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Iterated Hyperbolic Mixed Problems

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Introduction

Mixed problems for hyperbolic equations have been investigated by many authors, but mostly in the case when the equations are of second order or the case of one space dimension. In 1962, S. Agmon [1] established a priori estimates for solutions of general mixed problems of higher order hyperbolic equations with constant coefficients in a half space. In the case of variable coefficients, there are recent works of S. Mizohata [2] and of S. Miyatake [3]. In this paper, we study the conditions for solvability of higher order mixed problems by means of iteration procedure. We confined ourselves here to the case of half space, but it is not difficult to see that our method is also applicable to general bounded or unbounded domains.

In §1, we summarize L^2-energy method in elliptic general boundary value problems, due to Schechter [4]. We also clarify the dependence on the parameters in order to apply his results to unbounded domains. Here we consider even the case where boundary operators are not normal. The existence theorem is proved in appendix, using singular integral operators.

In §2, we shall study the iterated hyperbolic mixed problems on the basis of energy inequalities.

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§1. Preliminary (Energy of Elliptic Type)

1. Fundamental energy inequalities.

In this section we consider polynomials of one variable:

$$P(\xi)=a_0\xi^m+a_1\xi^{m-1}+\cdots+a_m=a_0\prod_{j=1}^{m}(\xi-\xi_j)\quad (a_0\neq 0),$$

$$P'(\xi)=a'_0\xi^m+a'_1\xi^{m-1}+\cdots+a'_m=a'_l\prod_{j=1}^{m-l}(\xi-\xi'_j)\quad (a'_0=\cdots=a'_{l-1}=0,\ a'_l\neq 0).$$

We denote

$$\tilde{u}(\xi)=\int_0^\infty e^{-ix\xi}u(x)dx,\qquad \omega_k=D^k u(0)\ \text{ for } u(x)\in\mathcal{D}(R^1),$$

where $D=\frac{1}{i}\frac{d}{dx}$, then we have

$$\widetilde{D^k u}(\xi)=\xi^k\tilde{u}(\xi)+i(\omega_{k-1}+\xi\cdot\omega_{k-2}+\cdots+\xi^{k-1}\omega_0).$$

Let us denote

$$\widehat{P}(\xi,s)=a_0 s^{m-1}+(a_0\xi+a_1)s^{m-2}+\cdots+(a_0\xi^{m-1}+\cdots+a_{m-1}),$$

$$\widehat{P}(\xi,\omega)=P(\xi,\omega_0,\cdots,\omega_{m-1})=a_0\omega_{m-1}+(a_0\xi+a_1)\omega_{m-2}+\cdots$$
$$\cdots+(a_0\xi^{m-1}+\cdots+a_{m-1})\omega_0,$$

then we have

$$\widetilde{P(D)u}(\xi)=P(\xi)u(\xi)+i\widehat{P}(\xi,\omega).$$

Now we denote

$$\mathcal{P}(\xi,s)=\mathcal{P}_{PP'}(\xi,s)=P(\xi)\widehat{P}'(\xi,s)-P'(\xi)\widehat{P}(\xi,s),$$

$$R=R_{PP'}=a_0^{m-l}a_l'^{m}\prod_{i=1}^{m}\prod_{j=1}^{m-l}(\xi_i-\xi'_j),$$

then we have

Lemma 1.1. *$\mathcal{P}$ is represented by*

$$\mathcal{P}(\xi,s)=\sum_{i,j=1}^{m}q_{ij}\xi^{m-i}s^{m-j}=(\xi^{m-1},\cdots,1)Q\begin{pmatrix}s^{m-1}\\ \vdots\\ 1\end{pmatrix},$$

where

$$q_{ij}=\sum_{\substack{h+t=i+j-1\\ h\geqslant\max(i,j)\\ t\leqslant\min(i,j)-1}}a_h a'_t-a'_h a_t,$$

$$\det Q=(-1)^{\frac{1}{2}m(m+1)}a_0^l R.$$

Proof. From the definition of $\mathcal{P}$, we have

$$\mathcal{P}(\xi, s)=(a_0\xi^m+a_1\xi^{m-1}+\cdots+a_m)\{a_0's^{m-1}+(a_0'\xi+a_1')s^{m-2}+\cdots$$
$$\cdots+(a_0'\xi^{m-1}+\cdots+a_{m-1}')\}$$
$$-(a_0'\xi^m+a_1'\xi^{m-1}+\cdots+a_m')\{a_0s^{m-1}+(a_0\xi+a_1)s^{m-2}+\cdots$$
$$\cdots+(a_0\xi^{m-1}+\cdots+a_{m-1})\}$$
$$=\{(a_1\xi^{m-1}+\cdots+a_m)a_0'-(a_1'\xi^{m-1}+\cdots+a_m')a_0\}s^{m-1}$$
$$+\{(a_2\xi^{m-2}+\cdots+a_m)(a_0'\xi+a_1')-(a_2'\xi^{m-1}+\cdots+a_m')(a_0\xi+a_1)\}s^{m-2}$$
$$+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+\{a_m(a_0'\xi^{m-1}+\cdots+a_{m-1}')-a_m'(a_0\xi^{m-1}+\cdots+a_{m-1})\}$$
$$=\sum_{\substack{h+t=i+j-1\\ h\geqslant j>t}}(a_ha_t'-a_h'a_t)\xi^{m-i}s^{m-j}.$$

Now suppose that $R=0$, that is, $\{P, P'\}$ has a common root ξ_0, then we have $\mathcal{P}(\xi_0, s)=0$ identically, which means that $\det Q=0$. Remarking that $\det Q$ is a polynomial of degree $m(m-l)$ with respect to $(\xi_1, \cdots, \xi_m, \xi_1', \cdots, \xi_{m-l}')$, we have

$$\det Q=cR,$$

where c is independent of $(\xi_1, \cdots, \xi_m, \xi_1', \cdots, \xi_{m-l}')$. It is shown easily that $c=(-1)^{\frac{1}{2}m(m+1)}a_0'^{l}$.

For the present we consider

$$P(\xi)=\xi^m+a_1\xi^{m-1}+\cdots+a_m\,,$$
$$P'(\xi)=\xi^m+a_1'\xi^{m-1}+\cdots+a_m'\,,$$

where we assume that

$$|a_j|,\ |a_j'|\leqslant Cr^j \qquad j=1, 2, \cdots, m \ \ (C>0,\ r>0).$$

Lemma 1.2. *We assume that*

$$|R_{PP'}|\geqslant cr^{m^2},$$

then we have

$$|P(\xi)|+|P'(\xi)|\geqslant c'(|\xi|+r)^m \qquad \text{for } \xi\in C^1,$$

where $c'=c'(C, c)$. The converse is also holds.

Proof. Assume that

$$|R_{PP'}| \geqslant cr^{m^2},$$

then we have

$$|\xi_i - \xi'_j| \geqslant c_0(C, c)r \qquad i, j = 1, 2, \cdots, m,$$

$$|P(\xi)| + |P'(\xi)| = \prod_{j=1}^{m} |\xi - \xi_j| + \prod_{j=1}^{m} |\xi - \xi'_j| \geqslant (\tfrac{1}{2}c_0 r)^m \quad \text{for } \xi \in C^1.$$

On the other hand, we have

$$|P(\xi)| + |P'(\xi)| \geqslant C_0(C)(|\xi| + r)^m \qquad \text{for } |\xi| \geqslant C_1(C)r.$$

Conversely assume that

$$|P(\xi)| + |P'(\xi)| \geqslant c'(|\xi| + r)^m \qquad \text{for } \xi \in C^1.$$

Let $\xi = \xi_j$, then we have

$$|P'(\xi_j)| \geqslant c'(|\xi_j| + r)^m \geqslant c' r^m,$$

therefore

$$|R| = |\prod_{j=1}^{m} P'(\xi_j)| \geqslant c'^m r^{m^2}.$$

Lemma 1.3. *We assume that*

$$|P(\xi)| + |P'(\xi)| \geqslant c(|\xi| + r)^m \qquad \textit{for } \xi \in R^1 \ (c > 0),$$

then we have for $u \in \mathcal{E}^m_{L^2}(0, \infty)$

$$\frac{c^2}{2}\int_0^\infty \{|D^m u(x)|^2 + \cdots + r^{2m}|u(x)|^2\}dx$$
$$\leqslant \int_0^\infty \{|P(D)u(x)|^2 + |P'(D)u(x)|^2\}dx$$
$$+ C'\{r|\omega_{m-1}|^2 + \cdots + r^{2m-1}|\omega_0|^2\},$$

where $C' = C'(C, c)$.

Proof. Let

$$P_k(\xi) = \frac{c}{\sqrt{2}} r^k \xi^{m-k},$$

then

$$|P(\xi)|^2 + |P'(\xi)|^2 - \sum_{k=0}^{m} |P_k(\xi)|^2 \geqslant \frac{c^2}{2}(|\xi| + r)^{2m}.$$

Now we have

$$
\begin{aligned}
&|\widetilde{P(D)u}(\xi)|^2+|\widetilde{P'(D)u}(\xi)|^2-\sum_{k=0}^{m}|\widetilde{P_k(D)u}(\xi)|^2\\
&=|P(\xi)u(\xi)+i\widehat{P}(\xi,\omega)|^2+|P'(\xi)\hat{u}(\xi)+i\widehat{P'}(\xi,\omega)|^2\\
&\quad-\sum_{k=0}^{m}|P_k(\xi)\hat{u}(\xi)+i\widehat{P}_k(\xi,\omega)|^2\\
&=(|P|^2+|P'|^2-\sum_{k=0}^{m}|P_k|^2)|\hat{u}|^2+2\operatorname{Re}\{i(\widehat{P}\overline{P}+\widehat{P}'\overline{P}'-\sum_{k=0}^{m}\widehat{P}_k\overline{P}_k)\bar{u}\}\\
&\quad+(|\widehat{P}|^2+|\widehat{P}'|^2-\sum_{k=0}^{m}|\widehat{P}_k|^2)\\
&=(|P|^2+|P'|^2-\sum_{k=0}^{m}|P_k|^2)\left|u+i\frac{\widehat{P}\overline{P}+\widehat{P}'\overline{P}'-\sum\widehat{P}_k\overline{P}_k}{|P|^2+|P'|^2-\sum|P_k|^2}\right|^2\\
&\quad+\frac{|\mathcal{P}_{PP'}|^2-\sum|\mathcal{P}_{PP_k}|^2-\sum|\mathcal{P}_{P'P_k}|^2+\sum_{j<k}|\mathcal{P}_{P_jP_k}|^2}{|P|^2+|P'|^2-\sum|P_k|^2}.
\end{aligned}
$$

From the representation of $\mathcal{P}$ in Lemma 1. 1,

$$
\begin{aligned}
&\int_{-\infty}^{\infty}\{|\widetilde{P(D)u}(\xi)|^2+|\widetilde{P'(D)u}(\xi)|^2-\sum_{k=0}^{m}|\widetilde{P_k(D)u}(\xi)|^2\}d\xi\\
&\geqslant -C'(C,c)\int_{-\infty}^{\infty}\frac{(\sum_{i,j=1}^{m}|\xi|^{m-i}r^{i+j-1}|\omega_{m-j}|)^2}{(|\xi|+r)^{2m}}d\xi\\
&\geqslant -C''(C,c)\sum_{j=1}^{m}(r^{j-\frac{1}{2}}|\omega_{m-j}|)^2.
\end{aligned}
$$

Lemma 1. 4. *We assume that*

$$|R_{PP'}|\geqslant cr^{m^2},$$

then we have for $u\in\mathcal{E}^m_{L^2}(0,\infty)$

$$c'\{r|\omega_{m-1}|^2+\cdots+r^{2m-1}|\omega_0|^2\}\leqslant\int_0^\infty\{|P(D)u(x)|^2+|P'(D)u(x)|^2\}dx,$$

where $c'=c'(C,c)$.

Proof. In the same way as in Lemma 1. 3, we have

$$
\begin{aligned}
&|\widetilde{P(D)u}(\xi)|^2+|\widetilde{P'(D)u}(\xi)|^2\\
&\quad=(|P|^2+|P'|^2)\left|u+i\frac{\widehat{P}\overline{P}+\widehat{P}'\overline{P}'}{|P|^2+|P'|^2}\right|^2+\frac{|\mathcal{P}_{PP'}|^2}{|P|^2+|P'|^2}.
\end{aligned}
$$

From Lemma 1. 2, we have

$$|P|^2+|P'|^2\geqslant c'(C,c)(|\xi|+r)^{2m}\qquad\text{for }\xi\in R^1.$$

Hence we denote

$$|P|^2+|P'|^2=\prod_{j=1}^{m}(\xi-\xi_j'')\cdot 2\prod_{j=m+1}^{2m}(\xi-\xi_j'')=P_+''P_-'',$$

where

$$\operatorname{Im}\xi_j''>c_0(C,c)r>0 \qquad j=1,2,\cdots,m.$$

Then we have

$$\int_{-\infty}^{\infty}\frac{|\mathcal{P}(\xi,\omega)|^2}{|P(\xi)|^2+|P'(\xi)|^2}d\xi=(\bar{\omega}_{m-1},\cdots,\bar{\omega}_0)Q^*$$

$$\times\left(\int_{-\infty}^{\infty}\frac{\xi^{2m-i-j}}{|P(\xi)|^2+|P'(\xi)|^2}d\xi\right)_{i,j=1,\cdots,m}Q\begin{pmatrix}\omega_{m-1}\\ \vdots\\ \omega_0\end{pmatrix}$$

$$=(\bar{\omega}_{m-1},\cdots,\bar{\omega}_0)Q^*E_0Q\begin{pmatrix}\omega_{m-1}\\ \vdots\\ \omega_0\end{pmatrix},$$

where we have

$$\det E_0=\frac{(2\pi i)^m}{\prod_{j=1}^{m}P_-''(\xi_j'')}.$$

Now we denote

$$\Omega_{m-j}=r^{j-\frac{1}{2}}\omega_{m-j} \qquad j=1,\cdots,m$$

and

$$(\bar{\omega}_{m-1},\cdots,\bar{\omega}_0)Q^*E_0Q\begin{pmatrix}\omega_{m-1}\\ \vdots\\ \omega_0\end{pmatrix}=(\bar{\Omega}_{m-1},\cdots,\bar{\Omega}_0)E_1\begin{pmatrix}\Omega_{m-1}\\ \vdots\\ \Omega_0\end{pmatrix},$$

then we have, from Lemma 1.1, that the absolute value of each element of E_1 is bounded by $C'(C,c)$, E_1 is positive, and $|\det E_1|>c_1(C,c)>0$.

Lemma 1.5. *We assume that* $\operatorname{Im}\xi_j<0$ $(j=1,\cdots,m)$ *and*

$$|P(\xi)|\geqslant c(|\xi|+r)^m \qquad \textit{for } \xi\in R^1.$$

Then we have, for $u\in\mathcal{E}_{L^2}^m(0,\infty)$,

$$\int_0^\infty|P(D)u(x)|^2dx\geqslant c^2\int_0^\infty\{|D^mu(x)|^2$$

$$+r^2|D^{m-1}u(x)|^2+\cdots+r^{2m}|u(x)|^2\}dx.$$

Proof. The unique solution of $P(D)u(x)=f(x)$ $(x>0)$ is given by

$$u(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\frac{\tilde{f}(\xi)}{P(\xi)}d\xi \qquad (x>0),$$

then we have

$$r^j D^{m-j}u(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{ix\xi}\frac{r^j\xi^{m-j}\tilde{f}(\xi)}{P(\xi)}d\xi \qquad (x>0).$$

Therefore we have

$$\int_0^\infty\sum_{j=0}^m|r^jD^{m-j}u(x)|^2dx=\frac{1}{2\pi}\int_{-\infty}^{\infty}\sum_{j=0}^m\left|\frac{r^j\xi^{m-j}\tilde{f}(\xi)}{P(\xi)}\right|^2d\xi\leqslant\frac{1}{c^2}\int_0^\infty|f(x)|^2dx.$$

Here we introduce

$$M_K(\xi)=\xi+iKr \qquad (K>0),$$

then we have from Lemma 1.5

$$c_K\int_0^\infty\sum_{j=0}^m|r^jD^{s-j}u(x)|^2dx\leqslant\int_0^\infty|M_K(D)^s u(x)|^2dx$$
$$\leqslant c'_K\int_0^\infty\sum_{j=0}^m|r^jD^{s-j}u(x)|^2dx.$$

Finally we consider the general case when

$$P(\xi)=\xi^m+a_1\xi^{m-1}+\cdots+a_m=\prod_{j-1}^m(\xi-\xi_j),$$
$$P'(\xi)=\xi^{m'}+a'_1\xi^{m'-1}+\cdots+a'_{m'}=\prod_{j=1}^{m'}(\xi-\xi'_j),$$

where we assume that

$$|a_j|\leqslant Cr^j \ \ (j=1,\cdots,m), \quad |a'_j|\leqslant Cr^j \ \ (j=1,\cdots,m').$$

Proposition 1.1. *We assume that*

$$|P(\xi)|^{m'}+|P'(\xi)|^m\geqslant c(|\xi|+r)^{mm'} \quad \text{for } \xi\in R^1 \ \ (c>0).$$

Then we have for $u\in\mathcal{E}^s_{L^2}(0,\infty)$

$$\int_0^\infty\sum_{j=0}^s|r^jD^{s-j}u(x)|^2dx\leqslant C_s\int_0^\infty\{\sum_{j=0}^{s-m}|r^jD^{s-m-j}P(D)u(x)|^2$$
$$+\sum_{j=0}^{s-m'}|r^jD^{s-m'-j}P'(D)u(x)|^2\}dx+C_s\{r|\omega_{s-1}|^2+\cdots+r^{2s-1}|\omega_0|^2\},$$

where $s\geqslant\max(m,m')$ and $C_s=C_s(C,c)$.

Proof. Let

$$Q(\xi)=M_1(\xi)^{s-m}P(\xi),$$
$$Q'(\xi)=M_1(\xi)^{s-m'}P'(\xi),$$

and apply Lemma 1.3 to $\{Q, Q'\}$.

Proposition 1.2. *We assume that P, P' are decomposed in such a way that*

$$P=P_0P_{(-)}, \qquad P'=P_0'P_{(-)}',$$

where all the roots of $P_{(-)}$ and $P_{(-)}'$ have negative imaginary parts and

$$|P_{(-)}(\xi)|\geqslant c(|\xi|+r)^{m-\mu}, \quad |P_{(-)}'(\xi)|\geqslant c(|\xi|+r)^{m'-\mu'} \quad \text{for } \xi\in R^1,$$
$$|R_{P_0P_0'}|\geqslant cr^{\mu\mu'}.$$

Then we have if u belongs to $\mathcal{E}_{L^2}^s(0,\infty)$,

$$c_s\int_0^\infty\sum_{j=0}^{s}|r^jD^{s-j}u(x)|^2dx$$
$$\leqslant\int_0^\infty\{\sum_{j=0}^{s-m}|r^jD^{s-m-j}P(D)u(x)|^2+\sum_{j=0}^{s-m'}|r^jD^{s-m'-j}P'(D)u(x)|^2\}dx,$$

where $s\geqslant\max(m, m')$ and $c_s=c_s(C,c)$.

Proof. We have

$$|\xi_i|,\ |\xi_j'|\leqslant C_0(C)r \qquad i=1,\cdots,m, \quad j=1,\cdots,m'.$$

We denote

$$Q(\xi)=M_{2C_0}(\xi)^{s-\mu}P_0(\xi),$$
$$Q'(\xi)=M_{3C_0}(\xi)^{s-\mu'}P_0'(\xi),$$

then

$$|R_{QQ'}(\xi)|\geqslant(C_0r)^{s2-\mu\mu'}|R_{P_0P_0'}|\geqslant cC_0^{s2-\mu\mu'}r^{s2}.$$

Therefore we have from Lemma 1.4

$$c_0(C,c)\int_0^\infty\sum_{j=0}^{s}|r^jD^{s-j}u(x)|^2dx$$
$$\leqslant\int_0^\infty\{\sum_{j=0}^{s-\mu}|r^jD^{s-\mu-j}P_0(D)u(x)|^2+\sum_{j=0}^{s-\mu'}|r^jD^{s-\mu'-j}P_0'(D)u(x)|^2\}dx.$$

On the other hand, we have from Lemma 1.5

$$\int_0^\infty |M_1(D)^{s-m}P(D)u(x)|^2dx=\int_0^\infty |M_1(D)^{s-m}P_{(-)}(D)P_0(D)u(x)|^2dx$$
$$\geqslant c_1(C,c)\int_0^\infty \sum_{j=0}^{s-\mu}|r^jD^{s-\mu-j}P_0(D)u(x)|^2dx,$$
$$\int_0^\infty |M_1(D)^{s-m'}P'(D)u(x)|^2dx$$
$$\geqslant c_1(C,c)\int_0^\infty \sum_{j=0}^{s-\mu'}|r^jD^{s-\mu'-j}P'_0(D)u(x)|^2dx.$$

2. Hyperbolic polynomials.

Let $P(\tau, \xi, \eta_1, \eta_2, \cdots, \eta_{n-1})$ be a homogeneous polynomial of degree m, where the coefficients of τ^m and ξ^m are not zero, which we denote

$$P(\tau,\xi,\eta)=\sum_{i+j+|\nu|=m} p_{ij\nu}\tau^i\xi^j\eta^\nu$$
$$=p_0\tau^m+p_1(\xi,\eta)\tau^{m-1}+\cdots+p_m(\xi,\eta)=p_0\prod_{j=1}^m(\tau-\tau_j(\xi,\eta))\quad (p_0\neq 0)$$
$$=a_0\xi^m+a_1(\tau,\eta)\xi^{m-1}+\cdots+a_m(\tau,\eta)=a_0\prod_{j=1}^m(\xi-\xi_j(\tau,\eta))\quad (a_0\neq 0).$$

We assume that $\{\tau_j(\xi,\eta)\}_{j=1,\cdots,m}$ are real for $(\xi,\eta)\in R^n$, which we say that $P(\tau,\xi,\eta)$ is hyperbolic with respect to τ. Then μ of $\{\tau_1,\cdots,\tau_m\}$ are negative and the others positive for $\xi>0$ and $\eta=0$, therefore μ of $\{\xi_1,\cdots,\xi_m\}$ have positive imaginary parts and the others negative ones for $\operatorname{Im}\tau<0$ and $\eta\in R^{n-1}$, which we denote by $\{\xi_1^+,\cdots,\xi_\mu^+\}$ and $\{\xi_1^-,\cdots,\xi_{m-\mu}^-\}$ respectively. Here we denote

$$P_+(\tau,\xi,\eta)=\prod_{j=1}^{\mu}(\xi-\xi_j^+(\tau,\eta)),\qquad P_-(\tau,\xi,\eta)=a_0\prod_{j=1}^{m-\mu}(\xi-\xi_j^-(\tau,\eta)).$$

Next we consider another hyperbolic polynomial P' of homogeneous degree m':

$$P'(\tau,\xi,\eta)=\sum_{i+j+|\nu|=m'} p'_{ij\nu}\tau^i\xi^j\eta^\nu=p'_0\prod_{j=1}^{m'}(\tau-\tau'_j(\xi,\eta))$$
$$=a'_0\prod_{j=1}^{m'}(\xi-\xi'_j(\tau,\eta))=P'_+(\tau,\xi,\eta)P'_-(\tau,\xi,\eta).$$

We denote the resultant of $\{P, P'\}$ with respect to τ by

$$R^0_{PP'}(\xi,\eta)=R^0(\xi,\eta)=p_0^{m'}p_0'^{m}\prod_{i=1}^{m}\prod_{j=1}^{m'}(\tau_i(\xi,\eta)-\tau'_j(\xi,\eta)),$$

and that of $\{P_+, P'_+\}$ with respect to ξ by

$$R_{P_+P'_+}(\tau, \eta) = R^+(\tau, \eta) = \prod_{i=1}^{\mu} \prod_{j=1}^{\mu'} (\xi_i^+(\tau, \eta) - \xi_j^+(\tau, \eta)).$$

Here we assume that

$$\text{(I)} \quad \inf_{\substack{(\xi, \eta) \in R^n \\ |\xi| + |\eta| = 1}} |R^0(\xi, \eta)| = k_1 \neq 0,$$

$$\text{(II)} \quad \inf_{\substack{\operatorname{Im}\tau < 0, \eta \in R^{n-1} \\ |\tau| + |\eta| = 1}} |R^+(\tau, \eta)| = k_2 \neq 0.$$

Now we denote

$$\max(|p_{ij\nu}|, |p'_{ij\nu}|, |p_0|^{-1}, |p'_0|^{-1}, |a_0|^{-1}, |a'_0|^{-1}) = K,$$

then we have

Lemma 2.1. (I) *is equivalent to the following: there exists* $h > 0$ *such that*

$$\text{(I)}' \quad \inf_{\substack{\tau \in C^1, \xi \in C^1, \eta \in R^{n-1} \\ |\tau| + |\xi| + |\eta| = 1 \\ |\operatorname{Im}\xi| < h}} \{|P(\tau, \xi, \eta)|^{m'} + |P'(\tau, \xi, \eta)|^{m}\} = k(h) > 0.$$

More precisely, if we assume (I), *then there exists* $h = h(K, k_1) > 0$ *and* $k(h) < h$, *and if we assume* (I)′, *then* $k_1 > c'(K, k(h)) > 0$.

Proof. Assume (I). Since $R^0_{P^{m'}P'^m} = (R^0_{PP'})^{mm'}$, we have from Lemma 1.2

$$\inf_{\substack{\tau \in C^1, (\xi, \eta) \in R^n \\ |\tau| + |\xi| + |\eta| = 1}} \{|P(\tau, \xi, \eta)|^{m'} + |P'(\tau, \xi, \eta)|^{m}\} \geqslant c(K, k_1) > 0.$$

Now we denote, for $\xi \in C^1$, $\operatorname{Re}\xi = \alpha$, $\operatorname{Im}\xi = \beta$, then we have

$$P(\tau, \alpha + i\beta, \eta) = P(\tau, \alpha, \eta) + \beta P_1(\tau, \alpha, \beta, \eta),$$

$$P'(\tau, \alpha + i\beta, \eta) = P'(\tau, \alpha, \eta) + \beta P'_1(\tau, \alpha, \beta, \eta),$$

where

$$|P_1(\tau, \alpha, \beta, \eta)| \leqslant C(K)(|\tau| + |\xi| + |\eta|)^{m-1},$$

$$|P'_1(\tau, \alpha, \beta, \eta)| \leqslant C(K)(|\tau| + |\xi| + |\eta|)^{m'-1}.$$

Therefore we have

$$|P(\tau, \xi, \eta)|^{m'} + |P'(\tau, \xi, \eta)|^{m}$$

$$\geqslant (|P(\tau, \alpha, \eta)|^{m'} + |P'(\tau, \alpha, \eta)|^{m}) - C'(K)|\beta|(|\tau| + |\xi| + |\eta|)^{mm'-1},$$

and then there exists $c' = c'(K, k_1) > 0$ such that

$$\inf_{\substack{\tau\in C^1, \xi\in C^1, \eta\in R^{n-1}\\ |\tau|+|\xi|+|\eta|=1\\ |\mathrm{Im}\,\xi|<C'}} \{|P(\tau,\xi,\eta)|^{m'}+|P'(\tau,\xi,\eta)|^{m}\}>c'.$$

The converse is shown easily.

Lemma 2.2. *We assume* (I) *and* (II). *Then* P, P' *are decomposed into*

$$P=P_0P_{(-)},\qquad P'=P_0'P_{(-)}',$$

where

$$\inf_{\substack{\mathrm{Im}\,\tau<0, \eta\in R^{n-1}\\ |\tau|+|\eta|=1}} |R_{P_0P_0'}(\tau,\eta)|\geqq c',$$

all the roots of $P_{(-)}$, $P_{(-)}'$ *have negative imaginary parts, and*

$$\inf_{\substack{\mathrm{Im}\,\tau<0, (\xi,\eta)\in R^n\\ |\tau|+|\xi|+|\eta|=1}} |P_{(-)}(\tau,\xi,\eta)|,\qquad \inf_{\substack{\mathrm{Im}\,\tau<0, (\xi,\eta)\in R^n\\ |\tau|+|\xi|+|\eta|=1}} |P_{(-)}'(\tau,\xi,\eta)|\geqq c'$$

where $c'=c'(K, k_1, k_2)>0$.

Proof. From Lemma 2.1, we have the two cases for $\xi_j(\tau,\eta)$:

i) $|\mathrm{Im}\,\xi_j(\tau,\eta)|\geqq c'(|\tau|+|\eta|)$,

ii) $|\mathrm{Im}\,\xi_j(\tau,\eta)|<c'(|\tau|+|\eta|)$ and $|P'(\tau,\xi_j(\tau,\eta),\eta)|\geqq c'(|\tau|+|\eta|)^{m'}$,

for $\mathrm{Im}\,\tau<0$, $\eta\in R^{n-1}$. The analogous situation holds for ξ_j'. Now we denote

$$P_{(0)}(\tau,\xi,\eta)=\prod_{\mathrm{Im}\,\xi_j^-(\tau,\eta)>-c'(|\tau|+|\eta|)} (\xi-\xi_j^-(\tau,\eta)),$$

$$P_{(-)}(\tau,\xi,\eta)=a_0\prod_{\mathrm{Im}\,\xi_j^-(\tau,\eta)\leqq -c'(|\tau|+|\eta|)} (\xi-\xi_j^-(\tau,\eta)),$$

$$P_0(\tau,\xi,\eta)=P_+(\tau,\xi,\eta)P_{(0)}(\tau,\xi,\eta),$$

where the degree of P_0 may vary with (τ,η). We define $P_{(0)}'$, $P_{(-)}'$, P_0' in the same way. Since we have

$$R_{P_0P_0'}=R_{P_+P_+'}R_{P_+P_{(0)}'}R_{P_{(0)}P_+'}R_{P_{(0)}P_{(0)}'},$$

we have

$$\inf_{\substack{\mathrm{Im}\,\tau<0, \eta\in R^{n-1}\\ |\tau|+|\eta|=1}} |R_{P_0P_0'}(\tau,\eta)|\geqq c(K, k_1, k_2)>0.$$

Now we denote

$$\|u\|_{s,\tau}^2=\int_{R_+^n}\sum_{i+j+|\nu|=s}|\tau^i D_x^j D_y^\nu u(x,y)|^2 dxdy.$$

Proposition 2.1. *We assume* (I), *then we have, for* $u\in\mathcal{D}_{L^2}^s(R_+^n)$,

$$c_s\|u\|_{s,\tau}\leqslant\|P(\tau, D_x, D_y)u\|_{s-m,\tau}+\|P'(\tau, D_x, D_y)u\|_{s-m',\tau}\quad \textit{for } \operatorname{Im}\tau<0,$$

where $s\geqslant\max(m, m')$ *and* $c_s=c_s(K, k_1)>0$.

Proof. Remarking Lemma 2.1, we only apply Proposition 1.1.

Proposition 2.2. *We assume* (I) *and* (II), *then we have, for* $u\in\mathcal{E}_{L^2}^s(R_+^n)$,

$$c_s\|u\|_{s,\tau}\leqslant\|P(\tau, D_x, D_y)u\|_{s-m,\tau}+\|P'(\tau, D_x, D_y)u\|_{s-m',\tau}\quad \textit{for } \operatorname{Im}\tau<0,$$

where $s\geqslant\max(m, m')$ *and* $c_s=c_s(K, k_1, k_2)>0$.

Proof. Remarking Lemma 2.2, we only apply Proposition 1.2.

Finally we consider the case of variable coefficients:

$$P(x, y;\ \tau, \xi, \eta)=\sum_{i+j+|\nu|=m} p_{ij\nu}(x, y)\tau^i\xi^j\eta^\nu,$$

$$P'(x, y;\ \tau, \xi, \eta)=\sum_{i+j+|\nu|=m'} p'_{ij\nu}(x, y)\tau^i\xi^j\eta^\nu,$$

where $p_{ij\nu}(x, y)$, $p'_{ij\nu}(x, y)\in\mathcal{B}(R_+^n)$, and

$$\inf_{(x,y)\in R_+^n}\{|p_{m00}(x, y)|, |p_{0m0}(x, y)|, |p'_{m'00}(x, y)|, |p'_{0m'0}(x, y)|\}\neq 0.$$

Theorem 2.1. *We assume that* P, P' *are hyperbolic with respect to* τ *for each point* $(x, y)\in R_+^n$, *and*

(I) $$\inf_{\substack{(x,y)\in R_+^n\\ (\xi,\eta)\in R^n\\ |\xi|+|\eta|=1}}|R_{PP'}^0(x, y;\ \xi,\eta)|\neq 0$$

(II) $$\inf_{\substack{y\in R^{n-1}\\ \operatorname{Im}\tau<0,\ \eta\in R^{n-1}\\ |\tau|+|\eta|=1}}|R_{PP'}^+(0, y;\ \tau, \eta)|\neq 0.$$

Then we have for $u\in\mathcal{E}_{L^2}^s(R_+^n)$

$$\|P(x, y;\ \tau, D_x, D_y)u\|_{s-m,\tau}+\|P'(x, y;\ \tau, D_x, D_y)u\|_{s-m',\tau}\geqslant c_s\|u\|_{s,\tau}$$

for $\operatorname{Im}\tau<0$ *and* $|\tau|>C_s$, *where* $s\geqslant\max(m, m')$ *and* c_s, C_s *are positive constants.*

Proof. Using partition of unity in R_+^n, we apply Proposition 2.2 near the boundary of R_+^n and Proposition 2.1 in the interior.

3. Lopatinski's determinants.

Given a system of polynomials $\{A(\xi);\ B_1(\xi), \cdots, B_m(\xi)\}$, we define Lopatinski's determinant of $\{A;\ B_1, \cdots, B_m\}$ by

$$\mathrm{Lop}\{A;\ B_1, \cdots, B_m\} = \det\left(\frac{1}{2\pi i}\oint_\Gamma \frac{B_i(\xi)\xi^{m-j}}{A(\xi)}d\xi\right)_{i,j=1,\cdots,m},$$

where Γ is a simple closed curve enclosing all the roots of A.

Lemma 3.1. Let $A(\xi) = \prod_{j=1}^{m}(\xi - \xi_j)$, *then*

$$\mathrm{Lop}\{A;\ B_1, \cdots, B_m\} = \frac{1}{\prod_{i>j}(\xi_i - \xi_j)}\begin{vmatrix} B_1(\xi_1)\cdots B_1(\xi_m) \\ \cdots\cdots\cdots \\ B_m(\xi_1)\cdots B_m(\xi_m)\end{vmatrix}.$$

Moreover if $\{B_1, \cdots, B_m\}$ *are given by*

$$\begin{pmatrix} B_1(\xi) \\ \vdots \\ B_m(\xi)\end{pmatrix} = \begin{pmatrix} b_{11}\cdots b_{1m} \\ \cdots\cdots \\ b_{m1}\cdots b_{mm}\end{pmatrix}\begin{pmatrix}\xi^{m-1} \\ \vdots \\ 1\end{pmatrix} = \mathcal{B}\begin{pmatrix}\xi^{m-1} \\ \vdots \\ 1\end{pmatrix},$$

then

$$\mathrm{Lop}\{A;\ B_1, \cdots, B_m\} = (-1)^{\frac{1}{2}m(m-1)}\det\mathcal{B}.$$

Remark. Let

$$B_j(\xi) = C(\xi)C_j(\xi) \qquad (j = 1, 2, \cdots, m)$$

then

$$\mathrm{Lop}\{A;\ B_1, \cdots, B_m\} = \prod_{j=1}^{m} C(\xi_j)\mathrm{Lop}\{A;\ C_1, \cdots, C_m\}.$$

Proof.

$$\mathrm{Lop}\{A;\ B_1, \cdots, B_m\} = \det\left(\sum_k \frac{B_i(\xi_k)\xi_k^{m-j}}{\prod_{l\neq k}(\xi_k - \xi_l)}\right)_{i,j=1,\cdots,m}$$

$$= \begin{vmatrix} B_1(\xi_1)\cdots B_1(\xi_m) \\ \cdots\cdots\cdots \\ B_m(\xi_1)\cdots B_m(\xi_m)\end{vmatrix}\begin{vmatrix} \dfrac{\xi_1^{m-1}}{\prod_{l\neq 1}(\xi_1-\xi_l)} \cdots \dfrac{1}{\prod_{l\neq 1}(\xi_1-\xi_l)} \\ \cdots\cdots\cdots\cdots \\ \dfrac{\xi_m^{m-1}}{\prod_{l\neq m}(\xi_m-\xi_l)} \cdots \dfrac{1}{\prod_{l\neq m}(\xi_m-\xi_l)}\end{vmatrix}.$$

If $\begin{pmatrix} B_1 \\ \vdots \\ B_m\end{pmatrix} = \mathcal{B}\begin{pmatrix}\xi^{m-1} \\ \vdots \\ 1\end{pmatrix}$, $\quad \dfrac{1}{\prod_{i>j}(\xi_i - \xi_j)}\begin{vmatrix} B_1(\xi_1)\cdots B_1(\xi_m) \\ \cdots\cdots\cdots \\ B_m(\xi_1)\cdots B_m(\xi_m)\end{vmatrix}$

$$= \frac{1}{\prod_{i>j}(\xi_i - \xi_j)}\begin{vmatrix} b_{11}\cdots b_{1m} \\ \cdots\cdots \\ b_{m1}\cdots b_{mm}\end{vmatrix}\begin{vmatrix} \xi_1^{m-1}\cdots\xi_m^{m-1} \\ \cdots\cdots \\ 1\ \cdots\ 1\end{vmatrix} = (-1)^{\frac{1}{2}m(m-1)}\det\mathcal{B}.$$

Lemma 3.2. *Let*

$$A(\xi)=\prod_{j=1}^{m}(\xi-\xi_j), \qquad A'(\xi)=\prod_{j=1}^{m'}(\xi-\xi'_j),$$

and let $\{B'_1(\xi), \cdots, B'_{m'}(\xi)\}$ *be divisible by* $A(\xi)$. *Then*

$$\mathrm{Lop}\{A;\ B_1, \cdots, B_m\}\mathrm{Lop}\{A';\ B'_1, \cdots, B'_{m'}\}$$
$$=(-1)^{mm'}R_{AA'}\mathrm{Lop}\{AA';\ B_1, \cdots, B_m, B'_1, \cdots, B'_{m'}\}.$$

Remark. Let

$$B'_j(\xi)=A(\xi)C'_j(\xi) \qquad (j=1, \cdots, m')$$

then

$$\mathrm{Lop}\{A; B_1, \cdots, B_m\}\cdot\mathrm{Lop}\{A';\ C'_1, \cdots, C'_{m'}\}$$
$$=\mathrm{Lop}\{AA';\ B_1, \cdots, B_m, B'_1, \cdots, B'_{m'}\}.$$

Proof. Since $B'_i(\xi_j)=0$ $(i=1, \cdots, m',\ j=1, \cdots, m)$, we have from Lemma 3.1

$$\mathrm{Lop}\{AA';\ B_1, \cdots, B_m, B'_1, \cdots, B'_{m'}\}$$
$$=\frac{\det(B_i(\xi_j))_{i,j=1,\cdots,m}\det(B'_i(\xi'_j))_{i,j=1,\cdots,m'}}{\prod_{i>j}(\xi_i-\xi_j)\prod_{i,j}(\xi'_i-\xi_j)\prod_{i>j}(\xi'_i-\xi'_j)}$$
$$=\frac{(-1)^{mm'}}{R_{AA'}}\mathrm{Lop}\{A;\ B_1, \cdots, B_m\}\mathrm{Lop}\{A';\ B'_1, \cdots, B'_{m'}\}.$$

Hereafter we consider $\{A;\ B_1, \cdots, B_\mu\}$ in the following situations: Let

$$A(\xi)=\xi^m+a_1\xi^{m-1}+\cdots+a_m=A_+(\xi)A_-(\xi),$$
$$A_+(\xi)=\prod_{j=1}^{\mu}(\xi-\xi_j^+), \qquad A_-(\xi)=\prod_{j=1}^{m-\mu}(\xi-\xi_j^-),$$

where $\{\xi_j^+\}_{j=1,\cdots,\mu}$ have positive imaginary parts, $\{\xi_j^-\}_{j=1,\cdots,m-\mu}$ have negative ones and

$$|a_j|\leqslant Cr^j \qquad j=1, \cdots, m,$$
$$|A_+(\xi)|\geqslant c(|\xi|+r)^\mu \qquad \text{for} \quad \xi\in R^1,$$
$$|A_-(\xi)|\geqslant c(|\xi|+r)^{m-\mu} \qquad \text{for} \quad \xi\in R^1.$$

And let

$$\begin{pmatrix} B_1(\xi) \\ \vdots \\ B_\mu(\xi) \end{pmatrix}=\begin{pmatrix} b_{11}\cdots b_{1m} \\ \cdots\cdots\cdots \\ b_{\mu1}\cdots b_{\mu m} \end{pmatrix}\begin{pmatrix} \xi^{m-1} \\ \vdots \\ 1 \end{pmatrix},$$

where

$$|b_{ij}|\leqslant Cr^{r_i-(m-j)}.$$

Proposition 3.1. *We assume that*

$$|\mathrm{Lop}\{A_+;\ B_1,\cdots,B_\mu\}|\geqslant dr^{r_1+\cdots+r_\mu-\frac{1}{2}\mu(\mu-1)},$$

then we have for $u\in\mathcal{E}^s_{L^2}(0,\infty)$

$$c_s\int_0^\infty\sum_{j=0}^{s}|r^{s-j}D^ju(x)|^2dx\leqslant\int_0^\infty\sum_{j=0}^{s-m}|r^{s-m-j}D^jA(D)u|^2dx+\sum_{j=1}^{\mu}|r^{s-r_j-\frac{1}{2}}B_j(\omega)|^2,$$

where $s\geqslant m$, $c_s=c_s(C,c,d)>0$.

Proof. Let us denote

$$B_k(\xi)=\xi^{k-\mu-1}A_+(\xi)\quad(k=\mu+1,\cdots,m),\qquad\begin{pmatrix}B_1\\ \vdots\\ B_m\end{pmatrix}=\mathcal{B}\begin{pmatrix}\xi^{m-1}\\ \vdots\\ 1\end{pmatrix},$$

then we have from Lemma 3.1

$$\mathrm{Lop}\{A_-;\ B_{\mu+1},\cdots,B_m\}=\frac{1}{\prod\limits_{i>j}(\xi_i^--\xi_j^-)}\begin{vmatrix}A_+(\xi_1^-)\cdots & A_+(\xi_{m-\mu}^-)\\ \xi_1^-A_+(\xi_1^-)\cdots & \xi_{m-\mu}^-A_+(\xi_{m-\mu}^-)\\ \cdots\cdots & \cdots\cdots\end{vmatrix}$$

$$=A_+(\xi_1^-)\cdots A_+(\xi_{m-\mu}^-)=(-1)^{\mu(m-\mu)}R_{A_+A_-}$$

and

$$\mathrm{Lop}\{A;\ B_1,\cdots,B_m\}=(-1)^{\frac{1}{2}m(m-1)}\det\mathcal{B}.$$

Therefore we have from Lemma 3.2

$$\mathrm{Lop}\{A_+;\ B_1,\cdots,B_\mu\}=(-1)^{\mu(m-\mu)+\frac{1}{2}m(m-1)}\det\mathcal{B}.$$

Now let

$$\begin{pmatrix}r^{m-r_1-\frac{1}{2}} & & \\ & \ddots & \\ & & r^{m-r_m-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}B_1\\ \vdots\\ B_m\end{pmatrix}=H\begin{pmatrix}r^{\frac{1}{2}} & & \\ & \ddots & \\ & & r^{m-\frac{1}{2}}\end{pmatrix}\begin{pmatrix}\xi^{m-1}\\ \vdots\\ 1\end{pmatrix},$$

then we have $|\det H|\geqslant d$, and then every elements of H^{-1} are bounded by $C'(C,d)$. Therefore we have

$$\sum_{j=1}^{m}r^{j-\frac{1}{2}}|\omega_{m-j}|\leqslant C'(C,d)\sum_{j=1}^{m}r^{m-r_j-\frac{1}{2}}|B_j(\omega)|.$$

On the other hand, since

$$r|B_j(\omega)|^2\leqslant\int_0^\infty(|DB_j(D)u(x)|^2+r^2|B_j(D)u(x)|^2)dx,$$

we have

$$\sum_{j=\mu+1}^{m}r^{2(m-r_j-\frac{1}{2})}|B_j(\omega)|^2\leqslant\sum_{j=\mu+1}^{m}r^{2(m-r_j-1)}\left\{\int_0^\infty(|D^{j-\mu}A_+(D)u(x)|^2\right.$$

$$+r^2|D^{j-\mu-1}A_+(D)u(x)|^2)dx\Big\} \leqslant 2\sum_{j=0}^{m-\mu} r^{2(m-\mu-j)}\int_0^\infty |D^j A_+(D)u(x)|^2 dx\,.$$

Then we have from Lemma 1.3

$$c_0(C, c, d)\int_0^\infty \sum_{j=1}^{m} |r^{m-j}D^j u(x)|^2 dx \leqslant \int_0^\infty |A(D)u(x)|^2 dx$$
$$+\sum_{j=1}^{\mu} |r^{m-r_j-\frac{1}{2}}B_j(\omega)|^2 + \int_0^\infty \sum_{j=0}^{m-\mu} |r^{m-\mu-j}D^j A_+(D)u(x)|^2 dx,$$

and we have from Lemma 1.5

$$c_1(C, c)\int_0^\infty \sum_{j=0}^{m-\mu} |r^{m-\mu-j}D^j A_+(D)u(x)|^2 dx \leqslant \int_0^\infty |A(D)u(x)|^2 dx.$$

For $s>m$, we put AM^{s-m} instead of A.

Next we consider an adjoint system of $\{A;\, B_1, \cdots, B_\mu\}$. Let us denote

$$\overline{A}(\xi) = \xi^m + \overline{a}_1\xi^{m-1} + \cdots + \overline{a}_m\,,$$
$$\widehat{A}(\xi, s) = (\xi^{m-1}, \cdots, 1)\begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ \vdots & & 0 & 1 & a_1 \\ \vdots & \cdot^{\cdot^{\cdot}} & & \cdot^{\cdot^{\cdot}} & \vdots \\ 0 & \cdot^{\cdot^{\cdot}} & & & \vdots \\ 1 & a_1 & \cdots & & a_{m-1} \end{pmatrix}\begin{pmatrix} s^{m-1} \\ \vdots \\ 1 \end{pmatrix} = (\xi^{m-1}, \cdots, 1)\mathcal{A}\begin{pmatrix} s^{m-1} \\ \vdots \\ 1 \end{pmatrix},$$

and

$$(u, v) = \int_0^\infty u(x)\overline{v(x)}dx.$$

Since

$$(D^k u, v) = i\{D^{k-1}u(0)\overline{v(0)} + D^{k-2}u(0)\overline{Dv(0)} + \cdots + u(0)\overline{D^{k-1}v(0)}\}$$
$$+ (u, D^k v),$$

we have

$$(A(D)u, v) - (u, \overline{A}(D)v) = i\overline{(D^{m-1}v(0), \cdots, v(0))}\mathcal{A}\begin{pmatrix} D^{m-1}u(0) \\ \vdots \\ u(0) \end{pmatrix}.$$

Now we assume that

$$\mathrm{Lop}\{A_+;\, B_1, \cdots, B_\mu\} \neq 0,$$

which implies that $\det\mathcal{B}\neq 0$, then we denote

$$\begin{pmatrix} B_1'(\xi) \\ \vdots \\ B_m'(\xi) \end{pmatrix} = (\mathcal{B}^{-1})^* \mathcal{A}^* \begin{pmatrix} \xi^{m-1} \\ \vdots \\ 1 \end{pmatrix}.$$

Here we have

Proposition 3. 2. *We assume that*

$$|\mathrm{Lop}\{A_+;\ B_1, \cdots, B_\mu\}| \geqslant dr^{r_1+\cdots+r_\mu-\frac{1}{2}\mu(\mu-1)},$$

then there exists a system of polynomials $\{B_j, B'_j\}_{j=1,\cdots,m}$, *and it satisfies that*

$$(A(D)u, v)-(u, \bar{A}(D)v)=i\sum_{j=1}^{m} B_j u(0)\overline{B'_j v(0)} \quad \textit{for } u, v\in\mathcal{E}^m_{L^2}(0, \infty),$$

and

$$|\mathrm{Lop}\{\bar{A}_-;\ B'_{\mu+1}, \cdots, B'_m\}| \geqslant c_0(C, d)>0.$$

Proof. To show the last statement, let

$$u(x)=e^{i\xi_k^+ x}, \qquad v(x)=e^{i\overline{\xi_l^-} x},$$

then we have

$$\sum_{j=1}^{m} B_j(\xi_k^+)\overline{B'_j(\overline{\xi_l^-})}=0,$$

that is,

$$\begin{pmatrix} B_1(\xi_1^+)\cdots B_m(\xi_1^+) \\ \cdots\cdots\cdots\cdots \\ B_1(\xi_\mu^+)\cdots B_m(\xi_\mu^+) \end{pmatrix}\overline{\begin{pmatrix} B'_1(\overline{\xi_1^-})\cdots B'_1(\overline{\xi_{m-\mu}^-}) \\ \cdots\cdots\cdots\cdots \\ B'_m(\overline{\xi_1^-})\cdots B'_m(\overline{\xi_{m-\mu}^-}) \end{pmatrix}}=0.$$

Since $B_k(\xi_1^+)=\cdots=B_k(\xi_\mu^+)=0$ $(k=\mu+1, \cdots, m)$, we have

$$B'_k(\overline{\xi_1^-})=\cdots=B'_k(\overline{\xi_{m-\mu}^-})=0 \qquad (k=1, \cdots, \mu),$$

that is, $\{B'_1(\xi), \cdots, B'_\mu(\xi)\}$ are all divisible by $\bar{A}_-(\xi)$. Now let

$$B'_j(\xi)=\bar{A}_-(\xi)C'_j(\xi) \qquad (j=1, \cdots, \mu),$$

where the absolute values of the coefficients of $\{C'_j(\xi)\}$ are bounded by a positive constant $K=K(C, d)$ if $r=1$, then we have from Lemma 3. 1 and Lemma 3. 2

$$\begin{aligned}&\mathrm{Lop}\{\bar{A}_+; C'_1, \cdots, C'_\mu\}\ \mathrm{Lop}\{\bar{A}_-;\ B'_{\mu+1}, \cdots, B'_m\}\\ &\quad=\mathrm{Lop}\{\bar{A};\ B'_{\mu+1}, \cdots, B'_m, B'_1, \cdots, B'_\mu\}=(-1)^{\mu(m-\mu)}\mathrm{Lop}\{\bar{A};\ B'_1, \cdots, B'_m\}\\ &\quad=(-1)^{\mu(m-\mu)}(-1)^{\frac{1}{2}m(m-1)}\det((\mathcal{B}^{-1})^*\mathcal{A}^*)=(-1)^{\mu(m-\mu)}\overline{(\det\mathcal{B})^{-1}}.\end{aligned}$$

Finally we consider equivalent boundary conditions. We say that $\{B_1(\xi), \cdots, B_\mu(\xi)\}$ and $\{\widetilde{B}_1(\xi), \cdots, \widetilde{B}_\mu(\xi)\}$ are equivalent, if the condition $\{B_j(D)u|_{x=0}=0\ (j=1, \cdots, \mu)\}$ is equivalent to the condition $\{B_j(D)u|_{x=0}$

$=0\ (j=1, \cdots, \mu)\}$ for $u \in C^\infty(R^1)$.

Lemma 3.3. *Let $\{B_j(\xi)\}_{j=1,\cdots,\mu}$ and $\{\widetilde{B}_j(\xi)\}_{j=1,\cdots,\mu}$ be sets consisting of linearly independent polynomials of order less than m. Then, in order that $\{B_j(\xi)\}$ and $\{\widetilde{B}_j(\xi)\}$ are equivalent, it is necessary and sufficient that there exists a regular constant matrix*

$$C=\begin{pmatrix} C_{11} \cdots C_{1\mu} \\ \vdots \quad \vdots \\ C_{\mu 1} \cdots C_{\mu\mu} \end{pmatrix}$$

such that

$$\begin{pmatrix} B_1(\xi) \\ \vdots \\ B_\mu(\xi) \end{pmatrix} = C \begin{pmatrix} \widetilde{B}_1(\xi) \\ \vdots \\ \widetilde{B}_\mu(\xi) \end{pmatrix}.$$

Proof. Let us denote

$$B_j(\xi)=\sum_{k=1}^m b_{jk}\xi^{m-k}, \quad \widetilde{B}_j(\xi)=\sum_{k=1}^m \widetilde{b}_{jk}\xi^{m-k} \quad (j=1, \cdots, \mu).$$

Then, in order that $\{B_j(\xi)\}$ and $\{\widetilde{B}_j(\xi)\}$ are equivalent, it is necessary and sufficient that

$$\begin{pmatrix} b_{11} \cdots b_{1m} \\ \vdots \quad \vdots \\ b_{\mu 1} \cdots b_{\mu m} \end{pmatrix} \begin{pmatrix} \omega_{m-1} \\ \vdots \\ \omega_0 \end{pmatrix} = 0$$

is equivalent to

$$\begin{pmatrix} \widetilde{b}_{11} \cdots \widetilde{b}_{1m} \\ \vdots \quad \vdots \\ \widetilde{b}_{\mu 1} \cdots \widetilde{b}_{\mu m} \end{pmatrix} \begin{pmatrix} \omega_{m-1} \\ \vdots \\ \omega_0 \end{pmatrix} = 0$$

for $(\omega_{m-1}, \cdots, \omega_0) \in C^m$, that is, the space generated by $\{b_j=(b_{j1}, \cdots, b_{jm})\}_{j=1,\cdots,\mu}$ is equal to the space generated by $\{\widetilde{b}_j=(\widetilde{b}_{j1}, \cdots, \widetilde{b}_{jm})\}_{j=1,\cdots,\mu}$. Hence we have

$$b_j=\sum_{k=1}^{\mu} c_{jk}\widetilde{b}_k \qquad j=1, \cdots, \mu.$$

Corollary. *Let $\widetilde{B}_j(\xi)$ be a normal set such that*

$$\widetilde{b}_{j1}=\cdots=\widetilde{b}_{j\,m-s_j-1}=0, \widetilde{b}_{j\,m-s_j}=1 \qquad (j=1, \cdots, \mu),$$

and

$$s_i \neq s_j \qquad \text{if} \quad i \neq j,$$

and let

$$|b_{jk}| \leqslant C_1, \quad |\widetilde{b}_{jk}| \leqslant C_2 \qquad (j=1, \cdots, \mu,\ k=1, \cdots, m).$$

Then there exists a positive number $\delta=\delta(C_1, C_2)$ *such that*

$$|\mathrm{Lop}\{A_+;\ \widetilde{B}_1, \cdots, \widetilde{B}_\mu\}| \geqslant \delta |\mathrm{Lop}\{A_+;\ B_1, \cdots, B_\mu\}|.$$

Proof. Let us assume $s_1 > s_2 > \cdots > s_\mu$ without loss of generality. From the Lemma 3.3, we have

$$\begin{pmatrix} b_{1\,m-s_1} & b_{1\,m-s_2} \cdots b_{1\,m-s_\mu} \\ b_{2\,m-s_1} & b_{2\,m-s_2} \cdots b_{2\,m-s_\mu} \\ \cdots\cdots & \cdots\cdots \\ b_{\mu\,m-s_1} & b_{\mu\,m-s_2} \cdots b_{\mu\,m-s_\mu} \end{pmatrix} = C \begin{pmatrix} 1 & \widetilde{b}_{1\,m-s_2} & \widetilde{b}_{1\,m-s_3} \cdots \widetilde{b}_{1\,m-s_\mu} \\ & 1 & \widetilde{b}_{2\,m-s_3} \cdots \widetilde{b}_{2\,m-s_\mu} \\ & & \ddots \quad \vdots \\ 0 & & \quad 1 \end{pmatrix},$$

therefore we have

$$|c_{ij}| \leqslant K(C_1, C_2) \qquad (i, j=1, \cdots, \mu),$$

hence

$$|\det C| \leqslant K_1(C_1, C_2).$$

On the other hand, we have

$$\mathrm{Lop}\{A_+;\ B_1, \cdots, B_\mu\} = \det C \cdot \mathrm{Lop}\{A_+;\ \widetilde{B}_1, \cdots, \widetilde{B}_\mu\},$$

then

$$|\mathrm{Lop}\{A_+;\ \widetilde{B}_1, \cdots, \widetilde{B}_\mu\}| \geqslant \frac{1}{K_1(C_1, C_2)} |\mathrm{Lop}\{A_+;\ B_1, \cdots, B_\mu\}|.$$

4. λ-elliptic ($\lambda<0$) operator in R^n_+

Let us consider

$$L(x, y;\ \lambda, \xi, \eta) = \{A(x, y;\ \lambda, \xi, \eta);\ B_1(y;\ \lambda, \xi, \eta), \cdots, B_m(y;\ \lambda, \xi, \eta)\},$$

where $\{A, B_1, \cdots, B_m\}$ are homogeneous polynomials of degree $\{2m, r_1, \cdots, r_m\}$ with respect to (λ, ξ, η):

$$A = \xi^{2m} + a_1(x, y;\ \lambda, \eta)\xi^{2m-1} + \cdots + a_{2m}(x, y;\ \lambda, \eta),$$
$$B_j = \xi^{r_j} + b_{j\,m-r_j+1}(y;\ \lambda, \eta)\xi^{r_j-1} + \cdots + b_{jm}(y;\ \lambda, \eta),$$

where the coefficients of A and B_j belong to $\mathscr{B}(R^n_+)$.

Now we assume that

$$\inf_{\substack{(x,y)\in R^n_+ \\ \lambda>0,\ (\xi,\eta)\in R^n \\ |\lambda|+|\xi|+|\eta|=1}} |A(x, y;\ \lambda, \xi, \eta)| \neq 0,$$

then we can decompose

$$A=A_+A_-=\prod_{j=1}^{m}(\xi-\xi_j^+(x,y;\lambda,\eta))\prod_{j=1}^{m}(\xi-\xi_j^-(x,y;\lambda,\eta)).$$

We denote Lopatinski's determinant of $\{A_+(0,y;\lambda,\xi,\eta);B_1(y;\lambda,\xi,\eta),\cdots,B_m(y;\lambda,\xi,\eta)\}$ with respect to ξ by $\mathrm{Lop}\{A_+;B_1,\cdots,B_m\}(y;\lambda,\eta)$. We say that L is λ-elliptic ($\lambda>0$) in the half space R_+^n, if

$$\inf_{\substack{(x,y)\in R_+^n\\ \lambda>0,(\xi,\eta)\in R^n\\ |\lambda|+|\xi|+|\eta|=1}}|A(x,y;\lambda,\xi,\eta)|\neq 0,$$

and

$$\inf_{\substack{y\in R^{n-1}\\ \lambda>0,\eta\in R^{n-1}\\ |\lambda|+|\eta|=1}}|\mathrm{Lop}\{A_+;B_1,\cdots,B_m\}(y;\lambda,\eta)|\neq 0.$$

We denote

$$L(x,y;\lambda,D_x,D_y)u(x,y)=\{A(x,y;\lambda,D_x,D_y)u(x,y);$$
$$B_1(y;\lambda,D_x,D_y)u(0,y),\cdots,B_m(y;\lambda,D_x,D_y)u(0,y)\},$$

then L is a bounded operator from $\mathcal{E}_{L^2}^{s+2m}(R_+^n)$ to

$$\mathcal{E}_{L^2}^{s}(R_+^n)\times\prod_{j=}^{m}\mathcal{E}_{L^2}^{s+2m-r_j-\frac{1}{2}}(R^{n-1})\,(=\mathfrak{H}_s).$$

Let $F=\{f;g_1,\cdots,g_m\}$ belong to $\mathfrak{H}_s$, we introduce norms of F in $\mathfrak{H}_s$ by

$$|||F|||_{s,\lambda}=||f||_{s,\lambda}+\sum_{j=1}^{m}\langle g_j\rangle_{s+2m-r_j-\frac{1}{2},\lambda},$$

where s is a non-negative integer, α is a real number, and

$$||f||_{s,\lambda}^2=\int_{R_+^n}\sum_{i+j+|\nu|=s}|\lambda^i D_x^j D_y^\nu f(x,y)|^2dxdy,$$

$$\langle g\rangle_{\alpha,\lambda}^2=\int_{R^{n-1}}(|\lambda|^2+|\eta|^2)^\alpha|\hat{g}(\eta)|^2d\eta,$$

where

$$g(\eta)=\int_{R^{-}}e^{-iy\eta}g(y)dy.$$

Theorem 4.1. *We assume that L is λ-elliptic ($\lambda>0$) in R_+^n, then we have*

$$||u||_{2m+s,\lambda}\leqslant C_s(|||Lu|||_{s,\lambda}+||u||)\qquad for\quad \lambda>0,$$

where $u\in\mathcal{E}_{L^2}^{2m+s}(R_+^n)$ ($s\geqslant 0$).

Proof. Using partition of unity in R^n_+, we apply Proposition 3.1 near the boundary of R^n_+ and Proposition 2.1 in the interior.

Let us say that $Lu=F$ is solvable at λ, if for any $F\in\mathfrak{H}_0$ there exists a unique solution $u\in\mathcal{E}^{2m}_{L^2}(R^n_+)$.

Theorem 4.2. *We assume that L is λ-elliptic $(\lambda>0)$ in R^n_+, then $Lu=F$ is solvable for $\lambda>\lambda_0$, where λ_0 is a positive constant.*

The proof is given in the appendix.

§2. Iterated Hyperbolic Mixed Problems

1. Hyperbolic energy.

Given a system of $m+1$ homogeneous polynomials with respect to (τ, ξ, η):

$$L(x, y;\ \tau, \xi, \eta)=\{A(x, y;\ \tau, \xi, \eta);\ B_1(y;\ \tau, \xi, \eta), \cdots, B_m(y;\ \tau, \xi, \eta)\}$$
$$(x>0,\ y\in R^{n-1}),$$

whose degrees are $\{2m;\ r_1, \cdots, r_m\}$ $(0\leqslant r_j\leqslant 2m-1)$. We say that L is τ-elliptic $(\mathrm{Im}\,\tau<0)$, if $L(x, y;\ \lambda e^{-i\theta}, \xi, \eta)$ is λ-elliptic $(\lambda>0)$ for each $(0<\theta<\pi)$. We say that L has the energy inequality of hyperbolic type, if

$$\|u\|_{2m-1,\tau}\leqslant\frac{C}{|\mathrm{Im}\,\tau|}|||Lu|||_{0,\tau}\quad \text{for } \mathrm{Im}\,\tau<-\gamma_0\ (\gamma_0>0).$$

Now we consider

$$L_l(x, y;\ \tau, \xi, \eta)=\{A_l(x, y;\ \tau, \xi, \eta);\ B_{l1}(y;\ \tau, \xi, \eta), \cdots, B_{lm_l}(y;\ \tau, \xi, \eta)\}$$

$(l=1, 2, \cdots, N)$, where the degree of A_l is $2m_l$ and the degree of B_{lj} is r_{lj}. Here we assume that

i) each L_l is τ-elliptic $(\mathrm{Im}\,\tau<0)$,

ii) each L_l has the energy inequality of hyperbolic type,

iii)

$$\text{(I)}\quad \inf_{\substack{(x,y)\in R^n_+\\ (\xi,\eta)\in R^n\\ |\xi|+|\eta|=1}} \Big|\prod_{i<j} R^0_{\Lambda_i A_j}(x, y;\ \xi, \eta)\Big|\neq 0,$$

(II) $$\inf_{\substack{y\in R^{n-1}\\ \operatorname{Im}\tau<0,\eta\in R^{n-1}\\ |\tau|+|\eta|=1}} |\prod_{i<j} R^{+}_{A_iA_j}(0, y;\ \tau, \eta)| \neq 0.$$

Now we denote

$$A(x, y;\ \tau, \xi, \eta)=\prod_{l=1}^{N} A_l(x, y;\ \tau, \xi, \eta),$$
$$Q_k(x, y;\ \tau, \xi, \eta)=\prod_{l\neq k} A_l(x, y;\ \tau, \xi, \eta) \quad (k=1, 2, \cdots, N),$$
$$B_{m_1+\cdots+m_{l-1}+k}(x, y;\ \tau, \xi, \eta)=B_{lk}(y;\ \tau, \xi, \eta)Q_l(x, y;\ \tau, \xi, \eta)$$
$$(k=1, 2, \cdots, m_l,\ l=1, 2, \cdots, N),$$

and

$$L(x, y;\ \tau, \xi, \eta)$$
$$=\{A(x, y;\ \tau, \xi, \eta);\ B_1(0, y;\ \tau, \xi, \eta), \cdots, B_m(0, y;\ \tau, \xi, \eta)\},$$

where $m=m_1+\cdots+m_N$ and $r_{m_1+\cdots+m_{l-1}+k}=2(m-m_l)+r_{lk}$.

Lemma 1.1.

$$\operatorname{Lop}\{A_+;\ B_1, \cdots, B_m\}=\prod_{i<j} R_{A_i^+A_j^+} \prod_{i\neq j} R_{A_i^+A_j^-} \prod_{l=1}^{N} \operatorname{Lop}\{A_l^+;\ B_{l1}, \cdots, B_{lm_l}\}.$$

Proof. Let's denote for $k=2, 3, \cdots, N$

$$Q_1^{(k)}=A_2A_3\cdots A_k,$$
$$Q_2^{(k)}=A_1A_3\cdots A_k,$$
$$\cdots\cdots\cdots\cdots$$
$$Q_k^{(k)}=A_1A_2\cdots A_{k-1},$$
$$A^{(k)}=A_1A_2\cdots A_k(=Q_{k+1}^{(k+1)}),$$
$$\{B_1^{(k)}, B_2^{(k)}, \cdots, B_{m_1+\cdots+m_k}^{(k)}\}=\{B_{11}Q_1^{(k)}, \cdots, B_{1m_1}Q_1^{(k)}, \cdots$$
$$\cdots, B_{k1}Q_k^{(k)}, \cdots, B_{km_k}Q_k^{(k)}\},$$

then

$$Q_j^{(N)}=Q_j\ (j=1, \cdots, N), \quad \{A^{(N)};\ B_1^{(N)}, \cdots, B_m^{(N)}\}=\{A;\ B_1, \cdots, B_m\}.$$

Now we assume that

$$\operatorname{Lop}\{A^{(k)};\ B_1^{(k)}, \cdots, B_{m_1+\cdots+m_k}^{(k)}\}$$
$$=\prod_{i<j\leq k} R_{A_i^+A_j^+} \prod_{\substack{i\neq j\\ i j,\leq k}} R_{A_i^+A_j^-} \prod_{l=1}^{k} \operatorname{Lop}\{A_l^+;\ B_{l1}, \cdots, B_{lm}\}.$$

On the other hand, we have

$$\begin{aligned}
&\mathrm{Lop}\{A_+^{(k)};\ B_1^{(k+1)}, \cdots, B_{m_1+\cdots+m_k}^{(k+1)}\} \\
&= \mathrm{Lop}\{A_+^{(k)};\ B_1^{(k)}A_{k+1}, \cdots, B_{m_1+\cdots+m_k}^{(k)}A_{k+1}\} \\
&= R_{A_+^{(k)}A_{k+1}}\mathrm{Lop}\{A_+^{(k)};\ B_1^{(k)}, \cdots, B_{m_1+\cdots+m_k}^{(k)}\},
\end{aligned}$$

$$\begin{aligned}
&\mathrm{Lop}\{A_{k+1}^+;\ B_{m_1+\cdots+m_k+1}^{(k+1)}, \cdots, B_{m_1+\cdots+m_{k+1}}^{(k+1)}\} \\
&= \mathrm{Lop}\{A_{k+1}^+;\ B_{k+1\,1}A^{(k)}, \cdots, B_{k+1\,m_{k+1}}A^{(k)}\} \\
&= R_{A_{k+1}^+A^{(k)}}\mathrm{Lop}\{A_{k+1}^+;\ B_{k+1\,1}, \cdots, B_{k+1\,m_{k+1}}\},
\end{aligned}$$

and we have from Lemma 3. 2 in §1.

$$\begin{aligned}
&\mathrm{Lop}\{A_+^{(k)};\ B_1^{(k+1)}, \cdots, B_{m_1+\cdots+m_k}^{(k+1)}\}\mathrm{Lop}\{A_{k+1}^+;\ B_{m_1+\cdots+m_k+1}^{(k+1)}, \cdots, B_{m_1+\cdots+m_{k+1}}^{(k+1)}\} \\
&\qquad = R_{A_{k+1}^+A_+^{(k)}}\mathrm{Lop}\{A_+^{(k+1)};\ B_1^{(k+1)}, \cdots, B_{m_1+\cdots+m_{k+1}}^{(k+1)}\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&R_{A_{k+1}^+A_+^{(k)}}\mathrm{Lop}\{A_1^{(k+1)};\ B_1^{(k+1)}, \cdots, B_{m_1+\cdots+m_{k+1}}^{(k+1)}\} \\
&= R_{A_+^{(k)}A_{k+1}}R_{A_{k+1}^+A^{(k)}}\prod_{i<j\leq k}R_{A_i^+A_j^+}\prod_{\substack{i\neq j\\ i,j\leq k+1}}R_{A_i^+A_j^-}\prod_{l=1}^{k+1}\mathrm{Lop}\{A_l^+;\ B_{l1}, \cdots, B_{lm}\} \\
&= R_{A_{k+1}^+A_+^{(k)}}\prod_{i<j\leq k+1}R_{A_i^+A_j^-}\prod_{\substack{i\neq j\\ i,j\leq k+1}}R_{A_i^+A_j^-}\prod_{l=1}^{k+1}\mathrm{Lop}\{A_l^+;\ B_{l1}, \cdots, B_{lm}\}.
\end{aligned}$$

Lemma 1.2. *We assume that* $\{A_l\}_{l=1,\cdots,N}$ *are hyperbolic polynomials and satisfy the condition* iii). *Then we have for* $u\in\mathcal{E}_{L^2}^{2m+s}(R_+^n)$

$$c_s\|u\|_{s+2m,\tau}\leqslant\sum_{l=1}^N\|Q_l(x, y;\ \tau, D_x, D_y)u\|_{s+2m_l,\tau}\quad\text{for } \mathrm{Im}\,\tau<0,\ |\tau|>C_s,$$

where $s\geqslant-\min\{2m_1, \cdots, 2m_N\}$.

Proof. The case when $N=2$ is shown in Theorem 2.1 in §1. We have

$$\begin{aligned}
Q_1^{(k)}(x, y;\ \tau, D_x, D_y) &= A_k(x, y;\ \tau, D_x, D_y)Q_1^{(k-1)}(x, y;\ \tau, D_x, D_y) \\
&\qquad + \text{lower order terms}, \\
Q_2^{(k)}(x, y;\ \tau, D_x, D_y) &= A_k(x, y;\ \tau, D_x, D_y)Q_2^{(k-1)}(x, y;\ \tau, D_x, D_y) + \text{l.o.t.}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
Q_{k-1}^{(k)}(x, y;\ \tau, D_x, D_y) &= A_k(x, y;\ \tau, D_x, D_y)Q_{k-1}^{(k-1)}(x, y;\ \tau, D_x, D_y) + \text{l.o.t.}, \\
Q_k^{(k)}(x, y;\ \tau, D_x, D_y) &= A_1(x, y;\ \tau, D_x, D_y)Q_1^{(k-1)}(x, y;\ \tau, D_y, D_y) + \text{l.o.t.} \\
&= A_2(x, y;\ \tau, D_x, D_y)Q_2^{(k-1)}(x, y;\ \tau, D_x, D_y) + \text{l.o.t.}
\end{aligned}$$

$$=\cdots\cdots\cdots\cdots\cdots$$

$$=A_{k-1}(x, y;\tau, D_x, D_y)Q_{k-1}^{(k-1)}(x, y;\tau, D_x, D_y)+\text{l.o.t.}$$

Then we have for $1\leqslant j\leqslant k-1$

$$\|Q_j^{(k)}u\|_{s+2m_j,\tau}+\|Q_k^{(k)}u\|_{s+2m_k,\tau}+\|u\|_{s+2(m_1+\cdots+m_k)-1,\tau}$$
$$\geqslant c_s\|Q_j^{(k-1)}u\|_{s+2m_j+2m_k,\tau},$$

therefore we have

$$\sum_{j=1}^{k}\|Q_j^{(k)}u\|_{s+2m_j,\tau}\geqslant c_s'\sum_{j=1}^{k-1}\|Q_j^{(k-1)}u\|_{s+2m_j+2m_k,\tau}.$$

Here we have only to use the mathematical induction for k.

Proposition 1.1. *We assume the conditions* i), ii), iii) *for* $\{L_l\}_{l=1,\cdots,N}$. *Then L becomes also τ-elliptic* ($\mathrm{Im}\,\tau<0$) *and has the energy inequality of hyperbolic type.*

Proof. τ-ellipticity ($\mathrm{Im}\,\tau<0$) of L follows Lemma 1.1.

To show the hyperbolic energy inequality for L, we denote

$$A(x, y;\tau, D_x, D_y)$$
$$=A_1(x, y;\tau, D_x, D_y)Q_1(x, y;\tau, D_x, D_y)+A_1'(x, y;\tau, D_x, D_y)$$
$$=\cdots\cdots\cdots\cdots\cdots$$
$$=A_N(x, y;\tau, D_x, D_y)Q_N(x, y;\tau, D_x, D_y)+A_N'(x, y;\tau, D_x, D_y)$$

$$B_{m_1+\cdots+m_{l-1}+k}(x, y;\tau, D_x, D_y)$$
$$=B_{lk}(y;\tau, D_x, D_y)Q_l(x, y;\tau, D_x, D_y)+B'_{m_1+\cdots+m_{l-1}+k}(x, y;\tau, D_x, D_y),$$

where the order of A' is less than $2m$ and the order of B_j' is less than r_j. Now we denote

$$A(x, y;\tau, D_x, D_y)u=f,$$
$$B_j(x, y;\tau, D_x, D_y)u|_{x=0}=g_j \quad (j=1, \cdots, m),$$
$$Q_l(x, y;\tau, D_x, D_y)u=u_l \quad (l=1, \cdots, N),$$

then we have

$$A_l u_l=f-A_l'u,$$
$$B_{lk}u_l|_{x=0}=g_{m_1+\cdots+m_{l-1}+k}-B'_{m_1+\cdots+m_{l-1}+k}u|_{x=0} \quad (k=1, \cdots, m_l).$$

Since each L_l has the energy inequality of hyperbolic type, we have

$$\|u_l\|_{2m_l-1,\tau}\leqslant\frac{C}{|\mathrm{Im}\,\tau|}\{\|f\|+\|A'_l u\|+\sum_{k=1}^{m}\langle g_{m_1+\cdots+m_{l-1}+k}\rangle_{2m_l-r_{lk}-\frac{1}{2},\tau}$$
$$+\sum_{k=1}^{m_l}\langle B'_{m_1+\cdots m_{l-1}+k}u\,|_{x=0}\rangle_{2m_l-r_{lk}-\frac{1}{2}},$$
$$\leqslant\frac{C'}{|\mathrm{Im}\,\tau|}\{\|f\|+\sum_{j=m_l+\cdots+m_{l-1}+1}^{m_1+\cdots+m_l}\langle g_j\rangle_{2m-r_j-\frac{1}{2}}+\|u\|_{2m-1,\tau}\}$$

for $\mathrm{Im}\,\tau<-\gamma_0$ $(l=1,\cdots,N)$. Summing up them with respect to l, we apply Lemma 1.2.

Example. Let us consider

$$A_a(\tau,\xi,\eta)=\tau^2-2\tau\xi-\{a\xi^2+(a+1)\eta^2\}\quad(a>0)$$
$$=\{\tau-(\xi+\sqrt{(a+1)(\xi^2+\eta^2)})\}\{\tau-(\xi-\sqrt{(a+1)(\xi^2+\eta^2)})\}.$$

The common root of $\{A_a, A_{a'}\}$ $(a\neq a')$ with respect to ξ must satisfy

$$a'A_a-aA_{a'}=(a'-a)(\tau^2-2\tau\xi-\eta^2)=0,$$

therefore

$$\mathrm{Im}\,\xi=\frac{\eta^2+|\tau|^2}{2|\tau|^2}\,\mathrm{Im}\,\tau,$$

that is, no common root is on the upper half plane, whenever $\mathrm{Im}\,\tau<0$.

Now let

$$L_j=\{A_{a_j};1\}\qquad(j=1,\cdots,N),$$

where $0<a_1<a_2<\cdots<a_N$. Then $\{L_j\}_{j=1,\cdots,N}$ satisfy i), ii), iii).

2. Hyperbolic energy (continued).

In the preceding section, we assumed i), ii), iii) on L, but (II) of iii) seems too strong restriction on A. In this section, we assume i), ii) on L. If we try to drop (II) of iii), A is forced to be combined with another appropriate boundary conditions $\{C_j\}_{j=1,\cdots,m}$ of different type from $\{B_j\}_{j=1,\cdots,m}$. Let us find sufficient condition on $\{C_j\}$, in order to get the hyperbolic energy for $M=\{A;C_1,\cdots,C_m\}$, where the degree of C_j is λ_j. For the simplicity, we assume that $M(0,y;\tau,\xi,\eta)$ is independent of y for $|y|>R$.

For each point (y,τ,η), $A_l(0,y;\tau,\xi,\eta)$ is written by

$$A_l(0,y;\tau,\xi,\eta)=S_l(y;\tau,\xi,\eta)P_l(y;\tau,\xi,\eta),$$

where $\{S_l, P_l\}$ are polynomials of ξ, the roots of S_l are also the roots of $Q_l=\prod_{k\neq l} A_k$, and the roots of P_l are not the roots of Q_l. We denote $S=\prod_{l=1}^{N} S_l$, the degree of S_l is μ_l and the degree of S is μ.

Here we assume

iii)′ (0) M is τ-elliptic ($\operatorname{Im}\tau<0$),

(I) $\inf_{\substack{(x,y)\in R_+^n\\(\xi,\eta)\in R^n}} |\prod_{i<j} R^0_{A_iA_j}(x, y; \xi, \eta)| \neq 0$,

(II) $\operatorname{Lop}\{S_+(y; \tau, \xi, \eta); C_{j_1}(y; \tau, \xi, \eta), \cdots, C_{j_{\mu_+}}(y; \tau, \xi, \eta)\} \neq 0$

for $y\in R^{n-1}$, $\operatorname{Im}\tau\leqslant 0$, $\eta\in R^{n-1}$ ($\mu_+=$degree of S_+).

Lemma 2.1. *We assume* iii)′, *then we have for* $u\in\mathcal{E}^{2m+s}_{L^2}(R^n_+)$

$$c_s\|u\|_{s+2m,\tau}\leqslant\sum_{l=1}^{N}\|Q_l(x, y; \tau, D_x, D_y)u\|_{s+2m_l,\tau}$$
$$+\sum_{j=1}^{m}\langle C_j(y; \tau, D_x, D_y)u|_{x=0}\rangle_{s+2m-\lambda_j,\tau}$$

for $\operatorname{Im}\tau<0$, $|\tau|>C_s$ $(s\geqslant 0)$.

Proof. Let us denote

$$D=\{y\in R^{n-1}, \operatorname{Im}\tau\leqslant 0, \eta\in R^{n-1};\ |y|\leqslant R,\ |\tau|+|\eta|=1\},$$

and let $(y_0, \tau_0, \eta_0)\in D$. Since $S_l(y_0; \tau_0, \xi, \eta_0)$ and $P_l(y_0; \tau_0, \xi, \eta_0)$ have no common root, there exists a neighbourhood V of (y_0, τ_0, η_0) in D, where $\widetilde{S}_l(y; \tau, \xi, \eta)$ and $\widetilde{P}_l(y; \tau, \xi, \eta)$ are polynomials in ξ with smooth coefficients in V, the degrees of them are invariable and

$$\widetilde{S}_l(y_0; \tau_0, \xi, \eta_0)=S_l(y_0; \tau_0, \xi, \eta_0),$$
$$\widetilde{P}_l(y_0; \tau_0, \xi, \eta_0)=P_l(y_0; \tau_0, \xi, \eta_0),$$
$$A_l(y; \tau, \xi, \eta)=\widetilde{S}_l(y; \tau, \xi, \eta)\widetilde{P}_l(y; \tau, \xi, \eta),$$

and moreover

$$\inf_{(y,\tau,\eta)\in V} |\prod_{i<j} R^{\pm}_{\widetilde{P}_i\widetilde{P}_j}(y; \tau, \eta)| \neq 0,$$
$$\inf_{(y,\tau,\eta)\in V} |\operatorname{Lop}\{\widetilde{S}_+(y; \tau, \xi, \eta); C_{j_1}(y; \tau, \xi, \eta), \cdots, C_{j_{\mu_+}}(y; \tau, \xi, \eta)\}| \neq 0,$$

where $\widetilde{S}=\prod_{l=1}^{N}\widetilde{S}_l$. Since D is compact, D is covered by finite number of $\{V_j\}$ with the same properties as above.

At first we assume that A is independent of y, and that support of $\tilde{u}(\tau, x, \eta)$ is contained in V. Since $\widetilde{S}_l Q_l=\widetilde{S}\widetilde{T}_l$, where $\widetilde{T}_l=\prod_{k\neq l}\widetilde{P}_k$, we have

$$c_s\int_{i+j+|\nu|=s+2m_l-\mu_l}\sum|\tau^i\eta^\nu D_x^j\widetilde{T}_l(\tau, D_x, \eta)\widetilde{S}(\tau, D_x, \eta)\tilde{u}(\tau, x, \eta)|^2dxd\eta$$
$$\leqslant\|Q_l(\tau, D_x, D_y)u(\tau, x, y)\|^2_{s+2m_l,\tau},$$

therefore we have from Lemma 1.2

$$c_s'\int_{i+j+|\nu|=s+2m-\mu}\sum|\tau^i\eta^\nu D_x^j\widetilde{S}(\tau, D_x, \eta)\tilde{u}(\tau, x, \eta)|^2dxd\eta$$
$$\leqslant\sum_{l=1}^{N}\|Q_l(\tau, D_x, D_y)u(\tau, x, y)\|^2_{s+2m_l,\tau}.$$

Then we have from Proposition 3.1 in §1

$$c_s''\|u\|_{s+2m,\tau}\leqslant\sum_{l=1}^{N}\|Q_l(\tau, D_x, D_y)u\|_{s+2m_l,\tau}+\sum_{k=1}^{\mu_+}\langle C_{j_k}(\tau, D_x, D_y)u|_{x=0}\rangle_{s+2m-\lambda_{j_k},\tau}.$$

In the general case, we use two kinds of the partition of unity for R^n_+ and for $\{\mathrm{Im}\,\tau\leqslant 0, \eta\in R^{n-1};\ |\tau|+|\eta|=1\}$. We refer to appendix about the treatment of the partition of unity in (τ, η)-space.

Proposition 2.1. *We assume* ii) *and* iii)$'$ *on* M, *and moreover*

$$\begin{pmatrix}B_1(y;\tau,\xi,\eta)\\ \vdots\\ B_m(y;\tau,\xi,\eta)\end{pmatrix}=\begin{pmatrix}c_{11}(y;\tau,\eta)\cdots c_{1m}(y;\tau,\eta)\\ \cdots\cdots\cdots\cdots\\ c_{m1}(y;\tau,\eta)\cdots c_{mm}(y;\tau,\eta)\end{pmatrix}\begin{pmatrix}C_1(y;\tau,\xi,\eta)\\ \vdots\\ C_m(y;\tau,\xi,\eta)\end{pmatrix},$$

where $c_{ij}(y;\tau,\eta)$ *are homogeneous polynomials of* (τ,η) *with coefficients in* $\mathscr{B}(R^{n-1})$. *Then* M *has hyperbolic energy inequality.*

Proof. It is similar to the proof of Proposition 1.1, only we remark that

$$\sum_{j=1}^{m}\langle B_j(y;\tau, D_x, D_y)u|_{x=0}\rangle_{2m-r_j-\frac{1}{2},\tau}\leqslant C\sum_{j=1}^{m}\langle C_j(y;\tau, D_x, D_y)u|_{x=0}\rangle_{2m-\lambda_j-\frac{1}{2},\tau}$$

and we use Lemma 2.1 instead of Lemma 1.1.

Example. Let us consider

$$A_a(\tau,\xi,\eta)=\tau^2-a\alpha(\xi,\eta)\qquad(a>0),$$

where $\alpha(\xi, \eta)=\xi^2+h(\eta)\xi+k(\eta)$ is positive definite. The common root of $\{A_a, A_{a'}\}$ $(a\neq a')$ only appears on $\tau=0$.

Now let

$$L_j=\{A_{a_j};\ 1\} \quad j=1, \cdots, N \quad (0<a_1<\cdots<a_N),$$

$$A=\prod_{j=1}^{N} A_{a_j}, \quad B_j=\prod_{l\neq j} A_{a_l} \ (j=1, \cdots, N), \quad C_j=\alpha^{j-1} \ (j=1, \cdots, N),$$

$$M=\{A;\ C_1, \cdots, C_N\},$$

then

$$\begin{pmatrix} B_1 \\ \vdots \\ B_N \end{pmatrix}=\begin{pmatrix} \tau^{2N}-\sum_{j=2}^{N} a_j\tau^{2(N-1)}\cdots(-1)^N\prod_{j=2}^{N} a_j \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \tau^{2N}-\sum_{j=1}^{N-1} a_j\tau^{2(N-1)}\cdots(-1)^N\prod_{j=1}^{N-1} a_j \end{pmatrix}\begin{pmatrix} 1 \\ \vdots \\ \alpha^{N-1} \end{pmatrix},$$

and

$$\begin{cases} S_l=1, & P_l=A_{a_l} \quad \text{for } \tau\neq 0, \\ S_l=A_{a_l}, & P_l=1 \quad \text{for } \tau=0. \end{cases}$$

Let $A_+(\xi)=\prod_{j=1}^{N}(\xi-\xi_j)$, then from Lemma 3.1 in §1,

$$\mathrm{Lop}\{A_+;\ 1, \alpha, \cdots, \alpha^{N-1}\}=\prod_{i>j}\frac{\alpha(\xi_i)-\alpha(\xi_j)}{\xi_i-\xi_j}=\prod_{i>j}\{(\xi_i+\xi_j)+h(\eta)\}\neq 0.$$

If $\mathrm{Im}\,\xi_j>0$ $(j=1, \cdots, N)$. Then M satisfies ii) and iii)$'$.

3. Existence theorem.

We say that $Lu=F$ is analytically solvable in the region $\mathrm{Im}\,\tau<-\gamma$, if $Lu=F$ is not only solvable in $\mathrm{Im}\,\tau<-\gamma$, but u is analytic in $\mathrm{Im}\,\tau<-\gamma$ with values $\mathcal{E}^{2m-1}_{L^2}(R^n_+)$, whenever F is analytic in $\mathrm{Im}\,\tau<-\gamma$ with values in $\mathfrak{H}_0$.

Proposition 3.1. *We assume that*

i) *L is τ-elliptic* $(\mathrm{Im}\,\tau<0)$,

ii) *L has the energy inequality of hyperbolic type in* $\mathrm{Im}\,\tau<-\gamma$. *Then $Lu=F$ is analytically solvable in* $\mathrm{Im}\,\tau<-\gamma$.

Proof. From i), applying Theorem 4.2 in §1, L is solvable in $\mathrm{Im}\,\tau<0$ and $|\tau|>C_{\arg\tau}$. Now we fix a solvable point τ_0 $(\mathrm{Im}\,\tau_0<-\gamma)$, and make Taylor's expansion on L and F at τ_0. Denoting $\tau=\tau_0+\mu$, we have

$$L=L_0+\mu L_1+\cdots+\mu^{2m}L_{2m},$$

$$F=F_0+\mu F_1+\mu^2 F_2+\cdots \qquad \text{for} \quad |\mu|<\rho_0(=-\mathrm{Im}\,\tau_0-\gamma).$$

Then $L_0 u=F_0$ is solvable and

$$|||L_k u|||_{0,\tau_0}\leqslant C\|u\|_{2m-1,\tau_0} \qquad (k=1,2,\cdots,2m),$$

where C is independent of τ_0.

Now we solve the problems:

$$L_0 u_0=F_0,$$

$$L_0 u_1=F_1-L_1 u_0,$$

$$\cdots\cdots\cdots\cdots$$

$$L_0 u_k=F_k-(L_1 u_{k-1}+\cdots+L_{2m}u_{k-2m}),$$

$$\cdots\cdots\cdots\cdots$$

We have from ii)

$$\begin{aligned}\|u_k\|_{2m-1,\tau_0}&\leqslant\frac{C}{|\mathrm{Im}\,\tau_0|}\{|||F_k|||_{0,\tau_0}+|||L_1 u_{k-1}|||_{0,\tau_0}+\cdots+|||L_{2m}u_{k-2m}|||_{0,\tau_0}\}\\&\leqslant C'\{|||F_k|||_{0,\tau_0}+\|u_{k-1}\|_{2m-1,\tau_0}+\cdots+\|u_{k-2m}\|_{2m-1,\tau_0}\},\end{aligned}$$

where C, C' are independent of τ_0. Then

$$\begin{aligned}\sum_{k=0}^{l}|\mu|^k\|u_k\|_{2m-1,\tau_0}\leqslant C'\{\sum_{k=0}^{l}|\mu|^k|||F_k|||_{0,\tau_0}+(|\mu|\\+\cdots+|\mu|^{2m})\sum_{k=0}^{l-1}|\mu|^k\|u_k\|_{2m-1,\tau_0}\}.\end{aligned}$$

Let $\rho_1<\min\left(1,\frac{1}{2mC'},\rho_0\right)$, then

$$\sum_{k=0}^{\infty}|\mu|^k\|u_k\|_{2m-1,\tau_0}\leqslant C(\rho_1)\sum_{k=0}^{\infty}|\mu|^k|||F_k|||_{0,\tau_0} \quad \text{for} \quad |\mu|<\rho_1.$$

Then $\sum_{k=0}^{l}\mu^k u_k$ converges to u in $\mathcal{E}_{L^2}^{2m-1}(R_+^n)$ for $|\mu|<\rho_1$.

On the other hand

$$\begin{aligned}L\sum_{k=0}^{l}\mu^k u_k=F_0+\mu F_1+\cdots+\mu^l F_l+\mu^{l+1}(L_1 u_l+L_2 u_{l-1}+\cdots+L_{2m}u_{l-2m+1})\\+\mu^{l+2}(L_2 u_l+\cdots+L_{2m}u_{l-2m+2})+\cdots+\mu^{l+2m}L_{2m}u_l,\end{aligned}$$

then

$$|||L\sum_{k=0}^{l}\mu^k u_k-\sum_{k=0}^{l}\mu^k F_k|||_{0,\tau_0}\leqslant 2mC\sum_{k-l-2m+1}^{l}|\mu|^k\|u_k\|_{2m-1,\tau}\xrightarrow[l\to\infty]{}0$$

for $|\mu|\leqslant\rho_1$. From i), applying Theorem 4.1 in §1, u belongs to $\mathcal{E}_{L^2}^{2m}(R_+^n)$.

Finally we remark that ρ_1 depends only on ρ_0, relating to τ_0. So in this way, u can be extended analytically on the region $\mathrm{Im}\,\tau<-\gamma$.

Appendix

1. Singular integral operators with positive parameter λ.

At first we consider $H(\lambda,\xi)$, which is homogeneous of degree zero with respect to $(\lambda,\xi)\in R_+^1\times R^n$, and we denote

$$|H|_N=\sum_{k+|\nu|\leqslant N}\sup_{\substack{(\lambda,\xi)\in R_+^1\times R^n\\ \lambda^2+|\xi|^2=1}}\left|\left(\frac{\partial}{\partial\lambda}\right)^k\left(\frac{\partial}{\partial\xi}\right)^\nu H(\lambda,\xi)\right|.$$

Let us denote

$$\hat{u}(\xi)=\mathcal{F}[u(x)]=\int_{R^n}e^{-ix\xi}u(x)\,dx,$$

$$u(x)=\mathcal{F}^{-1}[\hat{u}(\xi)]=\frac{1}{(2\pi)^n}\int_{R^n}e^{ix\xi}\hat{u}(\xi)\,d\xi,$$

for $u\in\mathcal{S}(R^n)$ or more generally for $u\in\mathcal{S}'(R^n)$. Here we define

$$H(\lambda,D)u(x)=\mathcal{F}^{-1}[H(\lambda,\xi)\hat{u}(\xi)],$$

$$\Lambda^s(\lambda,D)u(x)=\mathcal{F}^{-1}[\{\Lambda(\lambda,\xi)\}^s\hat{u}(\xi)]=\mathcal{F}^{-1}[(\sqrt{\lambda^2+|\xi|^2}\,)^s\hat{u}(\xi)]$$

for $u\in\mathcal{S}(R^n)$ (s: complex number).

Now we denote

$$H_s(\lambda,x)=\mathcal{F}^{-1}[H(\lambda,\xi)\{\Lambda(\lambda,\xi)\}^s],$$

then we have

Lemma 1.1. *Let* $\gamma>0$,

$$|H_s(\lambda,x)|\leqslant C|H|_{n+[l_0+\gamma]+1}|x|^{-(n+\mathrm{Re}\,s)}(\lambda|x|)^{-(\gamma+l_0-\mathrm{Re}\,s)}$$

for $\mathrm{Re}\,s\leqslant l_0$, *where* $C=C(n,l_0,\gamma)$ *is independent of* s.

Proof. Since we have

$$x^\nu H_s(\lambda, x) = \mathcal{F}^{-1}\left[\left(i\frac{\partial}{\partial\xi}\right)^\nu \{H(\lambda, \xi)\Lambda(\lambda, \xi)^s\}\right]$$

and

$$\left|\left(i\frac{\partial}{\partial\xi}\right)^\nu \{H(\lambda, \xi)\Lambda(\lambda, \xi)^s\}\right| \leqslant C(n, |\nu|)\,|H|_{|\nu|}(\lambda + |\xi|)^{\mathrm{Re}\,s - |\nu|},$$

$$|e^{ix\xi} - 1| \leqslant 2|x\xi|^\theta \qquad (0 \leqslant \theta \leqslant 1),$$

we have for $|\nu| > n + \mathrm{Re}\,s$

$$|x^\nu H_s(\lambda, x)| = \left|\frac{1}{(2\pi)^n}\int (e^{ix\xi} - 1)\left(i\frac{\partial}{\partial\xi}\right)^\nu \{H(\lambda, \xi)\Lambda(\lambda, \xi)^s\}\, d\xi\right|$$
$$\leqslant C(n, l_0, \gamma)\,|H|_{n+[l_0+\gamma]+1}|x|^{1-(l_0+\gamma-[l_0+\gamma])}\lambda^{\mathrm{Re}\,s - l_0 - \gamma}.$$

Now we denote the norm of $a(x) \in \mathcal{B}^{N+\alpha}(R^n)$ by

$$|a|_{N+\alpha} = \sum_{|\nu| \leqslant N} \sup_{x \in R^n}\left|\left(\frac{\partial}{\partial x}\right)^\nu a(x)\right| + \sum_{|\nu| = N} \sup_{x, y \in R^n} \frac{\left|\left(\frac{\partial}{\partial x}\right)^\nu a(x) - \left(\frac{\partial}{\partial y}\right)^\nu a(y)\right|}{|x - y|^\alpha}.$$

Lemma 1.2. *Let N be a non-negative integer, $0 \leqslant \alpha < \beta < 1$, $a(x) \in \mathcal{B}^{N+\beta}(R^n)$. Then we have for $u \in \mathcal{S}(R^n)$*

$$H(\lambda, D)\Lambda^{N+\alpha}(\lambda, D)a(x)u(x)$$
$$= \sum_{|\nu| \leqslant N} \frac{N!}{(N - |\nu|)!\,\nu!} a^{(\nu)}(x) R^\nu(\lambda, D) H(\lambda, D)\Lambda^{N+\alpha-|\nu|}(\lambda, D)u(x)$$
$$+ \sum_{|\nu_0| + |\nu| \leqslant N} \int G_{\nu_0\nu}(\lambda, x, y)\lambda^{\nu_0} D^\nu u(y)\, dy,$$

where

$$a^{(\nu)}(x) = D^\nu a(x), \quad R^\nu(\lambda, \xi) = \left(\frac{\xi}{\Lambda(\lambda, \xi)}\right)^\nu.$$

Moreover, there exist positive constants C, ε, such that

$$|G_{\nu_0\nu}(\lambda, x, y)| \leqslant C|a|_{N+\beta}|H|_{n+1} K_\varepsilon(\lambda, x - y) \quad \text{for } \lambda \geqslant 1,$$

where

$$K_\varepsilon(\lambda, x) = \begin{cases} \lambda^{-\varepsilon}|x|^{-n+\varepsilon} & \text{for } |x| < 1, \\ \lambda^{-\varepsilon}|x|^{-n-\varepsilon} & \text{for } |x| > 1. \end{cases}$$

Proof. At first we consider $H\Lambda^{N+s}$ for $\mathrm{Re}\,s < 0$. Since

$$\Lambda(\lambda, \xi) = \frac{\lambda}{\Lambda(\lambda, \xi)}\lambda + \frac{\xi_1}{\Lambda(\lambda, \xi)}\xi_1 + \cdots + \frac{\xi_n}{\Lambda(\lambda, \xi)}\xi_n$$
$$= R_0(\lambda, \xi)\lambda + R_1(\lambda, \xi)\xi_1 + \cdots + R_n(\lambda, \xi)\xi_n,$$

we have

$$H(\lambda, D)\Lambda^{N+s}(\lambda, D)a(x)u(x)$$
$$=H(\lambda, D)\Lambda^{s}(\lambda, D)\sum_{\nu_0+|\nu|+N}\frac{N!}{\nu_0!\nu!}\{R_0(\lambda, D)\lambda\}^{\nu_0}\{R(\lambda, D)\}^{\nu}$$
$$\times\sum_{\mu+\kappa=\nu}\frac{\nu!}{\mu!\kappa!}D^{\mu}a(x)D^{\kappa}u(x)$$
$$=\sum_{\nu_0+|\mu|+|\kappa|=N}\frac{N!}{\nu_0!\mu!\kappa!}\int(HR_0^{\nu_0}R^{\mu+\kappa})_s(\lambda, x-y)a^{(\mu)}(y)\lambda^{\nu_0}D^{\kappa}u(y)dy$$
$$=\sum_{|\mu|\leqslant N}a^{(\mu)}(x)\frac{N!}{(N-|\mu|)!\mu!}HR^{\mu}\Lambda^{N+s-|\mu|}u(x)$$
$$+\sum_{\nu_0+|\kappa|\leqslant N}\int G^{s}_{\nu_0\kappa}(\lambda, x, y)\lambda^{\nu_0}D^{\kappa}u(y)dy.$$

From Lemma 1.1, we have

$$|G^{s}_{\nu_0\kappa}(\lambda, x, y)|$$
$$=\left|\sum_{|\mu|=N-\nu_0-|\kappa|}\frac{N!}{\nu_0!\mu!\kappa!}(HR_0^{\nu_0}R^{\mu+\kappa})_s(\lambda, x-y)\{a^{(\mu)}(y)-a^{(\mu)}(x)\}\right|$$
$$\leqslant C|H|_{n+[\alpha+\gamma]+1}|a|_{N+\beta}\lambda^{-(\gamma+\alpha-\mathrm{Re}\, s)}|x-y|^{-n-\gamma-\alpha+\beta}.$$

Then we have for $\mathrm{Re}\, s\leqslant\alpha$ and $\lambda\geqslant 1$, taking $\gamma=\gamma_0$ $(0<\gamma_0<\beta-\alpha)$,

$$|G^{s}_{\nu_0\kappa}|\leqslant C|H|_{n+1}|a|_{N+\beta}\lambda^{-\gamma_0}|x-y|^{-n+(\beta-\alpha-\gamma_0)},$$

taking $\gamma=\gamma_1$ $(\beta-\alpha<\gamma_1<1-\alpha)$,

$$|G^{s}_{\nu_0\kappa}|\leqslant C|H|_{n+1}|a|_{N+\beta}\lambda^{-\gamma_1}|x-y|^{-n-(\gamma_1-\beta+\alpha)}.$$

Now we put $\varepsilon=\min(\gamma_0, \beta-\alpha-\gamma_0, \gamma_1-\beta+\alpha)$, then we have

$$|G^{s}_{\nu_0\kappa}|\leqslant C|H|_{n+1}|a|_{N+\beta}K_{\varepsilon}(x-y) \quad \text{for } \mathrm{Re}\, s\leqslant\alpha.$$

From this estimate, we can use the method of analytic continuation for $\mathrm{Re}\, s\leqslant\alpha$, in the representation formula.

Remark. We define

$$K_{\varepsilon,N}(u)=\sum_{\nu_0+|\nu|\leqslant N}\int K_{\varepsilon}(\lambda, x-y)|\lambda^{\nu_0}D^{\nu}u(y)|dy,$$

then

$$\|K_{\varepsilon,N}(u)\|\leqslant C\lambda^{-\varepsilon}\|\Lambda^{N}(\lambda, D)u\|.$$

Corollary. *Let* $0<3\varepsilon<1-\alpha$, then we have for $u\in\mathcal{S}(R^n)$

$$\|H(\lambda, D)\Lambda^{N+\alpha}(\lambda, D)a(x)u(x)-a(x)H(\lambda, D)\Lambda^{N+\alpha}(\lambda, D)u(x)\|$$
$$\leqslant C_\varepsilon\lambda^{-\varepsilon}|a|_{N+\alpha+2\varepsilon}|H|_{n+1}\|\Lambda^N(\lambda, D)u\|.$$

Let $H(x;\lambda,\xi)$ *be homogeneous of degree* 0 *with respect to* $(\lambda,\xi)\in R^1_+\times R^n$, *and denote*

$$|H|_{\alpha,k}=|H|_{\mathcal{B}^{\alpha,k}(R^n,S^n_+)}=\sum_{\substack{|\mu|\leqslant[\alpha]\\ l+|\nu|\leqslant k}}\sup_{\substack{x\in R^n\\(\lambda,\xi)\in S^n_+}}\left|\left(\frac{\partial}{\partial x}\right)^{\mu}\left(\frac{\partial}{\partial\lambda}\right)^{l}\left(\frac{\partial}{\partial\xi}\right)^{\nu}H(x;\lambda,\xi)\right|$$
$$+\sum_{\substack{|\mu|=[\alpha]\\ l+|\nu|\leqslant k}}\sup_{\substack{x,y\in R^n\\(\lambda,\xi)\in S^n_+}}\frac{\left|\left(\frac{\partial}{\partial x}\right)^{\mu}\left(\frac{\partial}{\partial\lambda}\right)^{l}\left(\frac{\partial}{\partial\xi}\right)^{\nu}H(x;\lambda,\xi)-\left(\frac{\partial}{\partial y}\right)^{\mu}\left(\frac{\partial}{\partial\lambda}\right)^{l}\left(\frac{\partial}{\partial\xi}\right)^{\nu}H(y;\lambda,\xi)\right|}{|x-y|^{\alpha-[\alpha]}}.$$

Then there exists an extension $\widetilde{H}(x;\lambda,\xi)$ *of* $H(x;\lambda,\xi)$ *which is homogeneous of degree* 0 *with respect to* $(\lambda,\xi)\in R^1\times R^n$, *satisfying*

$$|\widetilde{H}|_{\alpha,k}=|\widetilde{H}|_{\mathcal{B}^{\alpha,k}(R^n,S^n)}\leqslant C|H|_{\alpha,k}.$$

Now we denote spherical harmonics on S^n of order m by $Y_{mk}(\lambda,\xi)$ $(k=1,2,\cdots,k_m)$, and briefly denote $Y_{mk}=\varphi_{k_0+k_1+\cdots+k_{m-1}+k}$. Then $\{\varphi_j\}_{j=1,2,\cdots}$ is a complete orthonormal system in $L^2(S^n)$.

Here we denote

$$a_j(x)=\int_{S^n}\widetilde{H}(x;\lambda,\xi)\varphi_j(\lambda,\xi)dS,$$

then we have

Lemma 1.3. *For* $\alpha, k\geqslant 0$,

$$\sum_{j=1}^{\infty}|a_j|_\alpha|\varphi_j|_k\leqslant C|\widetilde{H}|_{\alpha,\beta(k)},$$

where $\beta(k)=2\left(\left[\frac{3n+2k}{4}\right]+1\right)$.

Proof. We use the well known properties about $\{\varphi_j\}$:

$$|a_j|_\alpha\leqslant C|H|_{\alpha,2N}\lambda_j^{-N}\quad(N=0,1,2,\cdots),$$
$$|\varphi_j|_k\leqslant C\lambda_j^{\frac{1}{2}(k+\frac{1}{2}n+\varepsilon)}\quad(\varepsilon>0),$$
$$\sum_{j=1}^{\infty}\lambda_j^{-\frac{1}{2}n-\varepsilon}<+\infty\quad(\varepsilon>0).$$

Therefore we have

$$|a_j|_\alpha |\varphi_j|_k \leqslant C|\widetilde{H}|_{\alpha,2N}\lambda_j^{-N+\frac{1}{2}(k+\frac{1}{2}n+\varepsilon)},$$

whose sum converges for $N > \dfrac{2k+3n}{4}$.

Now remarking Lemma 1.3 and

$$\left\| a_j(x)\frac{1}{(2\pi)^n}\int e^{ix\xi}\varphi_j(\lambda,\xi)\hat{u}(\xi)d\xi \right\| \leqslant C|a_j|_0|\varphi_j|_0\|u\|,$$

we define for $\widetilde{H}\in\mathscr{B}^{0,\beta(0)}(R^n, S^n)$,

$$\widetilde{H}(x;\lambda,D)u(x)=\sum_{j=1}^{\infty}a_j(x)\frac{1}{(2\pi)^n}\int e^{ix\xi}\varphi_j(\lambda,\xi)\hat{u}(\xi)d\xi \quad \textit{for} \quad u\in\mathcal{S}(R^n)$$

then we have

$$\|\widetilde{H}(x;\lambda,D)u\| \leqslant C|\widetilde{H}|_{0,\beta(0)}\|u\|.$$

Since $\sum_{j=1}^{N}a_j(x)\varphi_j(\lambda,\xi)$ converges to $\widetilde{H}(x;\lambda,\xi)$ in $\mathscr{B}^{0,0}(R^n, S^n)$ as $N\to\infty$,

$$\sum_{j=1}^{N}a_j(x)\frac{1}{(2\pi)^n}\int e^{ix\xi}\varphi_j(\lambda,\xi)\hat{u}(\xi)d\xi$$

converges to

$$\frac{1}{(2\pi)^n}\int e^{ix\xi}\widetilde{H}(x;\lambda,\xi)\hat{u}(\xi)d\xi$$

in $\mathscr{B}^0(R^n)$ as $N\to\infty$, which implies that $\widetilde{H}(x;\lambda,D)=0$ for $\lambda>0$, if $\widetilde{H}(x;\lambda,\xi)=0$ for $\lambda>0$. Hence we define for $H\in\mathscr{B}^{0,\beta(0)}(R^n, S^n_+)$

$$H(x;\lambda,D)u(x)=\widetilde{H}(x;\lambda,D)u(x) \quad (\lambda>0) \ \textit{for} \ u\in\mathcal{S}(R^n),$$

which is independent of extensions. We say that $H(x;\lambda,D)$ is a singular integral operator with symbol $H(x;\lambda,\xi)$.

Lemma 1.4.

i) *Let $l_1\geqslant 0$, then there exist $C>0$, $\varepsilon>0$, such that*

$$\begin{aligned}&\|H_1(x;\lambda,D)\Lambda^{l_1}(\lambda,D)H_2(x;\lambda,D)\Lambda^{l_2}(\lambda,D)u(x)\\ &\quad -(H_1\circ H_2)(x;\lambda,D)\Lambda^{l_1+l_2}u(x)\|\\ &\leqslant C\lambda^{-\varepsilon}|H_1|_{0,\beta(n+1)}|H_2|_{l_1+2\varepsilon,\beta(0)}\|\Lambda^{[l_1]+l_2}u\| \quad \text{for } \lambda\geqslant 1,\end{aligned}$$

where $(H_1\circ H_2)(x;\lambda,D)$ is a singular integral operator with symbol $H_1(x;\lambda,\xi)H_2(x;\lambda,\xi)$.

ii) $\|H(x;\lambda,D)^* u(x)-H^\sharp(x;\lambda,D)u(x)\|\leqslant C\lambda^{-\varepsilon}|H|_{2\varepsilon,\beta(n+1)}\|u\|$ *for* $\lambda\geqslant 1$, *where* H^* *implies the adjoint of* H *in* $L^2(R^n)$, *and* $H^\sharp(x;\lambda,D)$ *is a singular integral operator with symbol* $\overline{H(x;\lambda,\xi)}$.

Proof. i) $H_1\Lambda^{l_1}H_2\Lambda^{l_2}u$

$$=\sum_j a_{1j}(x)\varphi_j(\lambda,D)\Lambda^{l_1}(\lambda,D)\sum_k a_{2k}(x)\varphi_k(\lambda,D)\Lambda^{l_2}(\lambda,D)u.$$

From Lemma 1.2, we have

$$\|a_{1j}\varphi_j\Lambda^{l_1}a_{2k}\varphi_k\Lambda^{l_2}u-a_{1j}a_{2k}\varphi_j\Lambda^{l_1}\varphi_k\Lambda^{l_2}u\|$$
$$\leqslant C\lambda^{-\varepsilon}|a_{1j}|_0|a_{2k}|_{l_1+2\varepsilon}|\varphi_j|_{n+1}|\varphi_k|_0\|\Lambda^{[l_1]+l_2}u\|.$$

Then we have only to apply Lemma 1.3.

ii) $H(x;\lambda,D)^*u(x)=\sum_j\varphi_j(\lambda,D)\overline{a_j(x)}u(x)$.

From Lemma 1.2, we have

$$\|\varphi_j\bar a_j u-\bar a_j\varphi_j u\|\leqslant C\lambda^{-\varepsilon}|a_j|_{2\varepsilon}|\varphi_j|_{n+1}\|u\|.$$

Next we consider $H(x;\lambda,D)$, relating to decomposition of unity.

Lemma 1.5. *Let* $l\geqslant 0$ *and* $\beta(x)=1$ *on the support of* $\alpha(x)$. *Then*

$$H(x;\lambda,D)\Lambda^l(\lambda,D)\alpha(x)u(x)$$
$$=\sum_{|\nu|\leqslant[l]}\frac{[l]!}{([l]-|\nu|)!\nu!}\alpha^{(\nu)}(x)H(x;\lambda,D)R^\nu(\lambda,D)\Lambda^{l-|\nu|}(\lambda,D)u(x)$$
$$+\beta(x)(Tu)(x)+T'(\beta u)(x),$$

where

$$|Tu(x)|,\ |T'u(x)|\leqslant C|\alpha|_{l+2\varepsilon}|\beta|_{l+2\varepsilon}|H|_{0,\beta(n+1)}K_{\varepsilon,[l]}(u)(x).$$

Proof. $H\Lambda^l\alpha u=\sum_j a_j\varphi_j\Lambda^l\alpha u.$

From Lemma 1.2, we have

$$a_j\varphi_j\Lambda^l\alpha u=a_j\sum_{|\nu|\leqslant[l]}C_{l\nu}\alpha^{(\nu)}R^\nu\varphi_j\Lambda^{l-|\nu|}u$$
$$+a_j\sum_{\nu_0+|\nu|\leqslant[l]}\int G_{\nu_0\nu}^{(\varphi_j\Lambda^l,\alpha)}(\lambda,x,y)\lambda^{\nu_0}D^\nu u(y)dy,$$

where

$$|G_{\nu_0\nu}^{(\varphi_j\Lambda^l,\alpha)}(\lambda,x,y)|\leqslant C|\alpha|_{l+2\varepsilon}|\varphi_j|_{n+1}K_\varepsilon(\lambda,x-y).$$

Now we denote

$$G_{\nu_0\nu}^{(H\Lambda^l,\alpha)}(\lambda,x,y)=\sum_{j=1}^\infty a_j(x)G_{\nu_0\nu}^{(A_j\Lambda^l,\alpha)}(\lambda,x,y),$$

then

$$H\Lambda^l\alpha u=\sum_{|\nu|\leqslant[l]}C_{l\nu}\alpha^{(\nu)}(x)H(x;\lambda,D)R^\nu(\lambda,D)\Lambda^{l-|\nu|}u(x)$$
$$+\sum_{\nu_0+|\nu|\leqslant[l]}\int G_{\nu_0\nu}^{(H\Lambda^l,\alpha)}(\lambda,x,y)\lambda^{\nu_0}D^\nu u(y)dy,$$

where

$$|G_{\nu_0\nu}^{(H\Lambda^l,\alpha)}(\lambda,x,y)|\leqslant C|\alpha|_{l+2\varepsilon}|H|_{0,\beta(n+1)}K_\varepsilon(\lambda,x-y).$$

Now

$$H\Lambda^l\alpha u=H\Lambda^l\alpha\beta u=\sum_{|\nu|\leqslant[l]}C_{l\nu}\alpha^{(\nu)}(x)H(x;\lambda,D)R^\nu(\lambda,D)\Lambda^{l-|\nu|}\beta(x)u(x)$$
$$+\sum_{\nu_0+|\nu|\leqslant[l]}\int G_{\nu_0\nu}^{(H\Lambda^l,\alpha)}(\lambda,x,y)\lambda^{\nu_0}D^\nu(\beta(y)u(y))dy$$
$$=\sum_{|\nu|\leqslant[l]}C_{l\nu}\alpha^{(\nu)}(x)H(x;\lambda,D)R^\nu(\lambda,D)\Lambda^{l-|\nu|}u(x)$$
$$+\sum_{|\nu|\leqslant[l]}C_{l\nu}\alpha^{(\nu)}(x)\sum_{\mu_0+|\mu|\leqslant[l]-|\nu|}\int G_{\mu_0\mu}^{(HR^\nu\Lambda^{l-|\nu|},\beta)}(\lambda,x,y)\lambda^{\mu_0}D^\mu u(y)dy$$
$$+\sum_{\nu_0+|\nu|\leqslant[l]}\int G_{\nu_0\nu}^{(H\Lambda^l,\alpha)}(\lambda,x,y)\lambda^{\nu_0}D^\nu(\beta(y)u(y))dy$$
$$=\sum_{|\nu|\leqslant[l]}C_{l\nu}\alpha^{(\nu)}HR^\nu\Lambda^{l-|\nu|}u$$
$$+\beta(x)\sum_{\mu_0+|\mu|\leqslant[l]}\int G_{\mu_0\mu}(\lambda,x,y)\lambda^{\mu_0}D^\mu u(y)dy$$
$$+\sum_{\nu_0+|\nu|\leqslant[l]}\int G'_{\nu_0\nu}(\lambda,x,y)\lambda^{\nu_0}D^\nu(\beta(y)u(y))dy,$$

where

$$|G_{\mu_0\mu}(\lambda,x,y)|=|\sum_{|\nu|\leqslant[l]-\mu_0-|\mu|}C_{l\nu}\alpha^{(\nu)}(x)G_{\mu_0\mu}^{(HR^\nu\Lambda^{l-|\nu|},\beta)}(\lambda,x-y)|$$
$$\leqslant C|\alpha|_{[l]}|\beta|_{l+2\varepsilon}|H|_{0,\beta(n+1)K\varepsilon}(\lambda,x-y),$$
$$|G'_{\nu_0\nu}(\lambda,x,y)|=|G_{\nu_0\nu}^{(H\Lambda^l,\alpha)}(\lambda,x,y)|\leqslant C|\alpha|_{l+2\varepsilon}|H|_{0,\beta(n+1)}K_\varepsilon(\lambda,x-y).$$

Let $\{\alpha_k(x)\}_{k=1,2,\cdots}$ be a decomposition of unity in R^n, such that

$$\sum_{k=1}^\infty\alpha_k(x)^2=1,\quad \alpha_k(x)=\alpha(x-x_k),$$
$$\alpha(x)\geqslant0,\quad \alpha(x)=0 \text{ for } |x|>\delta,\quad |\alpha|_{l+2\varepsilon}\leqslant K\ (0<3\varepsilon<[l]+1-l).$$

Lemma 1.6. *Let* $l\geqslant0$, $\gamma>0$, *then*

$$\sum_{k=1}^\infty\|H(x_k;\lambda,D)\Lambda^l(\lambda,D)\alpha_k(x)u(x)-\alpha_k(x)H(x;\lambda,D)\Lambda^l(\lambda,D)u(x)\|^2$$
$$\leqslant C|H|^2_{\gamma,\beta(n+1)}(\lambda^{-\varepsilon}\|\Lambda^{[l]}u\|+\delta^\gamma\|\Lambda^l u\|)^2\quad for\ \lambda\geqslant1,$$

where $C=C(K,\varepsilon)$.

Proof. Relating to $\{\alpha_k(x)\}$ we consider $\{\beta_k\}$, $\{\gamma_k\}$, suct that

$$\beta(x)\alpha(x)=\alpha(x), \qquad \gamma(x)\beta(x)=\beta(x),$$
$$\beta(x)\geqslant 0, \qquad \gamma(x)\geqslant 0,$$
$$|\beta|_{l+2\varepsilon}\leqslant K', \qquad |\gamma|_{l+2\varepsilon}\leqslant K',$$
$$\beta_k(x)=\beta(x-x_k), \qquad \gamma_k(x)=\gamma(x-x_k),$$
$$\sum_{k=1}^{\infty}\beta_k(x^2)\leqslant K'', \qquad \sum_{k-1}^{\infty}\gamma_k(x)^2\leqslant K''.$$

Now we denote

$$\begin{aligned}
&H(x_k;\lambda, D)\Lambda^l(\lambda, D)\alpha_k(x)u(x)-\alpha_k(x)H(x;\lambda, D)\Lambda^l(\lambda, D)u(x)\\
&\quad=\{H(x_k;\lambda, D)\Lambda^l(\lambda, D)\alpha_k(x)-\alpha_k(x)H(x_k;\lambda, D)\Lambda^l(\lambda, D)\}\beta_k(x)u(x)\\
&\qquad+\alpha_k(x)\{H(x_k;\lambda, D)-H(x;\lambda, D)\}\Lambda^l(\lambda, D)\beta_k(x)u(x)\\
&\qquad+\alpha_k(x)\{H(x;\lambda, D)\Lambda^l(\lambda, D)\beta_k(x)-\beta_k(x)H(x;\lambda, D)\Lambda^l(\lambda, D)\}u(x)\\
&\quad=I_{k1}+I_{k2}+I_{k3}\,.
\end{aligned}$$

We have

$$\|I_{k1}\|\leqslant C|H|_{0,\beta(n+1)}\lambda^{-\varepsilon}\|\Lambda^{[l]}(\beta_k u)\| \qquad \text{(from Lemma 1.4)},$$
$$\begin{aligned}
\|I_{k2}\|&\leqslant C|\alpha_k(x)H(x_k;\lambda,\xi)-\alpha_k(x)H(x;\lambda,\xi)|_{0,\beta(0)}\\
&\qquad\times\|\Lambda^l\beta_k u\|\leqslant C\delta^\gamma|H|_{\gamma,\beta(0)}\|\Lambda^l(\beta_k u)\|,
\end{aligned}$$
$$\begin{aligned}
\|I_{k3}\|&\leqslant C\{\sum_{1<|\nu|<[l]}\|\gamma_k HR^\nu\Lambda^{l-|\nu|}u\|+|H|_{0,\beta(n+1)}(\|\gamma_k K_{\varepsilon,[l]}(u)\|\\
&\qquad+K_{\varepsilon,[l]}(\gamma_k u)\|)\} \qquad \text{(from Lemma 1.5)}.
\end{aligned}$$

Now we have

$$\sum_{k=1}^{\infty}\|\gamma_k K_{\varepsilon,[l]}(u)\|^2\leqslant C\|K_{\varepsilon,[l]}(u)\|^2\leqslant C\lambda^{-2\varepsilon}\|\Lambda^{[l]}u\|^2,$$
$$\sum_{k=1}^{\infty}\|K_{\varepsilon,[l]}(\gamma_k u)\|^2\leqslant C\sum_{k=1}^{\infty}\lambda^{-2\varepsilon}\|\Lambda^{[l]}(\gamma_k u)\|^2\leqslant C\lambda^{-2\varepsilon}\|\Lambda^{[l]}u\|^2.$$

Moreover, since

$$\|\Lambda^l\beta_k u\|^2\leqslant C\{\sum_{|\nu|\leqslant[l]}\|\gamma_k^{(\nu)}R^\nu\Lambda^{l-|\nu|}u\|^2+\|\gamma_k K_{\varepsilon,[l]}(u)\|^2+\|K_{\varepsilon,[l]}(\gamma_k u)\|^2\},$$

we have

$$\sum_{k=1}^{\infty}\|\Lambda^l\beta_k u\|^2\leqslant C\|\Lambda^l u\|^2.$$

Here we have

$$\sum_{k=1}^{\infty}(\|I_{k1}\|+\|I_{k2}\|+\|I_{k3}\|)^2\leqslant C|H|^2_{\gamma,\beta(n+1)}(\lambda^{-\varepsilon}\|\Lambda^{[l]}u\|+\delta^\gamma\|\Lambda^l u\|)^2.$$

2. λ-elliptic ($\lambda<0$) operator in R_+^n with singular integral boundary conditions.

Let us consider

$$L(x, y;\lambda, \xi, \eta)=\{A(x, y;\lambda, \xi, \eta);B_1(y;\lambda, \xi, \eta), \cdots, B_m(y;\lambda, \xi, \eta)\},$$

where

$$A(x, y;\lambda, \xi, \eta)=\sum_{i+j+|\nu|=2m} a_{ij\nu}(x, y)\lambda^i\xi^j\eta^\nu (a_{ij\nu}(x, y)\in\mathscr{B}(R_+^n)),$$
$$=\xi^{2m}+a_1(x, y;\lambda, \eta)\Lambda(\lambda, \eta)+\cdots+a_{2m}(x, y;\lambda, \eta)\Lambda(\lambda, \eta)^{2m},$$
$$B_j(y;\lambda, \xi, \eta)=\sum_{k=1}^{2m} b_{jk}(y;\lambda, \eta)\Lambda^{r_j-(2m-k)}(\lambda, \eta)\xi^{2m-k}$$
$$(b_{jk}(y;\lambda, \eta)\in\mathscr{B}(R^{n-1}\times S_+^{n-1})),$$

and $L(x, y;\lambda, \xi, \eta)$ is λ-elliptic ($\lambda>0$) in R_+^n in the sense stated in §4 of §1. Now we define

$$A(x, y;\lambda, D_x, D_y)=\sum_{i+j+|\nu|=2m} a_{ij\nu}(x, y)\lambda^i D_x^j D_y^\nu,$$
$$B_j(y;\lambda, D_x, D_y)=\sum_{k=1}^{2m} b_{jk}(y;\lambda, D_y)\Lambda^{r_j-(2m-k)}(\lambda, D_y)D_x^{2m-k},$$
$$L(x, y;\lambda, D_x, D_y)$$
$$=\{A(x, y;\lambda, D_x, D_y);B_1(y;\lambda, D_x, D_y), \cdots, B_m(y;\lambda, D_x, D_y)\}.$$

Lemma 2.1. *Let*

$$B(y;\lambda, D_x, D_y)=b(y;\lambda, D_y)\Lambda^{r+\frac{1}{2}}(\lambda, D_y)D_x^s,$$

where b is a singular integral operator in R^{n-1} and r, s are non-negative integer, then we have

$$\sum_{k=1}^{\infty}\langle B(y_k;\lambda, D_x, D_y)\alpha_k(x, y)u(x, y)-\alpha_k(x, y)B(y;\lambda, D_x, D_y)u(x, y)\rangle^2$$
$$\leqslant C|b|_{\gamma,\beta(n)}^2(\lambda^{-\varepsilon}+\delta^\gamma)^2\|u\|_{r+s+1,\lambda}^2,$$

where $\{\alpha_k\}$ is a partition of unity in R^n stated in Section 1.

Proof.

$$B(y_k;\lambda, D_x, D_y)\alpha_k(x, y)u(x, y)-\alpha_k(x, y)B(y;\lambda, D_x, D_y)u(x, y)$$
$$=b(y_k;\lambda, D_y)\Lambda^{r+\frac{1}{2}}(\lambda, D_y)\sum_{t=0}^{s}\frac{s!}{(s-t)!t!}(D_x^{s-t}\alpha_k(x, y))D_x^t u(x, y)$$
$$-\alpha_k(x, y)b(y;\lambda, D_y)\Lambda^{r+\frac{1}{2}}(\lambda, D_y)D_x^s u(x, y)$$

$$=\sum_{t=0}^{s}\frac{s!}{(s-t)!t!}\{b(y_k;\lambda,D_y)\Lambda^{r+\frac{1}{2}}(\lambda,D_y)\alpha_k^{(s-t)}(x,y)D_x^t u(x,y)$$
$$-\alpha_k^{(s-t)}(x,y)b(y;\lambda,D_y)\Lambda^{r+\frac{1}{2}}(\lambda,D_y)D_x^t u(x,y)\}$$
$$+\sum_{t=0}^{s-1}\frac{s!}{(s-t)!t!}\alpha_k^{(s-t)}(x,y)b(y;\lambda,D_y)\Lambda^{r+\frac{1}{2}}(\lambda,D_y)D_x^t u(x,y).$$

Here we apply Lemma 1.6.

Corollary.

$$\sum_{k=1}^{\infty}|||L(x_k,y_k;\lambda,D_x,D_y)\alpha_k(x,y)u(x,y)$$
$$-\alpha_k(x,y)L(x,y;\lambda,D_x,D_y)u(x,y)|||_{s,\lambda}^2$$
$$\leqslant C_s(\lambda^{-\varepsilon}+\delta^{\gamma})^2\|u\|_{2m+s,\lambda}^2 \qquad \text{for } \lambda\geqslant 1 \quad (s\text{: non-negative integer}).$$

Here we have

Proposition 2.1. *Let L be λ-elliptic ($\lambda>0$) in R_+^n, then*

$$\|u\|_{2m+s,\lambda}\leqslant C_s(|||Lu|||_{s,\lambda}+\|u\|) \qquad \textit{for } \lambda>0,$$

where $u\in\mathcal{E}_{L^2}^{2m+s}(R_+^n)$ (s: non-negative integer).

Next we define the adjoint system of L. At first we define

$$B_j(y;\lambda,\xi,\eta)=\xi^{r_j-m}A_+(0,y;\lambda,\xi,\eta)=\sum_{k=1}^{2m}b_{jk}(y;\lambda,\eta)\Lambda^{r_j-(2m-k)}(\lambda,\eta)\xi^{2m-k},$$
$$B_j(y;\lambda,D_x,D_y)=\sum_{k=1}^{2m}b_{jk}(y;\lambda,D_y)\Lambda^{r_j-(2m-k)}(\lambda,D_y)D_x^{2m-k}$$
$$(r_j=j-1,\ j=m+1,m+2,\cdots,2m),$$

and together with $\{B_j\}_{j=1,2,\cdots,m}$,

$$\begin{pmatrix}\Lambda^{2m-1-r_1}B_1(y;\lambda,D_x,D_y)\\ \Lambda^{2m-1-r_2}B_2(y;\lambda,D_x,D_y)\\ \vdots\\ \Lambda^{2m-1-r_{2m}}B_{2m}(y;\lambda,D_x,D_y)\end{pmatrix}=\mathcal{B}\begin{pmatrix}D_x^{2m-1}\\ \Lambda(\lambda,D_y)D_x^{2m-2}\\ \vdots\\ \Lambda^{2m-1}(\lambda,D_y)\end{pmatrix},$$

where

$$\mathcal{B}=(\Lambda^{2m-1-r_j}(\lambda,D_y)b_{jk}(y;\lambda,D_y)\Lambda^{-(2m-1-r_j)}(\lambda,D_y))_{j,k=1,2,\cdots,2m}.$$

Let

$$\mathcal{B}=\mathcal{B}_0+\mathcal{B}_1,$$

where

$$\mathcal{B}_0=(b_{jk}(y;\lambda,D_y))_{j,k=1,\cdots,2m},$$

$$\langle \mathcal{B}_1 U\rangle_{s,\lambda} \leqslant C\lambda^{-\varepsilon}\langle U\rangle_{s,\lambda} \qquad (\lambda \geqslant 1).$$

Lemma 2.2. *There exists* $\mathcal{H}$ *for* $\lambda > \lambda_0$, *such that*

$$\begin{pmatrix} D_x^{2m-1} \\ \Lambda(\lambda, D_y) D_x^{2m-2} \\ \vdots \\ \Lambda^{2m-1}(\lambda, D_y) \end{pmatrix} = \mathcal{H} \begin{pmatrix} \Lambda^{2m-1-r_1}(\lambda, D_y) B_1(y; \lambda, D_x, D_y) \\ \Lambda^{2m-1-r_2}(\lambda, D_y) B_2(y; \lambda, D_x, D_y) \\ \vdots \\ \Lambda^{2m-1-r_{2m}}(\lambda, D_y) B_{2m}(y; \lambda, D_x, D_y) \end{pmatrix},$$

where

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_1,$$

$$\mathcal{H}_0(y; \lambda, \eta) = \mathcal{B}_0(y; \lambda, \eta)^{-1},$$

$$\langle \mathcal{H}_1 U\rangle_{s,\lambda} \leqslant C\lambda^{-\varepsilon}\langle U\rangle_{s,\lambda}.$$

Proof. Let

$$\mathcal{H}_0(y; \lambda, D_y)\mathcal{B}_0(y; \lambda, D_y) - I = \mathcal{R}_1,$$

then we have from Lemma 1.4

$$\langle \mathcal{R}_1 U\rangle_{s,\lambda} \leqslant C\lambda^{-\varepsilon}\langle U\rangle_{s,\lambda} \qquad (\lambda \geqslant 1).$$

Here we have

$$\mathcal{H}_0 \mathcal{B} - I = \mathcal{H}_0 \mathcal{B}_1 + \mathcal{R}_1,$$

where there exists $(I + \mathcal{H}_0 \mathcal{B}_1 + \mathcal{R}_1)^{-1}$ for $\lambda > \lambda_0$, and then we have

$$(I + \mathcal{H}_0 \mathcal{B}_1 + \mathcal{R}_1)^{-1}\mathcal{H}_0 \mathcal{B} = I \qquad \text{for } \lambda > \lambda_0.$$

Let

$$\begin{aligned} \mathcal{H} &= (I + \mathcal{H}_0 \mathcal{B}_1 + \mathcal{R}_1)^{-1}\mathcal{H}_0 = \mathcal{H}_0 - (\mathcal{H}_0 \mathcal{B}_1 + \mathcal{R}_1)(I + \mathcal{H}_0 \mathcal{B}_1 + \mathcal{R}_1)^{-1}\mathcal{H}_0 \\ &= \mathcal{H}_0 + \mathcal{H}_1. \end{aligned}$$

Let

$$\begin{aligned} A^{(*)} = D_x^{2m} &+ D_x^{2m-1}\Lambda(\lambda, D_y) a_1(x, y; \lambda, D_y)^* + \cdots \\ &\cdots + \Lambda^{2m}(\lambda, D_y) a_{2m}(x, y; \lambda, D_y)^*, \end{aligned}$$

and $(\ ,\ )$ (resp $\langle\ ,\ \rangle$) be the inner product in $L^2(R_+^n)$ (resp $L^2(R^{n-1})$).

Lemma 2.3. *Let* $u, v \in \mathcal{E}_{L^2}^{2m}(R_+^n)$. *Then*

$$(Au, v) - (u, A^{(*)}v)$$

$$= i\left\langle \mathcal{A} \begin{pmatrix} D_x^{2m-1} \\ \Lambda(\lambda, D_y) D_x^{2m-2} \\ \vdots \\ \Lambda^{2m-1}(\lambda, D_y) \end{pmatrix} u, \begin{pmatrix} \Lambda^{-(2m-1)}(\lambda, D_y) D_x^{2m-1} \\ \Lambda^{-(2m-2)}(\lambda, D_y) D_x^{2m-2} \\ \vdots \\ 1 \end{pmatrix} v \right\rangle_{x=0},$$

where

$$\mathcal{A}=\mathcal{A}_0(y;\lambda, D_y)+\mathcal{A}_1,$$

$$\mathcal{A}_0(y;\lambda,\eta)=\begin{pmatrix} 0 & 0 & \cdots\cdots & 0 & 1 \\ \vdots & & \cdot & 1 & a_1(0,y;\lambda,\eta) \\ \vdots & \cdot & \cdot & \cdot & \vdots \\ \vdots & \cdot & \cdot & & \vdots \\ 1 & a_1 & \cdots\cdots & & a_{2m-1}(0,y;\lambda,\eta) \end{pmatrix},$$

$$\langle \mathcal{A}_1 U\rangle_{s,\lambda}\leqslant C\lambda^{-\varepsilon}\langle U\rangle_{s,\lambda}.$$

Proof.

$$(Au, v)-(u, A^{(*)}v)=i\sum_{k=0}^{2m-1}\sum_{l=0}^{2m-k-1}\langle D_x^{2m-k-1-l}u, D_x^l\Lambda^k a_k^* v\rangle_{x=0}$$

$$=i\sum_{k=0}^{2m-1}\sum_{l=0}^{2m-k-1}\sum_{h=0}^{l}\langle D_x^{2m-k-1-l}u, \tbinom{h}{l}\Lambda^k(D_x^{l-h}a_k)^* D_x^h v\rangle_{x=0}$$

$$=i\sum_{s=0}^{2m-1}\sum_{h=0}^{s}\langle\Lambda^h\{\sum_{k=0}^{s-h}\tbinom{s-k}{h}(D_x^{s-h-k}a_k)\Lambda^{-(s-h-k)}\}\Lambda^{-h}\Lambda^s D_x^{2m-1-s}u, \Lambda^{-h}D_x^h v\rangle_{x=0}$$

$$=i\sum_{s=0}^{2m-1}\sum_{t-s}^{2m-1}\langle\alpha_{st}\Lambda^s D_x^{2m-1-s}u, \Lambda^{-(2m-1-t)}D_x^{2m-1-t}v\rangle_{x-0}$$

where we denote

$$\mathcal{A}=(\alpha_{st})_{s,t=0,1,\cdots,2m-1}.$$

Now we denote for $\lambda>\lambda_0$

$$\begin{pmatrix} B'_1 \\ \vdots \\ B'_{2m}\end{pmatrix}=\begin{pmatrix}\Lambda^{2m-1-r_1} & & \\ & \ddots & \\ & & \Lambda^{2m-1-r_{2m}}\end{pmatrix}(\mathcal{A}\mathcal{H})^*\begin{pmatrix}\Lambda^{-(2m-1)}D_x^{2m-1} \\ \vdots \\ 1\end{pmatrix},$$

$$L'=\{A^{(*)}; B'_{m+1},\cdots, B'_{2m}\},$$

and

$$|||L'u|||'_{s,\lambda}=\|A^{(*)}u\|_{s,\lambda}+\sum_{j=m+1}^{2m}\langle B'_j u\rangle_{2m-r_j-\frac{1}{2}+s,\lambda},$$

then we have

Proposition 2.2. *It holds for* $\lambda>\lambda_0$ *that*

$$(Au, v)-(u, A^{(*)}v)=i\sum_{j=1}^{2m}\langle B_j u, B'_j v\rangle_{x=0}$$

for $u, v\in\mathcal{E}_{L^2}^{2m}(R_+^n)$, *and*

$$\|u\|_{2m+s,\lambda}\leqslant C|||L'u|||'_{s,\lambda}$$

for $u\in\mathcal{E}_{L^2}^{2m+s}(R_+^n)$ (s: *non-negative integer*).

Proof. Let

$$A_0^{(*)}(x, y; \lambda, \xi, \eta) = \overline{A(x, y; \lambda, \xi, \eta)},$$

$$\begin{pmatrix} B'_{10}(y; \lambda, \xi, \eta) \\ \vdots \\ B'_{2m0}(y; \lambda, \xi, \eta) \end{pmatrix} = \begin{pmatrix} \Lambda^{2m-1-r_1}(\lambda, \eta) & & \\ & \ddots & \\ & & \Lambda^{2m-1-r_{2m}}(\lambda, \eta) \end{pmatrix} (\mathcal{A}_0 \mathcal{H}_0)^*(y; \lambda, \eta)$$

$$\times \begin{pmatrix} \Lambda^{-(2m-1)}(\lambda, \eta)\xi^{2m-1} \\ \vdots \\ 1 \end{pmatrix},$$

and

$$L'_0 = \{A_0^{(*)}; B'_{m+1\,0}, \cdots, B'_{2m\,0}\}.$$

Then we have from Proposition 3. 2 in §1 that L'_0 is λ-elliptic ($\lambda > 0$). Therefore we can apply Proposition 2. 1 on L'_0.

Weak existence theorem for L follows Proposition 2. 2, and Proposition 2. 1 implies that weak solution of L becomes strong solution, hence we have Theorem 4. 2 in §1.

References

[1] Agmon, S., Problèmes mixtes pour les équations hyperboliques d'ordre supérieur, Colloques sur les équations aux dérivées partielles, C.N.R.S. (1962), 13-18.

[2] Mizohata, S.. Quelques problèmes au bord, du type mixte, pour des équations hyperboliques, Séminaire subventionne par le C.N.R.S. (R.C.P. 126), Collège de France 1966-1967, 23-60.

[3] Miyatake, S., On the mixed problems for fourth order hyperbolic equations, Proc. Japan Acad. **44** (1968), 257-262.

[4] Schechter, M., Integral inequalities for partial differential operators and functions satisfying general boundary conditions, Comm. Pure Appl. Math. **12** (1959), 37-66.