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Uniqueness in Cauchy's Problem for Certain Fourth Order Elliptic Equations

By

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1. We consider in the N dimensional space R^N with coordinate $(x_1, \cdots, x_N)$ and with norm $r(=(\sum_{i=1}^{N} x_i^2)^{1/2})$. Let Γ be a smooth initial hypersurface containing the origin in its interior and let Ω be a domain whose boundary contains Γ. We consider the real elliptic operator of the form

$$L = a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + b_i\frac{\partial}{\partial x_i} + c, \tag{1.1}$$

where $a_{ij} \in C^{1+\alpha}(\bar{\Omega})$ $(\alpha > 0)$ and $b_i, c \in L^\infty(\bar{\Omega})$.

When L is of the form (1.1), uniqueness in Cauchy's problem for solutions of $Lu = 0$ was shown by several mathematicians (see cf. [1], [4], [5], [7], [9], [16]). On the other hand, when the coefficients of L in (1.1) are smooth, Landis [12] and Lavrentév [13] proved that any solution u of $Lu = 0$ satisfying the following two conditions vanishes identically in Ω:[1)]

(i) $u \in C^2(\bar{\Omega})$ and $Lu = 0$ in Ω,

(ii) $u, u_{x_i} = o(\exp(-r^{-\delta}))$ $(r \to 0)$ along Γ,

where δ is a positive constant depending only on L and Γ. Their method may be said to give an explicit estimate expressing a relation between the solution and the Cauchy data.

In this note we shall give another proof for their results. Our method will be applied to the elliptic system

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1) Previously Mergelyan [14] proved this result for harmonic functions.

(1.2) $$L_p u_p = F_p(x, u_q, u_{qx_i}) \quad p=1, \cdots, m,$$

where each L_p is the elliptic operator of the form (1.1) and each nonlinear part F_p satisfies

(1.3) $$|F_p(x, u_q, u_{qx_i}) - F_p(x, v_q, v_{qx_i})| \leqq \text{const} \sum_{q=1}^{m} \sum_{|\alpha| \leqq 1} |D^\alpha(u_q - v_q)|.$$ [2]

Theorem 1. *There is a positive number δ such that if the solutions $\{u_p\}$ and $\{v_p\}$ of (1.2) satisfy*

$$u_p, v_p \in C^2(\bar{\Omega}) \cap C^3(\Gamma)$$

and

$$u_p - v_p, u_{px_i} - v_{px_i} = o(\exp(-r^{-2\delta})) \quad (r \to 0) \quad \text{along } \Gamma$$
$$(p=1, \cdots, m, \quad i=1, \cdots, N),$$

then $u_p \equiv v_p$ in Ω, where δ depends only on $\{L_p\}$ and Γ.

Next we treat the solution u of the fourth order elliptic differential inequalities

(1.4) $$|L_1 L_2 u| \leqq \text{const} \sum_{|\alpha| \leqq 3} |D^\alpha u|,$$

where L_1, L_2 are of the form (1.1) whose coefficients are sufficiently smooth and the nonlinear part of the right of (1.4) satisfies the Lipschitz condition as in (1.3). Uniqueness for solutions of (1.4) was shown by several mathematicians (see cf. [10], [11], [15], [17], [18], [19]). In particular, Mizohata [15][3] proved that the solution $u \in C^4(\bar{\Omega})$ of (1.4) vanishes identically in Ω if the Cauchy data of u vanishes on Γ. He used the singular integral method developed by Calderón [3]. On the other hand, Pederson [17], Protter [18] and Shirota [19] proved that the solution $u \in C^4(\bar{\Omega})$ of (1.4) vanishes identically in Ω if $D^\alpha u$ ($|\alpha| \leqq 3$) tends to zero rapidly at an interior point of Ω. They used an integral estimate with a weight function having singularity at the point.

Now does the solution of (1.4) vanish identically, if its Cauchy

2) We write $\dfrac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = D^\alpha$ for any vector $\alpha = (\alpha_1, \cdots, \alpha_n)$.

3) He proved also for the more general case of fourth order elliptic equations.

data tend to zero rapidly at a point? We shall answer to this problem, that is, we can prove

Theorem 2. *There is a positive constant* δ *depending only on* L_1, L_2 *and* Γ *such that if the solution* $u \in C^4(\bar{\Omega}) \cap C^7(\Gamma)$ *of* (1.4) *satisfies*

$$D^\alpha u = o(\exp(-r^{-2\delta})) \quad (r \to 0, \ |\alpha| \leqq 3)$$

along Γ, *then* u *vanishes identically in* Ω.

Remark. If Γ is a spherical surface in the neighborhood of the origin and if L_1 (or L_2) is the Laplacian operator at the origin, then the constant δ in Theorem 2 can be taken as $\delta > N$.

2. Let us denote by S_d an open sphere with the center $(d/2, 0, \cdots, 0)$ and with the radius $d/2$. We put $\Omega_h = \{0 < x_1 < h\} \cap S_1$, $\Gamma_h = \{0 < x_1 < h\} \cap \partial S_1$ and $l_h = \{x_1 = h\} \cap S_1$. In this section we shall prove the following

Proposition 2.1. *If* $u \in C^m(\bar{\Omega}_a) \cap C^{2m-1}(\Gamma_a)$ *and* $D^\alpha u = o(\exp(-r^{-2\delta-\varepsilon}))$ $(r \to 0, |\alpha| \leqq m-1)$ *along* Γ_a *for any fixed positive numbers* δ *and* ε, *then there is a function* v *such that*

$$v \in C^m(\bar{\Omega}_a - \{0\}) \cap C^{m-1}(\bar{\Omega}_a), \tag{2.1}$$

$$D^\alpha v = D^\alpha u \quad \text{on } \Gamma_a \quad \text{for } |\alpha| \leqq m-1 \tag{2.2}$$

and

$$\int_{l_h} |D^\alpha v|^2 dx_2 \cdots dx_n = o(\exp(-h^{-\delta})) \quad (h \to 0) \ \text{for } |\alpha| \leqq m. \tag{2.3}$$

Before proving the proposition we prepare a lemma. Let us take a function $\varphi(x) \in C^\infty(R^1)$ such that $\int_{-\infty}^{\infty} \varphi(x) dx = 1$ and the carrier of $\varphi \subset \{|x| \leqq 1\}$. Set

$$f(x, s) = \frac{1}{s} \int_{-2s}^{2s} \varphi((x-y)/s) dy \quad (s > 0). \tag{2.4}$$

Then we easily see that

$$f(x, s) \in C^\infty(R^1 \times \{s > 0\})$$

and

$$f(x, s)=\begin{cases}1 & |x|\leqq s\\ 0 & |x|>3s.\end{cases}$$

Further we have

Lemma 2.1. *There is a constant* C_{α} *such that*

(2.5) $$|D^{\alpha}f(x, s)|\leqq C_{\alpha}s^{-|\alpha|} \quad \text{in} \quad R^{1}\times\{s>0\},$$

where $\alpha=(\alpha_{1}, \alpha_{2})$ *and* $D^{\alpha}=D_{x}^{\alpha_{1}}D_{s}^{\alpha_{2}}$.

Proof. Obviously,

$$\begin{aligned} f(x, s) &= \int_{(x/s)-2}^{(x/s)+2}\varphi(t)\,dt \\ &= \int_{-2}^{2}\varphi(t)\,dt+\int_{0}^{x/s}\varphi(t+2)\,dt+\int_{x/s}^{0}\varphi(t-2)\,dt. \end{aligned}$$

Thus, if we prove the inequality (2.5) for $\int_{0}^{x/s}\varphi(t+2)\,dt$ and $\int_{x/s}^{0}\varphi(t-2)\,dt$, then we shall complete the proof.

It is easily seen that

(2.6) $$D_{x}\int_{0}^{x/s}\varphi(t+2)\,dt=\varphi\left(\frac{x}{s}+2\right)\frac{1}{s}$$

(2.7) $$D_{s}\int_{0}^{x/s}\varphi(t+2)\,dt=-\varphi\left(\frac{x}{s}+2\right)\frac{x}{s^{2}}.$$

We have the same equality as (2.6) and (2.7) for $\int_{x/s}^{0}\varphi(t+2)\,dt$. Noting that $f(x, s)=0$ in $|x|\geqq 3s$, we have proved (2.5) for $|\alpha|=1$. If we differentiate (2.6) and (2.7) any times and note that $|x/s|\leqq 3$, we obtain (2.5) for any α.

Now we shall prove Proposition 2.1.

Proof of Proposition 2.1. Let us use the polar coordinate $(r, \theta_{1}, \cdots, \theta_{n-1})$ with the center $(\frac{1}{2}, 0, \cdots, 0)$ such that

$$\begin{aligned} &x_{1}-\tfrac{1}{2}=r\cos\theta_{1}\\ &x_{2}=r\sin\theta_{1}\cos\theta_{2}\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &x_{n-1}=r\sin\theta_{1}\sin\theta_{2}\cdots\sin\theta_{n-2}\cos\theta_{n-1}\\ &x_{n}=r\sin\theta_{1}\sin\theta_{2}\cdots\sin\theta_{n-2}\sin\theta_{n-1}. \end{aligned}$$

We denote $(\theta_1, \cdots, \theta_{n-1})$ simply by Θ. Set

$$\tilde{v}(x) = \sum_{p=0}^{m-1} \frac{1}{p!} (r-\tfrac{1}{2})^p (D_r^p u)(\tfrac{1}{2}, \Theta).$$

Then it is easily seen that

$$D_r \tilde{v}(x) = \sum_{p=l}^{m-1} \frac{1}{(p-l)!} (r-\tfrac{1}{2})^{p-l} (D_r^p u)(\tfrac{1}{2}, \Theta) \tag{2.8}$$

and

$$(D_r^l \tilde{v})(\tfrac{1}{2}, \Theta) = (D_r^l u)(\tfrac{1}{2}, \Theta) \qquad \text{for } l \leqq m-1. \tag{2.9}$$

Put $k(x_1) = \exp(-x^{-\delta-(\varepsilon/3)})$. And we set for the function $f(x, s)$ in (2.4)

$$\zeta(x) = f\left(\sqrt{x_1 - x_1^2} - \sqrt{\sum_{i=2}^{N} x_i^2}\,,\ k(x_1)/3\right).$$

From now on we denote D_x^α simply by D_x^l for $|\alpha| = l$. Then we see by Lemma 1.1

$$|D_x^l \zeta(x)| \leqq c_l (x_1 \cdots x_n)^{-c_l} k(x_1)^{-l} \tag{2.10}$$

in a neighborhood of the origin, where c_l is a constant depending only on l. Set $v(x) = \tilde{v}(x)\zeta(x)$. Then we have

$$D_x^l v \sim D_x^{l_1} \tilde{v} \cdot D_x^{l_2} \zeta \qquad (l = l_1 + l_2), \tag{2.11}$$

where the notation $\sim$ means that the left side is a linear combination of each term on the right side. We see from (2.11)

$$D_x^l v \sim D_r^{l_1'} D_\Theta^{l_1''} \tilde{v} \cdot D_x^{l_2} \zeta \qquad (l_1' + l_1'' = l_1,\ l_1 + l_2 = l \leqq m). \tag{2.12}$$

Combining (2.8) and (2.12), we obtain

$$D_x^l v \sim (r - \tfrac{1}{2})^{p-l_1'} D_\Theta^{l_1''} ((D_r^p u)(\tfrac{1}{2}, \Theta)) D_x^{l_2} \zeta \qquad (l_1' \leqq p \leqq m-1). \tag{2.13}$$

We note that

$$|r - \tfrac{1}{2}| \leqq k(x_1) \tag{2.14}$$

in the carrier of v (or ζ).

Now let us consider each term of the right side of (2.13). If

$l_1''+p\leqq m-1$, we have from (2.10), (2.14) and from the assumption on u,

$$(2.15)\qquad \begin{aligned}&|(r-\tfrac{1}{2})^{p-l_1'}D_\Theta^{l_1''}((D_r^p u)(\tfrac{1}{2},\Theta))D_x^{l_2}\zeta|\\ &\leqq c(x_1\cdots x_n)^{-c}k(x_1)^{p-l_1'-l_2}\exp(-x_1^{-\delta-(\varepsilon/2)})\\ &\leqq c\exp(-x_1^{-\delta}).\end{aligned}$$

If $l_1''+p>m-1$, we see

$$(2.16)\qquad \begin{aligned}&|(r-\tfrac{1}{2})^{p-l_1'}D_\Theta''^{1}((D_r^p u)(\tfrac{1}{2},\Theta))D_x^{l_2}\zeta|\\ &\leqq c(x_1\cdots x_n)^{-c}k(x_1)^{p-l_1'-l_2}\\ &\leqq c(x_1\cdots x_n)^{-c}k(x_1)^{p+l_1''-l}.\end{aligned}$$

In particular when $l_1''+p>m-1$ and $l<m$, we have

$$(2.17)\qquad \text{the left side of (2.16)}\leqq c\exp(-x_1^{-\delta}).$$

Combining (2.15), (2.16) and (2.17), we have shown (2.1). Further we have obtained in the general case $l\leqq m$

$$|D_x^l v|\leqq c(x_1\cdots x_n)^{-c}.$$

Since the carrier of v (or ζ) is concentrated on a neighborhood of Γ_a, we see $|D_x^l v|\leqq cx_1^{-c}$. Thus we obtain

$$\int_{I_h}|D_x v|^2 dx_2\cdots dx_n\leqq cx_1^{-c}k(x_1)\leqq c\exp(-x_1^{-\delta}).$$

Hence we have shown (2.3). Since the equality (2.2) is trivial, we have completed the proof.

3. In this section we see how the behavior of the solutions of (1.2) or (1.4) is controled by the Cauchy data. This section is essentially based on Mizohata's result [15]. We set $\varphi_n(x_1)=\left(x_1+\frac{1}{n}\right)^{-n}$ and write $\varphi_n(x_1)$ simply by φ.

Lemma 3.1 (Mizohata [15]). *Let L_1, L_2 be second order elliptic operators of the form (1.1) with sufficiently smooth coefficients in $\bar{\Omega}_a$. Let $w\in C^3(\bar{\Omega}_a)\cap C^4(\bar{\Omega}_a-\{0\})$. Assume that*

$$(3.1)\qquad D^\alpha w=0\quad \text{on } \Gamma_a\quad \text{for } |\alpha|\leqq 3$$

and

$$(3.2) \qquad \|(D^\alpha w)(\varepsilon)\|^{4)} \to 0 \qquad (\varepsilon \to 0) \text{ for } |\alpha| \leqq 4.$$

Then there is a positive constant n_0, h_0 and c independent of n and h such that if $n > n_0$ and $h < h_0$, then

$$(3.3) \qquad c\left\{\int_{\Omega_h} \varphi^2 |L_1 L_2 w|^2 dx + \tilde{c}^2 n^2 \varphi^2(h)\right\} \geqq \tilde{c} \sum_{|\alpha| \leqq 3} \int_{\Omega_h} \varphi^2 |D^\alpha w|^2 dx,$$

where $\tilde{c} = (h + n^{-1})^{-1}$.

Proposition 3.1. *Let u be in $C^4(\bar{\Omega}_a) \cap C^7(\Gamma_a)$ and be a solution of* (1.4) *in Ω_a. If for some $\varepsilon > 0$, $\delta > 1$,*

$$D^\alpha u = o(\exp(-r^{-2\delta-\varepsilon})) \qquad (r \to 0, \ |\alpha| \leqq 3) \text{ on } \Gamma_a,$$

then it holds that

$$(3\cdot4) \qquad \int_{\Omega_h} |D^\alpha u|^2 dx = o(\exp(-h^{-\delta})) \qquad (h \to 0, \ |\alpha| \leqq 3).$$

Proof. From Proposition 2.1 there is a function v such that

$$(3.5) \qquad v \in C^4(\bar{\Omega}_a - \{0\}) \cap C^3(\bar{\Omega}_a),$$

$$(3.6) \qquad D^\alpha v = D^\alpha u \quad \text{on } \Gamma_a \text{ for } |\alpha| \leqq 3$$

and

$$(3.7) \qquad \|(D^\alpha v)(h)\|^2 = o(\exp(-h^{-\delta-(\varepsilon/3)})) \qquad (h \to 0) \text{ for } |\alpha| \leqq 4.$$

We put $w = u - v$. Then w satisfies the assumption in Lemma 3.1. Hence the inequality (3.3) holds. Here we use the following relations

$$|L_1 L_2 w|^2 \leqq 2(|L_1 L_2 u|^2 + |L_1 L_2 v|^2)$$

and

$$|D^\alpha w|^2 \geqq \tfrac{1}{2} |D^\alpha u|^2 - |D^\alpha v|^2.$$

Since u is a solution of (1.4), the inequality (3.3) becomes for sufficiently large $\tilde{c}$

$$(3.8) \qquad c\left\{\int_{\Omega_h} \varphi^2 |L_1 L_2 v|^2 dx + \tilde{c} \sum_{|\alpha| \leqq 3} \int_{\Omega_h} \varphi^2 |D^\alpha v|^2 dx + \tilde{c}^2 n^2 \varphi^2(h)\right\}$$
$$\geqq \tilde{c} \sum_{|\alpha| \leqq 3} \int_{\Omega_h} \varphi^2 |D^\alpha u|^2 dx.$$

4) We write $\|u(h)\|^2 = \int_{l_h} |u|^2 dx_2 \cdots dx_n$.

By (3. 7) this inequality becomes

$$\left(\frac{h}{2}+\frac{1}{n}\right)^{2n} c\left\{n^{2n+1}\exp(-h^{-\delta-(\varepsilon/3)})+\tilde{c}^2 n^2\left(\frac{1}{n}+h\right)^{-2n}\right\}$$
$$\geqq \tilde{c}\sum_{|\alpha|\leqq 3}\int_{\Omega_{h/2}}|D^\alpha u|^2\,dx.$$

Let us take $h+\frac{1}{n}$ sufficiently small and nh sufficiently large. We easily see that in order to prove

$$\sum_{|\alpha|\leqq 3}\int_{\Omega_{h/2}}|D^\alpha u|^2\,dx=o\left(\exp\left(-\left(\frac{h}{2}\right)^{-\delta}\right)\right)\qquad (h\to 0), \tag{3.9}$$

it is sufficient to show that we can choose n in such a way that

$$n^{2n+1}\leqq\exp\left(\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1}\right) \tag{3.10}$$

and

$$n^4\left(\frac{hn+2}{2hn+2}\right)^{2n}\leqq\exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right)\qquad \varepsilon_1,\varepsilon'>0, \tag{3.11}$$

where ε_1 is a given number and ε will be determined later. If there is a positive number $\bar{\varepsilon}$ such that

$$n^{1+\bar{\varepsilon}}\leqq\left(\frac{h}{2}\right)^{-\delta-\varepsilon_1}, \tag{3.12}$$

then (3. 10) holds. Since nh is sufficiently large, if we show that

$$\left(\frac{3}{5}\right)^{2n}\leqq\exp\left(-\left(\frac{h}{2}\right)^{-\delta-\varepsilon'}\right), \tag{3.13}$$

then (3. 11) holds. Let us take positive numbers ε', $\bar{\varepsilon}$ such that

$$\delta+\varepsilon'<(\delta+\varepsilon_1)/1+\bar{\varepsilon}.$$

Noting that $1<2\log(5/3)$, we can take n in such a way that

$$\frac{1}{2\log(5/3)}\left(\frac{2}{h}\right)^{\delta+\varepsilon'}<n<\left(\frac{2}{h}\right)^{(\delta+\varepsilon_1)/1+\bar{\varepsilon}}. \tag{3.14}$$

It is easily seen that the inequality (3. 14) implies (3. 12), (3. 13) and that $nh\to\infty$. Thus we have proved (3. 9).

Secondly we consider the system of second order differential inequalities

(3.15) $$|L_p u_p| \leqq \text{const} \sum_{q=1}^{n} \sum_{|\alpha|\leq 1} |D^\alpha u_q|, \qquad p=1, \cdots, m,$$

where each L_p is the elliptic operator of the form (1.1).

Proposition 3.2. *Let* $u_1, \cdots, u_m$ *be in* $C^2(\bar{\Omega}) \cap C^3(\Gamma_a)$ *and be solutions of the elliptic system* (3.15) *in* Ω_a. *If for some* $\varepsilon>0$, $\delta>1$,

$$u_p, Du_p = o(\exp(-r^{-2\delta-\varepsilon})) \qquad (r\to 0)$$

on Γ_a *for* $1\leqq p\leqq m$, *then we have*

(3.16) $$\int_{\Omega_h} |u_p|^2 dx, \int_{\Omega_h} |Du_p|^2 dx = o(\exp(-h^{-\delta})) \qquad (h\to 0,\ 1\leqq p\leqq m).$$

The proof of this proposition is obtained in the same way as in Proposition 3.1. Thus we shall sketch the proof briefly. From Proposition 2.1 there are functions $v_p (p=1, \cdots, m)$ such that

$$v_p \in C^2(\bar{\Omega}_a - \{0\}) \cap C^1(\bar{\Omega}_a),$$
$$D^\alpha v_p = D^\alpha u_p \quad \text{on } \Gamma_a \text{ for } |\alpha| \leqq 1$$

and

$$\|(D^\alpha v_p)(h)\|^2 = o(\exp(-h^{-\delta-(\varepsilon/3)})) \qquad (h\to 0) \text{ for } |\alpha|\leqq 2.$$

Put $w_p = u_p - v_p$. Then in the same manner as in Lemma 3.1 we see that there is a positive constant c independent of n and h $(n, h>0)$ such that

$$c\left\{\int_{\Omega_h} \varphi^2 |L_p w_p|^2 dx + \tilde{c}^2 n^2 \varphi^2(h)\right\} \geqq \tilde{c} \sum_{|\alpha|\leqq 1} \int_{\Omega_h} \varphi^2 |D^\alpha w_p|^2 dx,$$

where $c = n + h^{-1}$. Then we have the inequality as in (3.8),

$$c\sum_{p=1}^{m}\left\{\int_{\Omega_h} \varphi^2 |L_p v_p|^2 dx + \tilde{c} \sum_{|\alpha|\leqq 1} \int_{\Omega_h} \varphi^2 |D^\alpha v_p|^2 dx + \tilde{c}^2 n^2 \varphi^2(h)\right\}$$
$$\geqq \tilde{c} \sum_{p=1}^{m} \sum_{|\alpha|\leqq 1} \int_{\Omega_h} \varphi^2 |D^\alpha u_p|^2 dx.$$

We proceed in the same way as in the proof of Proposition 3.1. Then we obtain (3.16).

4. In this section we shall prove the following

Proposition 4.1. *Let u be in $C^4(\overline{\Omega}_a)\cap C^7(\Gamma_a)$ and be a solution of* (1.4) *in Ω_a. If for some $\varepsilon>0$, $\delta>1$,*

$$D^\alpha u=o(\exp(-r^{-2\delta-\varepsilon})) \qquad (r\to 0,\ |\alpha|\leqq 3) \text{ on } \Gamma_a, \tag{4.1}$$

then, in $S_{1/2}\cap\Omega_a$, it holds that

$$D^\alpha u=o(\exp(-r^{-\delta})) \qquad (r\to 0,\ |\alpha|\leqq 3). \tag{4.2}$$

Before proving the proposition we prepare a lemma. Let $x^{(0)}$ be a fixed point in R^N. And let L_1, L_2 be second order elliptic operators with sufficiently smooth coefficients in a neighborhood of $x^{(0)}$. Further we assume that L_1, L_2 are real and homogeneous. Then we have

Lemma 4.1.[5] *Let u be a C^4 function in a neighborhood of $x^{(0)}$. Then there is a constant C independent of R ($R<1$) such that*

$$\sum_{|\alpha|\leq 3}|D^\alpha u(x^{(0)})|\leqq CR^{-p_1}\left\{\left(\sum_{|\alpha|\leq 3}\int_{|x-x^{(0)}|\leq R}|D^\alpha u|^2dx\right)^{p_2} + \left(\int_{|x-x^{(0)}|\leq R}|L_1L_2u|^2dx\right)^{p_2}\right\}, \tag{4.3}$$

where p_1, p_2 depend only on N.

Proof. We may assume $x^{(0)}=0$. Let us write simply by C the constants independent of R. We take a C^∞ function such that

$$\phi(r)=\begin{cases}1 & \text{in } r\leqq R/2\\ 0 & \text{in } r\geqq R\end{cases}$$

and $|D^k\phi|\leqq CR^{-k}$. Put $v=\phi u$. Then we see

$$L_1L_2v=Q(D^k\phi\cdot D^{4-k}u)+\phi L_1L_2u, \tag{4.4}$$

where $Q(D^k\phi\cdot D^{4-k}u)$ is a linear combination of each term $D^k\phi\cdot D^{4-k}u$ $(1\leqq k\leqq 4)$.

Let $L_i^{(0)}$ $(i=1,2)$ be differential operators whose coefficients are those of L_i at $x^{(0)}$. Then there is a fundamental solution $E(x)$ of $L_1^{(0)}L_2^{(0)}$ such that

$$|D^\alpha E(x)|\leqq Cr^{4-N-|\alpha|}|\log r|. \tag{4.5}$$

5) From our proof it is easily seen that this lemma holds also for general fourth order elliptic operators.

Since $D_1^{(0)}L_2^{(0)}D^\alpha v = D^\alpha L_1 L_2 v + D^\alpha(L_1^{(0)}L_2^{(0)} - L_1 L_2)v$, we have

$$D^\alpha u(0) = \int_{r\leqq R} E(-x)\{D^\alpha L_1 L_2 v + D^\alpha(L_1^{(0)}L_2^{(0)} - L_1 L_2)v\}\,dx. \tag{4.6}$$

When $|\alpha|\leqq 3$, the equality (4.6) becomes

$$D^\alpha u(0) = (-1)^{|\alpha|}\int_{r\leqq R}(D^\alpha E)(-x)\{L_1 L_2 v + (L_1^{(0)}L_2^{(0)} - L_1 L_2)v\}\,dx. \tag{4.7}$$

We get by (4.4) and (4.5)

$$\left|\int_{r\leqq R} D^\alpha E(-x)L_1 L_2 v\,dx\right| \leqq cR^{-4}\int_{r\leqq R} r^{1-N}|\log r|\,(\sum_{k=0}^{3}|D^k u| + |L_1 L_2 u|)\,dx. \tag{4.8}$$

Noting that the coefficients of $L_1^{(0)}L_2^{(0)} - L_1 L_2$ have order $O(r)$ $(r\to 0)$, we have from (4.5) by the integration by parts

$$\left|\int_{r\leqq R} D^\alpha E(-x)(L_1^{(0)}L_2^{(0)} - L_1 L_2)v\,dx\right| \leqq C\int_{r\leqq R} r^{1-N}|\log r|\,|D^3 v|\,dx \leqq CR^{-3}\int_{r\leqq R} r^{1-N}|\log r|\,(\sum_{k=0}^{3}|D^k u|)\,dx. \tag{4.9}$$

Combining (4.7), (4.8) and (4.9), we obtain

$$|D^\alpha u(0)| \leqq CR^{-4}\int_{r\leqq R} r^{1-N}|\log r|\,(\sum_{k=0}^{3}|D^k u| + |L_1 L_2 u|)\,dx. \tag{4.10}$$

Put $m = \max_{r\leqq R}(\sum_{k=0}^{3}|D^k u|)$. Then by Hölder's inequality we have

$$\int_{r\leqq R} r^{1-N}|\log r|\,m^{-1}(\sum_{k=0}^{3}|D^k u|)\,dx \leqq \left(\int_{r\leqq R}(r^{1-N}|\log r|)^p\,dx\right)^{1/p}\left(\int_{r\leqq R}(m^{-1}\sum_{k=0}^{3}|D^k u|)^q\,dx\right)^{1/q}, \tag{4.11}$$

where $p^{-1}+q^{-1}=1$. We can take p, q in such a way that $q>2$ and the right side of (4.11) is finite. Since

$$(m^{-1}\sum_{k=0}^{3}|D^k u|)^q \leqq (m^{-1}\sum_{k=0}^{3}|D^k u|)^2,$$

the left side of (4.10) is estimated from above by

$$CR^{-p_1}\left(\int_{r\leqq R}\left(\sum_{k=0}^{3}|D^k u|^2\right)dx\right)^{p_2}.$$

Similarly we have

$$\int_{r\leqq R} r^{1-N}|\log r|\,|L_1L_2u|\,dx \leqq CR^{-p_1}\left(\int_{r\leqq R}|L_1L_2u|^2dx\right)^{p_2}.$$

Thus from (4.10) we have completed the proof.

Now we shall prove Proposition 4.1.

Proof of Proposition 4.1. For the point $x^{(0)}$ in $S_{1/2}$ we denote by $r_1(x^{(0)})$ the radius of a sphere tangent to S_1 whose center is $x^{(0)}$. It is easily seen that

$$r_1(x^{(0)})\sim x_1^{(0)} \qquad (x_1^{(0)}\to 0).$$

By Proposition 3.1 we have

$$\int_{\Omega_{r_1(x^{(0)})+x_1^{(0)}}}|D^\alpha u|^2dx = o(\exp(-x_1^{(0)-\delta-(\varepsilon/3)})) \tag{4.12}$$

$$(x_1^{(0)}\to 0,\ |\alpha|\leqq 3).$$

Let us apply Lemma 4.1 to the sphere with the center $x^{(0)}$. Then we obtain from (4.3) and (4.12)

$$D^\alpha u(x_0) = o(\exp(-x_1^{(0)-\delta})) \qquad (x_1^{(0)}\to 0,\ |\alpha|\leqq 3).$$

Thus we have completed the proof.

We note that Lemma 4.1 holds also for second order elliptic operators. Hence by Proposition 3.2 we have the following

Proposition 4.2. *Let u_p $(p=1,\cdots,m)$ be in $C^2(\overline{\Omega}_a)\cap C^3(\Gamma_a)$ and be solutions of* (1.2) *in Ω_a. Then, if for some $\varepsilon>0$, $\delta>1$,*

$$D^\alpha u_p = o(\exp(-r^{-2\delta-\varepsilon})) \qquad (r\to 0,\ |\alpha|\leqq 1,\ p\leqq m)$$

on Γ_a, then in $S_{1/2}\cap\Omega_a$

$$D^\alpha u_p = o(\exp(-r^{-\delta})) \qquad (r\to 0,\ |\alpha|\leqq 1,\ p\leqq m).$$

5. We prepare an energy estimate of Carleman's type for second order elliptic operators. We proceed along the method developed by Pederson [17], Protter [18] and Shirota [19].

Definition. If a function $v(x)$ satisfies the following conditions, we say that v belongs to $\mathcal{F}_{m,\delta}(S_d)$:

(i) $v\in C^{m-1}(R^N)\cap C^m(\overline{S}_d-\{0\})$ and the carrier of v is contained in $\overline{S}_d$.

(ii) For some $\varepsilon>0$,

$$D^\alpha v=o(\exp(-r^{-\delta-\varepsilon})) \qquad (r\to 0,\ |\alpha|<m) \text{ in } \overline{S}_d. \tag{5.1}$$

Now we set $\Phi(x)=x_1/r^2$. Then an easy computation shows that

$$\begin{aligned}&\Phi_{x_1}=\frac{r^2-2x_1^2}{r_4}, \quad \Phi_{x_i}=-2x_1x_i/r^4 \qquad (i\neq 1)\\ &\Phi_{x_1x_1}=(-2x_1)\left(\frac{3}{r^4}-\frac{4x_1^2}{r^6}\right)\\ &\Phi_{x_ix_i}=(-2x_1)\left(\frac{1}{r^4}-\frac{4x_i^2}{r^6}\right) \qquad (i\neq 1)\end{aligned} \tag{5.2}$$

and

$$|D^k\Phi|\leqq cr^{-k-1} \qquad (0\leqq k\leqq 2). \tag{5.3}$$

Let us note that the condition (5.1) implies

$$D_x^\alpha v=o(\exp(-\Phi^{\delta+\varepsilon})) \qquad (r\to 0,\ |\alpha|<m) \text{ in } \overline{S}_d. \tag{5.4}$$

Let L be a second order homogeneous elliptic operator of the form (1.1) defined in a neighborhood of the origin. And we assume $L=\Delta$ at the origin. Then we have

Proposition 5.1. *There are positive constants d_0, n_0 depending only on N and a_{ij} such that if $0<d\leqq d_0$, $n\geqq n_0$, and $\delta>N-1$ it holds for any $v\in\mathcal{F}_{2,\delta}(S_d)$*

$$\begin{aligned}&c\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(Lv)^2dx\\ &\quad\geqq n^3\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2dx+n\int\Phi^{-1}\exp(2n\Phi^\delta)|\nabla v|^2dx,\end{aligned} \tag{5.5}$$

where c is a constant depending on d_0 and n_0 but independent of n.

Proof. We can write

$$a_{ij}=\delta_{ij}+b_{ij}, \qquad |b_{ij}|\leqq cr. \tag{5.6}$$

Let us denote simply by c the positive constant independent of n and d. We see by (5.2)

$$|\nabla\Phi|^2=r^{-4}. \tag{5.7}$$

Let us put $g=\exp(-n\Phi^{\delta})$ and $v=zg$. It is easily seen that $z\in C^1(\overline{S}_d)\cap C^2(\overline{S}_d-\{0\})$ and the carrier of $z\subset\overline{S}_d$. Further we note that if $k+l\leqq 3$, and $k\neq 3$, $l\neq 3$,

$$D_x^k z D_x^l z=o(\exp(-\Phi^{\varepsilon/2})) \qquad (r\to 0) \text{ in } \overline{S}_d. \tag{5.8}$$

Obviously

$$g^{-1}Lv=Lz-2n\delta\Phi^{\delta-1}(\sum a_{ij}z_{x_i}\Phi_{x_j})+zg^{-1}Lg. \tag{5.9}$$

We use the following inequality for the right of (5.9)

$$(X+Y+Z)^2\geqq 2Y(X+Z).$$

Then

$$g^{-2}(Lv)^2\geqq -4n\delta\Phi^{\delta-1}(\sum a_{ij}z_{x_i}\Phi_{x_j})Lz \\ -4n\delta\Phi^{\delta-1}(\sum a_{ij}z_{x_i}\Phi_{x_j})zg^{-1}Lg. \tag{5.10}$$

Thus we have

$$\int r^4\Phi^{1-\delta}g^{-2}(Lv)^2dx\geqq -4n\delta\int r^4Lz(\sum_{i,j}a_{ij}z_{x_i}\Phi_{x_j})dx \\ -4n\delta\int r^4(\sum_{i,j}a_{ij}z_{x_i}\Phi_{x_j})zg^{-1}Lg\,dx, \tag{5.11}$$

where the integral domain is S_d. We write

$$\begin{aligned} &-4n\delta r^4Lz(\sum a_{ij}z_{x_i}\Phi_{x_j}) \\ &=-4n\delta r^4(\sum_i(1+b_{ii})z_{x_ix_i}+\sum_{i\neq j}a_{ij}z_{x_ix_j}) \\ &\qquad\times(\sum_i(1+b_{ii})z_{x_i}\Phi_{x_i}+\sum_{i\neq j}a_{ij}z_{x_i}\Phi_{x_j}) \\ &=-4n\delta r^4(\Delta z)(\nabla z\cdot\nabla\Phi)-4n\delta R, \end{aligned} \tag{5.12}$$

where R is the sum of the remained terms.

In general for any function $f(x)$ it holds

$$\int fz_{x_ix_i}z_{x_j}dx=-\int f_{x_i}z_{x_i}z_{x_j}dx+\tfrac{1}{2}\int f_{x_j}z_{x_i}^2dx. \tag{5.13}$$

Further we have

(5.14) $$\int f z_{x_i x_j} z_{x_k} dx = \tfrac{1}{2}\Big(-\int f_{x_j} z_{x_i} z_{x_k} dx + \int f_{x_k} z_{x_i} z_{x_j} dx - \int f_{x_i} z_{x_k} z_{x_j} dx\Big).$$

In fact

(5.15) $$\int f z_{x_i x_j} z_{x_k} dx = -\int f_{x_j} z_{x_i} z_{x_k} dx - \int f z_{x_i} z_{x_k x_j} dx,$$

(5.16) $$-\int f z_{x_i} z_{x_k x_j} dx = \int f_{x_k} z_{x_i} z_{x_j} dx + \int f z_{x_i x_k} z_{x_j} dx$$

and

(5.17) $$\int f z_{x_i x_k} z_{x_j} dx = -\int f_{x_i} z_{x_k} z_{x_j} dx - \int f z_{x_k} z_{x_j x_i} dx.$$

Combining (5.15), (5.16) and (5.17), we obtain (5.14).

Applying (5.13) and (5.14) to each of R, we see from (5.12)

(5.18) $$\Big|\int R\, dx \Big| \leqq cd \int \Phi^{-1} |\Delta z|^2 dx.$$

Now let us show the following inequality

(5.19) $$\int r^4 (\Delta z)(\nabla z \cdot \nabla \Phi) dx \leqq 0.$$

We get from (5.7)

$$r^4(\nabla z \cdot \nabla \Phi) = z_{x_1}(r^2 - 2x_1^2) - 2x_1 \sum_{i=2}^{N} z_{x_i} x_i .$$

Thus

$$\int r^4 (\Delta z)(\nabla z \cdot \nabla \Phi) dx$$
$$= \int z_{x_1 x_1} z_{x_1} (r^2 - 2x_1^2) dx + \sum_{i=2}^{N} \int z_{x_i x_i} z_{x_1} (r^2 - 2x_1^2) dx$$
$$- \int z_{x_1 x_1} 2x_1 \Big(\sum_{i=2}^{N} z_{x_i} x_i\Big) dx - \sum_{i=2}^{N} \int z_{x_i x_i} 2x_1 \Big(\sum_{j=2}^{N} z_{x_j} x_j\Big) dx$$
$$= J_1 + J_2 - J_3 - J_4 .$$

By (5.8) we can integrate by parts each J_i. Integrating by parts, we can verify

$$J_1 = \int z_{x_1}^2 x_1\, dx,$$

$$J_2 = -\int \Big(\sum_{i=2}^{N} z_{x_i}^2\Big) x_1\, dx - \int \Big(\sum_{i=2}^{N} 2x_i z_{x_i}\Big) z_{x_1} dx$$

$$J_3=(N-1)\int z_{x_1}^2 x_1\,dx-\int(\sum_{i=2}^{N}2x_i z_{x_i})z_{x_1}\,dx$$

and

$$J_4=\sum_{i=2}^{N}\int(N-2)z_{x_i}^2 x_1\,dx.$$

Combining these integrals, we get

$$\int r^4(\Delta z)(\nabla z\cdot\nabla\Phi)\,dx=(2-N)\sum_{i=1}^{N}\int z_{x_i}^2 x_1\,dx\leqq 0.$$

Hence the inequality (5.19) has been shown. We have from (5.12), (5.18) and (5.19)

$$\int r^4 Lz(\sum_{i,j}a_{ij}z_{x_i}\Phi_{x_j})\leqq cd\int\Phi^{-1}|\nabla z|^2\,dx. \tag{5.20}$$

Now we estimate the last integral in (5.11). A computation shows

$$g^{-1}Lg=\sum_{k,l}a_{kl}\{(n^2\delta^2\Phi^{2(\delta-1)}-n\delta(\delta-1)\Phi^{\delta-2})\Phi_{x_k}\Phi_{x_l}-n\delta\Phi^{\delta-1}\Phi_{x_k x_l}\}.$$

Thus

$$\begin{aligned}(5.21)\qquad & -4n\delta\int r^4(\sum_{i,j}a_{ij}z_{x_i}\Phi_{x_j})zg^{-1}Lg\,dx\\ &=2n\delta\sum_{i,j,k,l}\int(r^4a_{ij}\Phi_{x_j}n^2\delta^2a_{kl}\Phi^{2(\delta-1)}\Phi_{x_k}\Phi_{x_l})_{x_i}z^2\,dx\\ &\quad-2n\delta\sum_{i,j,k,l}\int(r^4a_{ij}\Phi_{x_j}n\delta(\delta-1)a_{kl}\Phi^{\delta-2}\Phi_{x_k}\Phi_{x_l})_x\,z^2\,dx\\ &\quad-2n\delta\sum_{i,j,k,l}\int(r^4a_{ij}\Phi_{x_j}n\delta a_{kl}\Phi^{\delta-1}\Phi_{x_k x_l})_{x_i}z^2\,dx.\end{aligned}$$

The first integral on the right of (5.21) contains the following three terms:

$$M_1=4n^3\delta^3(\delta-1)\sum_{i,j,k,l}\int r^4a_{ij}a_{kl}\Phi_{x_i}\Phi_{x_j}\Phi_{x_k}\Phi_{x_l}\Phi^{2\delta-3}z^2\,dx,$$

$$M_2=2n^3\delta^3\sum_{i,j,k,l}\int r^4a_{ij}a_{kl}\Phi_{x_j x}\,\Phi_{x_k}\Phi_{x_l}\Phi^{2(\delta-1)}z^2\,dx$$

and

$$M_3=2n^3\delta^3\sum_{i,j,k,l}\int r^4a_{ij}a_{kl}\Phi_{x_j}(\Phi_{x_k}\Phi_{x_l})_{x_i}\Phi^{2(\delta-1)}z^2\,dx.$$

From the positive definiteness of L and (5.7) we have

(5. 22) $$M_1 \geqq cn^3\delta^3(\delta-1)\int r^{-4}\Phi^{2\delta-3}z^2\,dx.$$

We decompose the integral M_2 into

(5. 23) $$M_2=2n^3\delta^3\sum_{i,j,k,l}\int r^4\delta_{ij}\delta_k\Phi_{x_ix_j}\Phi_{x_k}\Phi_{x_l}\Phi^{2(\delta-1)}z^2\,dx$$
$$+2n^3\delta^3\sum_{i,j,k,l}\int r^4(a_{ij}a_k-\delta_{ij}\delta_{kl})\Phi_{x_ix_j}\Phi_{x_k}\Phi_{x_l}\Phi^{2(\delta-1)}z^2\,dx.$$

Since we have assumed $a_{ij}=\delta_{ij}$ at the origin, the second integral of (5. 23) is absorbed by the right of (5. 22) if $d\leqq d_0$ for sufficiently small d_0. The first integral of (5. 23) becomes

$$2n^3\delta^3\int r^4\Delta\Phi(\nabla\Phi)^2\Phi^{2(\delta-1)}z^2\,dx.$$

Let us note that $\Delta\Phi=2(2-N)x_1/r^4$ by (5. 2). Then we see from (5. 7)

(5. 24) the first integral of (5. 23)
$$=4(2-N)n^3\delta^3\int x_1r^{-4}\Phi^{2(\delta-1)}z^2\,dx.$$

Since the constant c on the right of (5. 22) can be taken as arbitrarily near to 4 for sufficiently small d_0, the integral (5. 24) is absorbed by the right of (5. 22) for $\delta>N-1$. Thus we see that the term M_2 is absorbed by M_1.

Secondly we shall show that the term M_3 is absorbed by the right of (5. 22). Let us decompose M_3 in a similar way to (5. 23). We see

$$\sum_{i,j,k,l}\delta_{ij}\delta_{kl}\Phi_{x_j}(\Phi_{x_k}\Phi_{x_l})_{x_i}$$
$$=\sum_{i,k}\Phi_{x_i}(\Phi_{x_k}^2)_{x_i}=\sum_i\Phi_{x_i}(r^{-4})_{x_i}=-4\sum_i r^{-6}x_i\Phi_{x_i}.$$

On the other hand we have from (5. 2)

$$\sum_i x_i\Phi_{x_i}=-x_1r^{-2}.$$

Thus we get

$$\sum_{i,j,k,l}\delta_{ij}\delta_{kl}\Phi_{x_j}(\Phi_{x_k}\Phi_{x_l})_{x_i}\geqq 0.$$

Hence the term M_3 is absorbed by the right of (5. 22). More easily we can see that the remaining terms on the right of (5.21) are absorbed by the right of (5.22) if we take $n \geqq n_0$. Therefore combining (5.11), (5.20), (5.21) and (5.22), we obtain

$$
\begin{aligned}
(5.25) \qquad & n^3\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2\,dx \\
& \leqq c\Big(\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(Lv)^2 dx \\
& \quad + nd\int \Phi^{-1}\exp(2n\Phi^\delta)|\nabla v|^2 dx\Big).
\end{aligned}
$$

Let us put $q=\Phi^{-1}\exp(2n\Phi^\delta)$. Then

$$
(5.26) \qquad q_{x_i}^2 \leqq cn^2 r^{-4}\Phi^{2\delta-4}\exp(4n\Phi^\delta).
$$

And we see

$$
\int qvLv\,dx = -\int qa_{ij}v_{x_i}v_{x_j}dx - \int (qa_{ij})_{x_j}vv_{x_i}dx.
$$

Hence we have

$$
(5.27) \qquad \int q(\nabla v)^2 dx \leqq \Big|\int (qa_{ij})_{x_j}vv_{x_i}dx\Big| + \Big|\int qvLv\,dx\Big|.
$$

By Cauchy's inequality we get from (5.26)

$$
\begin{aligned}
(5.28) \qquad & |(qa_{ij})_{x_j}vv_{x_i}| = |\Phi^{-1/2}\exp(n\Phi^\delta)v_{x_i}|\cdot|\Phi^{1/2}\exp(-n\Phi^\delta)(qa_{ij})_{x_j}v| \\
& \leqq c\{\varepsilon\Phi^{-1}\exp(2n\Phi^\delta)|\nabla v|^2 + \varepsilon^{-1}n^2r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2\}.
\end{aligned}
$$

Here we take ε as sufficiently small. On the other hand we have

$$
\begin{aligned}
(5.29) \qquad & 2|qvLv| \leqq 2\sqrt{2r^{-4}\Phi^{\delta-2}}\cdot\sqrt{n^{-1}r^4\Phi^{1-\delta}}\exp(2n\Phi^\delta)|vLv| \\
& \leqq nr^{-4}\Phi^{\delta-2}\exp(2n\Phi^\delta)v^2 + n^{-1}r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(Lv)^2.
\end{aligned}
$$

Combining (5.27), (5.28) and (5.29), we have

$$
\begin{aligned}
(5.30) \qquad & \int \Phi^{-1}\exp(2n\Phi^\delta)(\nabla v)^2 dx \\
& \leqq c\Big\{n^2\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2 dx \\
& \quad + n^{-1}\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(Lv)^2 dx\Big\}.
\end{aligned}
$$

Substituting this inequality into (5.25), we obtain

(5.31)
$$n^3(1-cd)\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2\,dx$$
$$\leqq c\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(Lv)^2\,dx,$$

where d_0 is taken as more sufficiently small. By (5.30) and (5.31) we complete the proof.

Now let $\gamma_1, \cdots, \gamma_N$ be the fixed real numbers such that $\sum_{i=1}^{N}\gamma_i^2=1$. Then we put for $d>0$

$$\widetilde{S}_d=\{r^2<d(\gamma_1 x_1+\cdots+\gamma_N x_N)\}.$$

The set $\widetilde{S}_d$ is an open sphere with the center $\left(\frac{d}{2}\gamma_1, \cdots, \frac{d}{2}\gamma_N\right)$ and the radius $d/2$. Further we set $\widetilde{\Phi}(x)=\sum_{i=1}^{N}\gamma_i x_i/r^2$.

Definition. If a function $v(x)$ satisfies the following conditions, we say that v belongs to $\mathcal{F}_{2,d}(\widetilde{S}_d)$:

(i) $v\in C^1(R^N)\cap C^2(\overline{\widetilde{S}}_d-\{0\})$ and the carrier of v is contained in $\overline{\widetilde{S}}_d$.

(ii) For some $\varepsilon>0$

$$v,\ D_x v=o(\exp(-r^{-\delta-\varepsilon}))\qquad(r\to 0)\ \text{in}\ \overline{\widetilde{S}}_d.$$

Now we also assume that $L=\Delta$ at the origin as in Proposition 5.1. Then we have by an adequate orthogonal transformation for (5.5).

Corollary 5.1. *There are positive constants d_0, n_0 depending only on a_{ij} such that if $0<d\leqq d_0$, $n\geqq n_0$ and $\delta>N-1$, it holds for any $v\in\mathcal{F}_{2,\delta}(\widetilde{S}_d)$*

(5.32)
$$c\int r^4\widetilde{\Phi}^{1-\delta}\exp(2n\widetilde{\Phi}^\delta)(Lv)^2\,dx$$
$$\geqq n^3\int r^{-4}\widetilde{\Phi}^{2\delta-3}\exp(2n\widetilde{\Phi}^\delta)v^2\,dx+n\int\widetilde{\Phi}^{-1}\exp(2n\widetilde{\Phi}^\delta)|\nabla v|^2\,dx,$$

where c is the same constant as in Proposition 5.1.

We have assumed that $L=\Delta$ at the origin in Proposition 5.1 and Corollary 5.1. If we eliminate this assumption, the statement of Proposition 5.1 holds also, that is, we can prove

Proposition 5.2. *There are positive constants d_0, n_0 and δ_0*

depending only on a_{ij} *such that if* $0<d\leqq d_0$, $n\geqq n_0$ *and* $\delta>\delta_0$, *it holds for any* $v\in\mathcal{F}_{2,\delta}(S_d)$

$$(5.33)\qquad c\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(Lv)^2dx$$
$$\geqq n^3\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2dx+n\int\Phi^{-1}\exp(2n\Phi^\delta)|\nabla v|^2dx,$$

where c is a constant depending on d_0 *and* n_0 *but independent of n.*

Proof. Let $L^{(0)}$ be the operator whose coefficients are those of L at the origin. The operator $L^{(0)}$ is reduced to $\sum_{i=1}^N\lambda_iD_{x'_i}$ by an orthogonal transformation $x'=Tx(x_i=\sum_{j=1}^N s_{ij}x'_j)$. Secondly the operator $\sum_{i=1}^N\lambda_iD^2_{x'_i}$ is reduced to Δ by the transformation $x''=\Lambda x'(x'_i=\lambda_{ii}x'')$. On the other hand the sphere S_d is mapped into the following set:

$$(5.34)\qquad \sum_{i=1}^N\Big(\sum_{j=1}^N\lambda_js_{ij}x''_j\Big)^2<d\sum_{j=1}^N\lambda_js_{1j}x''_j.$$

Obviously there is a positive constant c such that

$$(5.35)\qquad c^{-1}\sum_{i=1}^Nx''^2_i\leqq\sum_{i=1}^N\Big(\sum_{j=1}^N\lambda_js_{ij}x''_j\Big)^2\leqq c\sum_{i=1}^Nx''^2_i.$$

We put

$$\gamma=\sqrt{\sum_{j=1}^N(\lambda_js_{1j})^2},\quad \gamma_j=\gamma^{-1}\lambda_js_{1j}\qquad(j=1,\cdots,N).$$

Setting $\tilde\Phi=\sum_{j=1}^N\gamma_jx''_j/\sum_{j=1}^Nx''^2_j$, we have the inequality (5.32) for any $v(x'')\in\mathcal{F}_{2,\delta}(\tilde S_d)$ $(0<d\leqq d_0)$ from Corollary 5.1, that is

$$(5.36)\qquad c\int\tilde r^4\tilde\Phi^{1-\delta}\exp(2n\tilde\Phi^\delta)(L''v)^2dx''$$
$$\geqq n^3\int\tilde r^{-4}\tilde\Phi^{2\delta-3}\exp(2n\tilde\Phi^\delta)v^2dx''+n\int\tilde\Phi^{-1}\exp(2n\tilde\Phi^\delta)|\nabla v|^2dx'',$$

where $\tilde r=(\sum_{i=1}^Nx'^2_i)$ and L'' is the transformed operator of L by $x''=\Lambda Tx$.

Let us note that if (5.34) holds, we have by (5.35)

$$\sum_{i=1}^Nx''^2_i<dc\gamma\sum_{j=1}^N\gamma_jx''_j.$$

Thus taking d'_0 sufficiently small, we see that if $v(x)\in\mathcal{F}_{2,\delta}(S_d)$ $(0<d\leqq d'_0)$, then $v(x'')\in\mathcal{F}_{2,\delta}(\tilde S_d)$ $(0<d\leqq d_0)$. Hence the inequality

(5.36) holds for the function $v(x'')$ of $v(x)\in\mathcal{F}_{2,\delta}(S_d)$ $(0<d\leqq d_0')$.

Now from (5.35) we easily see

$$\frac{r^{-1}(\sum_j\lambda_j s_{1j}x_j'')}{c\sum_i(\sum_j\lambda_j s_{ij}x_j'')^2}\leqq\tilde{\Phi}\leqq\frac{r^{-1}(\sum_j\lambda_j s_{1j}x_j'')}{c^{-1}\sum_i(\sum_j\lambda_j s_{ij}x_j)^2},$$

that is,

$$\frac{r^{-1}}{c}\frac{x_1}{r_2}\leqq\tilde{\Phi}\leqq\frac{r^{-1}}{c^{-1}}\frac{x_1}{r^2}. \tag{5.37}$$

And we see for another positive constant c

$$c^{-1}\sum_{i=1}^N x_i^2\leqq\sum_{i=}^N x_i''^2\leqq c\sum_{i=1}^N x_i^2. \tag{5.38}$$

Performing the inverse transformation $x=T^{-1}\Lambda^{-1}x''$ on the inequality (5.36), we apply the relations (5.37) and (5.38). Then we obtain (5.33) for any $v(x)\in\mathcal{F}_{2,\delta}(S_d)$ $(0<d\leqq d_0')$.

6. We prove the similar estimate to that in the previous section for fourth order elliptic operators. Let L_1 and L_2 be second order homogeneous operators of the form (1.1) with sufficiently smooth coefficients in a neighborhood of the origin. Then we have

Proposition 6.1. *There are positive constants d_0 and n_0 depending only on L_1 and L_2 such that if $0<d<d_0$, $n>n_0$ and $\delta>\delta_0$, it holds for any $v\in\mathcal{F}_{4,\delta}(S_d)$ that*

$$\begin{aligned} c\int r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(L_1L_2v)^2dx & \\ \geqq\int r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(\sum_{|\alpha|=3}|D^\alpha v|^2)dx & \\ +n^2\int r^8\Phi^{3-2\delta}\exp(2n\Phi^\delta)(\sum_{|\alpha|=2}|D^\alpha v|^2)dx & \\ +n^4\int\Phi^{-1}\exp(2n\Phi^\delta)(\nabla v)^2dx & \\ +n^6\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2dx. & \end{aligned} \tag{6.1}$$

Proof. We easily see

$$(6.2)\qquad \int qv\,\Delta v\,dx=-\int q(\nabla v)^2dx+\tfrac{1}{2}\int(\Delta q)v^2dx$$

and

$$(6.3)\qquad \int qv_{x_ix_i}v_{x_jx_j}dx=\int qv^2_{x_ix_j}dx-\tfrac{1}{2}\int(q_{x_ix_i}v^2_{x_j}-2q_{x_ix_j}v_{x_i}v_{x_j}+q_{x_jx_j}v^2_{x_i})dx.$$

Now we consider the integral

$$\int q(\Delta v)^2dx=\int(q\sum_{i=1}^N v^2_{x_ix_i}+q\sum_{i\neq j}v_{x_ix_i}v_{x_jx_j})dx.$$

Let us substitute (6.3) into the second term on the right. Then we have

$$\int q(\sum_{i,j}v^2_{x_ix_j})dx=\int q(\Delta v)^2dx+\int F(q_{x_ix_j},v_{x_k})dx,$$

where F is a sum of products of v_{x_k}, v_{x_l} and $q_{x_ix_j}$. Let $L_2^{(0)}$ be the operator whose coefficients of L_2 at the origin. Then by a coordinate transformation we get

$$(6.4)\qquad \int q(\sum_{i,j}v^2_{x_ix_j})dx\leqq\int q(L_2^{(0)}v)^2dx+\int F(q_{x_ix_j},v_{x_k})dx.$$

Obviously

$$\int q(L_2^{(0)}v)^2dx\leqq c\Big(\int q(L_2v)^2dx+\int q((L_2^{(0)}-L_2)v)^2dx\Big)$$
$$\leqq c\Big(\int q(L_2v)^2dx+d\int q(\sum_{i,j}v^2_{x_ix_j})dx\Big).$$

Thus if we take d sufficiently small, (6.4) becomes

$$(6.5)\qquad \int q(\sum_{i,j}v^2_{x_ix_j})dx\leqq c\Big(\int q(L_2v)^2dx+\int F(q_{x_ix_j},v_{x_k})dx\Big).$$

Hence we have

$$(6.6)\qquad \int q(\sum_{i,j,l}v^2_{x_ix_jx_l})dx\leqq c\Big(\int q(L_2v_{x_l})^2dx+\int F(q_{x_ix_j},v_{x_lx_k})dx\Big).$$

Now it is seen

$$(6.7)\qquad |L_2v_{x_l}|\leqq|(L_2v)_{x_l}|+|L_2v_{x_l}-(L_2v)_{x_l}|\leqq|(L_2v)_{x_l}|+c\sum_{i,j}|v_{x_ix_j}|.$$

Therefore we obtain from (6.6)

$$\int q(\sum_{i,j,l} v^2_{x_i x_j x_l})dx \leqq c\Big[\int q((L_2 v)_{x_l})^2 dx + \sum_{i,j}\int q v^2_{x_i x_j} + \sum_{i,j,k,l}\int |q_{x_i x_j}| v^2_{x_k x_l} dx\Big]. \tag{6.8}$$

Let us estimate the following from the above:

$$\int r^4 \Phi^{1-\delta} \exp(2n\Phi^\delta((L_2 v)^2 dx.$$

The integral can be rewritten in the form

$$\int r^{-4} \Phi^{2\delta-3} \exp(2n\Phi^\delta)(r^4 \Phi^{2-\frac{3}{2}\delta} L_2 v)^2 dx.$$

By Proposition 5.2 we see

$$\begin{aligned} &n^3 \int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)(r^4\Phi^{2-\frac{3}{2}\delta}L_2 v)^2 dx \\ &\quad + n\int \Phi^{-1}\exp(2n\Phi^\delta)|\nabla(r^4\Phi^{2-\frac{3}{2}\delta}L_2 v)|^2 dx \\ &\leqq c\int r^4 \Phi^{1-\delta}\exp(2n\Phi^\delta)(L_1(r^4\Phi^{2-\frac{3}{2}\delta}L_2 v))^2 dx. \end{aligned} \tag{6.9}$$

An easy computation shows

$$|\nabla(r^4\Phi^{2-\frac{3}{2}\delta})| \leqq cr^2\Phi^{1-\frac{3}{2}\delta}$$
$$|(r^4\Phi^{2-\frac{3}{2}\delta})_{x_i x_j}| \leqq c\Phi^{-\frac{3}{2}\delta}.$$

Thus taking n as sufficiently large (or d as sufficiently small) in (6.9), we have

$$\begin{aligned} &n^3\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(L_2 v)^2 dx \\ &\quad + n\int r^8\Phi^{3-3\delta}\exp(2n\Phi^\delta)|\nabla(L_2 v)|^2 dx \\ &\leqq c\Big\{\int r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(L_1 L_2 v)^2 dx \\ &\quad + \int r^8\Phi^{3-4\delta}\exp(2n\Phi^\delta)|\nabla(L_2 v)|^2 dx \\ &\quad + \int r^4\Phi^{1-4\delta}\exp(2n\Phi^\delta)(L_2 v)^2 dx\Big\}. \end{aligned}$$

Hence we obtain

$$(6.10)\qquad n^3\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(L_2v)^2dx + n\int r^8\Phi^{3-3\delta}\exp(2n\Phi^\delta)|\nabla(L_2v)|^2dx \leqq c\int r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(L_1L_2v)^2dx.$$

Combining Proposition 5.2 and (6.10), we see

$$(6.11)\qquad \int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2dx + n^{-2}\int\Phi^{-1}\exp(2n\Phi^\delta)|\nabla v|^2dx + n^{-5}\int r^8\Phi^{3-3\delta}\exp(2n\Phi^\delta)|\nabla(L_2v)|^2dx \leqq cn^{-6}\int r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(L_1L_2v)^2dx.$$

We set in (6.5)

$$q=n^{-4}r^8\Phi^{3-2\delta}\exp(2n\Phi^\delta).$$

Then it is easily seen that

$$|q_{x_ix_j}|\leqq n^{-2}r^4\Phi\exp(2n\Phi^\delta).$$

Hence for a sufficiently small d we have by (6.5), (6.10) and (6.11)

$$(6.12)\qquad \int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v^2dx + n^{-2}\int\Phi^{-1}\exp(2n\Phi^\delta)|\nabla v|^2dx + n^{-4}\int r^8\Phi^{3-2\delta}\exp(2n\Phi^\delta)(\sum v^2_{x_ix_j})dx \leqq cn^{-6}\int r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(L_1L_2v)^2dx.$$

Secondly we set in (6.8)

$$q=n^{-6}r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta).$$

Then we easily see

$$|q_{x_ix_j}|\leqq cn^{-4}r^8\Phi^{3-2\delta}\exp(2n\Phi^\delta).$$

Thus combining (6.8), (6.5), (6.11) and (6.12), we obtain the estimate (6.1).

7. Now we shall prove Theorem 1 and 2. First we shall prove Theorem 2.

Proof of Theorem 2. We may assume that $\Gamma=\Omega_a$.

Let δ and d_0 be the positive numbers in Proposition 6. 1. Let the assumption on u be such that

$$D^\alpha u=o(\exp(-r^{-2\delta-\varepsilon})) \qquad (r\to 0,\ \varepsilon>0,\ |\alpha|\leqq 3)$$

along Γ. Then by Proposition 4. 1 we see

$$(7.1)\qquad D^\alpha u=o(\exp(-r^{-\delta-(\varepsilon/3)})) \qquad (r\to 0,\ |\alpha|\leqq 3) \text{ in } S_{1/2}\cap\Omega_a.$$

Now we take a C^∞ function $\eta(t)$ such that

$$\eta(t)=\begin{cases}1 & 0\leqq t\leqq d_0/2\\ 0 & t\geqq d_0.\end{cases}$$

And put $\zeta(x)=\eta(r^2/x_1)$. Then we easily see

$$D^k(r^2/x_1)=D^k(x_1^{-1}r^2)\leqq cr^{-2(k+1)}.$$

Hence it holds

$$(7.2)\qquad |D^k\zeta|\leqq c_k r^{-c_k},$$

where c_k is a constant depending on k. Set $v(x)=\zeta(x)u(x)$. We note that $v=u$ in $S_{d_0/2}$. Then by (7. 1) and (7. 2) it is seen that $v\in\mathcal{F}_{4,\delta}(S_{d_0})$. We can assume that the constant on the right of (1. 4) is sufficiently small. Applying Proposition 6. 1 to v, we have from (1. 4)

$$(7.3)\qquad c\int_{S_{d_0}-S_{d_0/2}} r^{12}\Phi^{5-4\delta}\exp(2n\Phi^\delta)(L_1L_2v)^2\,dx \geqq n^6\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)u^2\,dx.$$

Here we note that for some positive constant c

$$r^{12}\Phi^{5-4\delta}\leqq c\leqq r^{-4}\Phi^{2\delta-3}$$

and

$$\Phi\geqq 2/d_0 \quad \text{in } S_{d_0/2}$$
$$\Phi\leqq 2/d_0 \quad \text{in } S_{d_0}-S_{d_0/2}.$$

Hence (7. 3) is reduced to

$$cn^{-6}\int_{S_{d_0}-S_{d_0/2}}(L_1L_2v)^2dx\geqq\int_{S_{d_0/2}}u^2dx.$$

Let n tend to infinity. Then $u=0$ in $S_{d_0/2}$. Therefore u vanishes identically by the well known result with respect to the uniqueness in Cauchy's problem.

Proof of Theorem 1. It is sufficient to prove the existence of δ such that the solutions $\{u_p\}$ of (3.15) satisfying

$$u_p\in C^2(\bar{\Omega})\cap C^3(\Gamma)$$

and

$$u_p,\ u_{px_i}=o(\exp(-r^{-2\delta-\varepsilon}))\qquad (r\to 0,\ \varepsilon>0)\ \text{along}\ \Gamma$$
$$(p=1,\cdots,m,\ i=1,\cdots,N)$$

vanish identically in Ω.

We obtain from Proposition 5.2

$$\text{(7.4)}\qquad c\int r^4\Phi^{1-\delta}\exp(2n\Phi^\delta)(L_pv_p)^2dx$$
$$\geqq n^3\int r^{-4}\Phi^{2\delta-3}\exp(2n\Phi^\delta)v_p^2dx$$
$$+n\int\Phi^{-1}\exp(2n\Phi^\delta)|\nabla v_p|^2dx.$$

Set $v_p(x)=\zeta(x)u_p$ for ζ in (7.1) and for the solutions u_p. Then by Proposition 4.2, we see $v_p\in\mathcal{F}_{2,\delta}(S_{d_0})$. Summing up (7.6) with respect to p and noting that u_p are solutions of (3.15), we see in a similar way to that in the proof of Theorem 2

$$cn^{-3}\int_{S_{d_0}-S_{d_0/2}}\sum_{p=1}^{m}(L_pv_p)^2dx\geqq\sum_{p=1}^{m}\int_{S_{d_0/2}}u_p^2dx.$$

Thus tending n to infinity, we have completed the proof.

Finally we give some corollaries.

Corollary 7.1. *Let u be a solution in $\bar{\Omega}_a$ of*

$$|Lu|\leqq C(x_1^{-2}|u|+x_1^{-1}\sum_{i=1}^{n}|u_{x_i}|),$$

where L is an operator of the form (1.1). *Then there is a positive number δ depending only on L such that if*

$$u, Du = o(\exp(-r^{-\delta})) \qquad (r \to 0) \text{ in } S_{1/2} \cap \overline{\Omega}_a,$$

then u vanishes identically.

Proof. We proceed in a similar manner to that in the proof of Theorem 1. Put $v(x) = \zeta(x)u(x)$ for ζ in (7.2). Then we have

$$\begin{aligned}
&\int_{S_{d_0}} r^4 \Phi^{1-\delta} \exp(2n\Phi^\delta)(Lv)^2 dx \\
&\leqq c\int_{S_{d_0/2}} r^4 \Phi^{1-\delta} \exp(2n\Phi^\delta)(x_1^{-4}|u|^2 + x_1^{-2}\sum_{i=1}^{N}|u_{x_i}|^2) dx \\
&\quad + c\int_{S_{d_0}-S_{d_0/2}} r^4 \Phi^{1-\delta} \exp(2n\Phi^\delta)(Lv)^2 dx.
\end{aligned}$$

The first integral on the right of this inequality is estimated from the above by

$$\begin{aligned}
&c\Big(\int_{S_{d_0/2}} r^{-4} \Phi^{1-\delta} \exp(2n\Phi^\delta)|u|^2 dx \\
&\quad + \int_{S_{d_0/2}} \Phi^{1-\delta} \exp(2n\Phi^\delta)|\nabla u|^2 dx.
\end{aligned}$$

Thus combining this inequality with Proposition 5.2, we obtain

$$\begin{aligned}
&cn^{-3}\int_{S_{d_0}-S_{d_0/2}} r^4 \Phi^{1-\delta} \exp(2n\Phi^\delta)(Lv)^2 dx \\
&\geqq \int_{S_{d_0/2}} r^{-4} \Phi^{2\delta-3} \exp(2n\Phi^\delta) u^2 dx.
\end{aligned}$$

Hence we complete the proof in the same way as in Theorem 1.

Similarly to Corollary 7.1 we can easily prove the following

Corollary 7.2. *Let u be a solution in $\overline{\Omega}_a$ of*

$$|L_1 L_2 u| \leqq c \sum_{|\alpha| \leqq 3} x_1^{|\alpha|-3} |D^\alpha u|,$$

where L_1 and L_2 are of the form (1.1) whose coefficients are sufficiently smooth. Then there is a positive number δ depending only on L_1 and L_2 such that if for $|\alpha| \leqq 3$

$$D^\alpha u = o(\exp(-r^{-\delta})) \qquad (r \to 0) \text{ in } S_{1/2} \cap \Omega_a,$$

then u vanishes identically.

References

[1] Aronszajn, N., A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order, J. Math. Pures Appl. **36** (1957), 235-249.

[2] Bernstein, I. S., On the unique continuation problem of elliptic partial differential equations, J. Math. & Mech. **10** (1961), 579-606.

[3] Calderôn, A. P., Uniqueness in the Cauchy problem for partial differential equations, Amer. J. Math. **80** (1958), 16-36.

[4] Carleman, T , Sur un problême d'unicité pour les systêmes d'équations aux dériveés partiells à deux variables indêpendants, Arkiv Mat. **26** B (1938), 1-9

[5] Cordes, H. O., Über die Bestimmtheit der Lösungen elliptischer Differentialgleichungen durch Anfangsvorgaben, Nachr. Akad. Wiss. Göttingen, **11** (1956), 239-258.

[6] Douglis, A., On uniqueness in Cauchy problems for elliptic systems of equations, Comm. Pure Appl. Math. **13** (1960), 593-607.

[7] Hartman, P. and A. Wintner, On the local behavior of solutions of non-parabolic partial differential equations, III. Approximations by spherical harmonics, Amer. J. Math. **77** (1955), 453-483.

[8] Hayashida, K., On the uniqueness in Cauchy's problem for elliptic equations, RIMS Kyoto Univ. Ser. A, **2** (1967), 429-449.

[9] Heinz. E., Über die Eindeutigkeit beim Cauchyschen Anfangswert-problem einer elliptischen Differentialgleichung zweiter Ordnung, Nachr. Akad. Wiss. Göttingen, **1** (1955), 1-12.

[10] Hörmander, L., On the uniqueness of the Cauchy problem I-II, Math. Scand. **6** (1958), 213-225; **7** (1959), 177-190.

[11] Kumano-Go, H., Unique continuation for elliptic equations, Osaka Math. J. **15** (1963), 151-172.

[12] Landis, E. M., On some properties of elliptic equations, Dokl. Akad. Nauk SSSR, **107** (1956), 640-643 (Russian).

[13] Lavrentév, M. M., On Cauchy's boundary value problem for linear elliptic equations of the second order, Dokl. Akad. Nauk SSSR, **112** (1957), 195-197 (Russian).

[14] Mergelyan, S. N., Harmonic approximation and approximate solution of Cauchy's problem for Laplace equation, Dokl. Akad. Nauk SSSR, **107** (1956), 644-647 (Russian).

[15] Mizohata, S., Unicité du prolongement des solutions des équation elliptiques du quatriême ordre, Proc. Japan Acad. **34** (1958), 687-692.

[16] Müller, C., On the behavior of solutions of the differential equations $\Delta U=F(x, U)$ in the neighborhood of a point, Comm. Pure Appl. Math. **7** (1954), 505-516.

[17] Pederson, R. N., On the unique continuation theorem for certain second and fourth order elliptic equations, Comm. Pure Appl. Math. **9** (1958), 67-80.

[18] Protter, M H., Unique continuation for elliptic equations, Trans. Amer. Math. Soc. **95** (1960), 81-91.

[19] Shirota, T., A remark on the unique continuation theorem for certain fourth order elliptic equations, Proc. Japan Acad. **36** (1960), 571-573.