equiv (\Omega, \mathrm{U}(g^{\chi(Y)})\Omega)$ in terms of $c(g, Y)$, $Q(g) \equiv Q(g, X)$, $\phi(g) \equiv \phi(g, X)$ and $\mathrm{P}(Y)$. We have

$$\mathrm{E}(g^{\chi(Y)}) = e^{-\frac{1}{2}(\phi(g), \mathrm{P}(Y)\phi(g)) + ic(g, Y)} \tag{5.2}$$

and hence define

$$F(g, Y) = -\frac{1}{2}(\phi(g), \mathrm{P}(Y)\phi(g)) + ic(g, Y). \tag{5.3}$$

$e^{F(g, Y)}$ is continuous in g for fixed Y by Lemma 2.4. We also have $e^{F(1, Y)} = \mathrm{E}(1) = 1$. Because we have a freedom of adding an integral multiple of 2π to $c(g, Y)$, we set $c(1, Y) = 0$. Then $F(1, Y) = 0$. From (4.20) and (4.21), we have $\phi(g^{-1}, Y) = -Q(g, Y)\phi(g, Y)$. Hence the unitarity of $Q(g, Y)$ implies (8). Next we compute

$$G(g_i, g_j; Y) \equiv F(g_i g_j^{-1}, Y) - F(g_i Y) - F(g_j^{-1} Y). \tag{5.4}$$

From the unitarity of $Q(g)$ and equation (4.17), we have

(5.5) $$\|P(Y)[\phi(g_1g_2)-\phi(g_2)]\|^2 = \|P(Y)Q(g_2)^*\phi(g_1)\|^2 = \|Q(g_2)^*P(Y)\phi(g_1)\|^2 = \|P(Y)\phi(g_1)\|^2 .$$

Hence

(5.6) $$-(\phi(g_1),\ P(Y)\phi(g_1))+(\phi(g_1g_2),\ P(Y)\phi(g_1g_2)) + (\phi(g_2),\ P(Y)\phi(g_2)) = 2\,\mathrm{Re}\,(\phi(g_1g_2),\ P(Y)\phi(g_2)) .$$

Setting $g_2=g_j$, $g_1=g_i g_j^{-1}$, we have

(5.7) $$\mathrm{Re}\,[F(g_ig_j^{-1},\ Y)-F(g_i,\ Y)-F(g_j,\ Y)] = \mathrm{Re}\,(\phi(g_i),\ P(Y)\phi(g_j)) .$$

By setting $g_i=1$, we have $\mathrm{Re}\,F(g^{-1},\ Y)=\mathrm{Re}\,F(g,\ Y)$. Hence

(5.8) $$\mathrm{Re}\,[F(g_ig_j^{-1},\ Y)-F(g_i,\ Y)-F(g_j^{-1},\ Y)] = \mathrm{Re}\,(\phi(g_i),\ P(Y)\phi(g_j)) .$$

Next we consider (4.22). We have

(5.9) $$c(g_1g_2,\ Y)-c(g_1,\ Y)-c(g_2,\ Y) = \mathrm{Im}\,(\phi(g_1),\ P(Y)\phi(g_2^{-1}))$$

modulo 2π. Hence, we obtain modulo 2π

(5.10) $$\mathrm{Im}\,[F(g_ig_j^{-1},\ Y)-F(g_i,\ Y)-F(g_j^{-1},\ Y)] = \mathrm{Im}\,(\phi(g_i),\ P(Y)\phi(g_j)) .$$

Combining (5.7) and (5.10), we have

(5.11) $$F(g_ig_j^{-1},\ Y)-F(g_i,\ Y)-F(g_j^{-1},\ Y) = (\phi(g_i),\ P(Y)\phi(g_j))$$

modulo $2\pi i$. By taking $H(g_i,\ g_j\ ;\ Y)$ to be the right hand side, we have the properties (2)-(5) for H.

Now let F and H be given. We introduce the free complex linear space K_0 over $B_X^{\mathrm{E}}\times G$, denoting a general element by

(5.12) $$\sum_{n=1}^{N} c_n\Psi(Y_n,\ g_n)\ , \qquad Y_n\in B_X^{\mathrm{E}},\ g_n\in G .$$

We introduce an inner product by linearity and

(5.13) $$(\Psi(Y_1,\ g_1),\ \Psi(Y_2,\ g_2)) = H(g_1,\ g_2\ ;\ Y_1\cap Y_2) .$$

After identifying all elements with 0 norm as 0, we obtain a prehilbert space. We denote its completion by $\mathfrak{L}$. We define $\phi(g)$, $c(g,\ Y)$, $P(Y)$ and $Q(g)$ by

(5.14) $\phi(g)=\Psi(X,g),\quad c(g,Y)=\mathrm{Im}\,F(g,Y)$

(5.15) $\mathrm{P}(Y)\Psi(Y',g)=\Psi(Y\cap Y',g)$

(5.16) $Q(g)\Psi(Y,g')=\Psi(Y,g'g^{-1})-\Psi(Y,g^{-1})\,.$

First we note that $\phi(1)=0$, because $\|\Psi(X,1)\|^2=H(1,1;X)=0$. By linearity, we want to extend $\mathrm{P}(Y)$ and $Q(g)$ to a dense subset. Because of (4) and (5), $\mathrm{P}(Y)$ brings 0 always to 0 and hence the linear extension is possible. Further (4) and (5) imply that $\mathrm{P}(Y)$ is bounded. From the definition, we see $\mathrm{P}(Y)^*=\mathrm{P}(Y)$ and $\mathrm{P}(Y)^2=\mathrm{P}(Y)$. Namely $\mathrm{P}(Y)$ is a projection. (4) then tells us that $Y\to\mathrm{P}(Y)$ is a projection valued measure.

From the continuity of $H(g_i,g_j;Y)$ in g_i and g_j, we see that $\Psi(Y,g)$ is strongly continuous in g. Hence $\phi(g)$ and $Q(g)$ are strongly continuous in g.

For $Q(g)$, the structure of H and the definition (5.16) of Q imply

(5.17)
$$\begin{aligned}&(Q(g)\Psi(Y_2,g_2),\ Q(g)\Psi(Y_1,g_1))\\&=(\Psi(Y_2,g_2g^{-1})-\Psi(Y_2,g^{-1}),\ \Psi(Y_1,g_1g^{-1})-\Psi(Y_1,g^{-1}))\\&=H(g_2g^{-1},g_1g^{-1};Y_2\cap Y_1)-H(g^{-1},g_1g^{-1};Y_2\cap Y_1)\\&\quad-H(g_2g^{-1},g^{-1};Y_2\cap Y_1)+H(g^{-1},g^{-1};Y_2\cap Y_1)\\&=F(g_2g_1^{-1},Y_2\cap Y_1)-F(g_1^{-1},Y_2\cap Y_1)-F(g_2,Y_2\cap Y_1)\\&=(\Psi(Y_2,g_2),\ \Psi(Y_1g_1))\,.\end{aligned}$$

Here we have used $F(1,Y)=0$, which follows from (1) with $g_i=1$ and (3). (5.17) holds up to $2n\pi i$, $n=0,\pm1,\cdots$. However, the continuity of $Q(g)$ implies $n=0$. Hence $Q(g)$ can be extended to the whole space as an isometric operator by linearity and continuity. From the definition $Q(g_1)Q(g_2)=Q(g_1g_2)$ holds on a total set $\Psi(Y,g)$ and hence on all vectors. Since $Q(1)=1$, $Q(g^{-1})Q(g)=Q(g)Q(g^{-1})=1$. Hence $Q(g)^{-1}$ exists, which implies that $Q(g)$ is unitary. Further $Q(g)^*=Q(g^{-1})$. By definition $Q(g)$ commutes with all $\mathrm{P}(Y)$. Since $\mathrm{Re}\,F(1,Y)=0$, (5.13), (5.15) and (8) imply $(\phi(g),\ \mathrm{P}(Y)\phi(g))=-2\,\mathrm{Re}\,F(g,Y)$. If we define $\mathrm{U}(g^{\chi(Y)})$ by (4.15), (4.16) and (4.17) with $\phi(g,X)=\phi(g)$, $Q(g,X)=Q(g)$, then the equation (5.1) is satisfied.

The function $c(g,Y)$ is σ-additive in Y mod 2π by (7). It satisfies (4.22) due to (5.14), (1), (5.13) and (5.15). It is continuous in g due to (6). Thus $\mathrm{U}(g^{\chi(Y)})$ is a continuous unitary representation of $\mathrm{C}_p(G,X)$. Q.E.D.

We remark that the absolute value part of (7) is a consequence of other assumptions. We also note that $\mathrm{Im}\, F(g, Y) = -\mathrm{Im}\, F(g^{-1}, Y)$ mod 2π follows from (1) and (5).

In the rest of this section, we rederive the form of the functional E(g) in the preceding theorem by an elementary method under the assumption that the functional is given by

$$\mathrm{E(g)} = \exp \int_X F(\mathrm{g}(x))\, \mathrm{d}\mu(x)\,. \tag{5.18}$$

Here $F(g)$ is assumed to be continuous in g, $F(1)=0$ and μ is a continuous positive finite measure on X.

Lemma 5.2. *Let A_{ij}, $i, j=1, \cdots, n$, be a matrix. The matrix $(\exp cA_{ij})$ is positive semidefinite for all positive c if and only if A is hermitian and PAP is positive semidefinite where $P_{ij}=\delta_{ij}-\frac{1}{n}$.*

Proof. (1) The sufficiency: We have

$$\exp cA_{ij} = e^{\alpha}(e^{\beta_i})^* (\exp c(PAP)_{ij})(e^{\beta_j})\,, \tag{5.19}$$

where

$$\alpha = -\frac{c}{n^2}\sum_{ij} A_{ij} \text{ (real)}\,, \tag{5.20}$$

$$\beta_j = \frac{c}{n}\sum_i A_{ij}\,. \tag{5.21}$$

Hence

$$\sum x_i^* (\exp cA_{ij}) x_j = e^{\alpha} \sum y_i^* (\exp c(PAP)_{ij}) y_j \tag{5.22}$$

where

$$y_j = x_j e^{\beta_j}\,. \tag{5.23}$$

By Lemma 8.2 of [3], the positive semidefiniteness of $(PAP)_{ij}$ implies the same for $\exp c(PAP)_{ij}$. Therefore we have the positive semidefiniteness of $\exp cA_{ij}$ from (5.22).

(2) The necessity: Let $Px=x$, namely $\sum x_i=0$. Then

$$\sum_{ij} x_i^* A_{ij} x_j = \lim_{c\to 0} c^{-1} \sum_{ij} x_i^* e^{cA_{ij}} x_j \geq 0\,. \tag{5.24}$$

Since P is a projection, we have the positive semidefiniteness of PAP.

It follows that $B_{ij} \equiv \exp c(PAP)_{ij}$ is hermitian. Since $f(c) \equiv$

$\sum_{ij} \exp cA_{ij}$ must be real, $\alpha = -(c/n^2)f'(0)$ is real. Since $\exp cA_{ij} = B_{ij}\exp(\alpha+\gamma_i+\beta_j)$ is hermitian, where $\gamma_i \equiv (c/n)\sum_j A_{ij}$; we have $\gamma_j^* = \beta_j + 2n\pi i$. From the continuity in c, we have $n=0$. Hence $A_{ij} = \alpha+\gamma_i+\beta_j+(PAP)_{ij}$ is hermitian. Q.E.D.

We now consider

$$\mathrm{E}(g^{\chi(Y)}) = \exp F(g)\mu(Y)\,. \tag{5.25}$$

By the Lemma 5.2, the positivity condition for E implies

$$\sum_{i,j} x^* F(g_i g_j^{-1}) x_j \geq 0 \tag{5.26}$$

whenever $\sum x_i = 0$. Let $i, j = 0, 1, \cdots, n$, $g_0 = 1$, $x_0 = -\sum_{i=1}^n x_i$. Then we have

$$\sum_{i,j=1}^n x_i^* F(g_i g_j^{-1}) x_j - \Big(\sum_{i=1}^n x_i^*\Big) \sum_{j=1}^n F(g_j^{-1}) x_j - \sum_{i=1}^n x_i^* F(g_i)\Big(\sum_{i=1}^n x_i\Big) \geq 0\,. \tag{5.27}$$

Namely $F(g_i g_j^{-1}) - F(g_i) - F(g_j^{-1})$ is positive semidefinite.

Conversely, assume that $F(g_i g_j^{-1}, x) - F(g_i, x) - F(g_j^{-1}, x)$, $i, j = 1, \cdots, n$ is positive semidefinite for any $\{g_i\}$, n and x. By setting $n=1$, $x_1 = 1$, we have $F(1, x) = 0$. Let μ be a finite positive measure on X. Then

$$F(\mathrm{g}) \equiv \int F(\mathrm{g}(x), x)\,\mathrm{d}\mu(x)\,, \qquad \mathrm{g} \in \mathrm{C}(X, G) \tag{5.28}$$

have the property that $H(\mathrm{g}_i, \mathrm{g}_j) \equiv F(\mathrm{g}_i \mathrm{g}_j^{-1}) - F(\mathrm{g}_i) - F(\mathrm{g}_j^{-1})$ is positive semidefinite and $F(1) = 0$. Then, for any x_i satisfying $\sum x_i = 0$, we have

$$\sum x_i^* H(\mathrm{g}_i, \mathrm{g}_j) x_j = \sum x_i^* F(\mathrm{g}_i \mathrm{g}_j^{-1}) x_j \geq 0\,. \tag{5.29}$$

Hence $\mathrm{E}(\mathrm{g}_i \mathrm{g}_j^{-1})$ is positive semidefinite and $\mathrm{E}(1) = 1$. In particular, if $F(g, x)$ is constant in x, we have the positivity for (5.18).

§6. Standard Examples for $\phi(g, Y)$

To analyze $\phi(g, Y)$, we introduce the following spaces.

Definition 6.1. *Let $h(t)$ be a function of the class $\mathcal{D}$ such that its Fourier transform $\tilde{h}(\lambda) = \int e^{it\lambda} h(t)\,\mathrm{d}t$ satisfies $\tilde{h}(0) = 0$, $1 > \tilde{h}(\lambda) \geq 0$,* *for* $\lambda \neq 0$ *and* $\tilde{h}''(0) \neq 0$. *Let* $\mathfrak{l}_i$ *be a linearly independent basis of* $\mathfrak{g}$, *and*

$$\mathrm{K} = \sum \mathrm{K}(\mathfrak{l}_i),\ \mathrm{K}(\mathfrak{l}) = 1 - \int Q(e^{t\mathfrak{l}})\, h(t)\, dt\,. \tag{6.1}$$

Let $\mathrm{K} = \int_0^\infty \lambda E(\mathrm{d}\lambda)$ *and* $\mathrm{K}^{-1/2}$ *be the inverse of the mapping* $\mathrm{K}^{1/2}$ *from* $(1 - E([0]))\mathfrak{L}$ *into* $\mathfrak{L}$.

The space $D^{\pm}$ is the range of $\mathrm{K}^{\mp 1/2}$ in $\mathfrak{L}$ equipped with the topology induced by a new norm $||\psi||_{\pm} \equiv ||\mathrm{K}^{\pm 1/2}\psi||$ and $\bar{D}^{+}$ is the completion of D^{+}. The space D_0 is the largest subspace of $\mathfrak{L}$ on which $Q(g) = 1$ for all g.

Lemma 6.2. *$D^{\pm}$ as a topological linear space does not depend on the choice of h and $\{\mathfrak{l}_i\}$. D_0 is the eigenspace belonging to the eigenvalue 0 of* K. *$(Q(g)-1)\Psi$ for any $g \in G$ and $\Psi \in \mathfrak{L}$ belongs to D^{-} and*

$$||(Q(g)-1)\Psi||_{-} \leq d(g)\, ||\Psi|| \tag{6.2}$$

where $d(g)$ does not depend on Ψ. For every Ψ, $||(Q(g)-1)\Psi||_{-} \to 0$ as $g \to 1$.

Proof. We first characterize a vector in the range of $\mathrm{K}(\mathfrak{l}_i)^{1/2}$. Consider the spectral decomposition of one parameter family of unitary operators

$$Q(e^{t\mathfrak{l}}) = \int e^{it\lambda} \mathrm{d}E_{\mathfrak{l}}(\lambda)\,. \tag{6.3}$$

Then

$$\mathrm{K}(\mathfrak{l}) = \int [1 - \tilde{h}(\lambda)]\, \mathrm{d}E_{\mathfrak{l}}(\lambda)\,. \tag{6.4}$$

By assumption $1 - \tilde{h}(\lambda) \geq 0$, where the equality holds only at $\lambda = 0$, $1 - \tilde{h}(\lambda) \to 1$ as $\lambda \to \infty$ and of order λ^2 at $\lambda = 0$. Hence a vector ψ is in the domain of $\mathrm{K}(\mathfrak{l})^{-1/2}$ if and only if $\int \lambda^{-2} \mathrm{d}(\psi, E_{\mathfrak{l}}(\lambda)\psi) < \infty$. In particular $(Q(e^{t\mathfrak{l}}) - 1)\psi$ is in the domain of $\mathrm{K}(\mathfrak{l})^{-1/2}$ because $\lambda^{-2}|e^{i\lambda t} - 1|^2$ is a bounded function of λ. We also know that $\mathrm{K}(\mathfrak{l})\psi = 0$ if and only if $Q(e^{t\mathfrak{l}})\psi = \psi$ for all t. Next, since all $\mathrm{K}(\mathfrak{l}_i) \geq 0$, we have $\mathrm{K} \geq \mathrm{K}(\mathfrak{l}_i)$.

Now we see that the eigenspace of K belonging to the eigenvalue 0 is the intersection of the same for $\mathrm{K}(\mathfrak{l}_i)$ and hence consists of ψ such that $Q(e^{tI_i})\psi=\psi$. Since e^{tI_i} generates G, we have proved the assertion on D_0.

Next we prove that $(Q(e^{tI_i})-1)\psi$ is in D^-. Since $K(I_i)\geq 0$, we have $(\varphi, K\varphi)\geq(\varphi, K(I_i)\varphi)$ for any φ. Hence $\|K(I_i)^{1/2}K^{-1/2}\varphi_1\|^2\leq\|\varphi_1\|^2$ where $\varphi_1=K^{1/2}\varphi$. Hence $K(I_i)^{1/2}K^{-1/2}$ has a bounded closure with norm$\leqq 1$, which we denote by R_i. The domain of R_i is $D_0^{\perp}$. If T is a bounded linear operator and S is a linear operator with a dense domain, then $(TS)^*=S^*T^*$ ([5] p. 297). Since the ranges of $K(I_i)^{1/2}$ and $K^{-1/2}$ are contained in $D_0^{\perp}$, we consider the restrictions $T=K(I_i)^{1/2}|D_0^{\perp}$ and $S=K^{-1/2}|D_0^{\perp}$ on the space $D_0^{\perp}$. Then

$$R_i^*\varphi=(TS)^*\varphi = K^{-1/2}K(I_i)^{1/2}\varphi$$

for $\varphi\in D_0^{\perp}$ where R_i^* is the adjoint of R_i on $D_0^{\perp}$. Therefore,

$$\|K^{-1/2}K(I_i)^{1/2}\varphi\|\leq\|\varphi\|$$

for $\varphi\in D_0^{\perp}$. Since $K(I_i)^{1/2}\varphi=0$ for $\varphi\in D_0$, we see that $K(I_i)^{1/2}\varphi$ for any φ is in the domain of $K^{-1/2}$ and the above inequality holds.

Since

$$\rho(t) = \sup_{\lambda} 2(1-\cos t\lambda)(1-\tilde{h}(\lambda))^{-1}<\infty\,, \tag{6.5}$$

$\varphi_3=[Q(e^{tI_i})-1]\psi$ for any ψ is in the domain of $K(I_i)^{-1/2}$. Hence setting $\varphi_2=K(I_i)^{-1/2}\varphi_3$, we see that φ_3 is in the domain of $K^{-1/2}$ and

$$\|[Q(e^{tI_i})-1]\psi\|_-^2 = \|K^{-1/2}\varphi_3\|^2\leq\|K(I_i)^{-1/2}\varphi_3\|^2\leq\rho(t)\|\psi\|^2\,. \tag{6.6}$$

Any $g\in G$ can be written as a product $e^{I(1)}\cdots e^{I(N)}$ where $I(p)$ is of the form $t_pI_{i(p)}$ for some $i(p)$. Then, writing $g(p)=e^{I(p)}\cdots e^{I(N)}$ and $g(N+1)=1$, we have

$$(Q(g)-1)\psi = \sum_{p=1}^{N}\{Q(e^{I(p)})-1\}Q(g(p+1))\psi\,, \tag{6.7}$$

$$\|\{Q(g)-1\}\psi\|_-\leq\{\sum_{p=1}^{N}\rho(t_p)^{1/2}\}\|\psi\|\,. \tag{6.8}$$

We thus have proved that $\{Q(g)-1\}\psi\in D_-$ and the equation (6.2). Near $g=1$, we may take $i(p)=p$ (running over a basis) and $g\to 1$ is equivalent to $\{t_p\}\to 0$. As $t\to 0$ and $\psi_s\to\psi$, $\delta_t(\psi)\equiv\|K(I_i)^{-1/2}[Q(e^{tI_i})-1]\psi\|\to 0$ and $\delta_t(\psi_s)\leq\rho(t)\|\psi_s-\psi\|+\delta_t(\psi)\to 0$. Hence (6.7) implies $\|(Q(g)-1)\psi\|_-\to 0$ as $g\to 1$ for every ψ in $\mathfrak{L}$.

Finally we prove that different choices of $\tilde{h}$ and $\{I_i\}$ give the same $D^{\pm}$. Let $\rho_\nu(t)$ and $\mathrm{K}_\nu(\mathfrak{l})$ be the ρ and $\mathrm{K}(\mathfrak{l})$ with $\tilde{h}=\tilde{h}_\nu$, $d_\nu(g)$ be the $d(g)$ with $\tilde{h}_\nu$ and $\{\mathfrak{l}_i^\nu\}$, $\mathrm{K}_\nu=\sum \mathrm{K}_\nu(\mathfrak{l}_i^\nu)$ where $\nu=1$ and 2. For $\varphi=\{Q(e^{t\mathfrak{l}})-1\}\psi$, we have

$$||\mathrm{K}_{\mathrm{I}}^{-1/2}\varphi|| \leq d_1(e^{t\mathfrak{l}})||\psi||\,. \tag{6.9}$$

Choose an a satisfying $0<a<2\pi/t$ and define $E_a\equiv E_{\mathfrak{l}}([-a, a])$. Then $R\equiv(Q(e^{t\mathfrak{l}})-1)^{-1}\mathrm{K}_2(\mathfrak{l})^{1/2}E_a$ is bounded and hence $||\psi||=||R\mathrm{K}_2(\mathfrak{l})^{-1/2}\varphi||\leq ||R||\,||\mathrm{K}_2(\mathfrak{l})^{-1/2}\varphi||$ if $E_a\psi=\psi$ and $E_0\psi=0$. If ψ runs over $(E_a-E_0)\mathfrak{L}$, then $\varphi=(Q(e^{t\mathfrak{l}})-1)\psi$ runs over E_a times the domain of $\mathrm{K}_2(\mathfrak{l})^{-1/2}$. (Note that $E_0\mathfrak{L}$ is orthogonal to the domain of $\mathrm{K}_2(\mathfrak{l})^{-1/2}$.) Hence we have

$$||\mathrm{K}_{\mathrm{I}}^{-1/2}\mathrm{K}_2(\mathfrak{l})^{1/2}\Phi|| \leq c(\mathfrak{l})||\Phi|| \tag{6.10}$$

for $\Phi=\mathrm{K}_2(\mathfrak{l})^{-1/2}\varphi$, which runs over $(E_a-E_0)\mathfrak{L}$. If $\Phi\in E_0\mathfrak{L}$, then the left hand side of (6.10) is 0 and hence (6.10) holds trivially.

Next consider ψ in $(1-E_a)\mathfrak{L}$. If A is a closed operator and ψ_n is a sequence of vectors in the domain of A such that $||A\psi_n||\leq a$ for all n and $\lim\psi_n=\psi$, then ψ is also in the domain of A and $||A\psi||\leq a$ due to the weak sequential compactness. If $\psi(t)$ is measureable in t, $\psi=\int\psi(t)\,\mathrm{d}t$ converges strongly, $||A\psi(t)||\leq a(t)$ and $\int a(t)\,\mathrm{d}t<\infty$, then ψ is in the domain of A and $||A\psi||\leq\int a(t)\,\mathrm{d}t$. By using this, we integrate (6.9) relative to $h_2(t)\,\mathrm{d}t$ where $d(\mathfrak{l})\equiv\int d_1(e^{t\mathfrak{l}})\,h_2(t)\,\mathrm{d}t<\infty$. We have $||\mathrm{K}_{\mathrm{I}}^{-1/2}\mathrm{K}_2(\mathfrak{l})\psi||\leq d(\mathfrak{l})||\psi||$. If we set $\Phi=\mathrm{K}_2(\mathfrak{l})^{1/2}\psi$, Φ runs over $(1-E_a)\mathfrak{L}$ when ψ runs over $(1-E_a)\mathfrak{L}$ and $||\psi||\leq||\mathrm{K}_2(\mathfrak{l})^{-1/2}\times(1-E_a)||\,||\Phi||$. Hence (6.10) with a new constant $c(\mathfrak{l})$ holds also for such Φ.

Thus we see that $\mathrm{K}_{\mathrm{I}}^{-1/2}\mathrm{K}_2(\mathfrak{l})^{1/2}$ is a bounded operator. Taking adjoint, we see that

$$||\mathrm{K}_2(\mathfrak{l})^{1/2}\mathrm{K}_{\mathrm{I}}^{-1/2}\varphi|| \leq a(\mathfrak{l})||\varphi|| \tag{6.11}$$

for any φ in the domain of $\mathrm{K}_{\mathrm{I}}^{-1/2}$. Setting $\mathrm{K}_{\mathrm{I}}^{-1/2}\varphi=\psi$, we have

$$(\psi,\ \mathrm{K}_2(\mathfrak{l})\psi) \leq a(\mathfrak{l})^2(\psi,\ \mathrm{K}_1\psi) \tag{6.12}$$

where $\psi\in D_0^{\perp}$. By adding this equation for $\mathfrak{l}=\mathfrak{l}_i^2$, $i=1,\cdots$, we have

$$(\psi,\ \mathrm{K}_2\psi) \leq a_1(\psi,\ \mathrm{K}_1\psi) \tag{6.13}$$

for some constant a_1 and for any ψ in the orthogonal complement of D_0. Since D_0 is the 0 eigenspace of both K_1 and K_2, an addition of vectors in D_0 to ψ does not change the inequality (6.10) and we have $K_2 \leq a_1 K_1$. Similarly, we have $K_1 \leq a_2 K_2$. Hence D^- is the same for $\nu=1$ and 2. From $K_1 \leq a_2 K_2$, we have

$$||K_1^{1/2} K_2^{-1/2} \psi||^2 \leq a_2 ||\psi||^2 \tag{6.14}$$

for $\psi \in K_2^{1/2}\mathfrak{L}$. Since $K_2^{1/2}\mathfrak{L}$ span $D_0^{\perp}$, $K_1^{1/2}K_2^{-1/2}$ is bounded on $D_0^{\perp}$.

Namely, $|(K_1^{1/2}\varphi, K_2^{-1/2}\psi)| \leq a_2^{1/2}||\varphi|| \, ||\psi||$ for any ψ in the domain of $K_2^{-1/2}$ and any φ. Hence $(K_1^{1/2}\varphi, K_2^{-1/2}\psi) = (\chi, \psi)$ for some χ in $D_0^{\perp}$, satisfying $||\chi|| \leq a_2^{1/2}||\varphi||$. Since $K_1^{1/2}\varphi$ is in $D_0^{\perp}$, we have $\chi = K_2^{-1/2}K_1^{1/2}\varphi$ and hence $||K_2^{-1/2}K_1^{1/2}\varphi|| \leq a_2^{1/2}||\varphi||$ for any φ. Therefore

$$||K_2^{-1/2}\psi|| \leq a_2^{1/2}||K_1^{-1/2}\psi|| \tag{6.15}$$

for any $\psi \in K_1^{1/2}\mathfrak{L}$. Similarly

$$||K_1^{-1/2}\psi|| \leq a_1^{1/2}||K_2^{-1/2}\psi|| \tag{6.16}$$

for any $\psi \in K_2^{1/2}\mathfrak{L}$. Since the domain of $K_2^{-1/2}K_1^{1/2}$ is $\mathfrak{L}$, the domain of $K_2^{-1/2}$ contains $K_1^{1/2}\mathfrak{L}$ which is the domain of $K_1^{-1/2}$. Similarly the domain of $K_2^{-1/2}$ contains the domain of $K_2^{-1/2}$ and hence the two are equal. Thus D^+ and its topology does not depend on the choice of h and $\mathfrak{l}_i$. Q.E.D.

Remark. In our preceding proof of the statement that $K(\mathfrak{l}_i)^{1/2}\varphi$ is in the domain of $K^{-1/2}$, we have actually provcd that for any bounded self-adjoint operators satisfying $A \geqq B \geqq 0$, the range of $A^{1/2}$ contains the range of $B^{1/2}$.

It is also possible to obtain $\bar{D}^+$ by the following procedure. Introduce a family of seminorms $||\psi||_{\mathfrak{l}} \equiv ||K(\mathfrak{l})^{1/2}\psi||$ to $D_0^{\perp}$ where $\mathfrak{l}$ runs over all elements in the Lie algebra. This defines a locally convex topology on $D_0^{\perp}$, which actually coincides with the topology of D^+. The completion of $D_0^{\perp}$ with respect to this topology gives $\bar{D}^+$.

Lemma 6.3. *Let $\mathfrak{H}$ be a Hilbert space, G be a locally compact group, $Q(g)$ be a continuous unitary representation of G on $\mathfrak{H}$, and $\phi(g)$ be a $\mathfrak{H}$-valued function on G satisfying* (4.24).

(1) *If $\phi(g)$ is weakly measurable* (*i.e.* $(\psi, \phi(g))$ *is measurable for every* ψ) *and is locally essentially bounded* (*with respect to invariant measures on* G), *then $\phi(g)$ is strongly continuous.*

(2) *If* $||\phi(g)||$ *is measurable, then* $\phi(g)$ *is locally bounded.*

(3) *If* $\phi(g)$ *is weakly measurable and* $\mathfrak{H}$ *is separable, then* $||\phi(g)||$ *is measurable.*

Proof. (1) First, we show that $\phi(g)$ is strongly continuous at $g=1$. By setting $g_1=1$ in (4.24), we have $\phi(1)=0$. Hence this amounts to proving $\lim_{g\to 1}||\phi(g)||=0$.

Let $\mathfrak{N}$ be a bounded neighbourhood of 1 such that $||\phi(g)||\leq R$ for almost all g in $\mathfrak{N}$. Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be neighbourhoods of 1 such that $\mathfrak{N}_1\mathfrak{N}_2\subset\mathfrak{N}$. Let f be a continuous function with support in $\mathfrak{N}_1$ such that $\int f(g_1)\mathrm{d}g_1=1$ and $g\in\mathfrak{N}_2$. From (4.24), we have

$$(\psi, \phi(g)) = \int[f(g_1g^{-1})-f(g_1)](\psi, \phi(g_1))\mathrm{d}g_1 + \int([1-Q(g)]\psi, \phi(g_1))\mathrm{d}g_1 f(g_1)\,, \tag{6.17}$$

where $\mathrm{d}g_1$ is the right invariant measure on G. Hence

$$|(\psi, \phi(g))|\leq R||\psi||\int|f(g_1g^{-1})-f(g_1)|\mathrm{d}g_1+||\psi||\,||[1-Q(g)^*]\phi||\,,$$

where $\phi=\int\phi(g)f(g)\mathrm{d}g$, which converges weakly due to the measurability and essential boundedness of $\phi(g)$ on the bounded support of continuous f. This then implies $|(\psi, \phi(g))|\,||\psi||^{-1}\to 0$ as $g\to 1$ uniformly in ψ. Namely $||\phi(g)||\to 0$.

From (4.24), we have $||\phi(g_1g_2)-\phi(g_2)||=||\phi(g_1)||$. Hence $\phi(g)$ is strongly continuous at any $g\in G$.

(3) This is obvious from $||\phi(g)||^2=\sum_j|(\psi_j, \phi(g))|^2$ for a countable orthonormal basis ψ_j.

(2) Let $\eta(g)\equiv||\phi(g)||$. It is measurable, $0\leq\eta(g)<\infty$ and

$$\eta(g_1)+\eta(g_2)\geq\eta(g_1g_2)\geq|\eta(g_1)-\eta(g_2)| \tag{6.18}$$

due to (4.24). Let Δ be a bounded measurable subset of G, $v(\Delta)$ be a right invariant measure and

$$\xi(\Delta, a) = v(\{g\in\Delta\,;\,\eta(g)<a\})/v(\Delta)\,. \tag{6.19}$$

Since $g'\in\Delta$, $\eta(g')\leq b$ implies $\eta(g'g)\geq\eta(g)-b$ by (6.18), we have (for $b=\eta(g)-a$)

$$\xi(\Delta g, a)\leq 1-\xi(\Delta, \eta(g)-a)\,. \tag{6.20}$$

If $\mathfrak{N}$ is a sufficiently small neighbourhood of 1, then the invariant measure of the complement of Δg in Δ can be made smaller than given $\varepsilon > 0$ for every $g \in \mathfrak{N}$. Then

$$\xi(\Delta g, a) \geq \xi(\Delta, a) - \varepsilon / v(\Delta) . \tag{6.21}$$

If $\eta(g)$ is locally unbounded at $g=1$, there exist g_j, $j=1, 2, \cdots$ such that $g_j \in \mathfrak{N}$ and $\lim \eta(g_j) = +\infty$. By $\eta(g) < \infty$, we have $\lim_{b \to \infty} \xi(\Delta, b) = 1$. Hence (6.20) and (6.21) imply, when g_j is substituted into g, $\xi(\Delta, a) - \varepsilon / v(\Delta) \leq 0$. This contradicts with $\lim_{a \to \infty} \xi(\Delta, a) = 1$. Hence $\eta(g)$ is locally bounded at $g=1$.

From the first equation of (6.18), $\sup_{g \in \mathfrak{N}} \eta(g_1 g) \leq \sup_{g \in \mathfrak{N}} \eta(g) + \eta(g_1)$. Hence $\eta(g)$ is locally bounded at any $g_1 \in G$. Q.E.D.

Lemma 6.4. *If* $\psi \in D^+$, *then* $\mathrm{K}\psi \in D^-$. *The closure of this mapping is a unitary mapping from* $\bar{D}^+$ *onto* D^-. *The sesquilinear form* $(\psi, \phi)_{\mathfrak{L}}$ *for* $\psi \in D^+$, $\phi \in D^-$ *can be extended to* $\psi \in \bar{D}^+$ *and it gives the duality between* D^+ *and* D^-. *Here* $(,)_{\mathfrak{L}}$ *is the inner product in* $\mathfrak{L}$.

Proof. By definition, $\|\psi\|_+^2 = \|\mathrm{K}^{1/2}\psi\|^2 = \|\mathrm{K}^{-1/2}\mathrm{K}\psi\|^2 = \|\mathrm{K}\psi\|_-^2$ for any $\psi \in D^+$ $(= D_0^{\perp})$. Hence K is isometric and defined on a dense set D^+ in $\bar{D}^+$. Further, $\mathrm{K}D^+$ is dense in D^-. Hence the closure of K is a unitary mapping from $\bar{D}^+$ onto D^-. For $\psi \in D^+$, $\phi \in D^-$, we have $(\mathrm{K}\psi, \phi)_- = (\psi, \phi)_{\mathfrak{L}}$. Since $\bar{D}^+$ is self-dual with respect to $(,)_+$, $\mathrm{K}\bar{D}^+ = D^-$ is dual to $\bar{D}^+$ with respect to $(,)_{\mathfrak{L}}$. Q.E.D.

We note that for any operator A which maps D^- into D^-, we can define A^* as an operator on $\bar{D}^+$ by $(A^*\psi, \phi)_{\mathfrak{L}} = (\psi, A\phi)_{\mathfrak{L}}$ for $\psi \in \bar{D}^+$, $\phi \in D^-$. This definition coincide with A^* in $\mathfrak{L}$ if ψ is in D^+.

Lemma 6.5. *For each* $\Psi \in \bar{D}^+$,

$$\phi(g) = [Q(g)^* - 1]\Psi \tag{6.22}$$

is in $Z^1(G, D_0^{\perp})$.

Proof. By (6.2), $Q(g) - 1$ is a bounded mapping from $\mathfrak{L}$ into D^-. The dual of $\mathfrak{L}$ and D^- with respect to $(,)_{\mathfrak{L}}$ is $\mathfrak{L}$ and $\bar{D}^+$. Hence $Q(g)^* - 1$ can be considered as an adjoint of $Q(g) - 1$ which must be a bounded mapping from $\bar{D}^+$ into $\mathfrak{L}$, with the norm bounded by

$d(g)$. Since $(Q(g)-1)D_0=0$, $(Q(g)^*-1)\bar{D}^+ \perp D_0$. Further,

$$(Q(g_1)-1)Q(g_2) = (Q(g_1g_2)-1)-(Q(g_2)-1) \tag{6.23}$$

and hence (4.24) holds. By Lemma 6.2, we have, for any $\Psi \in D_+$ and $\Phi \in \mathfrak{L}$,

$$([Q(g)^*-1]\Psi, \Phi) \leq ||\Psi||_+ ||[Q(g)-1]\Phi||_- \rightarrow 0$$

as $g \rightarrow 1$. Therefore $\phi(g)$ is weakly continuous at $g=1$. Then $\phi(g)$ is weakly continuous at any g by (4.24). We already know the local boundedness from $||\phi(g)|| \leq d(g)||\Psi||_+$. (It also follows from the weak continuity.) By Lemma 6.3 (1), $\phi(g)$ is strongly continuous.

Q.E.D.

Let H be a connected invariant subgroup of a Lie group G, $\hat{G}=G/H$, H_1 be the commutator group of H, H_2 be the maximal compact subgroup of the connected abelian group H/H_1 and $K \equiv (H/H_1)/H_2$. Let $\tau(g)$, $g \in G$ be defined by $\tau(g)h = ghg^{-1}$ for $h \in H$ and $\tau(g)k = [[\tau(g)h]]$ for $k=[[h]] \in K$ ($h \in H$, $[h] \in H/H_1$). Then for $h \in H$, $\tau(h)$ is the identity on K and hence we can define $\tau(\hat{g})=\tau(g)$ on K for $\hat{g}=gH \in \hat{G}$. Let $\gamma(\hat{g})$ be a measurable function $\hat{g} \in G/H$ with values in G such that $\gamma(1)=1$ and $\hat{g}=\gamma(\hat{g})H$. Let

$$\alpha(\hat{g}_1, \hat{g}_2) = [[\gamma(\hat{g}_1\hat{g}_2)^{-1}\gamma(\hat{g}_1)\gamma(\hat{g}_2)]] . \tag{6.24}$$

Let $k(g) \equiv [[\gamma(\hat{g})^{-1}g]]$ where $g \in G$, $\hat{g}=gH$. Then

$$\alpha(\hat{g}_1, \hat{g}_2) = k(g_1g_2)\{\tau(g_2)^{-1}k(g_1)^{-1}\}k(g_2)^{-1} \tag{6.25}$$

where $\hat{g}_1=g_1H$, $\hat{g}_2=g_2H$. From this, the following equality can easily be checked by using the commutativity of elements in K:

$$\alpha(\hat{g}_2, \hat{g}_3)\alpha(\hat{g}_1\hat{g}_2, \hat{g}_3)^{-1}\alpha(\hat{g}_1, \hat{g}_2\hat{g}_3) = \tau(\hat{g}_3)^{-1}\alpha(\hat{g}_1, \hat{g}_2) . \tag{6.26}$$

Hence α determines a cohomology $\hat{\alpha}$ in $H^2(\hat{G}, K)$. Let α_{γ_1} and α_{γ_2} be the α corresponding to two choices γ_1 and γ_2 for γ. Then $\gamma_1(\hat{g})\gamma_2(\hat{g})^{-1} \in H$ and hence $\delta(\hat{g})=[[\gamma_1(\hat{g})^{-1}\gamma_2(\hat{g})]]$ is a K-valued measurable function. We have

$$\alpha_{\gamma_1}(\hat{g}_1, \hat{g}_2) = \alpha_{\gamma_2}(\hat{g}_1, \hat{g}_2)\{\delta(\hat{g}_1\hat{g}_2)(\tau(\hat{g}_2)^{-1}\delta(\hat{g}_1))^{-1}\delta(\hat{g}_2)^{-1}\} . \tag{6.27}$$

Hence $\hat{\alpha}$ does not depend on the choice of γ and is determined by $\hat{G}$. We write as $\hat{\alpha}(\hat{G})$.

If K_1 is a $\tau(\hat{G})$ invariant connected subgroup of K and $\hat{\alpha}(\hat{G})+K_1$ is a 0 cohomology in K/K_1, we shall call K_1 as an annihilator. In this case, there exists K/K_1 valued measurable function $\delta_1(\hat{g})$ on $\hat{G}$, of which $[\alpha_{\gamma_1}(\hat{g}_1, \hat{g}_2)]$ for a given γ_1 is the coboundary: $[\alpha_{\gamma_1}(\hat{g}_1, \hat{g}_2)] = \delta_1(\hat{g}_1\hat{g}_2)[\tau(\hat{g}_2)^{-1}\delta_1(\hat{g}_1)^{-1}]\delta_1(g_2)^{-1}$. Then $\gamma_2(\hat{g})=\gamma_1(\hat{g})\delta_1(\hat{g})$ satisfies $[\alpha_{\gamma_2}(\hat{g}_1, \hat{g}_2)]=0$.

Theorem 6.6. *Let G be a Lie group, H be a connected invariant subgroup of G, $\hat{G}=G/H$, $\hat{g}$ be the coset containing $g\in G$, H_1 be the commutator subgroup of G, H_2 be the maximal compact subgroup of H/H_1, $[h]$ be the coset containing $h\in H$, $K=(H/H_1)/H_2$, $[[h]]$ be the coset containing $[h]\in H/H_1$, $\tau(\hat{g})$ be the representation of $\hat{G}$ on K induced from the inner automorphism of G, $\gamma(\hat{g})$ be a G-valued measurable function on $\hat{G}$ such that $\hat{g}=\gamma(\hat{g})H$ and $\gamma(1)=1$, $\alpha_\gamma(\hat{g}_1, \hat{g}_2) = [[\gamma(\hat{g}_1\hat{g}_2)^{-1}\gamma(\hat{g}_1)\gamma(\hat{g}_2)]]$, $\hat{\alpha}(\hat{G})$ be the cohomology class of α_γ in $H^2(\hat{G}, K)$, which is independent of γ, K_1 be an annihilator for $\hat{\alpha}(\hat{G})$, $[k]$ be the coset in K/K_1 containing $k\in K$, and γ_0 be a choice of γ satisfying $[\alpha_{\gamma_0}(\hat{g}_1, \hat{g}_2)]=1$ for all $\hat{g}_1, \hat{g}_2\in\hat{G}$, which exists.*

Let $\mathfrak{k}'$ be the Lie algebra of the connected abelian group K/K_1, μ be a $\tau(\hat{G})$ invariant nonnegative hermitian form on $\mathfrak{k}'+i\mathfrak{k}'$, $\mathfrak{L}_a$ be the quotient of $\mathfrak{k}'+i\mathfrak{k}'$ by vectors of vanishing μ-norm, and $\phi_a(g)= \log[[[\gamma_0(\hat{g})^{-1}g]]]$.

Further let $Q_b(\hat{g})$ be a continuous unitary representation of $\hat{G}$ on a Hilbert space $\mathfrak{L}_b$ and $\phi_b(\hat{g})\in Z^1(\hat{G}, \mathfrak{L}_a\oplus\mathfrak{L}_b)$.

Let

$$\mathfrak{L} = \mathfrak{L}_a\oplus\mathfrak{L}_b, \quad Q(g) = \tau(\hat{g})\oplus Q_b(\hat{g}), \tag{6.28}$$

$$\phi(g) = \phi_a(g)+\phi_b(\hat{g}). \tag{6.29}$$

Then $Q(g)$ is a continuous unitary representation of G, $Q(g)=1$ for $g\in H$, and $\phi(g)\in Z^1(G, \mathfrak{L})$.

Conversely, any such $Q(g)$ and $\phi(g)$ is of this form, i.e., the representation space can be decomposed as a direct sum of $Q(g)$ invariant subspaces $\mathfrak{L}_a$ and $\mathfrak{L}_b$, each of which is as described above.

Proof. Since K/K_1 is connected, commutative and contains no compact part, the exponential mapping from $\mathfrak{k}'$ onto K/K_1 is one to one and onto. Hence $\phi_a(g)$ is defined for all g. By the invariance of μ, $\tau(\hat{g})$ is unitary. The equation (4.24) for $\mathfrak{L}_a$ part follows from

(6.25) and $[\alpha_{\gamma_0}(g_1, g_2)]=1$. $\tau(\hat{g})$ is continuous by construction. The continuity of ϕ follows from (6.28) and the measurability of ϕ due to Lemma 6.3.

We now prove the converse part. For $h\in H$, we have from (4.24)

$$\phi(g) = \phi(gh)-\phi(h)\,. \tag{6.30}$$

Further

$$\begin{aligned} Q(g)^*\phi(h) &= \phi(hg)-\phi(g) \\ &= \phi(gg^{-1}hg)-\phi(g) = \phi(g^{-1}hg) \end{aligned} \tag{6.31}$$

Let $\mathfrak{L}_a$ be the subspace spanned by $\phi(h)$, $h\in H$. Then

$$Q(g)\phi(h) = \phi(ghg^{-1}) = \phi(\tau(g)h) \tag{6.32}$$

on this space. Hence it is invariant under $Q(G)$. Let $\mathfrak{L}=\mathfrak{L}_a\oplus\mathfrak{L}_b$, $\phi(g)=\phi_a'(g)\oplus\phi_b'(g)$. Then $\phi_b'(h)=0$ for $h\in H$. From (6.30), we have $\phi_b'(g)=\phi_b'(gh)$. Namely $\phi_b'(g)$ depends only on $\hat{g}=gH$. Since $Q(g)$ commute with the projection on $\mathfrak{L}_a$ and $\mathfrak{L}_b$, (4.24) holds for ϕ_a' and ϕ_b' separately. Hence we have the required property for $\mathfrak{L}_b$ part. We now turn our attention to $\phi_a'(g)$. From (6.30) we have

$$\phi_a'(h_1h_2h_1^{-1}h_2^{-1}) = 0 \tag{6.33}$$

$$\phi_a'(\exp 2\pi t\mathfrak{l}) = t\phi_a'(\exp 2\pi\mathfrak{l}) = 0 \tag{6.34}$$

if $h_i\in H$, $e^{t\mathfrak{l}}\in H$, and $\exp 2\pi\mathfrak{l}=1$. Hence $\phi_a'(h)=0$ for $h\in H_1$ and $\phi_a'(g)$ depends only on gH_1. Further $\phi_a'(h)=0$ for $[h]\in H_2$ and hence $\phi_a'(h)$ depends only on $[[h]]\in K$. From

$$\phi_a'(\exp t\mathfrak{l}) = t\phi_a'(\exp \mathfrak{l}) \tag{6.35}$$

$\phi_a'(h)$ is real linear when considered as a function of $\log h$.

Let K_1 be the set of $k\in K$ for which $\phi_a'(k)=0$. It is a connected subgroup of K due to (6.30) and (6.35). Obviously K_1 must be $\tau(G)$ invariant. The Lie algebra $\mathfrak{k}_1'$ of K/K_1 can be identified with a real linear subset of $\mathfrak{L}_a$, which spans $\mathfrak{L}_a$. The inner product of $\mathfrak{L}_a$ induces an inner product of $\mathfrak{k}_1'+i\mathfrak{k}_1'$, which is $\tau(G)$ invariant, and positive semidefinite. Since K is commutative, $\tau'(g)$ defined on K and hence on K/K_1 depends only on $\hat{g}=gH\in\hat{G}$. Hence we write it as $\tau(\hat{g})$. We may also write $\phi_a'(h)=\log[[[h]]]$ for $h\in H$ according to the above identification.

Finally, for each $\hat{g}\in\hat{G}$, we fix $\gamma(\hat{g})\in G$ such that $\hat{g}=\gamma(\hat{g})H$. We shall compute

$$
\begin{aligned}
(6.36) \qquad \log[\alpha(\hat{g}_1, \hat{g}_2)] &= \phi'_a(\gamma(\hat{g}_1\hat{g}_2)^{-1}\gamma(\hat{g}_1)\gamma(\hat{g}_2)) \\
&= \phi'_a(\gamma(\hat{g}_1)\gamma(\hat{g}_2)) + Q(\gamma(\hat{g}_1)\gamma(\hat{g}_2))^*\phi'_a(\gamma(\hat{g}_1\hat{g}_2)^{-1}) \\
&= \phi'_a(\gamma(\hat{g}_1)\gamma(\hat{g}_2)) + Q(\gamma(\hat{g}_1\hat{g}_2))^*\phi'_a(\gamma(\hat{g}_1\hat{g}_2)^{-1}) \\
&= \phi'_a(\gamma(\hat{g}_1)\gamma(\hat{g}_2)) - \phi'_a(\gamma(\hat{g}_1\hat{g}_2)) \\
&= \phi'_a(\gamma(\hat{g}_2)) + Q(\hat{g}_2)^*\phi'_a(\gamma(\hat{g}_1)) - \phi'_a(\gamma(\hat{g}_1\hat{g}_2))\,.
\end{aligned}
$$

The orthogonal projection on $\mathfrak{k}'$, in the real Hilbert space $\mathfrak{L}_a$ with respect to the real part of the complex inner product of $\mathfrak{L}_a$, commutes with $Q(g)$, $g\in G$. Applying it on (6.36) and using (6.31), we see that $[\alpha(\hat{g}_1, \hat{g}_2)]$ is cohomologous to 0. Thus K_1 must be an annihilator.

We now define ϕ_a as in the theorem and $\phi_b(g)=(\phi'_a(g)-\phi_a(g))\oplus\phi'_b(g)$. We have $\phi_b(h)=0$ for $h\in H$. Q.E.D.

We now study further structure of ϕ'_a. For this purpose, it is convenient to decompose at the start the representation $Q(g)$ of G on the finite dimensional $\mathfrak{L}_a$ into irreducible representations $Q_j(g)$ on $P_j\mathfrak{L}_a$ where P_j are projections. Then $P_j\phi'_a\in Z^1(G, P_j\mathfrak{L}_a)$ and we can discuss each $P_j\phi'_a$. Equivalently, we shall assume that $Q(g)$, $g\in G$ is already an irreducible representation on $\mathfrak{L}_a$.

Let H_{a1} be the identity component of the subgroup H_a consisting of all $g\in G$ such that $Q(g)=1$ on the subspace $\mathfrak{L}_a$. Let H_{01} be the set of all $h\in H_{a1}$ such that $\phi'_a(h)=0$. Since $Q(g_1gg_1^{-1})=Q(g_1)Q(g_1^{-1})=1$ if $Q(g)=1$, H_a and H_{a1} must be invariant subgroup of G. If $h\in H_{01}$, then

$$(6.37) \quad \phi'_a(ghg^{-1}) = Q(g)\phi'_a(gh)+\phi'_a(g^{-1}) = Q(g)\phi'_a(g)+\phi'_a(g^{-1}) = 0\,.$$

Hence $ghg^{-1}\in H_{01}$ and H_{01} is an invariant subgroup of G.

By the same reasoning as the proof of the previous theorem, $\hat{H}=H_{a1}/H_{01}$ is faithfully represented by a real linear subset $\mathfrak{L}_{a1}$ of $\mathfrak{L}_a$ ((6.30) and (6.35)) and hence it is abelian, does not contain any compact subgroup and is connected. $Q(g)$ on $\mathfrak{L}_{a1}$ coincides with the adjoint representation $\mathrm{Ad}\,(g)$ on H_{a1}.

Let P be the orthogonal projection on $\mathfrak{L}_{a1}$ in the real Hilbert space $\mathfrak{L}_a$ with respect to the real part of the complex inner product of $\mathfrak{L}_a$. Let $\phi''_a(g)=P\phi'_a(g)$. Since $\mathfrak{L}_{a1}$ is invariant under $Q(g)$, ϕ''_a

belongs to $Z^1(G, \mathfrak{L}_{a1})$. Let H_0 be the subgroup of G consisting of all $g\in H_a$ such that $\phi_a''(g)=0$. By (6.37), it is an invariant subgroup. H_a/H_0 is faithfully represented by $\mathfrak{L}_{a1}$ and is identifiable with $\hat{H}$. $G_a=G/H_a$ is faithfully represented by $g_a=gH_a\rightarrow \mathrm{Ad}\,(g)$ and H_a/H_0 is maximal abelian in G/H_0. Since G_a is a connected Lie group with a faithful finite dimensional unitary representation, it is a direct product of a compact group and R^m.

For each $g\in G$, there exists an $h\in H_{a1}$ such that $\phi_a'(h)=\phi_a''(g)$ and $\xi(g)\equiv hH_0\in\hat{H}$ is uniquely determined. Let $\hat{H}\times G_a$ be the semidirect product with the multiplication law

$$(h_1, g_1)(h_2, g_2) = (h_1\tau(g_1)h_2, g_1g_2) \tag{6.38}$$

for $h_1, h_2\in\hat{H}$, $g_1, g_2\in G_a$. Then the mapping $g\rightarrow(\xi(g), gH_a)$ is a homomorphism onto $\hat{H}\times G_a$. If $g_1H_a=g_2H_a$, then $g_1^{-1}g_2\in H_a$ and $Q(g_1)=Q(g_2)$. If $\xi(g_1)=\xi(g_2)$ in addition, then $\phi_a''(g_1^{-1}g_2)=Q(g_2)^*\phi_a''(g_1^{-1})+\phi_a''(g_2)=Q(g_1)^*\phi_a''(g_1^{-1})+\phi_a''(g_2)=\phi_a''(g_2)-\phi_a''(g_1)=0$. Hence $g_1^{-1}g_2\in H_0$. Therefore $gH_0\in G/H_0\rightarrow(\xi(g), gH_a)$ is an isomorphism of G/H_0 onto $\hat{H}\times G_a$.

We now have the following structure: (1) a commutative group $\hat{H}$ isomorphic to R^n, (2) a real inner product μ on $\hat{H}$, (3) a connected subgroup G_a of the orthogonal group on the real Hilbert space $L_2(H, \mu)$, and (4) an invariant subgroup H_0 of G such that G/H_0 is isomorphic to the semidirect product $\hat{H}\times G_a$.

Conversely, suppose that such structure is given and $gH_0\rightarrow(\xi(g), \eta(g))$ is the isomorphism of G/H_0 onto $\hat{H}\times G_a$. Then $Q(g)=\eta(g)$ is orthogonal matrix on $L_2(\hat{H}, \mu)$, $\xi(g)\in Z^1(G, L_2(\hat{H}, \mu))$ and

$$H_a = \{g; Q(g) = 1,\ g\in G\}\,, \tag{6.39}$$

$$H_0 = \{g; \xi(g) = 0,\ g\in H_a\}\,, \tag{6.40}$$

$\hat{H}=H_a/H_0$ and $G_a=G/H_a$.

Finally we discuss $\phi_a'(g)-\phi_a''(g)=\phi_a'''(g)$, which belongs to $Z^1(G, \mathfrak{L}_a)$. If $g\in H_{a1}$, then $\phi_a'''(g)=0$, namely $\phi_a'''(g)$, $g\in H_a$ depends only on $gH_{a1}\in G_a/H_{a1}$. We see from (6.31) and the continuity of $\phi_a'''(h)$ that $Q(g)=1$ for all $g\in G$ on $\phi_a'''(h)$, $h\in H_a$. (H_a/H_{a1} is countable.) Since Q is assumed to be an irreducible representation, either $\phi_a'''(h)=0$ for all $h\in H_a$ or $Q(g)=1$ for all $g\in G$ and $H_a=H_{a1}$. In either case we may assume $\phi_a'''(h)=0$ for $h\in H_a$.

Now $Q(g)$ and $\phi_a'''(g)$ depends only on $gH_a \in G/H_a$ and Q is a faithful unitary representation of $G/H_a = G_a$. If G_a has no nontrivial connected invariant abelian subgroup, then G_a is compact and hence ϕ_a''' is a coboundary by Theorem 7.1. Suppose that G_a has an invariant abelian subgroup G_c isomorphic to R^m, $m \neq 0$. The projection $E_{G_c}(0)$ on $Q(G_c)$ invariant vectors commutes with $Q(g)$, $g \in G$. Since Q is an irreducible and faithful representation of G_a, $E_{G_c}(0) = 0$. Since $\mathfrak{L}_a$ is of finite dimension, $\phi_a''(g)$ is coboundary due to Theorem 7.3.

Summarizing, we have

Theorem 6.7. (1) *Let G be a connected Lie group, H_a and H_0 be invariant subgroups of G such that (i) $H_a \supset H_0$ and $\hat{H} \equiv H_a/H_0$ is isomorphic to R^n, (ii) there exists an isomorphism $g \in G/H_0 \to (\xi(g), \eta(g)) \in \hat{H} \times G_a$, from G onto the semidirect product of $\hat{H}$ with $G_a \equiv G/H_a$, where the adjoint representation of G on $\hat{H}$ canonically induces the action of G_a on $\hat{H}$, (iii) G_a is a direct product of a compact group and R^m, (iv) $\hat{H}$ is maximal abelian in G/H_0, and (v) there exists* ad(G) *invariant inner product μ on $\hat{H} + i\hat{H}$. Then $\phi(g) \equiv \xi(gH_0) \in L_2(\hat{H} + i\hat{H}, \mu) \equiv \mathfrak{L}$ is in $Z^1(G, \mathfrak{L})$ relative to the unitary representation* $Q(g) = \mathrm{ad}(g)$,

$$H_a = \{g ; Q(g) = 1, \ g \in G\} , \tag{6.41}$$

$$H_0 = \{g ; \phi(g) = 0, \ g \in H_a\} . \tag{6.42}$$

(2) *Let G be a connected Lie group, $Q(g)$ be an irreducible finite dimensional continuous unitary representation on $\mathfrak{L}$, $\phi \in Z^1(G, \mathfrak{L})$. Let H_a and H_0 be defined by* (6.41) *and* (6.42). *Then, (a) H_a and H_0 are invariant subgroups satisfying (i)-(iv) of* (1), *(b) $\phi(h)$ depends only on $hH_0 \in H_a/H_0 \equiv \hat{H}$ and ϕ gives an isomorphism of $L_2(\hat{H} + i\hat{H}, \mu)$ onto $\mathfrak{L}$ for an appropriate μ, and (c) $\phi(g) - \xi(gH_0)$ is a coboundary.*

(3) *Let G be a connected Lie group, H be a connected invariant subgroup, $Q(g)$ be continuous unitary representation of G on $\mathfrak{L}$ without identity subrepresentation, $Q(g) = 1$ for $g \in H$, $\phi \in Z^1(G, \mathfrak{L})$. Assume that $\mathfrak{L}$ is spanned by $\phi(h)$, $h \in H$. Let H_a and H_0 be defined by* (6.41) *and* (6.42). *Then the conclusion (a), (b) and (c) of (2) hold.*

In the above, $L_2(\hat{H} + i\hat{H}, \mu)$ denotes the quotient of $\hat{H} + i\hat{H}$ by the subspace of vectors with vanishing μ norm.

The proof of (3) can be obtained by the following slight modification of the proof of (2). Since an identity subrepresentation is assumed to be absent, we have $H_a = H_{a1}$. We have $G_a = G_K \times G_c$ where G_K is compact, $G_c = R^m$ and $\times$ denotes the direct product. As before, $(1-E_{Gc}(0))\phi_a'''$ is a coboundary due to Theorem 7.3. On $E_{Gc}(0)\phi_a'''$, $Q(g)=1$ for $g \in G_c$ and hence $Q(g)=1$ for all $g \in G$ on $E_{Gc}(0)\phi_a'''(h)$, $h \in G_c$ by (6.31). By the absence of the identity subrepresentation, $E_{Gc}(0)\phi_a'''(G_c)=0$. Hence $E_{Gc}(0)\phi_a'''$ is a cycle for the compact group G_K and must be a coboundary by Theorem 7.1.

Q.E.D.

§7. Determination of Cocycles

Theorem 7.1. *For a compact group* G, $Z^1(G, \mathfrak{L}) = B^1(G, \mathfrak{L})$.

Proof. Since G is compact and $\phi(g)$ is continuous on G, $\phi(g)$ is uniformly bounded. We integrate (4.24) with respect to g_1 using the invariant measure on G and obtain

$$Q(g_2)^* \int \phi(g_1) \mathrm{d}\mu(g_1) = \int \phi(g_1 g_2) \mathrm{d}\mu(g_1) - \phi(g_2) \tag{7.1}$$

If we set

$$\Omega = \int \phi(g) \mathrm{d}\mu(g) \tag{7.2}$$

and use the invariance of μ, we obtain

$$\phi(g_2) = (1 - Q(g_2)^*)\Omega \tag{7.3}$$

where Ω is a vector independent of g_2.

Lemma 7.2. *If G is abelian and $Q(g)$, $g \in G$ does not contain the identity representation, then* $Z^1(G, \mathfrak{L}) = B^1(G, \bar{D}^+)$.

Proof. The basic equation for our discussion is

$$(Q(g_1)^* - 1)\phi(g_2) = (Q(g_2)^* - 1)\phi(g_1) \,. \tag{7.4}$$

This equation is obtained as the difference of the equation (4.24) and the same equation with g_1 and g_2 interchanged, where the commutativity $g_1 g_2 = g_2 g_1$ has been used.

Let $E_{\mathrm{I}_1}(\Delta)$ be the spectral projection for $Q(e^{t\mathrm{I}_1}) = \int e^{it\lambda} E_{\mathrm{I}_1}(\mathrm{d}\lambda)$ and $\mathrm{K}(\mathrm{I}_1)$ be the operator introduced in §6. Then $\mathrm{K}(\mathrm{I}_1)^{-1}$ is bounded for

$E_{\mathfrak{l}_1}((-\infty, \infty)-(-\varepsilon, \varepsilon))$ for any $\varepsilon>0$. Hence

$$[1-E_{\mathfrak{l}_1}((-\varepsilon, \varepsilon))]\phi(g) = (1-Q(g)^*)\Omega_1\,, \tag{7.5}$$

where

$$\Omega_1 = \mathrm{K}(-\mathfrak{l}_1)^{-1}[1-E_{\mathfrak{l}_1}((-\varepsilon, \varepsilon))]\int h(t)\,\phi(e^{t\mathfrak{l}_1})\,\mathrm{d}t$$

is a vector in the Hilbert space $\mathfrak{L}$ independent of g. Since $\mathfrak{l}_1$ is arbitrary, we see that

$$(1-E(\Delta))\phi(g) = (1-Q(g)^*)\Omega_\Delta \tag{7.6}$$

for a vector Ω_Δ in $(1-E(\Delta))\mathfrak{L}$ where E is the joint spectral projection for $Q(e^{t_j\mathfrak{l}_j})$ and Δ is any neighbourhood of the origin.

We now define a functional Ω over a certain set of vectors in $\mathfrak{L}$ by

$$(\Omega, \psi) = (\phi(g), \chi) \tag{7.7}$$

if

$$\psi = (1-Q(g))\chi\,. \tag{7.8}$$

First we show that

$$\psi = (1-Q(g_1))\chi_1 = (1-Q(g_2))\chi_2 \tag{7.9}$$

implies

$$(\phi(g_1), \chi_1) = (\phi(g_2), \chi_2)\,. \tag{7.10}$$

From (7.9) and (7.6) we have

$$\begin{aligned}(\phi(g_1), [1-E(\Delta)]\chi_1) &= (\Omega_\Delta, (1-Q(g_1))\chi_1) = (\Omega_\Delta, \psi)\\ &= (\Omega_\Delta, (1-Q(g_2))\chi_2) = (\phi(g_2), [1-E(\Delta)]\chi_2)\,.\end{aligned}$$

By taking the limit of Δ shrinking to 0 and using the assumption $E(0)=0$, we obtain (7.10). Thus (7.7) does not depend on how ψ is expressed in the form of eq. (7.8). Furthermore

$$([1-E(\Delta)]\psi, \Omega) = (\psi, \Omega_\Delta), \quad \psi\in\mathfrak{L}\,, \tag{7.11}$$

for any neighbourhood Δ of the origin.

Next we show that Ω can be extended to be linear. Let

$$\sum_{i=1}^{N}(1-Q(g_i))\chi_i = 0\,. \tag{7.12}$$

Then (7.6) implies

$$\sum_{i=1}^{N}(\phi(g_i),\ [1-E(\Delta)]\chi_i) = 0\,.$$

By taking the limit $\Delta\to 0$, we have again

$$\sum_{i=1}^{N}(\phi(g_i),\ \chi_i) = 0\,. \tag{7.13}$$

Hence Ω has a linear extension, which we denote again by Ω.

Next, we use the fact that $K(I_i)^{-1/2}[1-Q(e^{tI_i})]$ is bounded and $[1-Q(e^{tI_i})]^{-1}K(I_i)^{1/2}E(\Delta)$ is also bounded for a fixed t if Δ is sufficiently small. If

$$\psi = (1-Q(e^{tI_i}))\chi\,, \tag{7.14}$$

ψ is in the domain of $K(I_i)^{-1/2}$ and any vector ψ in $E(\Delta)$ times the domain of $K(I_i)^{-1/2}$ can be obtained by (7.14) with the following χ:

$$\chi = [1-Q(e^{tI_i})]^{-1}K(I_i)^{1/2}(K(I_i)^{-1/2}E(\Delta)\psi)\,. \tag{7.15}$$

In particular, there is a constant a_i such that

$$||E(\Delta)\chi|| \leq a_i||K(I_i)^{-1/2}E(\Delta)\psi||\,. \tag{7.16}$$

Hence

$$|(\Omega,\ E(\Delta)\psi)| \leq a_i||\phi(e^{tI_i})||\ ||K(I_i)^{-1/2}E(\Delta)\psi||\,. \tag{7.17}$$

Let $I_1\cdots I_n$ span the Lie algebra. Since G is commutative, $K(I_i)$ commutes with each other.

If we split up Δ into mutually disjoint n regions Δ_i, $i=1,\cdots,n$ such that $(p_1\cdots p_n)\in\Delta_i$ implies $p_i\geq p_j$ for any j, then $E(\Delta_i)$ are mutually orthogonal projections with the sum $E(\Delta)$. We have

$$||K^{-1/2}E(\Delta_i)\psi||^2 \geq (1/n)||K(I_i)^{-1/2}E(\Delta_i)\psi||^2\,. \tag{7.18}$$

Therefore if ψ is in the domain of $K^{-1/2}$, then $E(\Delta_i)\psi$ is in the domain of $K(I_i)^{-1/2}$. From (7.17) and (7.18), we have

$$|(\Omega,\ E(\Delta_i)\psi)| \leq n^{1/2}a_i||\phi(e^{tI_i})||\ ||K^{-1/2}E(\Delta_i)\psi||\,.$$

Hence

$$|(\Omega,\ E(\Delta)\psi)| \leq a||E(\Delta)\psi||_-\,, \tag{7.19}$$

$$a = n\max_i a_i||\phi(e^{tI_i})||\,.$$

We have the estimate (7.19) for any ψ in D^-. Together with (7.11), we see that Ω is in the dual of D^-, namely

$$\Omega \in \bar{D}^+ . \tag{7.20}$$

From (7.7) and (7.8), we have

$$\phi(g) = (1-Q(g)^*)\Omega , \tag{7.21}$$

which is the required result. Q.E.D.

Theorem 7.3. *Let H be an invariant abelian subgroup of G and $\phi \in Z^1(G, \mathfrak{L})$. Then*

$$\phi(g) = \phi_1(g)+\phi_2(g), \quad \phi_1(g) = (1-Q(g)^*)\Omega \tag{7.22}$$

where $\phi_2(g) \in Z^1(G, E_H(0)\mathfrak{L})$, $\Omega \in \bar{D}^+(H)$ and $(1-Q(g)^*)\Omega \in \mathfrak{L}$ for all $g \in G$, $E_H(0)$ is the projection on the subspace of vectors invariant under H and $\bar{D}^+(H)$ is the $\bar{D}^+$ for H.

Proof. Since H is an invariant subgroup, $E_H(0)$ commutes with all $Q(g)$, $g \in G$. Thus

$$\phi_1(g) = (1-E_H(0))\phi(g) \tag{7.23}$$

$$\phi_2(g) = E_H(0)\phi(g) \tag{7.24}$$

are both in $Z^1(G, \mathfrak{L})$. By the previous lemma, $\phi_1(h)$, $h \in H$ is of the form

$$\phi_1(h) = (1-Q(h)^*)\Omega \tag{7.25}$$

where Ω is in $\bar{D}^+(H)$. We shall show that

$$\phi_1(g) = (1-Q(g)^*)\Omega . \tag{7.26}$$

From the definition equation for a cocycle, we obtain

$$(1-Q(h)^*)\phi_1(g) = \phi_1(h)-Q(g)^*\phi_1(ghg^{-1}) . \tag{7.27}$$

If we put

$$\begin{aligned} \chi &\equiv (1-Q(h))\psi-(1-Q(ghg^{-1}))Q(g)\psi \\ &= (1-Q(g))(1-Q(h))\psi \in D^-(H) , \end{aligned} \tag{7.28}$$

we obtain, from (7.25) for $h \in H$ and $ghg^{-1} \in H$ and (7.27),

$$(\chi, \Omega) = ([1-Q(h)]\psi, \phi_1(g)) . \tag{7.29}$$

Or somewhat differently written,

$$([1-Q(g)]\chi_1, \Omega) = (\chi_1, \phi_1(g)) \tag{7.30}$$

where $\chi_1=[1-Q(h)]\psi$, $\psi\in\mathfrak{L}$.

If we make the simultaneous spectral decomposition of $Q(h)$, $h\in H$, then $K(H)$ is a multiplication of a function $\sum_i(1-\tilde{h}(p_i))\equiv \mathrm{K}(p)$. Since $Q(g)\mathrm{K}(\mathfrak{l})Q(g)^*=\mathrm{K}(\mathrm{Ad}\,(g)\mathfrak{l})$, $\mathrm{K}(H)_g\equiv Q(g)^*\mathrm{K}(H)Q(g)$ is a multiplication of a function $\mathrm{K}(\mathrm{Ad}\,(g)p)$. Since $\mathrm{Ad}\,(g)$ is non singular and continuous in g, $K(H)^{1/2}K(H)_g^{-1/2}$ and $K(H)^{-1/2}K(H)_g^{1/2}$ are locally bounded and hence uniformly bounded on a compact set. This implies that $Q(g)$ maps $D^-(H)$ into $D^-(H)$, $D^+(H)$ into $D^+(H)$ and is uniformly bounded for g in a compact set, with respect to the norms of $D^-(H)$ and $D^+(H)$, respectively.

Since the linear combinations of $[1-Q(h)]\psi$, $\psi\in\mathfrak{L}$ are dense in $D^-(H)$ as is seen from the simultaneous spectral decomposition of $Q(h)$, $h\in H$, we see that (7.30) holds for all $\chi_1\in D^-(H)$ and

$$(\chi_1, (1-Q(g)^*)\Omega) = (\chi_1, \phi_1(g)) . \tag{7.31}$$

Q.E.D.

Lemma 7.4. *If H is an invariant subgroup of G and $Z^1(H, \mathfrak{L}) = B^1(H, \mathfrak{L})$, then $\phi\in Z^1(G, \mathfrak{L})$ is always of the form*

$$\phi(g) = (1-Q(g)^*)\Omega+\phi_2(g) , \tag{7.32}$$

$\Omega\in(1-E_H(0))\mathfrak{L}$, $\phi_2(g)\in Z^1(G/H, E_H(0)\mathfrak{L})$ where $E_H(0)$ is the subspace of vectors invariant under H.

Proof. The proof is exactly the same as the previous one. In the present case, $Z^1(H, \mathfrak{L})=B^1(H, \mathfrak{L})$ is 0 on $E_H(0)\mathfrak{L}$ and hence $\phi_2(g)$ is in $Z^1(G/H, E_H(0)\mathfrak{L})$.

Determination of cocycles. Assume G is a connected Lie group. From the above theorems, we can analyze given $\phi(g)$, $g\in G$ in the following way. Take maximal invariant abelian connected subgroup G_1 of G and apply Theorem 7.3, Theorem 6.6 and Theorem 6.7. The problem is then reduced to $G/G_1=\hat{G}_1$. Continue this procedure until $\hat{G}$ has no invariant connected abelian subgroup. If the original G is solvable, then $\hat{G}$ is trivial and the problem is completely solved. Otherwise we are left with a semisimple group. If it has invariant compact subgroup, we can apply Theorem 7.1 and Lemma 7.4 and proceed until $\hat{G}$ has no invariant compact subgroup.

We have not solved the problem for a semisimple Lie group.

Summarizing, we have the following.

Theorem 7.5. *Let G be a Lie group and G_j, $j=1, \cdots, n$ be an ascending sequence of invariant connected subgroups of G such that $G_1=1$, each G_j/G_{j-1} $(j=2, \cdots, n)$ is abelian and G/G_n is semisimple. Let $\mathfrak{L}$ be a Hilbert space, Q be a continuous unitary representation of G on $\mathfrak{L}$ and $\phi \in Z^1(G, \mathfrak{L})$.*

Then there exist mutually orthogonal invariant subspaces $\mathfrak{L}_j^t$, $j=1, \cdots, n$ and $\mathfrak{L}_j^a$, $j=1, \cdots, n$ of $\mathfrak{L}$ such that

(7.33) $$\mathfrak{L} = (\bigoplus_{j=1}^{n} \mathfrak{L}_j^t) \oplus (\bigoplus_{j=1}^{n} \mathfrak{L}_j^a),$$

(7.34) $$Q(g) = (\oplus Q_j^t(g)) \oplus (\oplus Q_j^a(g)),$$

(7.35) $$\phi(g) = (\oplus \phi_j^t(g)) \oplus (\oplus \phi_j^a(g)),$$

(7.36) $$Q_j^t(g) = Q_{j-1}^a(g) = 1 \quad for \quad g \in G_j,\ j=2, \cdots, n,$$

(7.37) $$\phi_j^t(g) = \hat{\phi}_j^t(gG_j), \quad j=1, \cdots, n,$$

(7.38) $$\hat{\phi}_j^t \in B^1(G/G_j,\ \bar{D}^+(G_{j+1}/G_j)), \quad j=1, \cdots, n-1,$$

(7.39) $$\hat{\phi}_n^t \in Z^1(G/G_n, \mathfrak{L}_n),$$

(7.40) $$\phi_j^a(g) = 0 \quad for \quad g \in G_j,\ j=1, \cdots, n-1,$$

(7.41) $$Q_n^a(g) = 1 \quad for \quad g \in G.$$

L_j^a is finite dimensional and is spanned by $\phi_j^a(g)$, $g \in G_{j+1} (G_{n+1}=G)$, Q_j^a, $j=1, \cdots, n-1$ has no identity subrepresentation. (The structure of ϕ_j^a is given by Theorem 6.7.)

If G is solvable and $\mathfrak{L}$ has a finite dimension, then we can reformulate Theorem 7.5 as follows:

Corollary 7.6. *Let G be a solvable Lie group, $\mathfrak{L}$ be a Hilbert space of finite dimension, Q be a continuous unitary representation of G on $\mathfrak{L}$ and $\phi \in Z^1(G, \mathfrak{L})$.*

Then there exists an ascending sequence of invariant connected subgroups G_j of G and mutually orthogonal invariant subspaces $\hat{\mathfrak{L}}_j$ $(j=1, \cdots, n)$ such that $G_1=\{1\}$, $G_n=G$, G_j/G_{j-1} is abelian $(j=2, \cdots, n)$, $\mathfrak{L}=\oplus \hat{\mathfrak{L}}_j$, $Q(g)=\oplus \hat{Q}_j(g)$, $\phi(g)=\sum \hat{\phi}_j(g)$, $\hat{Q}_j(g)=1$ for $g \in G_{j+1}$, $\hat{\phi}_j(g)=0$ for $g \in G_j$, $\hat{\phi}_j(g) \in \hat{\mathfrak{L}}_j$ $(j=1, \cdots, n-1)$, $\hat{\mathfrak{L}}_j$ is spanned by $\hat{\phi}_j(g)$, $g \in G_{j+1}$

$(j=1, \cdots, n-1)$, $\hat{\phi}_n(g)$ is a coboundary on $\mathfrak{L}$ and $\hat{Q}_j(g)$ is either an identity representation or without any identity subrepresentation $(j=1, \cdots, n-1)$. (The structure of $\hat{\phi}_j$ is given by Theorem 6.7.)

To obtain this form, we proceed in exactly the same manner as Theorem 7.5. All ϕ_j^b are coboundaries and all coboundaries are lumped together as $\hat{\phi}_n$ in the present Corollary. Each Q_j^a in Theorem 7.5 can be split into a direct sum of an identity representation and a representation without any identity subrepresentation. Denoting $G_{j0}=\{g\in G_j,\ \phi_j(g)=0\}$, G_j/G_{j0} is split into a direct sum of two invariant subgroups G_{jA} and G_{jB} of G/G_{j0}, where G_A consists of some central elements of G/G_{j0} and G_B does not contain any central elements of G/G_{j0}. If we inflate the ascending sequence $\{G_j\}$ of Theorem 7.5 by inserting the invariant subgroup $\{g;\ gG_0\in G_{jA}\}$ between G_{j-1} and G_j, we obtain the structure in Corollary 7.6 where G_j and n are different from those in Theorem 7.5.

§8. Determination of $c(g)$

When $\phi(g)$ is determined, $c(g)$ is to be determined from

$$
\begin{aligned}
(8.1)\qquad c(g_1g_2)-c(g_1)-c(g_2) &= -\mathrm{Im}\,(\phi(g_1g_2),\ \phi(g_2)) \\
&= \mathrm{Im}\,(\phi(g_1),\ \phi(g_2^{-1}))\quad (\mathrm{mod}\ 2\pi)\,.
\end{aligned}
$$

Obviously, if $c_0(g)$ is one solution a general solution for $e^{ic(g)}$ is obtained by multiplying $e^{ic_0(g)}$ with an arbitrary unitary character of G. In the following, we omit mod 2π. All equations involving $c(g)$ linearly are to be understood modulo 2π.

Lemma 8.1. *If $Z^2(G, R)=B^2(G, R)$, then (8.1) has a solution.*

Proof. Sufficient to prove that the righthand side of (8.1) belongs to $Z^2(G, R)$. Let $d(g_1, g_2)=(\phi(g_1), \phi(g_2^{-1}))$. Then

$$
\begin{aligned}
(8.2)\qquad & d(g_1,\ g_2g_3)-d(g_1g_2,\ g_3) \\
&= \{(\phi(g_1),\ Q(g_2)\phi(g_3^{-1}))+(\phi(g_1),\ \phi(g_2^{-1}))\} \\
&\quad - \{(Q(g_2)^*\phi(g_1),\ \phi(g_3^{-1}))+(\phi(g_2),\ \phi(g_3^{-1}))\} \\
&= d(g_1,\ g_2)-d(g_2,\ g_3)\,.
\end{aligned}
$$

This proves that $\mathrm{Im}\, d(g_1, g_2)\in Z^2(G, R)$.

For a simply connected semisimple group and a compact group,

$Z^2(G, R) = B^2(G, R)$.

Theorem 8.2. *If* $\phi_1 \in B^1(G, \mathfrak{L})$ *and* $\phi_2 \in Z^1(G, \mathfrak{L})$, *then both* $(\phi_1(g_1), \phi_2(g_2^{-1}))$ *and* $(\phi_2(g_1), \phi_1(g_2^{-1}))$ *are coboundaries.*

Proof. Let $\phi_1(g) = (1 - Q(g)^*)\Omega$, $\Omega \in \mathfrak{L}$, $c_1(g) = (\Omega, \phi_2(g^{-1}))$, $c_2(g) = (\phi_2(g), \Omega)$. Then

$$(\phi_1(g_1), \phi_2(g_2^{-1})) = c_1(g_1) + c_1(g_2) - c_1(g_1 g_2), \tag{8.3}$$

$$(\phi_2(g_1), \phi_1(g_2^{-1})) = c_2(g_1) + c_2(g_2) - c_2(g_1 g_2). \tag{8.4}$$

To treat the case where Ω is outside of $\mathfrak{L}$, we need a few preparations.

Lemma 8.3. *Let* $g = e^{t\mathrm{I}}$, $Q(g) = \int e^{it\lambda} \mathrm{d}E(\lambda)$, $E_A \equiv E([-A, A])$. *Then*

$$\Omega_A(\mathrm{I}) \equiv \lim_{t \to 0} t^{-1} E_A \phi(g) \tag{8.5}$$

exists in $\mathfrak{L}$ *for any* $\phi \in Z^1(G, \mathfrak{L})$. $\omega(\mathrm{I}) \equiv E(0)\Omega_A(\mathrm{I})$ *is then independent of* A *and*

$$\phi(g) = (1 - Q(g)^*)\Omega + t\omega(\mathrm{I}) \tag{8.6}$$

where Ω *is in* $\bar{D}_+$ *for one parameter group* $\{e^{t\mathrm{I}}\}$. $(1 - E_A)\Phi \in \mathfrak{L}$ *and* $(-\mathrm{d}Q(\mathrm{I}) E_A)^* \Omega = \Omega_A(\mathrm{I}) - \omega(\mathrm{I})$ *where* $\mathrm{d}Q(\mathrm{I}) = \left(\frac{d}{dt} Q(e^{t\mathrm{I}}) \text{ at } t = 0\right)$. *If* $f \in \mathcal{D}$ *and* $\int f(t)\,\mathrm{d}t = 0$, *then* $Q(f)\Omega$ *is in* $\mathfrak{L}$ *where* $Q(f) = \int Q(t) f(t)\,\mathrm{d}t$. *Let* D_G *be the set of* $\int Q(g) \Psi f(g)\,\mathrm{d}g$ *for* $\Psi \in \mathfrak{L}$ *and* f *in the class* $\mathcal{D}$. *For* $\psi \in D_G$, *the limit*

$$(\psi, \Omega(\mathrm{I})) = \lim_{t \to 0} (\psi, t^{-1}\phi(g)) \tag{8.7}$$

exists and satisfies

$$\Omega(\lambda_1 \mathrm{I}_1 + \lambda_2 \mathrm{I}_2) = \lambda_1 \Omega(\mathrm{I}_1) + \lambda_2 \Omega(\mathrm{I}_2) \tag{8.8}$$

$$\mathrm{d}Q(\mathrm{I}_1)\Omega(\mathrm{I}_2) - \mathrm{d}Q(\mathrm{I}_2)\Omega(\mathrm{I}_1) = \Omega([\mathrm{I}_1, \mathrm{I}_2]). \tag{8.9}$$

Proof. Since $Q(g)E_A$, $g = e^{t\mathrm{I}}$ is holomorphic in t, it follows from (6.17) (ψ replaced by $E_A\psi$) that $\phi_A(e^{t\mathrm{I}})$ is strongly C^∞ in t, where $\phi_A(g) = E_A\phi(g)$. Since $\phi_A(1) = 0$, we have $\|\phi_A(e^{t\mathrm{I}})\| \le ta$ for $|t| \le T$ and for a constant a.

Since $\phi \in Z^1(G, \mathfrak{L})$, we have

$$\phi(e^{t\mathfrak{l}}) = \sum_{j=0}^{n-1} Q(e^{-(jt/n)\mathfrak{l}})\phi(e^{(t/n)\mathfrak{l}}) . \tag{8.10}$$

Therefore, we have

$$||\phi_A(e^{t\mathfrak{l}}) - n\phi_A(e^{(t/n)\mathfrak{l}})|| \leq nc(t)||\phi(e^{(t/n)\mathfrak{l}})|| , \tag{8.11}$$

$$c(t) = \sup_{0\leq s\leq t} ||(Q(e^{-s\mathfrak{l}}) - 1)E_A|| . \tag{8.12}$$

Hence

$$||\phi_A(e^{t\mathfrak{l}}) - n\phi_A(e^{(t/n)\mathfrak{l}})|| \leq (1-c(t))^{-1}c(t)||\phi_A(e^{t\mathfrak{l}})|| . \tag{8.13}$$

Since $c(t)$ is of order t for small t, we have for sufficiently small t and a constant b,

$$||\phi_A(e^{t\mathfrak{l}}) - (n^{-1})^{-1}\phi_A(e^{t\mathfrak{l}/n})|| < bt^2 . \tag{8.14}$$

Hence

$$||(t/n)^{-1}\phi_A(e^{t\mathfrak{l}/n}) - (t/m)^{-1}\phi_A(e^{t\mathfrak{l}/m})|| < 2bt . \tag{8.15}$$

Further, we obtain from $\phi(g_1) - \phi(g_2) = Q(g_2)^*\phi(g_1 g_2^{-1})$

$$||\phi_A(e^{t\mathfrak{l}/n}) - \phi_A(e^{s\mathfrak{l}})|| \leq ||\phi_A(e^{(t/n-s)\mathfrak{l}})|| < a|t/n-s| . \tag{8.16}$$

Let $t/n \geq s \geq t/(n+1)$. Then

$$||(t/n)^{-1}\phi_A(e^{t\mathfrak{l}/n}) - s^{-1}\phi_A(e^{s\mathfrak{l}})|| \leq 2(n+1)^{-1}a . \tag{8.17}$$

Hence by taking $t=\varepsilon$ and $s, s' < \varepsilon^2$, we have

$$||s^{-1}\phi_A(e^{s\mathfrak{l}}) - s'^{-1}\phi_A(e^{s'\mathfrak{l}})|| < c\varepsilon \tag{8.18}$$

for some constant c. Therefore $s^{-1}\phi_A(e^{s\mathfrak{l}})$ is a Cauchy sequence and has a limit in $\mathfrak{L}$.

If we denote the one parameter subgroup $\{e^{t\mathfrak{l}}; -\infty < t < \infty\}$ tentatively by H, then $\phi \in Z^1(G, \mathfrak{L})$ implies $\phi \in Z^1(H, \mathfrak{L})$. Hence by Lemma 7.2, we have (8.6). From (8.6), $\omega(\mathfrak{l}) = E(0)\Omega_A(\mathfrak{l})$ is independent of A.

Next by integrating with the weight $h(t)$, we have

$$\int \phi(e^{t\mathfrak{l}})\, h(t)\, \mathrm{d}t = \mathrm{K}(\mathfrak{l})\,\Omega \in \mathfrak{L} . \tag{8.19}$$

Hence

$$(1-E_A)\Omega = \mathrm{K}(\mathfrak{l})^{-1}(1-E_A)\int \phi(^{t\mathfrak{l}})\, h(t)\, \mathrm{d}t \in \mathfrak{L} . \tag{8.20}$$

Because $t^{-1}(1-Q(e^{t\mathfrak{l}}))E_A\psi$ converges to $-\mathrm{d}Q(\mathfrak{l})E_A\psi$ in $D^-(H)$, we also have $(-\mathrm{d}Q(\mathfrak{l})E_A)^*\Omega=\Omega_A(\mathfrak{l})-\omega(\mathfrak{l})\in\mathfrak{L}$ from (8.6). From these two conclusions, $Q(f)\Omega\in\mathfrak{L}$ for $\int f(t)\,\mathrm{d}t=0$, $f\in\mathcal{D}$ follows because $Q(f)(1-E_A)$ and $\mathrm{d}Q(\mathfrak{l})^{-1}Q(f)E_A$ are both bounded.

We now come to the second half of the lemma (which is incidentally not used in later discussions). It is known that D_G is a common domain of any polynomial of infinitesimal generators of $Q(g)$. By splitting Ω in (8.6) into $(1-E_A)\Omega$ and $E_A\Omega$, we immediately have the existence of (8.7). At the same time,

(8.21) $$\phi(g_1g_2)=Q(g_2)^*\phi(g_1)+\phi(g_2)\,,$$

with $g_j=\exp t\mathfrak{l}_j$ implies (8.8) in the first order in t_1, t_2, because $\phi(1)=0$. By using (8.21) repeatedly, we have

(8.22) $$\begin{aligned}\phi(g_1g_2g_1^{-1})&=Q(g_1)Q(g_2)^*\phi(g_1)+Q(g_1)\phi(g_2)+\phi(g_1^{-1})\\&=Q(g_1)\{(Q(g_2)^*-1)\phi(g_1)+\phi(g_2)\}\,.\end{aligned}$$

Hence

(8.23) $$\Omega(g_1\mathfrak{l}_2-\mathfrak{l}_2)=(Q(g_1)-1)\,\Omega(\mathfrak{l}_2)-Q(g_1)\,\mathrm{d}Q(\mathfrak{l}_2)\,\phi(g_1)\,.$$

Taking the first order in t_1, we have (8.9). Q.E.D.

We remark that it is possible to obtain (8.21) in a neighbourhood of the identity from (8.8) and (8.9) provided we have the convergences of several sequences.

Lemma 8.4. *Let $\phi\in Z^1(G,\mathfrak{L})$. If g is in the commutator subgroup of G, then $\phi(g)\in D^-(G)$.*

Proof. We first note that $D^{\pm}$ is invariant under $Q(g)$ for any $g\in G$. The reason is as follows: If $\mathfrak{l}_1\cdots\mathfrak{l}_n$ is a basis of $\mathfrak{g}$, then $\mathrm{Ad}(g)\mathfrak{l}_1\cdots\mathrm{Ad}(g)\mathfrak{l}_n$ is also a basis of $\mathfrak{g}$ because g has an inverse g^{-1} and $\mathrm{Ad}(g)$ is nonsingular. Thus K and $Q(g)\mathrm{K}Q(g)^{-1}\equiv\mathrm{K}_g$ defines the same $D^{\pm}$ and the same topology by Lemma 6.2, which establishes the invariance of $D^{\pm}$ under $Q(g)$.

Next we note the following consequences of (4.24):

(8.24) $$\begin{aligned}\phi(g_1^{-1}g_2^{-1}g_1g_2)&=Q(g_2)^*\phi(g_1^{-1}g_2^{-1}g_1)+\phi(g_2)\\&=Q(g_2)^*Q(g_1)^*\phi(g_1^{-1}g_2^{-1})+Q(g_2)^*\phi(g_1)+\phi(g_2)\\&=Q(g_2)^*Q(g_1)^*Q(g_2)(\phi(g_1^{-1})-\phi(g_2))+Q(g_2)^*\phi(g_1)+\phi(g_2)\\&=Q(g_2)^*(1-Q(g_1^{-1}g_2g_1))\,\phi(g_1)+(1-Q(g_2^{-1}g_1^{-1}g_2))\,\phi(g_2)\end{aligned}$$

where we have used $Q(g)^*\phi(g^{-1})=\phi(1)-\phi(g)=-\phi(g)$. Since $(1-Q(g))\mathfrak{L}\in D^-$ for any g and D^- is invariant under $Q(g)$, we see that $\phi(g)\in D^-$ if $g=g_1^{-1}g_2^{-1}g_1g_2$. Further $\phi(g_ag_b)=Q(g_b)^*\phi(g_a)+\phi(g_b)$ and hence $\phi(g)\in D^-$ if g is a product of elements of the form $g_1^{-1}g_2^{-1}g_1g_2$. Therefore $\phi(g)\in D^-$ for all g in the commutator subgroup of G.

Lemma 8.5. *Let G_1 be the commutator subgroup of a connected Lie group G and $G/G_1=K_1\cdot K_2$ where K_1 is compact and K_2 does not contain a compact subgroup. If $\phi\in Z^1(G,\mathfrak{L})$, $g\in G$, $[g]\in K_1$ where $[g]$ denote the class of g in G/G_1, then $\phi(g)\in D^-$.*

Proof. Let Lie algebras for G, G_1, K_1, K_2 be $\mathfrak{g}, \mathfrak{g}_1, \mathfrak{k}_1, \mathfrak{k}_2$. Then $\mathfrak{g} \bmod \mathfrak{g}_1=\mathfrak{k}_1+\mathfrak{k}_2$. Since $[g]\in K_1$, there exists $\mathfrak{l}\in\mathfrak{g}$, such that $[\mathfrak{l}]\in\mathfrak{k}_1$ and $[g]=\exp[\mathfrak{l}]$. Let $\exp t\mathfrak{l}=g(t)$. Then $[g(1)]=[g]$ and hence $g=g(1)g_1$, $g_1\in G_1$. By the previous lemma, $\phi(g_1)\in D^-$. Therefore, if $\phi(g(1))\in D^-$, then $\phi(g)=Q(g_1)^*\phi(g(1))+\phi(g_1)\in D^-$. We now show that $\phi(g(1))\in D^-$.

Let E_A be defined as before in terms of the spectral projection of $Q(g(t))$. First consider the case where there exists a t_0 such that $[g(t_0)]=1$ and hence $g(t_0)\in G_1$. Then $\phi(g(t_0))\in D^-$. Since $(1-E_A)\mathfrak{L}\subset D^-$, we have $\Psi\equiv E_A\phi(g(t_0))\in D^-$. If A is sufficiently small, then from the known structure of $\phi(g)$ for an abelian group $\{g(t); -\infty<t<+\infty\}$ (Lemma 7.2), we have

$$E_A\phi(g(t)) = \int_{-A-0}^{A+0} F(\lambda)\,\mathrm{d}E(\lambda)\,\Psi \tag{8.25}$$

where $F(\lambda)$ is a function of class $\mathscr{D}$, coinciding with $(1-e^{-it_0\lambda})^{-1}(1-e^{-it\lambda})$ for $\lambda\in[-A, A]$, $\lambda\neq 0$. Therefore

$$E_A\phi(g(t)) = \frac{1}{2\pi}\int \tilde{F}(s)\,\mathrm{d}s Q(g(-s))\,\Psi\,. \tag{8.26}$$

If we can show the continuity of $Q(g(-s))\Psi$ in D^-, then (8.26) is in D^- and we have $\phi(g(t))\in D^-$ due to $(1-E_A)\mathfrak{L}\subset D^-$.

The continuity follows from the following inequality and Lemma 6.2:

$$\begin{aligned} \|Q(g(-t))\Psi-Q(g(-t'))\Psi\|_- &= \|\mathrm{K}^{-1/2}Q(g(-t))(1-Q(g(t-t'))\Psi\| \\ &= \|\mathrm{K}_t^{-1/2}[1-Q(g(t-t'))]\Psi\| \\ &\leq a^{1/2}\|\mathrm{K}^{-1/2}\{1-Q(g(t-t'))\}\,\Psi\|\,. \end{aligned} \tag{8.27}$$

Here $\mathrm{K}_t = Q(g(t))\mathrm{K}Q(g(-t))$ and the local uniform bound $\|\mathrm{K}_t^{-1/2}\mathrm{K}^{1/2}\| \leq a^{1/2}$ is obtained as follows. We take $\mathrm{K}_1 = \mathrm{K}_t$ and $\mathrm{K}_2 = \mathrm{K}$ in the proof of Lemma 6.2. Then a_1 in (6.13) is a locally bounded function of t.

We now consider the general case. If $\mathfrak{l}_1 \cdots \mathfrak{l}_m$ are linearly independent basis of $\mathfrak{k}_1$ such that $\{e^{t\mathfrak{l}_j}\}$ for each j is compact, then by repeated use of

$$\phi(ge^{t\mathfrak{l}_j}) = Q(e^{t\mathfrak{l}_j})^*\phi(g) + \phi(e^{t\mathfrak{l}_j}), \tag{8.28}$$

and

$$Q(g')D^- \subset D^-, \tag{8.29}$$

we have

$$\phi(\exp(t_1\mathfrak{l}_1 + \cdots + t_m\mathfrak{l}_m)) \in D^- \tag{8.30}$$

for any $t_1 \cdots t_m$. Q.E.D.

Theorem 8.6. *If* $\phi \in B^1(G, \bar{D}^+)$, *then* $\mathrm{Im}(\phi(g_1g_2), \phi(g_2)) \in B^2(G, R)$, *namely* (8.1) *has a solution.*

Proof. Let G, G_1, K_1, K_2 be as in Lemma 8.5. Let $[g]$ be the class of g in G/G_1 and $[[g]]$ be the element of K_2 such that $[[g]] \equiv [g] \bmod K_1$, and $[[g]] = \exp(t(g), \mathfrak{l})$, where $(t(g), \mathfrak{l}) = \sum t_i(g)\mathfrak{l}_i$ and $\mathfrak{l}_i$ is a fixed basis of $\mathfrak{k}_2$. Then $t_i(g)$ is a representation of G on the additive group R. Let $\hat{\mathfrak{l}}_i$ be an element of $\mathfrak{g}$ such that $(\hat{\mathfrak{l}}_i \bmod \mathfrak{g}_1) = \mathfrak{l}_i$.

Next we note that if $g = g_1 \cdots g_n$ and $\phi(g_j) - \phi_j \in D^-$, then $(\phi(g) - \sum_j \phi_j) \in D^-$. This is because

$$\begin{aligned}\phi(g) = \sum_{j=1}^{n} \phi(g_j) &+ (Q(g_n)^* - 1)\phi(g_{n-1}) + (Q(g_{n-1}g_n)^* - 1)\phi(g_{n-2}) \\ &+ \cdots + (Q(g_2g_3 \cdots g_n)^* - 1)\phi(g_1)\end{aligned} \tag{8.31}$$

and $(Q(g)^* - 1)\phi \in D^-$ if $\phi \in \mathfrak{L}$.

Now $\phi(g) = (1 - Q(g)^*)\Omega$ for $\Omega \in \bar{D}^+$. By Lemma 6.4, $\phi \in Z^1(G, \mathfrak{L})$ and hence the previous lemmas are applicable. For $g \in G$, $g = g'e^{t_1(g)\mathfrak{l}_1} \cdots e^{t_n(g)\mathfrak{l}_n}$ for some $g' \in G_1K_1$. We already know from Lemma 8.5 that $\phi(g') \in D^-$. Next $\phi(e^{t\mathfrak{l}_i}) - t\phi(e^{\mathfrak{l}_i}) = B\Omega$, $B = (tQ(e^{\mathfrak{l}_i})^* - Q(e^{t\mathfrak{l}_i})^* + 1 - t)$. $\mathrm{K}(\mathfrak{l}_i)^{1/2}\bar{D}^+ \subset \mathfrak{L}$ and $\mathrm{K}(\mathfrak{l}_i)^{-1/2}B\mathrm{K}(\mathfrak{l}_i)^{-1/2}$ is bounded, $B\Omega$ is in the domain of $\mathrm{K}(\mathfrak{l}_i)^{-1/2}$ and hence $B\Omega \in D^-$. Thus we have

$$\phi(g)-\sum t_i(g)\phi(e^{I_i})\in D^- . \tag{8.32}$$

Let

$$c_1(g)\equiv(\Omega,\ \phi(g)-\sum t_i(g)\phi(e^{I_i})) . \tag{8.33}$$

Then, using $t(g_1 g_2)=t(g_1)+t(g_2)$, we have

$$\begin{aligned} c_1(g_1 g_2)-c_1(g_1)-c_1(g_2) &= (\Omega,\ \phi(g_1 g_2)-\phi(g_1)-\phi(g_2)) \\ &= (\Omega,\ (Q(g_2)^*-1)\phi(g_1)) \\ &= -(\phi(g_2^{-1}),\ \phi(g_1)) \end{aligned} \tag{8.34}$$

Hence $c(g)=\operatorname{Im} c_1(g)$ satisfies (8.1). Q.E.D.

Since Ω in Theorem 7.3 is not necessarily in $\bar{D}^+(G)$, we can not in general apply Theorem 8.6 to ϕ_j^t, $j=1, \cdots, n-1$ of Theorem 7.5. We also do not know the structure of a cocycle $\phi(g)$ for a semisimple group and hence we do not know whether $\operatorname{Im}(\phi_n^t(g_1), \phi_n^t(g_2^{-1}))$ is automatically a coboundary except when G/G_n is simply connected in Theorem 7.5. We shall leave these problems for a future study and now consider the case where $\phi(g)=\sum_{j=1}^{n}\phi_j(g)$ in Corollary 7.6. We want a condition on ϕ_j such that there exists $c(g)$ (mod 2π) satisfying

$$c(g_1 g_2)-c(g_1)-c(g_2)=\sum_{j=1}^{n-1}\operatorname{Im}(\phi_j(g_1),\ \phi_j(g_2^{-1})) . \tag{8.35}$$

In this case, the existence of c is not automatic and we shall obtain an interesting structure.

We shall analyze (8.35) by an inductive procedure.

Theorem 8.7. *Let $G_{(\alpha)}$, $G_{\alpha 0}$, H_α, $H_{\alpha 0}$ be invariant subgroups of a connected Lie group $G^{(\alpha)}$ such that $G_{(\alpha)}$ is connected, $H_\alpha\supset G_{(\alpha)}$, $H_\alpha\supset H_{\alpha 0}$ and $G_{\alpha 0}=H_{\alpha 0}\cap G_{(\alpha)}$. Let $Q_\alpha(g)$ be a continuous unitary representation of $G^{(\alpha)}$ on a finite dimensional space $\mathfrak{L}_\alpha$ and $\phi_\alpha\in Z^1(G^{(\alpha)}, \mathfrak{L}_\alpha)$. Assume that $\phi_\alpha(g)$, $g\in G_{(\alpha)}$ spans $\mathfrak{L}_\alpha$,*

$$H_\alpha=\{g;\ Q_\alpha(g)=1,\ g\in G^{(\alpha)}\} , \tag{8.36}$$

$$H_{\alpha 0}=\{g;\ \phi_\alpha(g)=0,\ g\in H_\alpha\} , \tag{8.37}$$

and $Q_\alpha(g)$ is either an identity representation or without any identity subrepresentation. Assume that there exists a real valued measurable *function* $c_\alpha(g)$ *on* $G^{(\alpha)}$ *satisfying*

$$c_\alpha(g_1 g_2)-c_\alpha(g_1)-c_\alpha(g_2) = \mathrm{Im}\,(\phi_\alpha(g_1),\ \phi_\alpha(g_2^{-1})) + F_\alpha(g_1 G_{(\alpha)},\ g_2 G_{(\alpha)}) \tag{8.38}$$

modulo 2π *where* F_α *is a continuous function on* $G^{(\alpha)}/G_{(\alpha)}\times G^{(\alpha)}/G_{(\alpha)}$ *satisfying*

$$F_\alpha(1, x) = F_\alpha(x, 1) = 0\,,\quad x\in G^{(\alpha)}/G_{(\alpha)}\,. \tag{8.39}$$

Then there exist subgroups K_α *and* G'_α of $G^{(\alpha)}$ *and* $\Omega_\alpha\in\mathfrak{L}_\alpha$ *such that* $G'_\alpha\supset H_{\alpha 0}$, $G_{(\alpha)}\cap G'_\alpha=G_{\alpha 0}$, $H_\alpha=G_{(\alpha)}G'_\alpha$, $K_\alpha\cap H_\alpha=H_{\alpha 0}$, $G^{(\alpha)}=H_\alpha K_\alpha$, $G_{(\alpha)}/G_{\alpha 0}$ *and* $G'_\alpha/H_{\alpha 0}$ *are isomorphic to* R^n *and* $R^{n'}$ *for some* $n'\leq n$, $K_\alpha/H_{\alpha 0}$ *is the direct product of a compact group and* R^m,

$$\phi_{\alpha 0}(g)\equiv \phi_\alpha(g)+(1-Q_\alpha(g)^*)\,\Omega_\alpha \tag{8.40}$$

belongs to $Z^1(G^{(\alpha)},\ \mathfrak{L}_\alpha)$, $\phi_{\alpha 0}(K_\alpha)=0$, $Q_\alpha(g_1)\phi_{\alpha 0}(g_2)=\phi_{\alpha 0}(g_1 g_2 g_1^{-1})$ *and* $\mathrm{Im}\,(\phi_{\alpha 0}(K_\alpha G'_\alpha),\ \phi_{\alpha 0}(K_\alpha G'_\alpha))=0$.

Further, let

$$c_{\alpha 0}(g) = c_\alpha(g)+(1/2)\,(\Omega_\alpha,\ (Q_\alpha(g)-Q_\alpha(g)^*)\,\Omega_\alpha)\,. \tag{8.41}$$

Then there exists a dual element x_α *of the Lie algebra* $\mathfrak{g}_{(\alpha)}$ *of* $G_{(\alpha)}$ *such that*

$$x_\alpha(\mathrm{Ad}\,(g)\,\mathfrak{l}) = x_\alpha(\mathfrak{l}) \tag{8.42}$$

for $\mathfrak{l}\in\mathfrak{g}_{\alpha 0}$, $g\in G^{(\alpha)}$ *as well as for* $\mathfrak{l}\in\mathfrak{g}_{(\alpha)}$, $g\in K_\alpha$ *and*

$$\mathrm{Im}\,(\phi_{\alpha 0}(g_1),\ \phi_{\alpha 0}(g_2)) = -(1/2)\,x_\alpha(\log g_1 g_2 g_1^{-1} g_2^{-1})\,, \tag{8.43}$$

where $g_1\in G_{(\alpha)}$, $g_2\in H_\alpha$, *log is taken modulo the commutator subgroup of* $G_{\alpha 0}$.

The subgroup $G_{\alpha 0}$ *must be connected and any* $g\in G_{\alpha 0}$ *can be written as* $g=e^{\mathfrak{l}}g_1$, $\mathfrak{l}\in\mathfrak{g}_{\alpha 0}$, g_1 *in the commutator subgroup of* $G_{\alpha 0}$. *For any such decomposition,*

$$c_\alpha(g) = c_{\alpha 0}(g) = x_\alpha(\mathfrak{l})\,. \tag{8.44}$$

$\exp ic_\alpha(g)$, $g\in G_{\alpha 0}$ *is a unitary character on* $G_{\alpha 0}$.

Any g *in* $G_{(\alpha)}$ *can be written as* $g=e^{\mathfrak{l}}g_1$, $\mathfrak{l}\in\mathfrak{g}_{(\alpha)}$, $g_1\in G_{\alpha 0}$. *For any such decomposition,*

$$c_\alpha(g) = c_{\alpha 0}(g) = x_\alpha(\mathfrak{l})+c_\alpha(g_1)\,. \tag{8.45}$$

Any g in $G^{(\alpha)}$ can be written as $g=g^{(3)}g^{(2)}g^{(1)}$, $g^{(1)}\in G_{(\alpha)}$, $g^{(2)}\in G'_{\alpha}$, $g^{(3)}\in K_{\alpha}$. For any such decomposition

$$c_{\alpha 0}(g) = c_{\alpha 0}(g^{(3)}g^{(2)})+c_{\alpha 0}(g^{(1)}) + (1/2)\, x_{\alpha}(\log g^{(2)}g^{(1)}\{g^{(2)}\}^{-1}\{g^{(1)}\}^{-1}), \tag{8.46}$$

where the argument in the last term belongs to $G_{\alpha 0}$.

Let $c_{\alpha+1}$ be the restriction of $c_{\alpha 0}$ to $K_{\alpha}G'_{\alpha}\equiv G^{(\alpha+1)}$. It satisfies $c_{\alpha+1}(g)=c_{\alpha 0}(g)$ for $g\in G_{\alpha 0}$ and

$$c_{\alpha+1}(g_1 g_2)-c_{\alpha+1}(g_1)-c_{\alpha+1}(g_2) = \hat{F}_{\alpha}(g_1 G_{\alpha 0},\, g_2 G_{\alpha 0}) \tag{8.47}$$

where $\hat{F}_{\alpha}(g_1 G_{\alpha 0},\, g_2 G_{\alpha 0})\equiv F_{\alpha}(g_1 G_{(\alpha)},\, g_2 G_{(\alpha)})$ for g_1, g_2 in $G^{(\alpha+1)}$. $G^{(\alpha+1)}/G_{\alpha 0}$ is isomorphic to $G^{(\alpha)}/G_{(\alpha)}$. $G^{(\alpha+1)}$ is connected.

Proof. From Theorem 6.7, there exists a measurable mapping ξ from $g\in G^{(\alpha)}$ to $\xi(g)\in H_{\alpha}$ such that $\xi(h)=h$ for $h\in H_{\alpha}$ and $\xi(g_1 g_2)H_{\alpha 0} = \{\xi(g_1)H_{\alpha 0}\}\{\tau(g_1)\xi(g_2)H_{\alpha 0}\}$. There also exists $\Omega_{\alpha}\in\mathfrak{L}_{\alpha}$ such that

$$\phi_{\alpha}(g) = \phi_{\alpha}(\xi(g))+(Q_{\alpha}(g)^{*}-1)\,\Omega_{\alpha}\,. \tag{8.48}$$

Let K_{α} be the set of g such that $\xi(g)H_{\alpha 0}=H_{\alpha 0}$. It is a subgroup of $G^{(\alpha)}$, isomorphic to $G^{(\alpha)}/H_{\alpha}$, $K_{\alpha}\supset H_{\alpha 0}$, $G^{(\alpha)}=H_{\alpha}K_{\alpha}$ and $\phi_{\alpha 0}(K_{\alpha})=0$. K_{α} is the direct product of a compact group and R^{m}, $H_{\alpha}/H_{\alpha 0}$ is isomorphic to R^{N} for some N, $G_{(\alpha)}/G_{\alpha 0}$ is isomorphic to R^{n} for $n\leq N\leq 2n$, $\phi_{\alpha 0}(g)=\phi_{\alpha}(\xi(g))$ and $Q_{\alpha}(g_1)\phi_{\alpha 0}(g_2)=\phi_{\alpha 0}(g_1 g_2 g_1^{-1})$. Since $\phi_{\alpha 0}(g)=\phi_{\alpha}(g)$ for $g\in H_{\alpha}$, $\phi_{\alpha 0}(G_{\alpha})$ spans $\mathfrak{L}_{\alpha}$ and $K_{\alpha}\cap H_{\alpha}=H_{\alpha 0}$. Let $n'=N-n$. We shall choose G'_{α} after we have analyzed $c_{\alpha}(g)$.

Since $G_{(\alpha)}$ is connected and $G_{(\alpha)}/G_{\alpha 0}$ is simply connected, $G_{\alpha 0}$ is also connected.

We now consider (8.38). First, if $g_1\in G_{\alpha 0}$, then the right hand side vanishes and we have

$$c_{\alpha}(g g_1) = c_{\alpha}(g_1 g) = c_{\alpha}(g)+c_{\alpha}(g_1),\quad g_1\in G_{\alpha 0},\ g\in G^{(\alpha)}\,. \tag{8.49}$$

In particular, $\exp ic_{\alpha}(g)$, $g\in G_{\alpha 0}$, is a unitary character on $G_{\alpha 0}$.

If $g_1\in G_{(\alpha)}$, then

$$c_{\alpha}(g_1 g_2)-c_{\alpha}(g_1)-c_{\alpha}(g_2) = -\mathrm{Im}\,(\phi_{\alpha}(g_1),\, Q_{\alpha}(g_2)\phi_{\alpha}(g_2))\,, \tag{8.50}$$

$$c_{\alpha}(g_2 g_1)-c_{\alpha}(g_1)-c_{\alpha}(g_2) = -\mathrm{Im}\,(\phi_{\alpha}(g_2),\, \phi_{\alpha}(g_1))\,, \tag{8.51}$$

where we have used $\phi_{\alpha}(g^{-1})=-Q_{\alpha}(g)\phi_{\alpha}(g)$ and $Q_{\alpha}(g_1)=1$. If we set $g_2=g_1^{-1}$ in (8.51) and use $c_{\alpha}(1)=0$, we have

$$c_\alpha(g_1^{-1}) = -c_\alpha(g_1), \ g_1 \in G_{(\alpha)}. \tag{8.52}$$

If $g_2 \in H_\alpha$, then $Q_\alpha(g_2)=1$ and we have from (8.50) and (8.51)

$$\frac{1}{2}\{c_\alpha(g_1 g_2)+c_\alpha(g_2 g_1)\} = c_\alpha(g_1)+c_\alpha(g_2) \tag{8.53}$$

mod π. Substituting g_1^{-1} into g_1 and g_1g_2 and g_2g_1 into g_2 in (8.53), we have

$$c_\alpha(g_1 g_2) = c_\alpha(g_1)+\frac{1}{2}\{c_\alpha(g_2)+c_\alpha(g_1 g_2 g_1^{-1})\} \tag{8.54}$$

$$c_\alpha(g_2 g_1) = c_\alpha(g_1)+\frac{1}{2}\{c_\alpha(g_1^{-1} g_2 g_1)+c_\alpha(g_2)\} \tag{8.55}$$

mod π where we have used (8.52). We set

$$k(g)g_1 \equiv g_1(g g_1 g^{-1})^{-1} \tag{8.56}$$

$$k'(g)g_1 \equiv (g^{-1} g_1 g)^{-1} g_1. \tag{8.57}$$

Since $Q_\alpha(g_2)=1$ for $g_2 \in H_\alpha$ and ϕ_α is faithful on $G_{(\alpha)}/G_{\alpha 0}$, $k(g)g_1$ and $k'(g)g_1$ are in $G_{\alpha 0}$. Substituting $k(g_2)g_1$ and $k'({}_2g)g_1$ into g_1 and g_2 into g in (8.49), we have

$$c_\alpha(g_1 g_2 g_1^{-1}) = c_\alpha(g_2)+c_\alpha(k(g_2)g_1) \tag{8.58}$$

$$c_\alpha(g_1^{-1} g_2 g_1) = c_\alpha(g_2)+c_\alpha(k'(g_2)g_1) \tag{8.59}$$

for $g_1 \in G_{(\alpha)}$, $g_2 \in H_\alpha$. By substituting (8.58) and (8.59) into (8.54) and (8.55), we obtain

$$c_\alpha(g_1 g_2) = c_\alpha(g_1)+c_\alpha(g_2)+\frac{1}{2}c_\alpha(g_1 g_2 g_1^{-1} g_2^{-1}) \tag{8.60}$$

$$c_\alpha(g_2 g_1) = c_\alpha(g_1)+c_\alpha(g_2)+\frac{1}{2}c_\alpha(g_2^{-1} g_1^{-1} g_2 g_1) \tag{8.61}$$

mod π for $g_1 \in G_{(\alpha)}$, $g_2 \in H_\alpha$. If $g_1 \in G_{\alpha 0}$ in (8.60), we obtain from (8.49),

$$0 = c_\alpha(g_1 g_2 g_1^{-1} g_2^{-1}) = c_\alpha(g_1)-c_\alpha(g_2 g_1 g_2^{-1}). \tag{8.62}$$

Thus we have restriction for c_α:

$$c_\alpha(g g_1 g^{-1}) = c_\alpha(g_1), \quad g_1 \in G_{\alpha 0}, \ g \in H_\alpha. \tag{8.63}$$

From (8.50) and (8.60), we have

$$\text{Im}\,(\phi_\alpha(g_1), \phi_\alpha(g_2)) = -(1/2)c_\alpha(g_1 g_2 g_1^{-1} g_2^{-1}), \quad g_1 \in G_{(\alpha)},\ g_2 \in H_\alpha \tag{8.64}$$

mod π. Since $Q_\alpha(g)$ is unitary and $Q_\alpha(g)\phi_\alpha(g_1) = \phi_\alpha(g g_1 g^{-1})$ for $g_1 \in H_\alpha$, we have

$$c_\alpha(g g_1 g^{-1}) = c_\alpha(g_1), \tag{8.65}$$

for $g \in G^{(\alpha)}$, $g_1 = g_2 g_3 g_2^{-1} g_3^{-1}$, $g_2 \in G_{(\alpha)}$, $g_3 \in H_\alpha$.

So far we have considered c_α and ϕ_α. Since $c_\alpha(g) = c_{\alpha 0}(g)$ and $\phi_\alpha(g) = \phi_{\alpha 0}(g)$ for $g \in H_\alpha$, all the results so far hold for $c_{\alpha 0}$ and $\phi_{\alpha 0}$ as well. We now consider $c_{\alpha 0}$ and $\phi_{\alpha 0}$ for general $g \in G^{(\alpha)}$.

If $g_2 \in K_\alpha$ in (8.48) and (8.51) where c_α and ϕ_α are replaced by $c_{\alpha 0}$ and $\phi_{\alpha 0}$, then the right hand sides are 0. We substitute $g_2^{-1} g_1 g_2$ into g_1 of (8.51) and subtract from (8.50). Then we obtain

$$c_\alpha(g_2^{-1} g_1 g_2) = c_\alpha(g_1), \quad g_1 \in G_{(\alpha)},\ g_2 \in K_\alpha. \tag{8.66}$$

Since $G^{(\alpha)} = H_\alpha K_\alpha$, (8.63) and (8.66) imply

$$c_\alpha(g^{-1} g_1 g) = c_\alpha(g_1), \quad g_1 \in G_{\alpha 0},\ g \in G^{(\alpha)}. \tag{8.67}$$

Let

$$y_\alpha(\mathfrak{l}) = \exp i c_\alpha(e^{\mathfrak{l}}), \quad \mathfrak{l} \in \mathfrak{g}_{(\alpha)}. \tag{8.68}$$

Since $c(g)$ is measurable and $\phi(g)$ is C^∞, we see that $c(g) \bmod 2\pi$ is C^∞ by integrating (8.51) over g_2 with a class $\mathcal{D}$ function. Define

$$x_\alpha(\mathfrak{l}) = (\mathrm{d}/\mathrm{d}t) y_\alpha(t\mathfrak{l})|_{t=0}, \quad \mathfrak{l} \in \mathfrak{g}_{(\alpha)}. \tag{8.69}$$

From the definition,

$$x_\alpha(t\mathfrak{l}) = t x_\alpha(\mathfrak{l}), \quad \mathfrak{l} \in \mathfrak{g}_{(\alpha)}. \tag{8.70}$$

Since $y_\alpha(0) = 1$, we have from (8.60)

$$y_\alpha(\mathfrak{l}) = \exp i x_\alpha(\mathfrak{l}). \tag{8.71}$$

Hence

$$c_\alpha(e^{\mathfrak{l}}) = x_\alpha(\mathfrak{l}). \tag{8.72}$$

Since $G_{(\alpha)}/G_{\alpha 0}$ is commutative, $[\mathfrak{l}_1, \mathfrak{l}_2] \in \mathfrak{g}_{\alpha 0}$ for any $\mathfrak{l}_1, \mathfrak{l}_2 \in \mathfrak{g}_{(\alpha)}$. Further

$$x_\alpha([\mathfrak{l}_1, \mathfrak{l}_2]) = 0, \quad \mathfrak{l}_1 \in \mathfrak{g}^{(\alpha)},\ \mathfrak{l}_2 \in \mathfrak{g}_{\alpha 0} \tag{8.73}$$

from (8.67). From (8.49), we have

(8.74) $$x_\alpha(\mathfrak{l}+\mathfrak{l}_1) = x_\alpha(\mathfrak{l})+x_\alpha(\mathfrak{l}_1)\,,\quad \mathfrak{l}\in\mathfrak{g}_{(\alpha)},\ \mathfrak{l}_1\in\mathfrak{g}_{\alpha 0}\,.$$

From (8.60), the Baker-Hausdorff formula, (8.74) and (8.73), we have

(8.75) $$x_\alpha(\mathfrak{l}_1+\mathfrak{l}_2) = x_\alpha(\mathfrak{l}_1)+x_\alpha(\mathfrak{l}_2)\,,\quad \mathfrak{l}_1,\ \mathfrak{l}_2\in\mathfrak{g}_{(\alpha)}\,.$$

From (8.70) and (8.75), we see that x_α is in the dual of $\mathfrak{g}_{(\alpha)}$.

(8.42) follows from (8.66) and (8.67). (8.44) follows from (8.49) and (8.72). (8.45) follows from (8.49). (8.46) follows from (8.60), (8.74) and (8.66) mod π. The argument in the last term of (8.46) is in $G_{\alpha 0}$ as long as $G'_\alpha\subset H_\alpha$. (8.43) follows from (8.50) and (8.60) mod π. The two sides of (8.43) are continuous in g_1. Since it holds for $g_1=1$ and $G_{(\alpha)}$ is connected, (8.43) holds. Since the continuity of c_α (mod 2π) follows from (8.38), (8.46) holds mod 2π by the same reason.

We now choose G'_α so that $H_\alpha\supset G'_\alpha\supset H_{\alpha 0}$, $H_\alpha=G_{(\alpha)}G'_\alpha$, $G_{(\alpha)}\cap G'_\alpha=G_{\alpha 0}$, $G'_\alpha/H_{\alpha 0}$ is isomorphic to $R^{n'}$ and $\operatorname{Im}(\phi_{\alpha 0}(K_\alpha G'_\alpha),\ \phi_{\alpha 0}(K_\alpha G'_\alpha))=0$. The last equation implies (8.47).

Let $\mathfrak{L}^{(1)}_\alpha$ be the real Hilbert space consisting of elements in $\mathfrak{h}_\alpha/\mathfrak{h}_{\alpha 0}$ equipped with an inner product

(8.76) $$\mu_\alpha(\hat{\mathfrak{l}}_1,\ \hat{\mathfrak{l}}_2) = \operatorname{Re}(\phi_{\alpha 0}(e^{\mathfrak{l}_1}),\ \phi_{\alpha 0}(e^{\mathfrak{l}_2}))$$

where $\hat{\mathfrak{l}}_j=\mathfrak{l}_j+\mathfrak{h}_{\alpha 0}\in\mathfrak{h}_\alpha/\mathfrak{h}_{\alpha 0}$, $j=1, 2$. Let

(8.77) $$\mu_\alpha(\hat{\mathfrak{l}}_1,\ \beta_\alpha\hat{\mathfrak{l}}_2) = \operatorname{Im}(\phi_{\alpha 0}(e^{\mathfrak{l}_1}),\ \phi_{\alpha 0}(e^{\mathfrak{l}_2}))\,.$$

The real matrix β_α on $\mathfrak{L}^{(1)}_\alpha$ is antisymmetric and $\|\beta_\alpha\|\le 1$. Let $\mathfrak{L}^{(2)}_\alpha$ be the subspace $\{\mathfrak{l}+\mathfrak{h}_{\alpha 0}\,;\ \mathfrak{l}\in\mathfrak{g}_{(\alpha)}\}$. We have $\dim\mathfrak{L}^{(2)}_\alpha=n$, $\dim\mathfrak{L}^{(1)}_\alpha=n+n'$.

Let $\beta'_\alpha=E\beta_\alpha E$ for the projection E on $\mathfrak{L}^{(2)}_\alpha$. Since β'_α is also real antisymmetric, there exists an orthonormal basis z_j, $j=1, \cdots, n$ in $\mathfrak{L}^{(2)}_\alpha$ such that $\beta'_\alpha z_{2j-1}=a_j z_{2j}$, $\beta'_\alpha z_{2j}=-a_j z_{2j-1}$, $a_j>0$ for $j=1, \cdots, m$ and $\beta'_\alpha z_j=0$ for $j>2m$ where m is some integer not exceeding $n/2$.

Let $\mathfrak{L}_{\alpha r}$ be the real linear subset of $\mathfrak{L}_\alpha$ spanned by $\phi_{\alpha 0}(e^{\mathfrak{l}})$, $\mathfrak{l}\in z_j$, $j>2m$; $\phi_{\alpha 0}(e^{\mathfrak{l}})$, $\mathfrak{l}\in z_{2j}$, $j\le m$; and $\phi_{\alpha 0}(e^{\mathfrak{l}})-ia_j\phi_{\alpha 0}(e^{\mathfrak{l}'})$, $\mathfrak{l}\in z_{2j-1}$, $\mathfrak{l}'\in z_{2j}$, $j\le m$. Then the inner product of elements in $\mathfrak{L}_{\alpha r}$ are all real. Since $\phi_{\alpha 0}(G_\alpha)$ span $\mathfrak{L}_\alpha$ (as a complex linear set), $\mathfrak{L}_\alpha$ must be the complexification of $\mathfrak{L}_{\alpha r}$.

Let $\mathfrak{L}_\alpha^{(3)}$ be the orthogonal complement of $\mathfrak{L}_\alpha^{(2)}$ in $\mathfrak{L}_\alpha^{(1)}$. For each $u \in \mathfrak{L}_\alpha^{(3)}$, let $s(u)$ be defined by

$$(8.78) \qquad s(u) = u + \sum_j{}' a_j(1-(a_j)^2)^{-1}(z_{2j}, \beta_\alpha u) z_{2j-1}$$

where the summation is over $j \leq m$ such that $a_j \neq 1$. Let $s(\mathfrak{L}_\alpha^{(3)}) = \mathfrak{L}_\alpha^{(4)}$.

By construction, $\mathfrak{L}_\alpha^{(2)}$ is a real linear subset of $\mathfrak{L}_\alpha^{(1)}$, $\mathfrak{L}_\alpha^{(3)} \cap \mathfrak{L}_\alpha^{(2)} = \{0\}$, and $\mathfrak{L}_\alpha^{(3)} + \mathfrak{L}_\alpha^{(2)} = \mathfrak{L}_\alpha^{(1)}$. Hence the same is true for $\mathfrak{L}_\alpha^{(4)}$. Further, by construction of $\mathfrak{L}_\alpha^{(3)}$ and by (8.78), $\mathfrak{l} \in z \in \mathfrak{L}_\alpha^{(4)}$ satisfies $\operatorname{Re}(\psi, \phi(e^{\mathfrak{l}})) = 0$ for all $\psi \in \mathfrak{L}_{\alpha r}$. Note that $\phi_{\alpha 0}(e^{\mathfrak{l}}) - i a_j \phi_{\alpha 0}(e^{\mathfrak{l}'}) = 0$ if $a_j = 1$ because it has a 0 norm. This implies that $\phi(e^{\mathfrak{l}}) \in i\mathfrak{L}_{\alpha r}$ for $\mathfrak{l} \in z \in \mathfrak{L}_\alpha^{(4)}$ and hence $\operatorname{Im}(\phi(e^{\mathfrak{l}}), \phi(e^{\mathfrak{l}'})) = 0$ for any $\mathfrak{l} \in z \in \mathfrak{L}_\alpha^{(4)}$ and $\mathfrak{l}' \in z' \in \mathfrak{L}_\alpha^{(4)}$.

Let G'_α be the subgroup of H_α generated by $H_{\alpha 0}$ and $e^{\mathfrak{l}}$, $\mathfrak{l} \in z \in \mathfrak{L}_\alpha^{(4)}$. $H_\alpha \supset G'_\alpha \supset H_{\alpha 0}$ holds by construction. Since $\mathfrak{L}_\alpha^{(3)} + \mathfrak{L}_\alpha^{(2)} = \mathfrak{L}_\alpha^{(1)}$ and $G'_\alpha \supset H_{\alpha 0}$, we have $H_\alpha = G_{(\alpha)} G'_\alpha$. Since $\mathfrak{L}_\alpha^{(4)} \cap \mathfrak{L}_\alpha^{(2)} = \{0\}$, $G_{(\alpha)} \cap G'_\alpha = G_{(\alpha)} \cap H_{\alpha 0} = G_{\alpha 0}$. $G'_\alpha / H_{\alpha 0}$ is isomorphic to $\mathfrak{L}_\alpha^{(4)} \sim R^{n'}$. $\operatorname{Im}(\phi(g_1), \phi(g_2)) = 0$ for $g_1, g_2 \in G'_\alpha$ by construction.

Since $G_{(\alpha)} \cap G^{(\alpha+1)} = G_{\alpha 0}$ and $G^{(\alpha+1)} G_{(\alpha)} = G^{(\alpha)}$, $G^{(\alpha)}/G_{(\alpha)}$ is isomorphic to $G^{(\alpha+1)}/G_{\alpha 0}$. Since $G^{(\alpha)}$ is connected, $G^{(\alpha)}/G_{(\alpha)}$ is connected. Since $G_{\alpha 0}$ is connected in addition, $G^{(\alpha+1)}$ is connected. Q.E.D.

By using this theorem, we can analyze a solution c of (8.35) in the following manner. Let G_j be as in Corollary 7.6, for a given connected solvable group G.

First we set $\alpha = 2$ in Theorem 8.7 and consider $G^{(\alpha)} = G$, $G_{(\alpha)} = G_2$. $\mathfrak{L}_\alpha$, Q_α, ϕ_α are $\hat{\mathfrak{L}}_1$, $\hat{Q}_1$, $\hat{\phi}_1$, respectively. H_α and $H_{\alpha 0}$ are defined by (8.36) and (8.37). $G_{\alpha 0}$ is defined as $H_{\alpha 0} \cap G_{(\alpha)}$. The function F_α is the sum $\sum_{j=2}^{n-1} \operatorname{Im}(\hat{\phi}_j(g_1), \hat{\phi}_j(g_2^{-1}))$. A solution c of (8.35) is taken as c_α. Then Theorem 8.7 is applicable.

There exists an element x_α in the dual of $\mathfrak{g}_{(\alpha)}$ such that $c_\alpha(e^{\mathfrak{l}}) = x_\alpha(\mathfrak{l})$ for $\mathfrak{l} \in \mathfrak{g}_{(\alpha)}$. The group $G^{(\alpha+1)}$ contains $G_{\alpha 0}$ and $G^{(\alpha+1)}/G_{\alpha 0}$ is isomorphic to G/G_2. Both $G^{(\alpha+1)}$ and $G_{\alpha 0}$ are connected. $c_\alpha(g)$ for a general g is given in terms of x_2, Ω_2 and c_3 through formulas (8.40) and (8.46) where Ω_2 is a vector in $\mathfrak{L}_1$ and c_3 has to satisfy (8.47). In addition, $c_3(e^{\mathfrak{l}}) = x_2(\mathfrak{l})$ for $\mathfrak{l} \in \mathfrak{g}_{20}$. $\operatorname{Im}(\phi_{\alpha 0}(g_1), \phi_{\alpha 0}(g_2))$ is fixed in terms of x_2 by (8.43). Since G_2 is abelian, this implies in particular that $\operatorname{Im}(\phi_2(g_1), \phi_2(g_2^{-1})) = 0$ for $g_1, g_2 \in G_2$ (or else c does not exist). (If $\phi_{\alpha 0}$ is given first, (8.43) is a restriction on x_2, for which a solution might not exist.)

Next we proceed to higher c_α. We make an inductive assumption that we are given a connected subgroup $G^{(\alpha)}$ of G, containing a connected invariant subgroup $G_{(\alpha-1)0}=G^{(\alpha)}\cap G_{\alpha-1}$ such that $G^{(\alpha)}/G_{(\alpha-1)0}$ is isomorphic to $G/G_{\alpha-1}$. We then look for c_α satisfying (8.38) with $\phi_\alpha=\hat{\phi}_{\alpha-1}$ and $F_\alpha=\sum_{j=\alpha}^{n-1}\mathrm{Im}(\hat{\phi}_j(g_1), \hat{\phi}_j(g_2^{-1}))$.

We apply Theorem 8.7 to the connected group $G^{(\alpha)}$ and its invariant subgroup $G_{(\alpha)}=G^{(\alpha)}\cap G_\alpha$. $G_{(\alpha)}/G_{(\alpha-1)0}$ is isomorphic to $G_\alpha/G_{\alpha-1}$, which is connected. Hence $G_{(\alpha)}$ is also connected. $\mathfrak{L}_\alpha$, Q_α, ϕ_α are taken to be $\hat{\mathfrak{L}}_{\alpha-1}$, $\hat{Q}_{\alpha-1}$, $\hat{\phi}_{\alpha-1}$, respectively. H_α and $H_{\alpha 0}$ are defined by (8.36) and (8.37). $G_{\alpha 0}$ is defined as $H_{\alpha 0}\cap G_{(\alpha)}$. The function F_α is as given above and we assume the existence of a solution c_α of (8.38).

We then obtain a group $G^{(\alpha+1)}$ such that it contains $G_{\alpha 0}=G^{(\alpha+1)}\cap G_\alpha$ (which automatically contains $G_{(\alpha-1)0}$) and $G^{(\alpha+1)}/G_{\alpha 0}$ is isomorphic to G/G_α. Both $G^{(\alpha+1)}$ and $G_{\alpha 0}$ are connected. There exist an element x_α in the dual of $\mathfrak{g}_{(\alpha)}$ satisfying (8.42), an element Ω_α in $\mathfrak{L}_{\alpha-1}$ and $c_{\alpha+1}$ satisfying (8.47) with $\hat{F}_\alpha=\sum_{j=\alpha}^{n-1}\mathrm{Im}(\hat{\phi}_j(g_1), \hat{\phi}_j(g_2^{-1}))$. $c_\alpha(g)$ for a general g is given in terms of x_α, Ω_α and $c_{\alpha+1}$ through formulas (8.40), (8.44), (8.45) and (8.46). $\mathrm{Im}(\phi_{\alpha 0}(g_1), \phi_{\alpha 0}(g_2))$ is determined by (8.43) in terms of x_α. x_α has to coincide with $x_{\alpha-1}$ on $\mathfrak{g}_{(\alpha-1)0}$ and $c_{\alpha+1}(e^{\mathrm{I}})$ has to coincide with $x_\alpha(\mathrm{I})$ for $\mathrm{I}\in\mathfrak{g}_{\alpha 0}$.

Proceeding recursively, we obtain the following structure:

(1) *Subgroups*. $G^{(\alpha)}$, $\alpha=2, \cdots, n$ is a descending sequence of connected subgroups of G and K_α, H_α, $G_{(\alpha)}$, G'_α, $G_{\alpha 0}$, $H_{\alpha 0}$ are subgroups of $G^{(\alpha)}$. They are interrelated with each other and with G_j of Corollary 7.6, by the relations: (i) $G_{(\alpha)}=G^{(\alpha)}\cap G_\alpha$, $G_{\alpha 0}=G^{(\alpha+1)}\cap G_\alpha$, $G_{(2)}\cdots G_{(\alpha)}=G_\alpha$, $G=G_{\alpha-1}G^{(\alpha)}$. (ii) $G_{(\alpha-1)0}\,g\rightarrow G_{\alpha-1}\,g$ gives isomorphisms of $G^{(\alpha)}/G_{(\alpha-1)0}$ and $G_{(\alpha)}/G_{(\alpha-1)0}$ onto $G/G_{\alpha-1}$ and $G_\alpha/G_{\alpha-1}$, respectively. (iii) $H_\alpha K_\alpha=G^{(\alpha)}$, $G'_\alpha K_\alpha=G^{(\alpha+1)}$, $G_{(\alpha)}G_\gamma^\alpha=H_\alpha$, $H_\alpha\cap G^{(\alpha+1)}=G'_\alpha$, $H_\alpha\cap K_\alpha=H_{\alpha 0}$, $G_{(\alpha)}\cap G'_\alpha=G_{(\alpha)}\cap G^{(\alpha+1)}=G_{(\alpha)}\cap K_\alpha=G_{(\alpha)}\cap H_{\alpha 0}=G_{\alpha 0}$, $G_{\alpha 0}\supset G_{(\alpha-1)0}$ (an ascending sequence). (iv) H_α, $G_{(\alpha)}$, $H_{\alpha 0}$, and $G_{\alpha 0}$ are invariant subgroups of $G^{(\alpha)}$. Besides $G^{(\alpha)}$, $G_{(\alpha)}$ and $G_{\alpha 0}$ are connected. $K_\alpha/H_{\alpha 0}$ is the direct product of a compact group and R^{m_α}, $H_\alpha/H_{\alpha 0}$ is isomorphic to R^{N_α} for some $N_\alpha\geq 0$, and is maximal abelian in $G^{(\alpha)}/H_{\alpha 0}$. (v) $G^{(2)}=G$, $G^{(n)}=G_{(n)}=H_n$ and $K_n=\{1\}$.

(2) *Hilbert spaces and cocycles.* There exists a real linear mapping p_α from the Lie algebra of H_α onto a total set in a complex Hilbert space $\mathfrak{L}_\alpha$ $(=\hat{\mathfrak{L}}_{\alpha-1})$, equipped with an inner product $\hat{\mu}$. The kernel of p_α is $\mathfrak{H}_{\alpha 0}$. On $\mathfrak{L}_\alpha$, there exists a continuous unitary representation $Q_\alpha(g)$ of $g\in G$ such that $Q_\alpha(g)=1$ for $g\in G_{\alpha-1}$ and $Q_\alpha(g)p_\alpha=p_\alpha \operatorname{Ad}(g)$ for $g\in G^{(\alpha)}$ where $\operatorname{Ad}(g)$ is the adjoint representation of $G^{(\alpha)}$ on the Lie algebra of H_α. For $g=g_1e^{\mathfrak{l}}g_3$, $g_1\in G_{\alpha-1}$, $\mathfrak{l}\in\mathfrak{h}_\alpha$, $g_3\in K_\alpha$,

$$\phi_\alpha(g) = p_\alpha\mathfrak{l}+(Q_\alpha(g)^*-1)\Omega_\alpha \tag{8.79}$$

is in $Z^1(G, \mathfrak{L}_\alpha)$ where $\Omega_\alpha\in\mathfrak{L}_\alpha$ is fixed.

$$\mu_\alpha(\mathfrak{l}_1, \mathfrak{l}_2) = \operatorname{Re}\hat{\mu}_\alpha(p_\alpha\mathfrak{l}_1, p_\alpha\mathfrak{l}_2) \tag{8.80}$$

is an $\operatorname{Ad}(G^{(\alpha)})$ invariant positive real inner product on $\mathfrak{h}_\alpha$ with the kernel $\mathfrak{h}_{\alpha 0}$.

$$\mu_\alpha(\mathfrak{l}_1, \beta_\alpha\mathfrak{l}_2) = \operatorname{Im}\hat{\mu}_\alpha(p_\alpha\mathfrak{l}_1, p_2\mathfrak{l}_2) \tag{8.81}$$

defines an antisymmetric operator β_α on real Hilbert space $\mathfrak{h}_\alpha/\mathfrak{h}_{\alpha 0}$ such that

$$\|\beta_\alpha\| \leq 1 . \tag{8.82}$$

β_α commutes with $\operatorname{Ad}(g)$, $g\in G^{(\alpha)}$.

(3) x_α is in the dual of $\mathfrak{g}_{(\alpha)}$ and satisfies

$$x_\alpha(\operatorname{Ad}(g)\mathfrak{l}) = x_\alpha(\mathfrak{l}) \tag{8.83}$$

for $\mathfrak{l}\in\mathfrak{g}_{\alpha 0}$, $g\in G^{(\alpha)}$ as well as for $\mathfrak{l}\in\mathfrak{g}_{(\alpha)}$, $g\in K_\alpha$. Further,

$$x_\beta|\mathfrak{g}_{\alpha 0} = x_\alpha|\mathfrak{g}_{\alpha 0} \quad (\beta>\alpha) . \tag{8.84}$$

(4) β_α is related to x_α through

$$\mu_\alpha(\mathfrak{l}_1, \beta_\alpha\mathfrak{l}_2) = -(1/2)x_\alpha([\mathfrak{l}_{11}, \mathfrak{l}_{21}]+[\mathfrak{l}_{12}, \mathfrak{l}_{21}]+[\mathfrak{l}_{11}, \mathfrak{l}_{22}]) \tag{8.85}$$

if $\mathfrak{l}_j=\mathfrak{l}_{j1}+\mathfrak{l}_{j2}$, $\mathfrak{l}_{j1}\in\mathfrak{g}_{(\alpha)}$, $\mathfrak{l}_{j2}\in\mathfrak{g}'_\alpha$, $j=1, 2$.

(5) c_α is given in terms of x_α, Ω_α and $c_{\alpha+1}$ by (8.40), (8.44), (8.45) and (8.46), where c_{n+1} is to be taken 0. $c\equiv c_2$ satisfies (8.35) with $\hat{\phi}_j=\phi_{j+1}$.

In the above analysis, we have shown that the pair c and ϕ necessarily leads to the structures (1)~(5). We have not listed all the restrictions, which could be extracted from our construction.

(An example is a relation between adjoint representations of G on Lie algebras of $G_{(\alpha)}$ and H_α.) However, the structure and its property extracted above as (1)∼(5) are sufficient in the sense that, starting from the subgroup structure (1), the dual elements x_α of (3) and real inner product μ_α, we can always obtain (in a unique manner) a pair $\phi(g)$ and $c(g)$. More precisely, we have

Theorem 8.8. *Let the structure of subgroups be given as in* (1) *above. Let* x_α *be given as in* (3) *above. Let* μ_α *be an* $\mathrm{Ad}(G^{(\alpha)})$ *invariant positive real inner product on* $\mathfrak{h}_\alpha$ *with the kernel* $\mathfrak{h}_{\alpha 0}$ *such that* β_α *defined by* (8.85) *satisfies* (8.82). *Then there exist* $\mathfrak{L}_\alpha$, Ω_α, Q_α, ϕ_α *and* c_α *which satisfy* (2), (4) *and* (5) *above.* $\mathfrak{L}_\alpha$, Q_α, $\phi_{\alpha 0}$ *and* $c_{\alpha 0}$ *are unique while* Ω_α *is arbitrary.*

Proof. Since (8.85) is antisymmetric in $\mathfrak{l}_1$ and $\mathfrak{l}_2$, the linear extension of

$$\hat{\mu}_\alpha(\mathfrak{l}_1, \mathfrak{l}_2) = \mu_\alpha(\mathfrak{l}_1, \mathfrak{l}_2) + i\mu_\alpha(\mathfrak{l}_1, \beta_\alpha \mathfrak{l}_2) \tag{8.86}$$

to the complexification $\mathfrak{h}_\alpha + i\mathfrak{h}_\alpha$ of $\mathfrak{h}_\alpha$, which is again denoted by $\hat{\mu}$, is a nonnegative inner product. The quotient of $\mathfrak{h}_\alpha + i\mathfrak{h}_\alpha$ by the kernel of $\hat{\mu}_\alpha$ norm, considered as a complex Hilbert space with an inner product $\hat{\mu}_\alpha$, will be denoted by $\mathfrak{L}_\alpha$. The natural homomorphism from real Hilbert space $\mathfrak{h}_\alpha$ onto a total set in $\mathfrak{L}_\alpha$ is denoted by p_α. The kernel of p_α is $\mathfrak{h}_{\alpha 0}$. (We do not necessarily demand that $p_\alpha G_{(\alpha)}$ be total in $\mathfrak{L}_\alpha$.)

Due to (8.83) and $\mathrm{Ad}(g)[\mathfrak{l}_1, \mathfrak{l}_2] = [\mathrm{Ad}(g)\mathfrak{l}_1, (g)\mathfrak{l}_2]$, we have $\mu_\alpha(\mathrm{Ad}(g)\mathfrak{l}_1, \beta_\alpha \mathrm{Ad}(g)\mathfrak{l}_2) = \mu_\alpha(\mathfrak{l}_1, \beta_\alpha \mathfrak{l}_2)$. Therefore,

$$\hat{\mu}_\alpha(\mathrm{Ad}(g)\mathfrak{l}_1, \mathrm{Ad}(g)\mathfrak{l}_2) = \hat{\mu}_\alpha(\mathfrak{l}_1, \mathfrak{l}_2), \quad g \in G^{(\alpha)}. \tag{8.87}$$

This shows the extence of a unique continuous unitary representation $Q_\alpha(g)$ of $g \in G^{(\alpha)}$ such that $Q_\alpha(g)p_\alpha = p_\alpha \mathrm{Ad}(g)$. $Q_\alpha(g) = 1$ if $g \in H_\alpha$.

Next we define $Q_\alpha(g) = Q_\alpha(g_2)$ if $g = g_1 g_2$, $g_1 \in G_{\alpha-1}$, $g_2 \in G^{(\alpha)}$. Since $G = G_{\alpha-1}G^{(\alpha)}$, any g has such a decomposition. Since $G_{\alpha-1} \cap G^{(\alpha)} = G_{(\alpha-1)0} \subset H_\alpha$, $Q_\alpha(g_2) = Q_\alpha(g_2')$ for a different decomposition $g = g_1' g_2'$. Since $g \in G^{(\alpha)} \to G_\alpha g$ is an isomorphism from $G^{(\alpha)}/G_{(\alpha)}$ to G/G_α and since $Q_\alpha(g_1 g_2) = Q_\alpha(g_2)$ for $g_1 \in G_\alpha$, the extended Q_α is a continuous unitary representation of G.

For $g=g_1e^{\mathfrak{l}}g_3$, $g_1\in G_{\alpha-1}$, $\mathfrak{l}\in\mathfrak{h}_\alpha$, $g_3\in K_\alpha$, we define

(8.88) $$\phi_{\alpha 0}(g) = p_\alpha \mathfrak{l}\,.$$

Any $g\in G$ has such a decomposition, because $G=G_{\alpha-1}H_\alpha K_\alpha$, $H_\alpha/H_{\alpha 0}$ is exponential and $H_{\alpha 0}\subset K_\alpha$. Since $G_{\alpha-1}\cap H_\alpha K_\alpha=G_{(\alpha-1)0}\subset G_{\alpha 0}$, $H_\alpha\cap K_\alpha=H_{\alpha 0}$, and $p_\alpha\mathfrak{l}=p_\alpha\mathfrak{l}'$ for $e^{\mathfrak{l}}=g_1e^{\mathfrak{l}'}g_3$, $g_1\in G_{\alpha 0}$, $g_3\in H_{\alpha 0}$, the above definition of $\phi_{\alpha 0}(g)$ is independent of the decomposition of g. (Note the following: For $\mathfrak{l}_1\in\mathfrak{g}_{\alpha 0}$, $\mathfrak{l}'\in\mathfrak{h}_\alpha$, $\mathrm{Ad}\,(e^{\mathfrak{l}_1})\mathfrak{l}'-\mathfrak{l}'=\sum_{n=1}^{\infty}(n!)^{-1}\,\mathrm{Ad}\,(\mathfrak{l}_1)^n\mathfrak{l}'\in\mathfrak{h}_{\alpha 0}$. Since $G_{\alpha 0}$ is connected, $\mathfrak{l}''=\mathrm{Ad}\,(g_1)\mathfrak{l}'$ satisfies $p_\alpha\mathfrak{l}''=p_\alpha\mathfrak{l}'$ for any $g_1\in G_{\alpha 0}$. Next, if $e^{\mathfrak{l}}=e^{\mathfrak{l}''}g_3'$ for $g_3'=g_1g_3\in H_{\alpha 0}$, $\mathfrak{l},\mathfrak{l}''\in\mathfrak{h}_\alpha$, then $\mathfrak{l}-\mathfrak{l}''\in\mathfrak{h}_{\alpha 0}$ because $H_\alpha/H_{\alpha 0}$ is isomorphic to R^{N_α}. Hence $p_\alpha\mathfrak{l}''=p_\alpha\mathfrak{l}$.)

We now prove that $\phi_{\alpha 0}\in Z^1(G,\mathfrak{L}_\alpha)$. We already know from earlier results that $\phi_{\alpha 0}$ restricted to $G^{(\alpha)}$ is in $Z^1(G^{(\alpha)},\mathfrak{L}_\alpha)$. We also know that, for $g\in G_{\alpha-1}$ and $g_1\in G^{(\alpha)}$, $Q_\alpha(g)=1$, $\phi_\alpha(g)=0$, $\phi_\alpha(gg_1)=\phi_\alpha(g_1)$ and $\phi_\alpha(g_1g)=\phi_\alpha(g_1gg_1^{-1}g_1)=\phi_\alpha(g_1)$. Here the last equality is due to $g_1gg_1^{-1}\in G_{\alpha-1}$ if $g\in G_{\alpha-1}$. Therefore $\phi_{\alpha 0}\in Z^1(G,\mathfrak{L}_\alpha)$.

We now define c_α assuming that $c_{\alpha+1}$ is already defined. To start the inductive procedure, we take $c_{n+1}\equiv 0$.

The quotient of $G_{\alpha 0}$ by its commutator subgroup, say $G'_{\alpha 0}$, is connected and abelian. Hence it is exponential and any $g\in G_{\alpha 0}$ can be written as $g=e^{\mathfrak{l}}g_1$, $\mathfrak{l}\in\mathfrak{g}_{\alpha 0}$, $g_1\in G'_{\alpha 0}$. Hence (8.44) defines $c_{\alpha 0}(g)$ for all $g\in G_{\alpha 0}$. Suppose $e^{\mathfrak{l}}e^{-\mathfrak{l}'}$ is in the commutator subgroup. Since $G_{\alpha 0}/G'_{\alpha 0}$ is exponential, $\mathfrak{l}-\mathfrak{l}'\in\mathfrak{g}'_{\alpha 0}$. From special case of (8.83) with $g\in G_{\alpha 0}$, we have $x_\alpha(\mathfrak{g}'_{\alpha 0})=0$. Hence the definition (8.44) is independent of the decomposition $g=e^{\mathfrak{l}}g_1$.

If $g=e^{\mathfrak{l}}g_1$, $g'=e^{\mathfrak{l}'}g_1'$, $\mathfrak{l},\mathfrak{l}'\in\mathfrak{g}_{\alpha 0}$, $g_1,g_1'\in G'_{\alpha 0}$, then $gg'=e^{\mathfrak{l}+\mathfrak{l}'}g_1''$ for some $g_1''\in G'_{\alpha 0}$ because $G_{\alpha 0}/G'_{\alpha 0}$ is abelian. Therefore

(8.89) $$c_{\alpha 0}(gg') = c_{\alpha 0}(g)+c_{\alpha 0}(g')\,,\quad g,g'\in G_{\alpha 0}\,.$$

Next, we define $c_{\alpha 0}(g)$ for $g\in G_{(\alpha)}$ by (8.45). If $g=e^{\mathfrak{l}}g_1=e^{\mathfrak{l}'}g_1'$, $\mathfrak{l},\mathfrak{l}'\in\mathfrak{g}_{(\alpha)}$, $g_1,g_1'\in G_{\alpha 0}$, then $e^{-\mathfrak{l}}e^{\mathfrak{l}'}=g_1(g_1')^{-1}\in G_{\alpha 0}$ and hence $\mathfrak{l}'-\mathfrak{l}\in\mathfrak{g}_{\alpha 0}$. Let $\mathfrak{l}''\equiv\int_0^1\mathrm{Ad}\,(e^{-\beta\mathfrak{l}})(\mathfrak{l}'-\mathfrak{l})\,d\beta\in\mathfrak{g}_{\alpha 0}$. Then $e^{-\mathfrak{l}}e^{\mathfrak{l}'}=e^{\mathfrak{l}''}g_2$ for some $g_2\in G'_{\alpha 0}$. Hence

(8.90) $$\begin{aligned} c_{\alpha 0}(g_1) &= c_{\alpha 0}(e^{\mathfrak{l}''}g_2)+c_{\alpha 0}(g_1') \\ &= x_\alpha(\mathfrak{l}'')+c_{\alpha 0}(g_1') \\ &= x_\alpha(\mathfrak{l}'-\mathfrak{l})+c_{\alpha 0}(g_1')\,, \end{aligned}$$

where the last equality follows from (8.83). Therefore, the definition (8.45) does not depend on the decomposition of g.

From (8.83), it follows that

$$c_{\alpha 0}(gg'g^{-1}) = c_{\alpha 0}(g') \tag{8.91}$$

for $g \in G^{(\alpha)}$, $g' \in G_{\alpha 0}$ as well as for $g \in K_\alpha$, $g' \in G_{(\alpha)}$. From the definition and (8.89), we also have

$$c_{\alpha 0}(gg') = c_{\alpha 0}(g) + c_{\alpha 0}(g'), \quad g \in G_{(\alpha)}, \ g' \in G_{\alpha 0}. \tag{8.92}$$

Suppose $g \in H_\alpha$, $g' = e^{\mathfrak{l}} g_1$, $\mathfrak{l} \in \mathfrak{g}_\alpha$, $g_1 \in G_{\alpha 0}$. Then $gg'g^{-1}(g')^{-1} = g_2 g_3$, $g_2 = e^{\mathrm{Ad}(g)\mathfrak{l}} e^{-\mathfrak{l}} \in G_{\alpha 0}$, $g_3 = e^{\mathfrak{l}}\{(gg_1 g^{-1}) g_1^{-1}\} e^{-\mathfrak{l}} \in G_{\alpha 0}$. By the same argument as in (8.90), we have $c_{\alpha 0}(g_2) = x_\alpha(\mathrm{Ad}(g)\mathfrak{l} - \mathfrak{l})$. From (8.83) and (8.89), we have $c_{\alpha 0}(g_3) = c_{\alpha 0}(gg_1 g^{-1} g_1^{-1}) = c_{\alpha 0}(gg_1 g^{-1}) + c_{\alpha 0}(g_1^{-1}) = c_{\alpha 0}(g_1) - c_{\alpha 0}(g_1) = 0$. Therefore

$$c_{\alpha 0}(gg'g^{-1}(g_1')^{-1}) = x_\alpha(\mathrm{Ad}(g)\mathfrak{l} - \mathfrak{l}). \tag{8.93}$$

If $\mathfrak{l}' \in \mathfrak{h}_\alpha$, then

$$x_\alpha(\mathrm{Ad}(e^{\mathfrak{l}'})\mathfrak{l} - \mathfrak{l}) = x_\alpha([\mathfrak{l}', \mathfrak{l}]) \tag{8.94}$$

due to (8.83). From (8.93) and the commutativity in $G_{(\alpha)}/G_{\alpha 0}$, it follows that

$$c_{\alpha 0}(g(g_1 g_2) g^{-1}(g_1 g_2)^{-1}) = c_{\alpha 0}(gg_1 g^{-1} g_1^{-1}) + c_{\alpha 0}(gg_2 g^{-1} g_2^{-1}) \tag{8.95}$$

if $g_1, g_2 \in G_{(\alpha)}$.

We now define $c_{\alpha 0}(g)$ for $g \in G^{(\alpha)}$ by making decomposition $g = g^{(3)} g^{(2)} g^{(1)}$, $g^{(1)} \in G_{(\alpha)}$, $g^{(2)} \in G'_\alpha$, $g^{(3)} \in K_\alpha$ and then by setting

$$c_{\alpha 0}(g) = c_{\alpha+1}(g^{(3)} g^{(2)}) + c_{\alpha 0}(g^{(1)}) + (1/2) c_{\alpha 0}(g^{(2)} g^{(1)} \{g^{(2)}\}^{-1} \{g^{(1)}\}^{-1}). \tag{8.96}$$

We want to show the independence of this definition on the decomposition of g.

Let $g = g_a^{(3)} g_a^{(2)} g_a^{(1)} = g_b^{(3)} g_b^{(2)} g_b^{(1)}$ be two decompositions of g. Since $K_\alpha \cap H_\alpha = H_{\alpha 0}$, $g_a^{(3)} = g_b^{(3)} g'$ for some $g' \in H_{\alpha 0}$. Let $g' g_a^{(2)} \equiv g_c^{(2)}$. Then

$$x_\alpha(\mathrm{Ad}(g_c^{(2)})\mathfrak{l}) = x_\alpha(\mathrm{Ad}(g_a^{(2)})\mathfrak{l}) \tag{8.97}$$

for any $\mathfrak{l} \in \mathfrak{g}_\alpha$ due to (8.83) and $g' \in H_{\alpha 0} \subset K_\alpha$. Therefore $c_{\alpha 0}(g)$ defined by two decompositions $g = g_a^{(3)} g_a^{(2)} g_a^{(1)} = g_b^{(3)} g_c^{(2)} g_a^{(1)}$ is the same. Next, we note that $g_c^{(2)} = g_b^{(2)} g''$, $g_b^{(1)} = g'' g_a^{(1)}$ for $g'' = (g_b^{(2)})^{-1} g_c^{(2)} = g_b^{(1)} (g_a^{(1)})^{-1} \in G_{(\alpha)} \cap G'_\alpha = G_{\alpha 0}$. From (8.92) and (8.91), we have

$$c_{\alpha 0}(g_b^{(1)}) = c_{\alpha 0}(g_a^{(1)}) + c_{\alpha 0}(g'') . \tag{8.98}$$

Since $g'' \in G_{\alpha 0} \subset G_{(\alpha+1)0}$, we have from the inductive assumption and its consequence (8.49)

$$c_{(\alpha+1)}(g_b^{(3)} g_c^{(2)}) = c_{(\alpha+1)}(g_b^{(3)} g_b^{(2)}) + c_{(\alpha+1)}(g'') . \tag{8.99}$$

We note that $c_{(\alpha+1)}(g'') = c_{(\alpha+1)0}(g'') = c_{\alpha 0}(g'')$ by our assumption (8.84). From (8.95), we have

$$\begin{aligned} c_{\alpha 0}(g_b^{(2)} g_b^{(1)} (g_b^{(2)})^{-1} (g_b^{(1)})^{-1}) &= c_{\alpha 0}(g_b^{(2)} g_a^{(1)} (g_b^{(2)})^{-1} (g_a^{(1)})^{-1}) \\ &\quad + c_{\alpha 0}(g_b^{(2)} g'' (g_b^{(2)})^{-1} (g'')^{-1}) \end{aligned} \tag{8.100}$$

where the second term vanishes due to the special case $\mathfrak{l}=0$ of (8.93).

Finally we note that $x_{\alpha 0}(\mathrm{Ad}\,(g_b^{(2)})\,\mathfrak{l}) = x_{\alpha 0}(\mathrm{Ad}\,(g''')^{-1}\,\mathrm{Ad}\,(g_c^{(2)})\,\mathfrak{l}) = x_{\alpha 0}(\mathrm{Ad}\,(g_c^{(2)})\,\mathfrak{l})$ for any $\mathfrak{l} \in G_{(\alpha)}$ due to $g''' \equiv g_c^{(2)} g'' (g_c^{(2)})^{-1} \in G_{\alpha 0} \subset K_\alpha$ and (8.83). Hence

$$c_{\alpha 0}(g_c^{(2)} g_a^{(1)} (g_c^{(2)})^{-1} (g_a^{(1)})^{-1}) = c_\alpha(g_b^{(2)} g_a^{(1)} (g_b^{(2)})^{-1} (g_a^{(1)})^{-1}) . \tag{8.101}$$

Combining (8.98), (8.99), (8.100) and (8.101), we have the independence of $c_{\alpha 0}(g)$ for two decompositions $g = g_b^{(3)} g_c^{(2)} g_a^{(1)} = g_b^{(3)} g_b^{(2)} g_b^{(1)}$.

Q.E.D.

To sum up our result, we can obtain all possible pairs ϕ and c from the subgroup structure (1), x_α and μ_α. x_α determines c and the imaginary part of the inner product of ϕ while μ_α determines the real part of the inner product of ϕ. In order that our construction works, it is necessary and sufficient for x_α and μ_α to satisfy the invariance (8.83), the mutual compatibility (8.84), the positivity of μ_α and the inequality (8.82).

§9. Examples

(1) *Abelian group.*

In this case we have both $B^1(G, \bar{D}^+)$ and $Z^1(G, L_2(\mathfrak{L}, \mu))$ where $\mathfrak{L} = \mathfrak{g} + i\mathfrak{g}$. Since $\mathrm{Ad}\,(G) = 1$, μ can be arbitrary positive definite real inner product. Imaginary part for $(\phi_1^a(g_1), \phi_1^a(g_2))$ is impossible and e^{ic} is a character. We write a general elements of G by $f = (f_1 \cdots f_n)$ and denote the group operation by vector addition. Then the general form of F in Theorem 5.1 is

$$F(f) = -\sum_{j,k=1}^{n} f_j \mu_{jk} f_k + i \sum_{j=1}^{n} c_j f_j + \int \Big(\exp i \sum_{j=1}^{n} f_j p_j - 1 - i \sum_{j=1}^{n} f_j p_j g(p)\Big) \frac{1+p^2}{p^2} \mathrm{d}\mu(p) \tag{9.1}$$

where μ_{jk} is a real positive semidefinite matrix, c_j is a real number, μ is a finite positive measure, $p^2 = \sum_{j=1}^{n} (p_j)^2$ and $g(p)$ is any smooth function with $g(0)=1$ and decreasing rapidly at $p \to \infty$. A different choice of g is equivalent to a different choice of c_j. The exact form of $\frac{(1+p^2)}{p^2}$ is not important, because any bounded change can be absorbed in μ.

If $X = R^N$, we may represent elements in $\mathrm{C}(G, X)$ by $\mathrm{f} \equiv (\mathrm{f}_1(x), \cdots, \mathrm{f}_n(x))$. Then the continuous tensor product part of a uniformly continuous σ factorizable separable functional E can be written as

$$\mathrm{E}(f) = \exp \int F_x(\mathrm{f}(x)) \mathrm{d}\nu(x) \tag{9.2}$$

where F_x is of the form (9.1) and μ_{jk}, c_j and μ depends on x, where ν is a continuous positive measure. (To obtain this form of an integral, we may use the standard reduction theory of $\mathfrak{L}$ relative to $\{\mathrm{P}(Y)\}$.)

The first term and the last integral in (9.1) defines two mutually orthogonal exponent space $\mathfrak{L}_a$ and $\mathfrak{L}_b$. The representation space is the $e^{\mathfrak{L}_a} \otimes e^{\mathfrak{L}_b}$ and the representing unitary operator is of the form $U_a(\mathrm{f}) \otimes U_b(\mathrm{f})$. $\mathfrak{L}_a$ is the direct sum of n copies of R^N with the inner product

$$(\mathrm{f}^1, \mathrm{f}^2) = 2 \sum_{jk} \int \mathrm{f}_j^1(x) \mu_{jk}(x) \mathrm{f}_k^2(x) \mathrm{d}\nu(x) . \tag{9.3}$$

In terms of creation and annihilation operators, $\mathrm{U}_a(f)$ can be written as

$$\mathrm{U}_a(\mathrm{f}) = \chi_a(\mathrm{f}) \exp i \{(\mathrm{a}_a^*, \mathrm{f}) + (\mathrm{f}, \mathrm{a}_a)\} , \tag{9.4}$$

$$(\mathrm{a}_a^*, \mathrm{f}) = \sum_j \int \mathrm{a}_{aj}(x)^* \mathrm{f}_j(x) \mathrm{d}x . \tag{9.5}$$

The second term of (9.1) gives the c number coefficients

$$\chi_a(f) = \exp i \int \sum_{j=1}^{n} c_j(x) f_j(x) \mathrm{d}x . \tag{9.6}$$

$\mathfrak{L}_b$ is the $\mathfrak{L}_2$ space on the space of points $(x_1 \cdots x_N, p_1 \cdots p_n)$, $x \in R^N$, $p \in R^n$ with respect to $d\mu_x(p) d\nu(x)$. $Q(f)$ is the multiplication of the function $\exp i \sum_{j=1}^{n} f_j(x) p_j$ and $\Omega \in \bar{D}^+$ is represented by a function $(1+p^2)^{1/2}/(p^2)^{1/2} \equiv \Omega(p)$. In terms of creation and annihilation operator, $U_b(f)$ can be written as

$$(9.7) \qquad U_b(f) = \exp i \int d\mu_x(p) d\nu(x) \{(a_b(x, p)^* + \Omega(p)) \times (a_b(x, b) + \Omega(p)) - \Omega(p)^2 g(p)\} \sum_{j=1}^{n} f_j(x) p_j .$$

The special case of (9.1) for $n=1$ is given in [2] and is known as Lévy-Kinchin formula. The equation (9.7) gives a representation of the Poisson process part essentially in terms of the number operator on the Hilbert space of the Gaussian process. (However, the creation and annihilation operators are displaced by $\Omega(p)$, and the number is weighted by p_j.)

This situation prevails for a general case, as can be seen from Lemma 4.2. The representing operator $U(e^{I\chi_Y})$ is of the form (9.7) for ϕ in $B^1(G, \bar{D}^+(G))$ and of the form (9.4) for ϕ_α^a and $g \in H_\alpha$.

(2) *Canonical commutation relation* (*CCR*).

The group G is the Heisenberg group given by the multiplication rule

$$(9.8) \qquad (a_1, b_1, c_1)(a_2, b_2, c_2) = (a_1+a_2, b_1+b_2, c_1+c_2+a_2 b_1)$$

where a, b, c are real numbers. The commutator group G_c of G coincide with the center of G and consists of $(0, 0, c)$. We may take it as our G_2. For *CCR*, there is an additional requirement that $(0, 0, c)$ is represented by a c-number e^{ic}. Thus $B^1(G, \bar{D}^+)$ part is actually $B^1(G/G_c, \bar{D}^+)$ and exactly the same as the commutative case. Also by the condition that $(0, 0, c)$ is represented by e^{ic}, $\phi_j(g)=0$ for $j=2$. For $j=3$, $G=G_3$ and hence $Q_3(g)=1$. Since c_2 is already fixed for G_2, the imaginary part of $(\phi_3(g_1), \phi_3(g_2))$ part is fixed to be $\frac{1}{2}(a_1 b_2 - a_2 b_1)$. The real part is arbitrary except for the positive definiteness condition. As a net result, we have

$$(9.9) \qquad F(a, b, c) = -\mu_{11} a^2 - \mu_{12} ab - \mu_{22} b^2 - \frac{i}{2} ab + ic + i(\alpha a + \beta b) + \int \{\exp i(ap_1 + bp_2) - 1 - i(ap_1 + bp_2) g(p)\} \frac{1+p^2}{p^2} d\mu(p) ,$$

(9.10) $$\mu_{11}>0\,,\quad 4\mu_{11}\mu_{22}\geq(\mu_{12})^2+(1/4)\,.$$

This form was obtained by more elementary consideration in [1]. For E(f), we have an expression of the form (9.2).

The exponent space can be split into a direct sum of 3 spaces $\mathfrak{L}^{(j)}$, $j=1,2,3$. The representation space is $e^{\mathfrak{L}^{(1)}}\otimes e^{\mathfrak{L}^{(2)}}\otimes e^{\mathfrak{L}^{(3)}}$ and the representing operator is $U_1(f)\otimes U_2(f)\otimes U_3(f)$. $\mathfrak{L}^{(1)}$ is the space of $\{f(x), g(x)\}$, both f and g being real, with an inner product

(9.11) $$(\{f_1(x), g_1(x)\}, \{f_2(x), g_2(x)\}) = 2\int\{f_1(x)f_2(x)\mu_{11}(x) + g_1(x)g_2(x)\mu'_{22}(x)+\frac{1}{2}(f_1(x)g_2(x)+f_2(x)g_1(x)\mu_{12}(x) + \frac{i}{4}(f_1(x)g_2(x)-f_2(x)g_1(x))\}\,d\nu(x)\,,$$

(9.12) $$\mu'_{22}(x) = \{4\mu_{11}(x)\}^{-1}|\mu_{12}(x)+\frac{i}{2}|\,.$$

This is related to the standard form of the Fock representation of *CCR* by a Bogoliubov transformation. $\mathfrak{L}^{(2)}$ is the space of $\{g(x)\}$, g being complex, with the inner product

$$(g_1(x), g_2(x)) = 2\int g_1(x)^*g_2(x)(\mu_{22}(x)-\mu'_{22}(x))\,d\nu(x)\,.$$

The $\phi(g)$ in this space corresponds to a real wave function. The splitting between $\mathfrak{L}^{(1)}$ and $\mathfrak{L}^{(2)}$ is rather arbitrary, for example we may leave a part of μ_{11} in $\mathfrak{L}^{(2)}$ and modify μ'_{22} accordingly. In any case the creation and annihilation operator of the original *CCR* is represented by a linear combination of two kinds of Fock creation and annihilation operators. If we choose the splitting appropriate, it can be arranged as a hole and a particle, namely the creation operator of *CCR* is represented as a sum of Fock creation operator of one space (particle) and Fock annihilation operator of another space (hole). A typical example is in [4]. The $\mathfrak{L}^{(3)}$ part is exactly the same as the commutative case where the relevant group is that of $(a, b, 0)$ and is two dimensional.

(3) *The current algebra over rotation group.*

We consider $G=SU(2)$ or $O(3)$. Then the group is compact and hence $Z^1(G, L)=B^1(G, L)$. In this case we have a representation

$U(g)=e^{iH(\mathfrak{l})}$, $g=e^{\mathfrak{l}}$ in a Fock space $e^{\mathfrak{L}}$ where $\mathfrak{L}=\int^{\oplus}\mathfrak{L}_x \mathrm{d}\nu(x)^{1/2}$, $Q(g)=\int^{\oplus}Q_x(g)$, $\Omega=\int^{\oplus}\Omega_x$ and

$$(9.13)\qquad H(\mathfrak{l}) = -i\int(\mathrm{a}^*(x)+\Omega(x),\ \mathrm{d}Q_x(\mathfrak{l})(\mathrm{a}(x)+\Omega(x)))\mathrm{d}\nu(x)\,.$$

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