Publ. RIMS, Kyoto Univ.
Vol. 6 (1970), 113-187

Holomorphic Extension Problem for Standard Real Submanifolds of Second Kind

By

Isao NARUKI

Contents

Received January 16, 1970.

Introduction

Recently real submanifolds of complex manifolds have been studied by several complex analysts, differential geometers. Ever since Lewy [5] studied the holomorphic extension problem for pseudo-convex hypersurfaces in C^2, Lewy [6], Wells [12], [13], Greenfield [2] obtained many results on the hull of holomorphy of lower-dimensional real submanifolds in complex manifolds. On the other hand Tanaka [9], [10], [11] studied the pseudo-conformal equivalence between real submanifolds as an example of geometry of differential systems and at the same time he introduced in a group-theoretic manner an ideal class of real submanifolds in complex vector spaces which are called 'standard real submanifolds'. In this paper we attempt to study the hulls of holomorphy for a special kind of standard real submanifolds, i.e., those of the second kind and further to solve the holomorphic extension problem for those real submanifolds. We shall now give a precise formulation of the holomorphic extension problem.

Let M be a real submanifold of a complex manifold $\tilde{M}$ and $T^C(M)$, $T^C(\tilde{M})$ the complexifications of the tangent bundles $T(M)$, $T(\tilde{M})$ of M, $\tilde{M}$ respectively. For any $p\in M$, $T_p^C(M)$ is a subspace of $T_p^C(\tilde{M})$ and we shall denote by $\mathcal{S}_p$ the intersection of $T_p^C(M)$ with $T_p^{(0,1)}(\tilde{M})$, that is,

$$\mathcal{S}_p = T_p^C(M) \cap T_p^{(0,1)}(M) .$$

Here we have set $T_p^{(0,1)}(\tilde{M}) = \{x+iIx;\ x\in T_p(\tilde{M})\}$. ($I$ is the automorphism of $T_p(\tilde{M})$ defining the complex structure of $T_p(\tilde{M})$.) We assume that dim $\mathcal{S}_p$ does not depend on p. Then we obtain a unique vector bundle $\mathcal{S}$ whose fiber at each p of M is $\mathcal{S}_p$. This subbundle $\mathcal{S}$ of $T^C(M)$ is called the *tangential Cauchy-Riemann bundle* of M. A continuous function f on an open subset U of M is called $\mathcal{S}$-holomorphic in U if it is a (distribution) solution of the following differential equation

$$\forall X \in \Gamma_U(\mathcal{S}):\ Xf=0 .$$

This equation is called the tangential Cauchy-Riemann equation of M. From definition of $\mathcal{S}$ it follows immediately that the restriction of a holomorphic function to the intersection of its domain with M is always $\mathcal{S}$-holomorphic. The converse is, however, not true in general. Thus it is an interesting problem to examine under various condition on M to what extent an S-holomorphic function can be continued into a holomorphic function. More precisely it may be formulated as the following

Problem H. *Let M, $\widetilde{M}$ be as above. Given an open subset U of M, find an open subset $\widetilde{U}$ of $\widetilde{M}$, whose closure contains U with the property that one can extend every $\mathcal{S}$-holomorphic function in U to a continuous function v in $U \cup \widetilde{U}$ so that v is holomorphic in $\widetilde{U}$.*

This is the holomorphic extension problem. The investigation of this may be divided into the following two steps. The first is to show that a holomorphic function u defined near U can be holomorphically continued to a neighbourhood of $U \cup \widetilde{U}$ and that this $\widetilde{U}$ can be chosen at once for all u. The next is to show that in the set of $\mathcal{S}$-holomorphic functions on U there densely exist restrictions to U of holomorphic functions near U. The first step may be done by the method of Wells and Greenfield, but we do not utilize this since a much simpler approach is possible when M is a standard real submanifold. (However it should be noted that our procedure is essentially based on the same idea as theirs, i.e., the analytic discs.) As opposed to the generality of the results obtained in [13], [2], we were able to make a fitting choice of U because the situation is simple in our present case. In fact, for a standard real submanifold M of second kind in a complex vector space N we can canonically assign a domain D of N whose closure contains M in such a way that, when M is the Šilov boundary of a non degenerate Siegel domain of the second kind, the corresponding domain is nothing but the Siegel domain itself, moreover, in this case, the above $\widetilde{U}$ can be taken so that $U \cup \widetilde{U}$ is an open subset of $M \cup D$. As for the second step our discussion here is goup-theoretic and is based on an approximation theorem which is an analogue of the famous theorem of Harish-Chandra, Nelson and Gårding in the representation theory, but our theorem is very special and much more complicated. In the near future the method we exploit will be developed into a theory of differential equations admitting a Lie transformation group and is also interesting in its own right.

Among standard real submanifolds of second kind two classes of them are of particular importance, one is the class of so called 'totally indefinite' ones and the other is the class of Šilov boundaries of non degenerate Siegel domains of the second kind. One of the most remarkable facts concerning totally indefinite standard real submanifolds is that every $\mathcal{S}$-holomorphic function on such a submanifold is the restriction of an entire holomorphic function in the ambient complex vector space, moreover it is also locally true, that is, every $\mathcal{S}$-holomorphic functon on an open subset of the real submanifold can be extended into a unique holomorphic function in a fixed open subset of the ambient space containing the domain of the original function. As a result we obtained in this case a hypoellipticity theorem in the real analytic category on the tangential Cauchy-Riemann equation which is, however, not elliptic. Further it is also proved that the space of $\mathcal{S}$-holomorphic functions on an open subset of a totally indefinite standard real submanifold is a Montel space, so that the finiteness theorem is obtained for $\mathcal{S}$-holomorphic vector bundles over compact locally standard real submanifolds. However, this theorem holds for a much wider class of real submanifolds [7]. The method of [7] is based on an estimate of Hörmander used in connection with the theory of hypoelliptic second order differential equations [3]. But the procedure used here leads us to a conjecture on Problem H for arbitrary totally indefinite real submanifolds. For two integers n, k such that $n\geq 2$, $0\leq k\leq n^2-1$, it is possible to construct a totally indefinite standard real submanifold of real dimension $2n+k$ in C^{n+k}, thus its real codimension being k.

Šilov boundaries of non-degenerate Siegel domains of the second kind are all contained in some wider class of standard real submanifolds, which is dual, in certain sense, for the class of totally indefinite ones.

For this dual class we also have solved Problem H in slight stronger form. However, the meaning of the result is not so clear in this case as in totally indefinite case. As examples of compact locally standard real submanifold of this type, there are the Šilov boundaries of the classical domains $I_{p,q}(p\neq q)$, which are all well-known [4], [8].

Because of the restriction of the argument to standard real submanifolds we have received an important simplification of the method, i.e. the homomorphisms between standard real submanifolds. (This was informed by Tanaka in connection with the problem on the existence of a bounded global $\mathcal{S}$-holomorphic function on standard real submanifolds.) In fact this eliminates the necessity to attack each standard real submanifold, at least as far as concerning our problem, and instead we may only study a special one denoted by D_0 (or M^0). This is the Šilov boundary of certain Siegel domain equivalent to a classical domain and for this reason the treatment is very handy.

Now we shall shortly describe the construction of this paper. In §1 we study general real submanifolds in complex manifolds and define standard real submanifolds. We study also elementary properties of standard real submanifolds of the second kind, for example, a simple method of their construction, homorphism, and so on. Most of the materials of this section are due to [10], [11]. In §2 we investigate the shape of the holomorphic hull of an open subset of D_0. In §3 we study some special kind of differential operators on a trivial $U(n)$-bundle which are invariant under the right (or left) operation of $U(n)$ in order to obtain an approximation theorem for solutions of those differential operators. The results can be directly extended also for right invariant differential operators on an arbitrary $U(n)$-bundle on a C^∞ manifold. However, we do not include this since it is trivial. In §4 we give a convenient parametrization of D_0 in order to apply the result of §3 to D_0. Further combining the result of §2 we solve Problem H for D_0 in somewhat stronger version. In §5 we extend the result of §4 for almost all standard real submanifolds of the second kind, i.e. the stable ones. First we shall concern ourselves with totally indefinite ones and mention the conjecture indicated already. Next we study the dual class of standard real submanifolds, and as a result we solve Problem H for Šilov boundaries of non-degenerate Siegel domains of the second kind.

Here we shall give some notational conventions frequently used. For two subsets A, B of a vector space W we denote the set $\{a+b; a\in A, b\in B\}$ by $A+B$. For two vector spaces V and W we shall regard, in canonical way, V and W as subspaces of the direct sum $V+W$. If we wish to distinguish between elements of V and those of W, $V+W$ will be considered to be also the direct product $V\times W$, and then (x,y) means $x+y$ when x, y lie in V, W respectively. The letter I will mostly be used to denote the linear automorphism of a complex vector space sending each element to that multiplied by $i=\sqrt{-1}$. If A is an endomorphism of a (real or complex) vector space V, $D_V(A)$ denotes the determinant of A. When A maps a subspace W of V into itself, then $D_W(A)$ means the determinant of $A|_W$. We also denote the determinant of a matrix A by $D(A)$. For a vector space V, I_V is the identity map of V while I_n is the identity matrix of type (n,n). In §§3~4 we use capital roman letter to denote matrices and capital gothic letters to denote matrix groups. In other sections Lie groups are usually denoted by capital roman letters. For two subsets A, B of a topological space Λ, to say that A is an open (resp. closed) subset of B means that A is open (resp. closed) in the relative topology of B induced from Λ. Unless smoothness is stated explicitly, we assume the differentiability of class C^∞. For a (real or complex) vector bundle E over a manifold M we shall denote by $\underline{E}$, (resp. $\underline{\underline{E}}$) the sheaves of germs of C^∞-sections (resp. continuous sections). For $p\in M$, E_p denotes the fiber of E over p. Similarly S_p is a stalk of S over p if S is a sheaf over M. $\Gamma_\Omega(S)$ is the set of sections over Ω and $\Gamma(S)$ denotes the set of global sections of S. For simplicity we use $\Gamma_\Omega(E)$, $\Gamma(E)$ instead of $\Gamma_\Omega(\underline{E})$, $\Gamma(\underline{E})$. For a complex manifold X, $H(X)$ denotes the set of holomorphic functions on X. For a relatively compact subset S of X we shall denote by $\widehat{S}_X$ the subset $\{x\in X\colon |f(x)|\leq\sup_{z\in S}|f(z)|\ f\in H(X)\}$ of X. Here the author apologizes to the reader for the disharmony of notations which arises from the comparative independence of each section. In fact it might be said that this paper grew out of many small papers.

In this occasion the author thanks to Prof. Tanaka for suggesting the idea of homomorphism and a number of stimulating conversations. He also thanks to Prof. Matsuura for his constant encouragement and critical advices.

§1. Standard Real Submanifolds

1.1. Definition of Standard Real Submanifolds. To motivate our discussion we shall begin with the study of general real submanifolds in complex manifolds.

Let M, $\widetilde{M}$, $\mathcal{S}$ be as in the introduction and define a distribution (in the sense of Chevalley) D on M setting

$$D_p=\{\mathrm{Re}\,x\colon\ x\in\mathcal{S}_p\}$$

where $\mathrm{Re}\,x$ denotes the real part of x. Assume that there exists a series of distributions $D=D^1\subseteq D^2\subset,\cdots,D^\mu=T(M)$ such that

$$\underline{D}^i=[\underline{D}^{i-1},\underline{D}]+\underline{D}^{i-1}\qquad(i\geq2)$$

where $\underline{D}^k$ denotes the sheaf of germs of smooth sections D^k. Then

$$[\underline{D}^j;\underline{D}^k]\subseteq\underline{D}^{j+k}.\tag{1}$$

Let $\mathfrak{g}^i$ denote the quotient bundle of D^i by D^{i-1} $(i\geq2)$ and set $\mathfrak{g}^1=D^1$, $\mathfrak{m}=\mathfrak{g}^1\oplus\mathfrak{g}^2\oplus\cdots\oplus\mathfrak{g}^\mu$. Then the bracket operation induces a Lie algebra structure of $\Gamma(\mathfrak{m})$, such that

$$[\underline{\mathfrak{g}}^j,\underline{\mathfrak{g}}^k]\subseteq\underline{\mathfrak{g}}^{j+k}$$

where we have put $\mathfrak{g}^k=0$ if $k>\mu$.

One can easily show that

$$[fX,gY]=fg[X,Y]\qquad f,g\in C^\infty(M)\qquad X,Y\in\Gamma(\mathfrak{m}).$$

Thus this Lie algebra structure is defined pointwisely, i.e., $\mathfrak{m}_p=\mathfrak{g}_p^1+\cdots+\mathfrak{g}_p^\mu$ $(p\in M)$ has also the canonical Lie algebra structure such that

$$[X_p, Y_p] = [X, Y]_p \quad X, Y \in \Gamma(\mathfrak{m}).$$

The properties of $\mathfrak{m}_p$ are the followings

(i) $\mathfrak{m}_p$ is generated by $\mathfrak{g}_p^1$

(ii) $[\mathfrak{g}_p^j, \mathfrak{g}_p^k] \subseteq \mathfrak{g}_p^{j+k}$

(iii) $[\mathrm{Re}\, x, \mathrm{Re}\, y] = [\mathrm{Re}\,(ix), \mathrm{Re}\,(iy)] \quad x, y \in S_p$.

Let I be the linear isomorphism $\mathrm{Re}\, x \to \mathrm{Re}\, -ix$ of D_p onto itself, then (iii) is equivalent to

(iii)' $[x, y] = [Ix, Iy] \quad x, y \in D_p$.

For $p \in M$ $\mathfrak{m}_p$ is called the *Levi-Tanaka algebra* of M at p. We assume that for $(p, q) \in M \times M$ there exists a Lie algebra isomorphism σ of $\mathfrak{m}_p$ onto $\mathfrak{m}_q$ such that $\sigma(\mathfrak{g}_p^i) = \mathfrak{g}_q^i$ and

$$\sigma(Ix) = I\sigma(x) \quad x \in \mathfrak{g}_p^1.$$

A real submanifold satisfying all hypotheses assumed so far is called a *strongly regular* submanifold.

From now on we shall discuss only strongly regular submanifolds. Now a question arises: Suppose that $\mathfrak{m} = \mathfrak{g}_1 + \cdots + \mathfrak{g}_\mu$ where $\mathfrak{g}_1$ is a complex vector space, is a finite-dimensional Lie algebra satisfying following properties

(i) $\mathfrak{m}$ is generated by $\mathfrak{g}_1$

(ii) $[\mathfrak{g}_j, \mathfrak{g}_k] = \mathfrak{g}_{j+k} \qquad (\mathfrak{g}_l = 0 \text{ if } l > \mu)$

(iii) $[Ix, Iy] = [x, y] \qquad x, y \in \mathfrak{g}_1$

where I is the linear isomorphism of $\mathfrak{g}_1$ defining the complex structure of $\mathfrak{g}_1$. Is there a strongly regular real submanifold M in certain complex manifold such that the Levi-Tanaka algebra at each point p of M is isomorphic to $\mathfrak{m}$? This is affirmatively answered in the following way.

Let $\mathfrak{m}^C$ be the complexification $\mathfrak{m} + i\mathfrak{m}$ of $\mathfrak{m}$, M^C the simply connected Lie group whose Lie algebra is $\mathfrak{m}^C$, and exp the exponential map of $\mathfrak{m}^C$ into M^C. Then exp is a holomorphic isomorphism of $\mathfrak{m}^C$ onto M^C since $\mathfrak{m}^C$ is nilpotent. Set

$$S = \{x + iIx;\, x \in \mathfrak{g}_1\}, \quad N = \overline{S} + \mathfrak{g}_2^C + \cdots + \mathfrak{g}_\mu^C.$$

S, $\overline{S}$ are abelian subalgebra of $\mathfrak{m}^C$ and N is an ideal of $\mathfrak{m}^C$. Set further $M'=\exp(\mathfrak{m})$, $H=\exp(S)$, $\boldsymbol{N}=\exp(N)$. M', H are Lie subgroups of M^C. $\boldsymbol{N}$ is a normal subgroup of M^C. Since exp is one to one and $\mathfrak{m}\cap S=(0)$, we obtain

$$M'\cap H=(e).$$

In other words the restriction to M' of the canonical projection π: $M^C\to M^C/H$ is one to one on M'. Then real submanifold $M=\pi(M')$ of complex manifold M^C/H is the desired real submanifold. To see this let σ_p be the isomorphism of $\mathfrak{m}^C$ onto $T_p(M^C)$ defined by the Maurer-Cartan form of M^C. Then the sequence

$$0\to S\xrightarrow{l}\mathfrak{m}^C\xrightarrow{d\pi_p\circ\sigma_p}T_{\pi(p)}(M^C/H)\to 0$$

is exact where l is the inclusion map. Denote by f_p the isomorphism of $\mathfrak{m}^C/S$ onto $T_{\pi(p)}(M^C/H)$ such that the following diagram is commutative:

$$\begin{array}{ccc} \mathfrak{m}^C & \xrightarrow{d\pi_p\circ\sigma_p} & T_{\pi(p)}(M^C/H) \\ \text{canonical}\downarrow & \nearrow f_p & \\ \mathfrak{m}^C/S & & \end{array}$$

Let $\mathfrak{p}$ denote the restriction to $\mathfrak{m}$ of the canonical projection $\mathfrak{m}^C\to\mathfrak{m}^C/S$. $\mathfrak{p}$ is an isomorphism into $\mathfrak{m}^C/S$ since $\mathfrak{m}\cap S=(0)$. For $p\in M'$ we have

$$\mathfrak{p}(\mathfrak{m})=f_p^{-1}(T_{\pi(p)}(M),$$

while we obtain by simple calculation

$$\mathfrak{p}(\mathfrak{m})^C\cap(\mathfrak{m}^C/S)^{(0,1)}=\mathfrak{p}^C(S)$$

where $\mathfrak{p}^C=\mathfrak{p}\otimes\boldsymbol{C}$: $\mathfrak{m}^C\to(\mathfrak{m}^C/S)^C$. (For a complex vector space V we shall denote by $V^{(0,1)}$ the subspace of V^C given by

$$V^{(0,1)}=\{x+iIx\colon x\in V\}$$

where I is the automorphism of V which defines the complex structure of V.)

Therefore we obtain

$$\sigma_p(S)=(d\bar{\pi}_p)^{-1}(\mathcal{S}_{\pi(p)})$$

where $\mathcal{S}$ is the tangential Cauchy-Riemann bundle of M and $\bar{\pi}$ is the restriction to M' of π. This implies that

$$(d\bar{\pi}_p)^{-1}(D_p^i) = \{\sigma_p(x) : x \in \mathfrak{g}_1 + \cdots + \mathfrak{g}_i\}$$

since $\mathfrak{m}$ can be regarded as the Lie algebra of left invariant vector fields M'. Therefore $\mathfrak{m}_p = \sum_{i=1}^{\mu} D_p^i / D_p^{i-1}$ can be canonically identified with $\mathfrak{m} = \sum_{i=1}^{\mu} \mathfrak{g}_i$ and this identification is a Lie algebra isomorphism.

A finite-dimensional Lie algebra $\mathfrak{m}$ with properties (i), (ii), (iii) is called *a fundamental algebra of the μ-th kind* and the real submanifold M constructed above is called *the standard real submanifold corresponding to* $\mathfrak{m}$, which will be denoted by $M(\mathfrak{m})$ hereafter.

Remark. Since $N \cap H = (e)$ the restriction to N of the canonical projection $M^C \to M^C/H$ is a holomorphic isomorphism onto M^C/H, the inverse of which we shall denote by T'. Since $\exp|N$ is a holomorphic isomorphism of N onto N, $T = (\exp)^{-1} \circ T'$ is also a holomorphic isomorphism of M^C/H onto N. Thus, if we identify M^C/H with N by T, M can be considered as a real submanifold of N, further extending the linear isomorphism of $\mathfrak{g}_1$ onto $\bar{S}$ given by $x \to \frac{1}{2}(x - iIx)$ to a linear isomorphism of $\mathfrak{g}_1 + \mathfrak{g}_2^C + \cdots + \mathfrak{g}_\mu^C$ onto N in obvious manner, we shall consider M as a real submanifold of $\mathfrak{g}_1 + \mathfrak{g}_2^C + \cdots + \mathfrak{g}_\mu^C$.

Remark. By $\rho(p)$ we shall denote the canonical left operation on M^C/H of an element $p \in M^C$. Then for $p \in M'$ $\rho(p)$ leaves M invariant. Thus the holomorphic transformation group $\rho(M')$ operates transitively on M. In other words M is homogeneous.

Above arguments show that standard real submanifolds are the most typical and important ones.

1.2. Elementary Study of Standard Real Submanifolds of the Second Kind. From now on we shall concern ourselves only with standard real submanifolds corresponding to fundamental Lie algebra of the second kind, which will be called shortly '2-standard real submanifolds.' First we give a method of construction of all 2-standard real submanifolds.

Let V be an n-dimensional complex vector space and I the linear isomorphism of V onto itself which defines the complex structure of V, i.e., I maps x into $\sqrt{-1}x$ for $x\in V$, let $H(V)$ denote the vector space of hermitian forms on V^*, i.e.,

$$H(V)=\{\alpha\in S^2_{\boldsymbol{R}}(V)\,;\ I\otimes I(\alpha)=\alpha\}$$

where $S^2_{\boldsymbol{R}}(V)$ denotes the vector space of symmetric elements of $V\underset{\boldsymbol{R}}{\otimes}V$. Further let π be a linear mapping of $H(V)$ onto a vector space W and set $\mathfrak{g}_1=V$, $\mathfrak{g}_2=W$, $\mathfrak{m}(\pi)=\mathfrak{g}_1+\mathfrak{g}_2$. Then we can define the bracket operation [,] on $\mathfrak{m}(\pi)\times\mathfrak{m}(\pi)$ so that $\mathfrak{m}(\pi)$ is a fundamental Lie algebra of second kind. For this we must set

$$[\mathfrak{g}_1,\mathfrak{g}_2]=[\mathfrak{g}_2,\mathfrak{g}_2]=(0)\,.$$

As to [,] on $\mathfrak{g}_1\times\mathfrak{g}_1$ we set

$$[x,y]=\frac{1}{4}\pi(x\otimes Iy-y\otimes Ix-Ix\otimes y+Iy\otimes x)\qquad x,y\in\mathfrak{g}_1\,.$$

It is easily checked that $\mathfrak{m}(\pi)$ satisfies properties (i), (ii), (iii). We shall denote by $M(\pi)$ the standard real submanifold of $\mathfrak{g}_1+\mathfrak{g}_2^{\boldsymbol{C}}$ corresponding to $\mathfrak{m}(\pi)$. By direct computation we have

$$M(\pi)=\{(x,y-\tfrac{1}{4}i[x,Ix])\in\mathfrak{g}_1+\mathfrak{g}_2^{\boldsymbol{C}}\colon\ x\in\mathfrak{g}_1,\ y\in\mathfrak{g}_2\}\,.$$

Here we write (x,z) for $x+z$, regarding $\mathfrak{g}_1+\mathfrak{g}_2^{\boldsymbol{C}}$ as $\mathfrak{g}_1\times\mathfrak{g}_2^{\boldsymbol{C}}$.

We can obtain any 2-standard submanifold in this way. To see this let $\mathfrak{m}=\mathfrak{g}_1+\mathfrak{g}_2$ be a fundamental Lie algebra of the second kind and set $V=\mathfrak{g}_1$, $W=\mathfrak{g}_2$. We define a linear mapping ϕ of W^* into $H(V^*)(=H(V)^*)$ setting

$$\phi(\alpha)(x,y)=\alpha([x,Iy])\,.$$

Then ϕ is one to one because of property (i) of $\mathfrak{m}$. Hence $\pi={}^t\phi$ is a linear mapping of $H(V)$ onto W, and we have $\mathfrak{m}=\mathfrak{m}(\pi)$. Details of the verification shall be left to the reader.

Now we shall introduce 'homomorphism' between standard real submanifolds, which makes it unnecessary to study all standard real submanifolds. (In fact it suffices to study only special one, namely

$M^0 = M(id_{H(V)})$.

Consider the following commutative diagram

$$\begin{array}{ccc} H(V) & \xrightarrow{\pi} & W \\ \| & & \downarrow \rho \\ H(V) & \xrightarrow{\pi'} & W' \end{array}$$

where π, π', ρ are all surjections. Then the linear mapping $\tilde{\rho} = id_V \oplus \rho^C$ of $N = V \oplus W^c$ onto $N' = V \oplus W'^C$ maps $M(\pi)$ onto $M(\pi')$, $\tilde{\rho}$ is obviously a holomorphic mapping and the restriction $\hat{\rho}$ to $M(\pi)$ of $\tilde{\rho}$ is called homomorphism induced by ρ. Now consider the following diagram:

$$\begin{array}{ccc} H(V) & \xrightarrow{id_{H(V)}} & H(V) \\ \| & & \downarrow \pi \\ H(V) & \xrightarrow{\pi} & W \end{array}$$

Then we obtain homomorphism $\hat{\pi}$ of M^0 onto $M(\pi)$. Thus every 2-standard real submanifold is the homomorphic image of M^0. This fact allows us to deduce many information concerning our problem for $M(\pi)$ from those for M^0. For example, if f is $\mathcal{S}$-holomorphic in an open set U of $M(\pi)$, then $\hat{\pi}^*(f)$ is also $\mathcal{S}$-holomorphic on $\hat{\pi}^{-1}(U)$. This is easily seen from

$$d\hat{\pi}_p(\mathcal{S}_p^0) = \mathcal{S}_{\hat{\pi}(p)}$$

where $\mathcal{S}^0$, $\mathcal{S}$ are the tangential Cauchy-Riemann bundles of M^0, $M(\pi)$ respectively. Conversely if, for a continuous function f on an open subset U of $M(\pi)$, $\hat{\pi}^*(f)$ is $\mathcal{S}$-holomorphc in $\hat{\pi}^{-1}(U)$, then f is also $\mathcal{S}$-holomorphic in U. Further, if f is the restriction of a holomorphic function in a neighbourhood of U, $\hat{\pi}^*(f)$ is also the restriction of a holomorphic function in a neighbourhood of $\hat{\pi}^{-1}(U)$. But it is not quite evident that f is the restriction of a holomorphic function defined near U whenever $\hat{\pi}^*(f)$ is the restriction of a holomorphic function defined near $\hat{\pi}^{-1}(U)$. Fortunately it is true. This follows from

Proposition 1.2.1. *Let $\mathcal{S}$ be the tangential Cauchy-Riemann* *bundle of a real submanifold M of a complex manifold M (assuming that S is well-defined). Suppose that (2) complex fiber dimension of $\mathcal{S}$+complex dimension of $\widetilde{M}$=real dimension of M. Then every holomorphic function u on a domain $\widetilde{U}$ of $\widetilde{M}$ such that $U\cap \widetilde{M}\neq\phi$ and $u|_{\widetilde{U}\cap M}=0$ must vanish identically in U.*

Proof. Condition (2) implies that for $p\in\widetilde{M}$ $T_p(\widetilde{M})$ is the unique complex subspace of itself containing $T_p(M)$. To show this let p_1 be the projection of $T_p^C(\widetilde{M})$ onto $T_p^{(1,0)}(\widetilde{M})$ in the direct sum $T_p^C(\widetilde{M}) = T_p^{(1,0)}(\widetilde{M})\oplus T_p^{(0,1)}(\widetilde{M})$ and $\check{p}_1$ the restriction to $T_p^C(M)$ of p_1. Then condition (2) implies that

$$\begin{aligned}&\text{dimension of } T_p^C(M)-\text{dimension of Kernel of } \check{p}_1\\ &=\text{dimension of } T_p^{(0,1)}(\widetilde{M})\end{aligned}$$

since $\mathcal{S}_p$=Kernel of $\check{p}_1$. Thus $\check{p}_1$ is onto; in other words $T_p^C(M) + T_p^{(0,1)}(\widetilde{M}) = T_p^C(\widetilde{M})$. Now let L be a complex subspace of $T_p(\widetilde{M})$ containing $T_p(M)$, then we have

$$\begin{aligned}&T_p^C(M)\subseteq L^C\\ &L=L^{(0,1)}\oplus L^{(0,1)}\end{aligned}$$

where $L^{(0,1)}=L^C\cap T_p^{(0,1)}(\widetilde{M})$, $L^{(0,1)}=L^C\cap T_p^{(0,1)}(\widetilde{M})$. Therefore

$$\begin{aligned}T_p^C(\widetilde{M}) &= T_p^C(M)+T_p^{(0,1)}(\widetilde{M}) = L^{(1.0)}+L^{(0,1)}+T_p^{(0,1)}(\widetilde{M})\\ &= L^{(1,0)}+T_p^{(0,1)}(\widetilde{M})\end{aligned}$$

which implies that $L^{(1,0)}=T_p^{(1,0)}(\widetilde{M})$. Consequently we have

$$L=T_p(\widetilde{M}).$$

Our assertion is proved.

Now let u be a holomorphic function in a domain $\widetilde{U}$ of $\widetilde{M}$ such that $\widetilde{U}\cap M\neq\phi$ and $u|_{\widetilde{U}\cap M}=0$. Suppose that u does not vanish identically. Then we obtain a series of proper analytic sets $V_1, V_2, \cdots$ of $\widetilde{U}$ such that $V_1=\{z\in\widetilde{U}: u(z)=0\}$, V_k is the set of singular points of V_{k-1} $(k\geq 2)$. Since $\dim V_k<\dim V_{k-1}$ and since $V_1\supseteq M\cap\widetilde{U}$, it holds for some i that

$$V_i\supseteq M\cap\widetilde{U},\ (V_i\setminus V_{i+1})\cap(M\cap\widetilde{U})\neq\phi.$$

(For $V_k \supseteq M \cap \widetilde{U}$, $(V_k \setminus V_{k+1}) \cap (M \cap \widetilde{U}) = \phi$ imply $V_{k+1} \supset M \cap \widetilde{U}$.) Choose

$$p \in (V_i \setminus V_{i+1}) \cap (M \cap \widetilde{U}).$$

Then

$$T_p(V_i) \supseteq T_p(M) \quad \text{since} \quad V_i \supseteq (M \cap \widetilde{U}).$$

As we have showed above this implies $T_p(V_i) = T_p(\widetilde{M})$ i.e., $\dim V_i = \dim \widetilde{M}$ which contradicts the properness of V_i. Thus $u=0$ is proved.

Q.E.D.

Let us now study the aspect of homomorphisms more in detail. Let $M'(\mathfrak{m}^0)$, $M'(\pi)$ be the simply connected Lie groups corresponding to $\mathfrak{m}^0(=\mathfrak{m}(id_{H(V)}))$, $\mathfrak{m}(\pi)$ respectively. Then the linear map $id_V \oplus \pi$ of $\mathfrak{m}^0$ onto $\mathfrak{m}(\pi)$ is a Lie algebra homomorphism. Let $\check{\pi}$ be the homomorphism of $M'(\mathfrak{m}_0)$ onto $M(\pi)$ induced by $id_V \oplus \pi$, the Lie algebra of the kernel $K(\pi)$ of $\check{\pi}$ is the kernel of π when one regards $H(V)$ as a subspace of $\mathfrak{m}^0 = V + H(V)$. By direct inspection we obtain

$$\begin{array}{ccccccc} 0 \to K(\pi) \to & M'(\mathfrak{m}^0) & \xrightarrow{\check{\pi}} & M'(\pi) & \to 0 \\ & \downarrow & & \downarrow & \\ & M^0 & \xrightarrow{\hat{\pi}} & M(\pi) & \end{array}$$

where the vertical arrows denote the canonical identifications. The first horizontal sequence is exact. On the other hand we can easily see that the transformation $\rho(\exp x)$ $(x \in H(V)^C)$ is the translation by x in $N^0 = V + H(V)^C$. Thus we obtain

$$\hat{\pi}^{-1}(\hat{\pi}(p)) = p + \operatorname{Ker} \pi \qquad p \in M^0$$

while

$$\check{\pi}^{-1}(\check{\pi}(p)) = p + (\operatorname{Ker} \pi)^C \qquad p \in M^0.$$

Remark. In the next section, by identifying $M'(\mathfrak{m}^0)$ with M^0 in canonical manner, we often regard M^0 as a group manifold. Then the left operation L_p of $p(p \in M^0)$ can be uniquely extended to an affine transformation $\rho(p')$ of N^0 when p' is identified with p under the identification.

Now let us prove our assertion stated before Proposition 1.2.1. Let U be a domain of $M(\pi)$ and f a continuous function on U such that $\hat{\pi}^*(f)$ is the restriction of a holomorphic function u' on a domain U' in N^0 containing $\hat{\pi}^{-1}(U)$. Let $\tau_h (h\in H(V))$ denote the translation $x\to x+h$ $(x\in N^0)$ and let ∂_h be the vector field which generates 1-parameter transformation group $(\tau_{th})_{t\in \boldsymbol{R}}$. Note $(\partial_h u')|_{\hat{\pi}^{-1}(S)}=\partial_h\hat{\pi}^*(f)=0$ for $h\in \mathrm{Ker}\,\pi$. Thus Proposition 1.2.1 implies

$$\partial_h u'=0 \qquad h\in \mathrm{Ker}\,\pi .$$

Note that u' is holomorphic. Hence it follows also

$$\partial_{ih} u'=0 \qquad h\in \mathrm{Ker}\,\pi .$$

Thus u' is constant on each component of the intersection of linear variety $p+(\mathrm{Ker}\,\pi)^C$ with U' for each $p\in U'$. This implies that, for any continuous map d of $V+\pi(H(V))^C$ into N^0 such that $\tilde{\pi}\circ d$ is the identity, the function $u=u'\circ d$ on $d^{-}(U')$ is holomorphic. If, moreover, $d(M(\pi))\subseteq M^0$, then $u|_U=f$. Thus the only task is to give such a map d. But this can be obtained in the following way. Let σ be a linear map of $W=\pi(H(V))$ into $H(V)$ such that $\pi\circ\sigma=id_W$. Set

$$d(x, y)=(x, \sigma^C(y+\tfrac{1}{4}i[x, Ix])-\tfrac{1}{4}i[x, Ix]_0) \qquad x\in V,\ y\in W^C$$

where $[\ ,\ ]$, $[\ ,\ ]_0$ are the bracket operations of $\mathfrak{m}(\pi)$ $\mathfrak{m}_0$, respectively. It is evident from $[\ ,\ ]=\pi[\ ,\ ]_0$ that d is the map with the desired properties. Q.E.D.

In the next section, instead of M^0, we shall study an equivalent real submanifold D_0 which is the Šilow boundary of certain Siegel domain of the second kind. The equivalence of D_0 with M^0, though almost obvious, will be firstly showed in §5.

§2. Holomorphic Hulls for Open Subsets of D_0

This section is divided into two parts 2.1 and 2.2. The first half consists of long, rather tedious but important preparation for the second half. For this reason the motivation of the first half is not clear for the readers who want to know the meanings of notations, lemmas, propositions and so on. Here we indicate only that all of these are made to be fitted for the application to D_0, D introduced at the beginning of the second half.

2.1. Basic Continuation Theorem. Fix an n-dimensional complex Hilbert space V with the inner product $(\ ,\ )$ and denote by $\mathcal{G}$ the vector space of linear operators of V into itself. For $A\in\mathcal{G}$ we shall denote by A^* its adjoint, i.e., $(Ax, y)=(x, A^*y)$ $x, y\in V$. As usual we call A hermitian if $A=A^*$. For a hermitian operator A, $A>0$ (resp. $A\geq 0$) means that $(Ax, x)>0$ (res. $(Ax, x)\geq 0$) for $0\neq x\in V$. For $y\in V$ the mapping $x\to(x, y)$ is a linear form on V, which we denote by y^*. Then $x\otimes y^*$ can be regarded as an element of $\mathcal{G}$ since $V\otimes V^*$ can be canonical identified with $\mathcal{G}$. Further we denote by $\mathfrak{l}$ the mapping of $\mathcal{G}\times V$ into $\mathcal{G}$ defined by

$$\mathfrak{l}(A, x)=\frac{1}{i}(A-A^*)-x\otimes x^* \qquad A\in\mathcal{G};\ x\in V.$$

$\mathfrak{l}(A, x)$ is obviously hermitian. Let S_1 denote the unit sphere of V and Δ the unit disc in C, i.e.,

$$S_1=\{x\in V\colon \|x\|=1\}$$
$$\Delta=\{\zeta\in \boldsymbol{C};\ |\zeta|\leq 1\}$$

where $\|x\|^1=(x, x)$. We shall shortly denote by $\mathcal{E}$ the product space $\mathcal{G}\times V\times S_1\times(-\infty, \infty)$ and by $\mathcal{E}^0$ the open, dense subset $\{(A, x, y, r);\ (\mathfrak{l}(A, x)y, y)\neq 0\}$ of $\mathcal{E}$. For later purposes we shall regard A, x, y, r as the projections of $\mathcal{E}$ onto $\mathcal{G}$, V, S_1, $(-\infty, \infty)$ respectively, i.e., $\varepsilon=(A(\varepsilon), x(\varepsilon), y(\varepsilon), r(\varepsilon))$ for $\varepsilon\in\mathcal{E}$, and further introduce some convenient notations setting

$$\left.\begin{aligned}\pi(\varepsilon)&=(A(\varepsilon), x(\varepsilon))\\ \tilde{y}(\varepsilon)&=\mathfrak{l}(\pi(\varepsilon))y(\varepsilon)\end{aligned}\right.\qquad \varepsilon\in\mathcal{E}.$$

Define a continuous function λ on $\mathcal{E}^0$ and a continuous mapping σ of $\mathcal{E}^0\times \boldsymbol{C}$ into $\mathcal{G}\times V$ setting

$$\begin{aligned}\lambda(\varepsilon)&=(|(\mathfrak{l}(\pi(\varepsilon)y(\varepsilon), y(\varepsilon))|^{-1}+r(\varepsilon)^2)^{1/2}\\ &=(|(\tilde{y}(\varepsilon), y(\varepsilon))|^{-1}+r(\varepsilon)^2)^{1/2}\end{aligned}$$

$$\sigma(\varepsilon,\zeta)=(A(\varepsilon)+i(\lambda(\varepsilon)\zeta-r(\varepsilon))y(\varepsilon)\otimes(x(\varepsilon)-r(\varepsilon)y(\varepsilon))^*,$$
$$x(\varepsilon)+(\lambda(\varepsilon)\zeta-r(\varepsilon))y(\varepsilon)).$$

Then σ has the following properties.

(1) σ_ε is a holomorphic mapping of $\boldsymbol{C}$ into $\mathcal{G}\times V$ where we denote by σ_ε the mapping $\Delta\ni\zeta\to\sigma_\varepsilon(\zeta)=\sigma(\varepsilon,\zeta)$.

(2) $$\pi(\varepsilon)\in\sigma_\varepsilon(\Delta).$$

In fact $\pi(\varepsilon)=\sigma_\varepsilon\left(\frac{r(\varepsilon)}{\lambda(\varepsilon)}\right)$ and $\left|\frac{r(\varepsilon)}{\lambda(\varepsilon)}\right|<1$. By simple computation we can obtain

(3) $$\mathfrak{l}(\sigma(\varepsilon,\zeta))=\mathfrak{l}(\pi(\varepsilon))$$
$$-(|\zeta|^2|(\hat{y}(\varepsilon),y(\varepsilon))|^{-1}+r(\varepsilon)^2(|\zeta|^2-1))\hat{y}(\varepsilon)\otimes y(\varepsilon)^*.$$

In particular, if $|\zeta|=1$,

(3)′ $$\mathfrak{l}(\sigma(\varepsilon,\zeta))=\mathfrak{l}(\pi(\varepsilon))-|(y(\varepsilon),y(\varepsilon))|^{-1}\tilde{y}(\varepsilon)\otimes y(\varepsilon)^*.$$

Set, for a subset S of $\mathcal{G}\times V$,

$$\widetilde{E}^0S=\{\varepsilon\in\mathcal{E}^0;\ \sigma_\varepsilon(\partial\Delta)\subseteq S\}\qquad E^0S=\pi(\widetilde{E}^0S)$$
$$ES=S\cup E^0S.$$

Then we have

(4) $$S\subset T\Rightarrow ES\subseteq ET\qquad \bigcup_\lambda ES_\lambda\subseteq E(\bigcup_\lambda S_\lambda)$$
$$E(\bigcap_\lambda S_\lambda)\subseteq\bigcap_\lambda ES_\lambda.$$

Further, for a bounded set S,

(5) $$ES\subseteq\hat{S}.$$

Here $\hat{S}$ means $\hat{S}_{\mathcal{G}\times V}$. (5) follows from (1) and (2).

Lemma 2.1.1. *If O is an open set of $\mathcal{E}\times V$, then $E^\circ O$, EO are open sets.*

In fact $\widetilde{E}^\circ O$ is an open set in $\mathcal{E}^\circ$ by the continuity of σ and π is an open mapping $\mathcal{E}$ onto $\mathcal{G}\times V$. Therefore $E^\circ O=\pi(E^\circ O)$ is open.

Now we are in a position to study holomorphic extendibility from S to ES for a subset S satisfying some conditions. Let $\varphi_t(t>0)$ be the holomorphic automorphism of $\mathcal{G}\times V$ defined by $\varphi_t(A, x) = (t^2A, tx)$. A subset S of $\mathcal{G}\times V$ is said to be *φ-star-shaped* if S contains 0 and if $\varphi_t S\subseteq S$ for $0<t<1$. All φ-star-shaped sets are connected. If S is φ-star-shaped, then ES is φ-star-shaped as is evident by the following relation:

$$\varphi_t\circ\sigma(A, x, y, r; \zeta) = \sigma(t^2A, tx, y, t^{-1}r, \sigma).$$

Now the key of our discussion is the following.

Proposition 2.1.1. *Let O be a φ-star-shaped open subset of $\mathcal{G}\times V$. Then, for any $f\in H(O)$, there exists uniquely Ef of $H(EO)$ such that $Ef|_0=f$.*

For the proof of this we need some notation and lemmas. For a holomorphic function f on O, we define a continuous function $\widetilde{E}^\circ f$ on $\widetilde{E}^\circ O$ setting

$$\widetilde{E}^\circ f(\varepsilon) = \int_0^{2\pi} f(\sigma_\varepsilon(e^{i\varphi}))P(\varphi, r(\varepsilon)/\lambda(\varepsilon))d\varphi$$

where $P(\varphi, z)$ is the Poisson kernel for the unit disc in $\boldsymbol{C}$, i.e.,

$$P(\varphi, z) = \frac{1-|z|^2}{2\pi|e^{i\varphi}-z|^2}.$$

Lemma 2.1.2. *Suppose that O is a bounded open set. If f is the restriction to O of a holomorphic function u in a neighbourhood of $\hat{O}$, then*

$$\widetilde{E}^\circ f(\varepsilon) = u(\pi(\varepsilon)).$$

Proof is evident if one observes that, for any $\varepsilon\in E^\circ O$,

$$\sigma_\varepsilon(\Delta)\subseteq\hat{O},$$

so that $f(\sigma_\varepsilon(e^{i\varphi}))$ is the boundary value to $\partial\Delta$ of the holomorphic function $u(\sigma_\varepsilon(\zeta))$ in a neighbourhood of Δ.

To formulate next lemma we need a new notion: Let Ω be a C^∞ manifold, $\mathcal{D}'(\Omega)$ the space of distributions on Ω. A mapping $t\to f(t)$ from an open interval (a, b) into $\mathcal{D}'(\Omega)$ is called an *analytic $\mathcal{D}'(\Omega)$-valued function* on (a, b) if, for any C^∞ density ψ with compact support, the function $\langle f(t), \psi\rangle$ is (real) analytic in (a, b).

Lemma 2.1.3. *Let $P(D)$ be a scalar differential operator on Ω and $f(t)$ be an analytic $\mathcal{D}'(\Omega)$-valued function on (a,b). Suppose $P(D)f(t)=0$ for t in a non-empty open subset of (a,b). Then $P(D)f(t)=0$ for any $t\in(a,b)$.*

Proof. Obvious.

Now define a function space on $E^\circ O$ setting

$$\mathcal{L}=\{f\in C^\infty(\widetilde{E}^\circ)O;\ ^\exists f'\in H(E^\circ O)^\forall \varepsilon\in\widetilde{E}^\circ O f'(\pi(\varepsilon))=f(\varepsilon)\}.$$

Lemma 2.1.4. *Let $f(t)$ be a continuous $\mathcal{D}^1(\widetilde{E}^\circ O)$-valued function on $(0,1]$ a nalytic in $(0,1]$ such that, for t in a non-empty open subset of $(0,1)$, $f(t)\in\mathcal{L}$. Then $f(1)\in\mathcal{L}$.*

Proof. This follows immediately from Lemma 6 since there exist scalar differential operators $P_r(D)$ $(r=1,2,\cdots,m)$ on $E^\circ O$ such that $\mathcal{L}$ is the set of solutions in $\mathcal{D}'(\widetilde{E}^\circ O)$ of the overdetermined elliptic system of differential equations

$$P_1(D)v=0,\cdots,\quad P_m(D)v=0\,.$$

The construction of such $P_r(D)$ is easy and shall be left to the reader.

Now we are ready to prove Proposition 2.1.1.

Proof of Proposition 2.1.1. Any φ-star-shaped open set O can be represented as the union of an increasing sequence of φ-star-shaped bounded open sets $O_1, O_2, \cdots$. It follows easily

$$EO=\bigcap_j EO_j$$

and each EO_j is connected as was remarked before. Therefore we may assume that O is bounded. For $z\in\boldsymbol{C}$, we denote by φ_z the holomorphic mapping of $\mathcal{G}\times V$ into itself

$$(A,x)\rightarrow(z^2A, zx).$$

Let $\mathfrak{d}_\varepsilon$ denote the open set in $\boldsymbol{C}$ given by

$$\mathfrak{d}_\varepsilon=\{z\in\boldsymbol{C}:\ \varphi_z\circ\sigma_\varepsilon(\partial\Delta)\subseteq O\}$$

and let ψ be a C^∞-function on $E^\circ O$ with compact support K. Then each $\mathfrak{d}_\varepsilon(\varepsilon\in K)$ contains a fixed neighbourhood $\mathfrak{d}_K$ of the $[0,1]$ since $\mathfrak{d}_\varepsilon$

depends continuously on ε and since $[0,1]\subseteq \mathfrak{d}_\varepsilon(\varepsilon\in\widetilde{E}^\circ O)$. Define a continuous function F' on $\mathfrak{d}_K\times\widetilde{E}^\circ O\times\partial\Delta$

$$F'(z,\varepsilon,e^{i\varphi})=\psi(\varepsilon)f\circ\varphi_z\circ\sigma_\varepsilon(e^{i\varphi})P(\varphi\cdot r(\varepsilon)/\lambda(\varepsilon)).$$

Then F' is obviously holomorphic with respect to z in $\mathfrak{d}_K$. Therefore, for a fixed volume $d\varepsilon$ on $\widetilde{E}^\circ O$,

$$\int_{\widetilde{E}^\circ O}F(z,\varepsilon)\psi(\varepsilon)d\varepsilon$$

is also analytic in $\mathfrak{d}_K$ when one puts

$$F(z,\varepsilon)=\int_0^{2\pi}f^\circ\psi_z\circ\sigma_\varepsilon(e^{i\varphi})P(\varphi,r(\varepsilon)/\lambda(\varepsilon))d\varphi$$

for (z,ε) such that $z\in\mathfrak{d}_\varepsilon$. In particular

$$\int_{\widetilde{E}^\circ O}F(t,\varepsilon)\psi(\varepsilon)d\varepsilon$$

is real analytic with respect to t in $(0,1)$, thus the continuous $\mathscr{D}'(E^\circ O)$-valued function

$$t\to F(t,\cdot)$$

on $[0,1]$ is analytic in $(0,1)$. Therefore in order to show $\widetilde{E}^\circ f=F(1,\cdot)\in\mathcal{L}$, it suffices to prove that, for sufficiently small t, $F(t,\cdot)$ is in $\mathcal{L}$ (Lemma 2.1.4.).

For this purpose let $u_t(0<t\leq 1)$ denote the holomorphic function defined on $\varphi_t^{-1}O=\varphi_{1/t}O$ setting

$$u_t(p)=f(\varphi_t(p))\qquad p\in\varphi_{1/t}O$$

and f_t the restriction to O of ut. Then we have

$$F(t,\varepsilon)=\widetilde{E}^\circ f_t(\varepsilon).$$

Therefore, for t such that $\varphi_{1/t}O$ contains $\widetilde{O}$, we have by Lemma 2.1.2,

$$F(t,\varepsilon)=u_t(\pi(\varepsilon)). \tag{6}$$

But, for sufficiently small t, $\hat{O}\subseteq\varphi_{1/t}O$ since $\hat{O}$ is bounded. Thus, for sufficiently small t, $F(t,\cdot)\in\mathcal{L}$ which proves that $\widetilde{E}^\circ f\in\mathcal{L}$. In other words there exists a holomorphic function $E^\circ f$ such that

$$\widetilde{E}^\circ f(\varepsilon) = E^\circ f(\pi(\varepsilon)).$$

Now what remains to be proved is that, *in the intersection of O and $E^\circ O$,*

$$E^\circ f = f. \tag{7}$$

To show this we set, for $\varepsilon \in \widetilde{E}^\circ O$ such that $\pi(\varepsilon) \in O$,

$$G(t) = F(t, \varepsilon) - f_t(\pi(\varepsilon)).$$

Then by (6) $G(t)$ vanishes when t is sufficiently small. Since, moreover, $G(t)$ is real analytic in t, $G(1)=0$ and (7) is proved. Thus, if one defines a holomorphic function Ef on EO setting

$$Ef(p) = \begin{cases} f(p) & (p \in O) \\ E^\circ f(p) & (p \in E^\circ O), \end{cases}$$

we obtain the desired holomorphic extension Ef of f. The uniqueness of Ef is obvious since EO is connected. Q.E.D.

For any subset S of $\mathcal{G} \times V$ let $H^\circ(S)$ denote the set of the restrictions to S of holomorphic functions defined on some neighbourhood of S. Then in terms of $H^\circ(S)$ we can give a more convenient formulation to Proposition 2.1.1.

Corollary 2.1.1. *If S is a φ-star-shaped subset of $\mathcal{G} \times V$, then there exists a linear operator E from $H^\circ(S)$ into $H^\circ(ES)$ such that*

$$Ef|_S = f$$

$$\sup_{p \in ES} |Ef(p)| = \sup_{p \in S} |f(p)|$$

for $f \in H^\circ(S)$. In particular the restriction map $H^\circ(ES) \to H^\circ(S)$ is onto.

Proof. Suppose $f \in H^\circ(S)$. Then there exists an open subset $U \supseteq S$ of $\mathcal{G} \times V$ and a holomorphic function g in U such that $g|_S = f$. Then the set

$$O = \{p \in \mathcal{G} \times V : 0 \leq {}^\forall t \leq 1 \varphi_t(p) \in U\}$$

is φ-star-shaped and open in $\mathcal{G} \times V$. Certainly O contains S. Thus we can assume that $U=O$. But then $Eg\in H^{\circ}(EO)$ and $Eg|O=g$ in the notation of Proposition 2.1.1. Set $Ef=Eg|_{ES}$. From the explicit construction of Eg, Ef is given by the formula

$$Ef(\pi(\varepsilon))=\int_0^{2\pi}f(\sigma_\varepsilon(e^{i\varphi}))P(\varphi, x(\varepsilon)/\lambda(\varepsilon))d\varphi \quad \varepsilon\in\widetilde{E}^{\circ}S.$$

Thus it follows

$$\sup_{p\in ES}|Ef(p)|=\sup_{p\in S}|f(p)|.$$

Q.E.D.

2.2. Application to D_0. From now on we shall apply the result obtained so far to the holomorphic continuation from the Šilow boundary of the Siegel domain

$$D=\{(A, x)\in\mathcal{G}\times V;\ \mathfrak{l}(A, x)>0\}.$$

Set for $k=0, 1, 2, \cdots$

$$D_k=\{(A, x);\ =\mathfrak{l}(A, x)\geq 0,\ \text{rank of } \mathfrak{l}(A, x)\leq k\}.$$

Each D_k is a closed subset of $\mathcal{G}\times V$. D_0 is the Šilow boundary of D (See Piatetski-Šapiro [7]). Further $\partial D=D_{n-1}$, $\overline{D}=D_n=D_{n+1}=\cdots$. Now set

$$\mathcal{E}_k^{\circ}=(D_k\times S_1\times(-\infty, \infty))\cap\mathcal{E}^{\circ}.$$

Then we have

Lemma 2.2.1. *Let ε be an element of $\mathcal{E}^{\circ}$. Then $\varepsilon\in\mathcal{E}_k^{\circ}$ if and only if $\sigma_\varepsilon(\partial\Delta)\subseteq D_{k-1}$.*

Proof. From (3)′ it follows that $\sigma_\varepsilon(\partial\Delta)\subseteq D_{k-1}$ if and only if the rank of the hermitian endomorphism

$$\mathcal{A}(\varepsilon)=\mathfrak{l}(\pi(\varepsilon))-|(y(\varepsilon), y(\varepsilon))|^{-1}y(\varepsilon)\otimes y(\varepsilon)^*$$

is less than k and $\mathcal{A}(\varepsilon)\geq 0$. Now suppose $\varepsilon\in\mathcal{E}_k^{\circ}$. Then $\mathfrak{l}(\pi(\varepsilon))\geq 0$, rank of $\mathfrak{l}(\pi(\varepsilon))\leq k$, and $(y(\varepsilon), y(\varepsilon))\neq 0$. Hence $(\dot{y}(\varepsilon), y(\varepsilon))=(\mathfrak{l}(\pi(\varepsilon))y(\varepsilon), y(\varepsilon))>0$, and thus

$$\mathcal{A}(\varepsilon)=\mathfrak{l}(\pi(\varepsilon))-(y(\varepsilon), y(\varepsilon))^{-1}\tilde{y}(\varepsilon)\otimes y(\varepsilon)^*.$$

Therefore $\mathcal{A}(\varepsilon)y(\varepsilon)=0$, while $\mathfrak{l}(\pi(\varepsilon)y(\varepsilon)=y(\varepsilon)\neq 0$. On the other hand $\mathfrak{l}(\pi(\varepsilon))-\mathcal{A}(\varepsilon)=(y(\varepsilon), y(\varepsilon))^{-1}y(\varepsilon)\otimes y(\varepsilon)^*\geq 0$. Thus, once $\mathcal{A}(\varepsilon)\geq 0$ is proved, it follows

$$\text{rank of } \mathcal{A}(\varepsilon) \leq k-1 ,$$

i.e.,

$$\sigma\varepsilon(\partial\Delta) \subseteq D_{k-1} .$$

To show $\mathcal{A}(\varepsilon) \geq 0$ we shall first observe that

$$(I(\pi(\varepsilon))(y(\varepsilon)+tx), y(\varepsilon)+tx) \geq 0$$

for $t \in \boldsymbol{C}$. Hence we obtain

$$(y(\varepsilon), y(\varepsilon))(I(\pi(\varepsilon))x, x) - |(y(\varepsilon), x)|^2 \geq 0 .$$

But the left-hand side is equal to $(\tilde{y}(\varepsilon), y(\varepsilon))(\mathcal{A}(\varepsilon)x, x)$. Therefore

$$(\mathcal{A}(\varepsilon)x, x) \geq 0 \qquad x \in V.$$

Thus $\mathcal{A}(\varepsilon) \geq 0$ and $\sigma\varepsilon(\partial\Delta) \subseteq D_{k-1}$ is proved.

Conversely suppose that $\mathcal{A}(\varepsilon) \geq 0$ and that rank of $\mathcal{A}(\varepsilon) \leq k-1$. The image of $I(\pi(\varepsilon))$ is contained in the sum of the image of $\mathcal{A}(\varepsilon)$ and the 1-dimensional subspace generated by $y(\varepsilon)$. Thus

$$\text{rank of } I(\pi(\varepsilon)) \leq \text{rank of } (\mathcal{A}(\varepsilon)) \text{ plus } 1 \leq k .$$

On the other hand

$$I(\pi(\varepsilon)) = A(\varepsilon) + |(y(\varepsilon), y(\varepsilon))|^{-1}\tilde{y}(\varepsilon)\otimes\tilde{y}(\varepsilon)^* \geq A(\varepsilon) \geq 0 .$$

Thus $\pi(\varepsilon) \in D_k$, that is, $\varepsilon \in \mathcal{E}_k^\circ$. Q.E.D.

By this lemma the restriction σ_k of σ to the set $\mathcal{E}_k^\circ \times \partial\Delta$ is a continuous map of $\mathcal{E}_k^\circ$ into D_{k-1}. Further if S is a subset of D_{k-1}, then $E^\circ S = \{\varepsilon \in \mathcal{E}_k^\circ ; \sigma\varepsilon(\partial\Delta) \subseteq S\}$. Note that the restriction π^k of π to $\mathcal{E}_k^\circ$ is an open mapping from $\mathcal{E}_k^\circ$ onto D_k. Thus we obtain

Lemma 2.2.2. *For an open subset S of D_{k-1}, $E^\circ S$ is open in D_k.*

Let E_k denote the k-th power of the set operation E, that is, $E_k S$ is defined as follows:

(i) $E_0 S = S$

(ii) $E_{k+1} S = E(E_k S)$.

Then we have

Lemma 2.2.3. $E_{k+1} S = S \cup E^\circ(E_k S)$

Proof. For $k=0$, there remains nothing to be proved. Assume that the case $k=j$ is proved. Then we have

$$\begin{aligned} E_{j+2}S &= E^{\circ}(E_{j+1}S)\cup E_{j+1}S \\ &= E^{\circ}(E_{j+1}S)\cup E^{\circ}(E_jS)\cup S \\ &= E^{\circ}(E_{j+1}S)\cup S \end{aligned}$$

since $E^{\circ}(E_jS)\subseteq E^{\circ}(E_{j+1}S)$. Thus the case $k=j+1$ is proved.

Q.E.D.

Proposition 2.2.1. *If the subset S of D_0 is open in D_0, then E_kS is an open subset of D_k.*

Proof. First we prove this proposition assuming the following lemma.

Lemma 2.2.4. *Under the assumption of Proposition 2.2.1, S is contained in the interior of E_kS.*

Let us now prove Proposition 2.2.1. For $k=0$ nothing remains to be proved. Suppose that E_jS is open in D_j. Then $E^{\circ}(E_jS)$ is open in D_{j+1} by Lemma 2.2.2. But by Lemma 2.2.3. we have

$$E_{j+1}S=S\cup E^{\circ}(E_jS).$$

Therefore by Lemma 2.2.4. $E_{j+1}S$ is open in D_{j+1}. Q.E.D.

Before proving Lemma 2.2.4 we shall recall the group-theoretic aspect of D_0, which will facilitate the proof of Lemma 2.2.4. For a point p of D_0 define an affine transformation $\rho(p)$ by setting

$$\rho(p)(A', x')=(A+A'+ix'\otimes x^*, x+x')$$

where $p=(A, x)$ $A\in\mathcal{G}$ $x\in V$. By simple computation

$$\mathfrak{l}(\rho(p)q)=\mathfrak{l}(p)+\mathfrak{l}(q)=\mathfrak{l}(q) \quad q\in\mathcal{G}\times V.$$

Thus $\rho(p)$ leaves each D_j invariant. In particular $\rho(p)p'\in D_0$ for $p, p'\in D_0$, so we can define a multiplication of a pair of elements of D_0 setting

$$p\cdot p'=\rho(p)p'.$$

This multiplication is associative

$$(p\cdot p')\cdot p''=p\cdot(p'\cdot p'').$$

This follows immediately from

$$x'\otimes x^*+x''\otimes(x+x')^*=x''\otimes(x')^*+(x'+x'')\otimes x^*.$$

If $(A, x)\in D_0$, then $(-A+ix\otimes x^*, -x)\in D_0$ and

$$(A, x)\cdot(-A+ix\otimes x^*, -x)=0.$$

Further $p\cdot 0=0\cdot p=p$. Therefore D_0 is a Lie group with this multiplication. The map $D_0\ni p\to\rho(p)\in AF_C(\mathcal{G}\times V)$ is a Lie group isomorphism into. The group $\rho(D_0)$ operates transitively on D_0. Further we have

$$E(\rho(p)S)=\rho(p)\cdot ES.$$

In fact

$$\sigma(\rho(p)q, y, r;\ \zeta)=\rho(p)\sigma(q, y, r;\ \zeta).$$

Thus, in order to prove Lemma 2.2.4, we may assume that $S\ni 0$ and it suffices to show that 0 is in the interior of E_kS in D_k.

Proof of Lemma 2.2.4. First we shall introduce a norm on $\mathcal{G}\times V$ setting:

$$\|p\|=\|A\|+\|x\|$$

where $p=(A, x)$ and $\|A\|$ denotes the operator norm of A. By B_δ we denote the open ball of radius δ in $\mathcal{G}\times V$:

$$B_\delta=\{p\in\mathcal{G}\times V;\ \|p\|<\delta\}.$$

Suppcse that $p\in\overline{D}$, $\|p\|<1$. Then $\|l(p)\|\leq 2(\|p\|+\|p\|^2)\leq 4\|p\|$. Therefore $\|\lambda(p, y, 0)l(p)y\|=\|l(p)y\|/_{\|l(p)^{1/2}y\|}\leq\|l(p)^{1/2}\|\leq 2\|p\|^{1/2}$, hence for $\zeta\in\Delta$

$$\begin{aligned}(8)\qquad \|\sigma(p, y, 0;\ \zeta)\|&\leq\|p\|+\|\lambda(p, y, 0)l(p)y\|(\|x\|+1)\\ &\leq\|p\|+2\|p\|^{1/2}(\|p\|+1)\leq 5\|p\|^{1/2}.\end{aligned}$$

Now suppose that $p\in(B_{(\delta/5)^2}\cap D_{j+1})\setminus D_0$ $(\delta<1)$. Then there exists $y\in S_1$ such that $(l(p)y, y)>0$ which implies $\sigma(p, y, 0, \partial\Delta)\subset D_j$. But by (8) we have $\sigma(p, y, 0;\ \partial\Delta)\subset B_\delta$. Thus $p\in E^\circ(B_\delta\cap D_j)$. Hence $B_{(\delta/5)^2}\cap D_{j+1}\setminus D_0\subset E^\circ(B_\delta\cap D_j)$. Since $B_{(\delta/5)^2}\cap D_0\subset B_\delta\cap D_j\subset E(B_\delta\cap D_j)$, we have

$$(10)\qquad B_{(\delta/5)^2}\cap D_{j+1}\subset E(B_\delta\cap D_j).$$

Choose a positive number $\eta<1$ such that

$$B_\eta \cap D_0 \subset S.$$

Then we have by (10)

$$B_{(\eta/5)^{2k}} \cap D_k \subset E_k S,$$

which implies that 0 is the interior of $E_k S$. Q.E.D.

For a σ-compact subset S of $\mathcal{G}\times V$ we shall also denote by $H(S)$ the closure of $H^\circ(S)$ in $C(S)$. Recall that any open subset of D_0 is generic (Proposition 1.2.1). Therefore by repeated use of Corollary 2.1.1 we obtain

Lemma 2.2.5. *Let S be a φ-star-shaped open subset of D_0. Then the restriction map $H(E_n S)\to H(S)$ is a topological isomorphism onto $H(S)$.*

Proof. We shall first prove this assuming that S is bounded. Set $S_t=\varphi_t S$ $0<t<1$. Then $S_{t'}$ is relatively compact in $S_{t''}$ when $t'<t''$. Since $\varphi_t E_n S=E_n S_t$, $E_n S_{t'}$ is also relatively compact in $E_n S_{t''}$ if $t'<t''$. Now, for $f\in H^\circ(S)$ we define $E_k f\in H^\circ(E_k S)$ by induction;

$$E_0 f=f$$
$$E_k f=E(E_{k-1}f) \quad (k\geq 1).$$

Here E is the linear map given in Corollary 2.1.1. According to the estimate in this corollary,

$$\sup_{p\in S_t}|f(p)|=\sup_{p\subset E_n S_t}|E_k f(p)| \quad 0<t\leq 1.$$

Thus the map $f\to E_n f$ is a topological isomorphism of $H^\circ(S)$ into $H^\circ(E_n S)$. This is also onto since the restriction map $H^\circ(E_n S)\to H^\circ(S)$ is one-to-one by Proposition 1.2.1. The map E_n is thus extended to a unique topological isomorphism of $H(S)$ onto $H(E_n S)$. But the inverse of E_n is the restriction map, and we have thus obtained the desired conclusion when S is bounded.

In case S is not bounded we set for $k=1, 2, \cdots$

$$S^k=S\cap\{(A, x)\in\mathcal{G}\times V;\ \|A\|+\|x\|^2<k\}.$$

Then it follows easily $E_nS=\bigcup_k E_nS^k$. Put

$$S^{(k)}=\varphi_{(k-1)/k}(S^k)\quad (k=1,2,\cdots).$$

Then it also holds $E_nS=\bigcup_k E_nS^{(k)}$. On the other hand

$$\overline{E_nS^{(k)}}=\overline{\varphi_{(k-1)/k}E_nS_k}\subset\varphi_{k/(k+1)}E_nS^k\subset\varphi_{k/(k+1)}E_nS^{k+1}=E_nS^{(k+1)}.$$

Thus $E_nS^{(k)}$ is relatively compact in $E_nS^{(k+1)}$. Here $S^{(k)}$ is obviously bounded, and we have thus reduced the second case to the first.

Q.E.D.

By lemmas 2.2.4 and 2.2.5 we can improve Proposition 1.2.1 for standard real submanifold D_0 as follows.

Lemma 2.2.6. *Let U be a domain in $\mathcal{G}\times V$ such that $U\cap D_0\neq\phi$. Then a continuous function u on $U\cap\overline{D}$ such that $u|_{U\cap D_0}=0$ and $u|_{U\cap D}\in H(U\cap D)$ must vanish identically in $U\cap\overline{D}$.*

Proof. Since $\rho(D_0)$ operates transitively on D_0, we may assume that $U\cap D_0\ni 0$. Take a sufficiently small $\delta>0$ so that $\overline{B}_\delta\subset U$ and set $S=B_\delta\cap D_0$. Then we have

$$ES\subset\hat{S}\subset\hat{\overline{B}}_\delta=\overline{B}_\delta.$$

By repeated use of this argument we obtain

$$E_nS\subset\overline{B}_\delta$$

and hence

$$E_nS\subset\overline{B}_\delta\cap\overline{D}.$$

Since $\overline{B}_\delta\subset U$, there exists a positive number η such that

$$\overline{B}_\delta+itI_V\subset U\quad 0\leq t\leq\eta$$

where I_V denotes the identity map of V. Therefore, for $0<t\leq\eta$,

$$E_nS+itI_V\subset(\overline{B}_\delta+itI_V)\cap(\overline{D}+itI_V)\subset U\cap D,$$

hence we obtain a continuous family $u_t\,(0<t<\eta)$ of elements of $H^\circ(E_nS)$ setting

$$u_t(A,x)=u(A+itI_V,x)$$

for $(A,x)\in E_nS$.

Certainly $u_t \to v$ at $t \to 0$ where we have set $v = u|_{E_n S}$. Thus $v \in H(E_n S)$. But the restriction map $H(E_n S) \to H(S)$ is one-to-one by Lemma 2.2.5. Thus $v=0$ which implies $u=0$ since $E_n S$ is an open subset of $\bar{D}$.

Q.E.D.

Lemma 2.2.6 is a very powerful device for piecing together local extensions into a global extension as will be explained in the following. Before proceeding we shall introduce some function spaces and sheaves.

The system $(H(S))_{S:\text{open in } D_0}$ together with restriction maps forms a presheaf on D_0 and we denote by $\mathcal{H}$ the sheaf induced from this presheaf. For any open subset S of D_0, $\underleftarrow{H}(S)$ is defined by

$$\underleftarrow{H}(S) = \{f \in C(S) : f|_{S'} \in H(S') \text{ for any relatively compact open subset } S' \text{ of } S\}.$$

A continuous function f on S is *D-holomorphic* in S if there exists an open subset U of $\mathcal{G} \times V$ containing S and a continuous function u on $U \cap \bar{D}$ such that (1) $u|_S = f$ (2) $u|_{U \cap D}$ is holomorphic. We denote by $H_D(S)$ the set of D-holomorphic functions on S and by $\mathcal{H}_D$ the sheaf of germs of D-holomorphic functions. Further for an open set V of $\bar{D}$ we denote by $\tilde{H}(V)$ the subspace of $C(V)$

$$\tilde{H}(V) = \{u \in C(V);\ u|_{V \cap D} \text{ is holomorphic}\}.$$

As an immediate consequences of Lemma 2.2.5 we obtain

Lemma 2.2.7. *$H(S) \subset \Gamma_S(\mathcal{H}_D)$ for an open set S of D_0.*

For the proof one may note that the system $\Pi = \{\rho(p)(B_\delta \cap D_0);\ p \in D_0, \delta > 0\}$ is a fundamental system of neighbourhoods of D_0 and that each element of Π is an image by $\rho(p)$ of a φ-star-shaped neighbourhood of D_0.

Lemma 2.2.8. *$\Gamma_S(\mathcal{H}_D) = H_D(S)$ for an open set S of D_0.*

To prove this we need a special neighbourhood system in $\bar{D}$ of a point of D_0. Let $h(z)$ $(z \in \mathcal{G})$ denote hermitian part of z, i.e.

$$h(z) = \frac{1}{2}(z + z^*)$$

and $C_{\delta,\eta}(z_0, x_0)$ $(\delta, \eta > 0)$ the neighbourhood of $(z_0, x_0) \in D_0$ given by

$$C_{\delta,\eta}(z_0, x_0) = \{(z, {}'x) \in \bar{D};\ \|h(z-z_0)\| < \delta,\ \|x-x_0\| < \delta,\ \|\mathfrak{l}(z, x)\| \leq \eta\}.$$

We also denote by $C_{\delta,0}$ the intersection of $C_{\delta,\eta}$ and D_0 (this is obviously independent of η). If $0 \leq \delta \leq \delta'$ $0 \leq \eta \leq \eta'$, it holds

$$C_{\delta,\eta}(z_0, x_0) \subset C_{\delta',\eta'}(z_0, x_0)$$

and $(C_{\delta,\eta}(z_0, x_0))_{\delta,\eta>0}$ is a complete system of neighbourhoods of (z_0, x_0).

Now consider the retracting deformation $\phi_t (0 \leq t \leq 1)$ of $\bar{D}$ onto D_0 given by

$$\phi_t(z, x) = \left(z - \frac{it}{2}\mathfrak{l}(z, x), x\right).$$

Then

$$\mathfrak{l}(\phi_t(z, x)) = (1-t)\mathfrak{l}(z, x)$$

$$h\left(z - \frac{it}{2}\mathfrak{l}(z, x)\right) = h(z).$$

Therefore, for $0 \leq t \leq 1$,

$$\phi_t C_{\delta,\eta}(z_0, x_0) = C_{\delta,(1-t)\eta}(z_0, x_0).$$

In particular, this implies that $C_{\delta,0}(z_0, x_0) \cap C_{\delta',0}(z_1, x_1) \neq \phi$ and $C_{\delta,\eta}(z_0, x^0) \cap C_{\delta',\eta'}(z_1, x_1)$ is connected whenever $C_{\delta,\eta}(z_0, x_0) \cap C_{\delta',\eta'}(z_1, x_1) \neq \phi$. In fact $C_{\delta,0}(z_0, x_0) \cap C_{\delta',0}(z_1, x_1)$ is the deformation retract by ϕ_t of $C_{\delta,\eta}(z_0, x_0) \cap C_{\delta',\eta'}(z_1, x_1)$ and $C_\delta(z_0, x_0) \cap C_{\delta',0}(z_1, x_1)$ is connected. The latter fact can be seen as follows; first consider the diffeomorphism φ of D_0 onto $\mathcal{G}_h \times V$ given by

$$\varphi(z, x) = (h(z), x)$$

where $\mathcal{G}_h$ is the vector space of hermitian endomorphisms. Then

$$\varphi(C_{\delta,0}(z_0, x_0)) = \{(z, x) \in \mathcal{G}_h \times V:\ \|z - h(z_0)\| < \delta,\ \|x - x_0\| < \delta\}$$

which is evidently convex and $\varphi(C_{\delta',0}(z_1, x_1))$ is also convex hence $\varphi(C_{\delta,0}(z_0, x_0) \cap C_{\delta',0}(z_1, x_1))$ is convex, a fortiori, it is connected.

Now we shall turn to the proof of Lemma 2.2.8.

Proof of Lemma 2.2.8. Suppose that $f \in \Gamma_S(\mathcal{H}_D)$. Then for any point p of S one can choose a neighbourhood $V(p)$ in $\bar{D}$ of p and an element u_p of $\widetilde{H}(V(p))$ such that $u_p|_{V(p)\cap S} = f|_{V(p)\cap S}$. We may assume that $V(p)=C_{\delta,\eta}(p)$ for some $\delta, \eta>0$ and that $V(p)\cap D_0\subset S$. We claim that

$$V(p)\cap V(p')\neq\phi\rightarrow u_p|_{V(p)\cap V(p')}=u_{p'}|_{V(p)\cap V(p')}.$$

As shown before, $V(p)\cap V(p')\cap D_0$ is non-empty if $V(p)\cap V(p')\neq\phi$. But $u_p-u_{p'}|_{V(p)\cap V(p')\cap D_0}=f-f=0$. Therefore we have by Lemma 2.2.6

$$u_p=u_{p'} \text{ in } V(p)\cap V(p')$$

since $V(p)\cap V(p')$ is connected. Thus our claim is proved. Thus if we define a continuous function u on $\bigcup_{p\in S} V(p)$ by setting

$$z\in V(p)\rightarrow u(z)=u_p(z),$$

then u is well-defined and $u|_S=f$ and u is holomorphic in $\bigcup_{p\in S} V(p)\cap D$. Thus $f\in H_D(S)$. Q.E.D.

Lemma 2.2.9. *$H_D(S)\subset \underleftarrow{H}(S)$ for an open set S of D_0.*

Proof. Let S' be a relatively compact open subset of S and U an open subset of $\mathcal{G}\times V$ containing S. Suppose further that $u\in\widetilde{H}(U\cap\overline{D})$. Then there exists a positive number η such that

$$S'+itI_V\subset U \text{ for } 0\leq t\leq\eta$$

since $\overline{S}'\subset S\subset U$. On the other hand if $t>0$,

$$S'+itI_V\subset D_0+itI_V\subset D.$$

Therefore $$S'+itI_V\subset U\cap D \quad (t>0).$$

Define a family $f_t(0<t\leq\eta)$ of elements of $H^\circ(S)$ setting

$$f_t(p)=u(p+itI_V) \quad p\in S'$$

Then f_t is well-defined and $f_t\rightarrow f=u|_{S'}$ as $t\rightarrow 0$. Thus $u|_S\in\underleftarrow{H}(S)$. Q.E.D.

In view of this lemma the natural restriction map

$$\widetilde{H}(U)\rightarrow\underleftarrow{H}(U\cap D_0)$$

is well defined when U is an open subset of $\overline{D}$ such that $U\cap D_0\neq\phi$. For any open subset S of D_0 we shall now assign an open subset $U_1(S)$ of $\overline{D}$ such that $S\subseteq U_1(S)$ and such that the restriction map

$$\widetilde{H}(U_1(S)) \to H(S)$$

is onto, that is,

$$H_D(S) = H(S) = H(U_1(S))|_S .$$

First we set for $\delta > 0$

$$Q_\delta = \{(A, x) \in D_0;\ \|A\| + \|x\|^2 < \delta^2\} .$$

Then Q_δ is a φ-star-shaped neighbourhood of 0 in D_0. In fact

$$\varphi_t Q_\delta = Q_{t\delta} .$$

Let S be an open subset of D_0. Put

$$\eta(p, S) = \sup\{\eta > 0;\ \rho(p)(Q_{2\eta}) \subseteq S\}$$

$$Q(p, S) = \rho(p)(Q_{\eta(p,S)}).$$

Then $Q(p, S)$ is a relatively compact open subset of S unless $S = D_0$. Hence, for $f \in H(S)$, there exists $u'_p \in H(E_n Q(p, S))$ such that

$$u'_p|_{Q(p,S)} = f|_{Q(p,S)}$$

by Lemma 2.2.5. Now denote by $\delta'(p, S)$ the supremum of the set

$$\{\delta > 0;\ C_{2\delta,2\delta}(p) \subset E_n Q(p, S)\}$$

and put

$$V_1(p, S) = C_{\delta'(p,S),\delta'(p,S)}(p)$$

$$U_1(S) = \bigcup_{p \in S} V_1(p, S) .$$

Then certainly $U_1(S)$ is an open subset of $\overline{D}$ such that $U_1(S) \cap D_0 = S$. Further is holds

$$u_p|_{V_1(p,S)\ D_0} = f|_{V_1(p,S)\ D_0}$$

where we have set $u_p = u'_p|_{V_1(p,S)}$. By the same reasoning as in the proof of Lemma 2.2.8 there exists $u \in \widetilde{H}(U_1(S))$ such that $u|_{V_1(p,S)} = u_p$. Of course $f = u|_S$ and we conclude

$$\widetilde{H}(S) = H(U_1(S))|_S ,$$

which, together with Lemmas 2.2.7–2.2.9, implies

$$H(S) = \Gamma_S(\mathcal{H}_D) = H_D(S) = \widetilde{H}(U_1(S))|_S .$$

From the first identity it follows also $\mathcal{H} = \mathcal{H}_D$.

To sum up, we have thus proved

Theorem 2.2.1. *For any open subset S of D_0*

$$H(S)=\Gamma_S(\mathcal{H})\,(=\Gamma_S(\mathcal{H}_D)=H_D(S))=\widetilde{H}(U_1(S))\,|_S$$

Here $U_1(S)$ is open in $\overline{D}$.

This is the main theorem of this section. However, in later discussion (especially in §5), it will be required to exploit another open subset with certain additional property instead of $U_1(S)$. The rest we shall devote to the preparation for §5. In the above notation we set for $p\in S$

$$\delta(p, S)=\min\left(\frac{1}{2}\delta'(p, S), 1\right)$$

$$V(p, S)=C_{\delta(p,S),\delta(p,S)}(p)$$

$$U(S)=\bigcup_{p\in S}V(p, S).$$

$U(S)$ is an open subset of $\overline{D}$ such that $S\subseteq U(S)\subseteq U_1(S)$, and we have

Lemma 2.2.10. *For any open subset S of D_0*

$$\overline{U(S)}\setminus\partial S\subseteq U_1(S).$$

Proof. First we shall prove that $\delta(p, S)$ is continuous with respect to $p\in S$. The continuity of $\eta(p, S)$ is obvious. Note that

$$Q(p, S)=\rho(p)Q_{\eta(p,S)}$$
$$=\rho(p)\varphi_{\eta(p,S)}(Q_1).$$

Thus the relation

$$C_{\delta,\delta}(p)\subseteq E_nQ(p, S)$$

is equivalent to

$$\varphi_{\eta(p,S)^{-1}}\rho(p^{-1})C_{\delta,\delta}(p)\subseteq E_nQ_1.$$

The left hand side depends continuously on p and δ, while the right hand side is the definite open subset of $\overline{D}$. Thus $\delta'(p, S)$ is continuous by its definition, and hence also $\delta(p, S)$ is continuous.

Next we prove that

$$q\in V(p, S)\Rightarrow d(p)\leq 3(d(q)+2)$$

where we have set $d(A, x) = \|A\| + \|x\|^2$ for $(A, x) \in \mathcal{G} \times V$. Set $p = (A, x)$, $q = (B, y)$. Since $\delta(p, S) \leq 1$, we have

$$\|h(A-B)\| < 1, \ \|l(q)\| \leq 1, \ \|x-y\| \leq 1$$

if $q \in V(p, S)$. Therefore

$$\begin{aligned} \|A\| &= \|h(A) + \tfrac{1}{2}(l(q) + ix \otimes x^*)\| \leq \|h(A)\| + \tfrac{1}{2}\|x\|^2 \\ &\leq \|h(B)\| + \|y\|^2 + \|h(A-B)\| + \|x-y\|^2 \\ &\leq \|B\| + \|y\|^2 + 2, \\ \|x\|^2 &\leq 2(\|y\|^2 + 1). \end{aligned}$$

Thus it follows

$$d(p) \leq 3(d(q) + 2).$$

Now suppose $q_0 \in \bar{U} \setminus \partial S$ and let $\{q_\nu\}_{\nu=1}^{\infty}$ be a sequence from $U(S)$ such that $q_\nu \to q_0 (\nu \to \infty)$. Since $U(S) = \bigcup_{p \in S} V(p, S)$, there exists a sequence $\{p_\nu\}_{\nu=1}^{\infty} \subseteq S$ such that

$$q_\nu \in V(p_\nu, S).$$

But then the inequality proved above implies that $\{p_\nu\}_{\nu=1}^{\infty}$ is bounded. Therefore we may assume that p_ν converges to $p_0 \in \bar{S}$ taking a suitable subsequence if necessary. However $p_0 \in \partial S$ contradicts $q_0 \notin \partial S$, for one can see immediately $\delta'(p_\nu, S) \to 0$ when $p_\nu \to p_0 \in \partial S$. Thus p_0 must lie in S. Hence the definition of $V(p, S)$ and the continuity of $\delta(p, S)$ imply that

$$q_0 \in \overline{V(p_0, S)}.$$

The right-hand side is contained in $V_1(p_0, S)$ since $\delta(p, S) \leq \frac{1}{2}\delta'(p, S)$. Therefore $q_0 \in U_1(S)$. Thus we have proved

$$\overline{U(S)} \setminus \partial S \subseteq U_1(S). \qquad \text{Q.E.D.}$$

The next lemma will also be used in §5.

Lemma 2.2.11. *Let H be a subspace of $\mathcal{G}_h$. Suppose that an open subset S of D_0 and $f \in \Gamma_S(\mathcal{H})$ are invariant by translations paallrel to H. Then f can be extended into a function u on $U_1(S) + H^C$ so that $u|_{U_1(S)} \in \widetilde{H}(U_1(S))$ and u is invariant by translations parallel to H^C. $U_1(S)$, $U(S)$ are invariant by translations parallel to H.*

Proof. The second assertion follows immediately from

$$C_{\delta,\eta}(p+h)=C_{\delta,\eta}(p)+h \quad h\in\mathcal{G}_k.$$

To prove the first assertion we shall first show that the intersection of the line

$$L_h(p)=\{p+ith;\ t\in\boldsymbol{R}\} \quad h\in\mathcal{G}_k$$

with $U(S)$ is connected. From definition of $C_{\delta,\eta}(q)$ it follows that $L_h(p)\cap C_{\delta,\eta}(q)$ is either empty or

$$\{p+ith;\ \mathfrak{l}(p)+2th\geq 0,\ \|\mathfrak{l}(p)+2th\|<\eta,\quad t\in\boldsymbol{R}\}.$$

Thus $U_1(S)\cap L_h(p)=\{p+ith;\ \mathfrak{l}(p)+2th\geq 0,\ \|\mathfrak{l}(p)+2th\|<\bar{\eta},\ t\in\boldsymbol{R}\}$ where $\bar{\eta}=\sup\{\delta'(q, S):\ L_h(p)\cap V_1(q, S)\neq\phi\}$. This is obviously con nected.

Let us now prove the existence of u with the required property. From Theorem 2.2.1 it follows that there exists $u'\in H(U_1(S))$ such that $u'|_S=f$. However, by assumption

$$\tau_h(U_1(S))=U_1(S)$$
$$\tau_h^*(f)=f$$

where $h\in H$ and τ_h is the translation $p\to p+h$. Thus Lemma 2.2.6 implies

$$\tau_h^*(u')=u' \quad h\in H. \tag{9}$$

Since u' is holomorphic in $U_1(S)\cap D$ and since $L_h(p)\cap U_1(S)$ is connected, we obtain for $h\in H$

$$u'(p+ih)=u'(p) \text{ if } p\in U_1(S)\cap D,\ p+ih\in U_1(S). \tag{10}$$

Now suppose only $p,\ p+ith\in U_1(S)$. Choose a positive definite $h_0\in\mathcal{G}_k$. Since $U_1(S)$ is an open subset of D, there exists $\delta>0$ such that

$$p(s)=p+ish_0\in U_1(S)$$
$$p(s)+ih\in U_1(S)$$

when $0\leq s\leq\delta$.

On the other hand

$$p(s),\ p(s)+ih\in D \quad 0<s.$$

Thus, by (10), we have for $0<s\leq\delta$

$$u'(p(s)+ih)=u'(p(s))$$

from which it follows

$$u'(p+ih)=u'(p).$$

Thus we have proved (10) provided only that p, $p+ih\in U_1(S)$ and $h\in H$. Therefore there exists a unique extension u of u' to $U_1(S)+iH=U_1(S)+H^C$ which is invariant by translation parallel to iH. Thus we conclude in view of (9) that u is the desired function.

Q.E.D.

It is not known for us whether the function u above is continuous or not. However, in §5 we shall prove using Lemma 2.2.10 the continuity of the restriction $u|_{U(S)+iH}$ provided that H contains no semi-definite endomorphism except 0.

§3. Approximation Theorems

3.1. Approximation Theorem on $U(n)$. In the preceding section V was an n-dimensional Hilbert space, $\mathcal{G}$ the endomorphism ring of V. Let $\{e_1, e_2, \cdots, e_n\}$ be a fixed orthonormal base of V. Set, for $Z\in\mathcal{G}$, $Z=\sum_{j,k} Z_{jk}e_j\otimes e_k^*$. Then the mapping $Z\to(Z_{jk})$ gives an identification of $\mathcal{G}$ with the ring of complex (n, n)-matrices, which we shall preserve in whole discussion of this section. Let $\boldsymbol{G}$ denote the group of invertible elements of $\mathcal{G}$, i.e., the general linear group of order n, and $\boldsymbol{K}$ the unitary group of V. Thus $\boldsymbol{K}$ in the abbreviation of $U(n)$. $\mathcal{G}$ is a Lie algebra by the usual cross-product and should be regarded as the Lie algebra of $\boldsymbol{G}$, and then the algebra of $\boldsymbol{K}$ is the Lie subalgebra of anti-hermitian endomorphisms of $\mathcal{G}$ denoted hereafter by $\mathcal{K}$.

Let dZ denote the $\mathcal{G}$-valued 1-form on $\boldsymbol{G}$ such that dZ_A gives the usual identification of the tangent space at A of $\mathcal{G}$ (or $\boldsymbol{G}$) with $\mathcal{G}$ for $A\in\boldsymbol{G}$, and let Z denote the $\mathcal{G}$-valued function on $\boldsymbol{G}$ given by the inclusion $\boldsymbol{G}\to\mathcal{G}$. Then the $\mathcal{G}$-valued form $\delta Z=Z^{-1}dZ$ is the Maurer-Cartan form of $\boldsymbol{G}$, that is

$$\delta Z(\widetilde{A})=A \qquad A\in\mathcal{G} \tag{1}$$

where $\widetilde{A}$ denotes the left invariant vector field on $\boldsymbol{G}$ which is the generator of the 1-parameter transformation $R_{\exp tA}$. (R_Z denotes the right multiplication by Z.) The δZ is left invariant and it holds

$$R_A^*(\delta Z)=ad(A^{-1})\delta Z\,. \tag{2}$$

In terms of the coordinate system (Z_{jk}) on $\boldsymbol{G}$, δZ can be written as follows:

$$\delta Z=(\sum_l (Z^{-1})_{jl}dZ_{lk})$$

where $(Z^{-1})_{jk}$, Z_{jk} must be considered as functions on $\boldsymbol{G}$.

The restriction $\iota^*(\delta Z)$ to $\boldsymbol{K}$ of δZ, where ι: $\boldsymbol{K}\to\boldsymbol{G}$ is the inclusion map, is $\mathcal{K}$-valued and is the Maurer-Cartan form of $\boldsymbol{K}$. In the following, we often use ω instead of $\iota^*(\delta Z)$.

Take a non-zero element v_0 of $\bigwedge^{n2}\mathcal{K}^*$ and define an n^2-form on $\boldsymbol{K}$

$$v_A=(A^t\omega_A)(v_0) \qquad A\in\boldsymbol{K}$$

where ${}^t\omega_A$ denote the transposed of ω_A. Then v is left invariant, v is also right invariant by (2) since

$$D_{\mathcal{K}}(ad(A^{-1}))=1 \qquad A\in\boldsymbol{K} \tag{3}$$

where $D_{\mathcal{K}}(f)$ denotes the determinant of an endomorphism f of $\mathcal{K}$. The formula (3) follows immediately from compactness of $\boldsymbol{K}$. Let $[\boldsymbol{K}]$ be the generator of $H_{n^2}(\boldsymbol{K}, Z)$ such that $\int_{[\boldsymbol{K}]}v>0$. Replacing v_0 by $cv_0(c>0)$ if necessary, we could choose v_0 so that

$$\int_{[\boldsymbol{K}]}v=1. \tag{4}$$

Let $\boldsymbol{T}$ denote the subgroup of $\boldsymbol{G}$ given by

$$\boldsymbol{T}=\{A\in\boldsymbol{K}\colon A_{jk}=0 \text{ for } (j, k) \text{ such that } j\neq k\},$$

and $\mathcal{T}$ the Lie algebra of $\boldsymbol{T}$, i.e.,

$$\mathcal{T}=\{A\in\mathcal{K};\ A_{jk}=0 \text{ for } j\neq k\}.$$

Set also

$$\mathcal{K}_1=\{A\in\mathcal{K}\colon A_{11}=A_{22}=\cdots=A_{nn}=0\}.$$

Then we have

(5) $$\mathcal{K}=\mathcal{K}_1\oplus\mathcal{T} \qquad \mathcal{K}^*=\mathcal{K}_1^*\oplus\mathcal{T}^*$$

(6) $$ad(T)\mathcal{K}_1=\mathcal{K}_1 \qquad T\in \boldsymbol{T}$$

(7) $$ad(T)\mathcal{T}=\mathcal{T} \qquad T\in \boldsymbol{T}.$$

As usual we shall make the canonical identification $\bigwedge(\mathcal{K}_1^*\oplus\mathcal{T}^*)=\bigwedge\mathcal{K}_1^*\otimes\bigwedge\mathcal{T}^*$, the right hand side of which should be interpreted as the tensor product of the two *graded* algebras $\bigwedge\mathcal{K}_1^*$ and $\bigwedge\mathcal{T}^*$. Then

(8) $$\overset{n^2}{\bigwedge}\mathcal{K}^*=(\overset{n(n-1)}{\bigwedge}\mathcal{K}_1^*)\otimes(\overset{n}{\bigwedge}\mathcal{T}^*).$$

Now choose $v_0'\in\overset{n(n-1)}{\bigwedge}\mathcal{K}_1^*$ and $v_0''\in\overset{n}{\bigwedge}\mathcal{T}^*$ so that

(9) $$v_0=v_0'\otimes v_0''$$

and define an n-form v^2 on $\boldsymbol{T}$ and an $n(n-1)$-form v' on $\boldsymbol{K}$

(10) $$v_T^2=\bigwedge{}^t\omega_T'(v_0'') \qquad T\in\boldsymbol{T}$$

(11) $$v_A'=\bigwedge{}^t\omega_A(\bigwedge{}^t\pi_1(v_0')) \qquad A\in\boldsymbol{K}$$

where π_1 is the projection of $\mathcal{K}$ onto $\mathcal{K}_1$ in the splitting (5) and ω' is the Maurer-Cartan form of $\boldsymbol{T}$ (i.e., the restriction to $\boldsymbol{T}$ of ω). Then v' is left invariant and satisfies the following relations:

(12) $$R_T^*(v')=D_{\mathcal{K}_1}(ad(T^{-1})\,|\,\mathcal{K}_1)v'=v'$$

(13) $$\widetilde{A}\,\lrcorner\, v'=0 \qquad A\in\mathcal{T}.$$

The last relation follows immediately from $A\lrcorner\bigwedge{}^t\pi_1(v_0')=0$. Therefore there uniquely exists an $(n-1)$-form v^1 *on* $\boldsymbol{K}/\boldsymbol{T}$ such that

(14) $$v'=\mathfrak{p}^*(v^1)$$

where $\mathfrak{p}$ is the canonical projection $\boldsymbol{K}\to\boldsymbol{K}/\boldsymbol{T}$. We could choose v_0' so that $\int_{[\boldsymbol{K}/\boldsymbol{T}]}v^1=1$ for a suitable generator $[\boldsymbol{K}/T]$ of $H_{n(n-1)}(\boldsymbol{K}/\boldsymbol{T},Z)$. (Note that $\boldsymbol{K}/\boldsymbol{T}$ is orientable since $\boldsymbol{K}/\boldsymbol{T}$ is simply-connected.)

Now set $N'=\boldsymbol{K}\times\boldsymbol{T}$, $N=\boldsymbol{K}/\boldsymbol{T}\times\boldsymbol{T}$. Then as made before we have the following canonical identifications:

$$\wedge T(N') = \wedge T(\boldsymbol{K}) \otimes \wedge T(\boldsymbol{T})$$
$$\wedge T(N) = \wedge T(\boldsymbol{K}/\boldsymbol{T}) \otimes \wedge T(\boldsymbol{T}).$$

Consider the map $\phi : N' \to \boldsymbol{K}$ defined by

$$\phi(A, T) = ATA^{-1}.$$

Since $\boldsymbol{T}$ is abelian, there exists a uniquely smooth mapping $\varphi : N \to \boldsymbol{K}$ for which the following diagram is commutative:

$$\begin{array}{ccc} N' & & \\ \hat{\mathfrak{p}} = \mathfrak{p} \times id_{\boldsymbol{T}} \downarrow & \searrow \phi & \\ N & \xrightarrow[\varphi]{} & \boldsymbol{K} \end{array}$$

Then our key lemma is the following:

Lemma 3.1.1. $\varphi^*(v) = \sigma v^1 \otimes v^2$ *where* σ *denotes the function on* $\boldsymbol{T}$ *defined by*

$$\sigma(T) = \prod_{j<k} |T_{jj} - T_{kk}|^2 = \prod_{j<k} |1 - T_{jj} \bar{T}_{kk}|^2 \quad \textit{for} \quad T \in \boldsymbol{T}.$$

Proof. Since $v' \otimes v^2 = \mathfrak{p}^*(v^1 \otimes v^2)$, it suffices to prove

$$\phi^*(v) = \sigma v' \otimes v^2. \tag{15}$$

For $(A, T) \in N'$ let $f_{(A,T)}$ be the linear mapping of $\mathcal{K} \oplus \mathcal{T}$ into $\mathcal{K}$ which makes the following diagram commutative:

$$(*)_1 \qquad \begin{array}{ccc} T_A(\boldsymbol{K}) \oplus T_T(\boldsymbol{T}) & \xrightarrow{\omega_A \oplus \omega_T} & \mathcal{K} \oplus \mathcal{T} \\ (d\phi)_{(A,T)} \downarrow & & \downarrow f_{(A,T)} \\ T_A(\boldsymbol{K}) & \xrightarrow[\omega_A]{} & \mathcal{K} \end{array}$$

To determine $f_{(A,T)}$ we set

$$A_1 = ATA^{-1}.$$

Then we have

$$\begin{aligned} dA_1 &= (dA)TA^{-1} + A(dT)A^{-1} - ATA^{-1}(dA)A^{-1} \\ &= AT(Ad(T^{-1}) - 1)\delta A \cdot A^{-1} + AT\delta TA^{-1} \end{aligned}$$

where $\delta A = A^{-1}dA$, $\delta T = T^{-1}dT$. Thus we obtain

$$\delta A_1 = A_1^{-1} dA_1 = ad(A)((Ad(T^{-1}) - 1)\delta A + \delta T).$$

Therefore, for $X\in\mathcal{K}$, $Y\in\mathcal{T}$,

$$f_{(A,T)}(X, Y)=ad(A)(Ad(T^{-1})-1)X+Y)$$
$$=ad(A)g_T(\pi_1\oplus id_{\mathcal{T}})(X, Y) \tag{16}$$

where g_T denotes the endomorphism $(Ad(T^{-1})-1)|_{\mathcal{K}_1}\oplus id_{\mathcal{T}}$ of $\mathcal{K}(=\mathcal{K}_1\oplus\mathcal{T})$. Hence

$$\begin{aligned}\wedge^t f_{(A,T)}(v_0) &= \wedge^t(\pi_1\oplus id_{\mathcal{T}})(\wedge^t g_T(v_0))\\ &=D_{\mathcal{K}}(g_T)((\wedge^t\pi_1)\otimes id_{\wedge\mathcal{T}})(v_0'\otimes v_0'')\\ &=\sigma(T)\wedge^t\pi_1(v_0')\otimes v_0''\end{aligned} \tag{17}$$

since $D_{\mathcal{K}}(g_T)=\sigma(T)$. Consequently we deduce from (10), (11), (17) and diagram $(*)_1$,

$$\phi^*(v)=\sigma v'\otimes v^2.$$

Thus Lemma 3.1.1 is proved.

Now let $\boldsymbol{K}_r$ be the set of elements of $\boldsymbol{K}$ whose characteristic values are mutually distinct, then $\varphi^{-1}(\boldsymbol{K}_r)=\boldsymbol{K}/\boldsymbol{T}\times\boldsymbol{T}$, where $\boldsymbol{T}_r=\boldsymbol{K}_r\cap\boldsymbol{T}$. Obviously the measures of $\boldsymbol{K}\backslash\boldsymbol{K}_r$, $N\backslash\varphi^{-1}(\boldsymbol{K}_r)$ are zero. Therefore for a suitable generator $[N]$ of $H_{n^2}(N, Z)$ we have

$$(n!)^{-1}\int_{[N]}\sigma v_1\otimes v^2=\int_{[\boldsymbol{K}]}v$$

since $\varphi|_{\varphi^{-1}(\boldsymbol{K}_r)}$ is an $n!$-hold covering. Thus applying Fubini's theorem, for the generator $[\boldsymbol{T}]$ of $H_n(\boldsymbol{T}, Z)$ such that $[\boldsymbol{K}/\boldsymbol{T}]\otimes[\boldsymbol{T}]=[N]$, we obtain

$$\int_{[\boldsymbol{T}]}\sigma v^2=n!. \tag{18}$$

However, for a suitable constant c,

$$v^2=c\overline{t_1t_2\cdots t_n}\,dt_1\cdots dt_n$$

where we denote by t_j the function $T\to T_{jj}$ $(T\in\boldsymbol{T})$. Consider the mapping s from $\boldsymbol{R}^n$ onto $\boldsymbol{T}$ defined by

$$s(\varphi_1,\cdots,\varphi_n)=\begin{pmatrix}e^{i\varphi_1} & & \\ & \ddots & \\ & & e^{i\varphi_n}\end{pmatrix}.$$

Then

$$s^*(\sigma v^2)=i^n c\prod_{j<k}|e^{i\varphi_j}-e^{i\varphi_k}|^2 d\varphi_1\wedge\cdots\wedge d\varphi_n.$$

Simple computation shows

$$\int_0^{2\pi}\cdots\int_0^{2\pi}\prod_{j<k}|e^{i\varphi_j}-e^{i\varphi_k}|\,d\varphi_1\cdots d\varphi_n=(2\pi)^n n!.$$

Therefore we have by (18) $c=(2\pi i)^{-n}$, and hence

(19) $$v^2=(2\pi i)^{-n}\overline{t_1\cdots t^n}\,dt_1\wedge\cdots\wedge dt_n.$$

Now let τ_ρ $(0<\rho<1)$ denote the diffeomorphism of $\boldsymbol{T}$ onto itself

$$T\to(T-\rho I_n)(I_n-\rho T)^{-1}.$$

Then simple calculation shows

$$\tau_\rho^*(\sigma v^2)=(\prod_{j=1}^n P(\rho,t_j))^n\sigma v^2$$

where

$$P(\rho,t)=(1-\rho^2)/|1-\rho t|^2 \quad \text{for } t \text{ such that } |t|\leq 1.$$

Set $\omega_\rho=\tau_\rho^*(\sigma v)/n!$, then we have the following identities:

(20) $$\omega_\rho=(\prod_{j=1}^n P(\rho,t_j))^n\omega_0$$

(21) $$\int_{[\boldsymbol{T}]}\omega_\rho=1$$

(23) $$n!\,v^1\otimes\omega_\rho=\varphi^*(\mathfrak{p}_\rho v)$$

where $\mathfrak{p}_\rho$ is the function on $\boldsymbol{K}$ given by

$$\mathfrak{p}_\rho(A)=(1-\rho^2)^{n^2}/|D(I_n-\rho A)|^{2n} \qquad A\in\boldsymbol{K}.$$

(21) follows from (18), and (23) follows from the fact that

$$(\prod_{j=1}^n P(\rho,t_j))^n=\varphi^*(\mathfrak{p}_\rho).$$

From (23) we can deduce

(24) $$\int_{[\boldsymbol{K}]}\mathfrak{p}_\rho v=1.$$

Let us now proceed to prove an approximation theorem for continuous functions on $\boldsymbol{T}$. Define a neighbourhood of the identity W_δ $(0<\delta<2)$ setting

$$W_\delta=\{T\in \boldsymbol{T};\ \mathrm{Re}(1-t_j)<\delta\}.$$

Then the first approximation theorem is the following

Theorem 3.1.1. *There exists a continuous function* $c(\delta,\rho)>0$ $(0<\delta<2,\ 0<\rho<1)$ *such that*

$$\lim_{\rho\to 1} c(\delta,\rho)=0 \qquad \textit{for fixed } \delta. \tag{25}$$

$$\left|\int_{[T]} f\omega_\rho\right|\leq \sup_{T\in W_\delta}|f(T)|+c(\delta,\rho)\|f\|_{\boldsymbol{T}} \qquad f\in c(\boldsymbol{T}) \tag{26}$$

where $\|f\|_{\boldsymbol{T}}=\sup_{T\in\boldsymbol{T}}|f(T)|$.

Proof. Set

$$\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_m)=(\prod_{j=1}^m P(\rho,e^{i\varphi_j}))^m\prod_{1\leq j<k\leq m}|e^{i\varphi_j}-e^{i\varphi_k}|^2.$$

Then by (21) we have

$$\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_m)d\varphi_1\cdots d\varphi_m=(2\pi)^m m! \tag{27}$$

and

$$(2\pi)^n n!\left|\int f\omega_\rho\right|=\left|\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}g(\varphi_1,\cdots,\varphi_n)\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_n)d\varphi_1\cdots d\varphi_n\right|$$

where $g=s^*(f)$.

If $1-\cos\varphi_j<\delta$ for $j=1,2,\cdots,n$, then

$$|g(\varphi_1,\cdots,\varphi_n)|\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_n)\leq\sup_{T\in W_\delta}|f(T)|\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_n). \tag{28}$$

If $1-\cos\varphi_1>\delta$, then

$$|g(\varphi_1,\cdots,\varphi_n)|\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_n)\leq C(1-\rho)(\rho\delta)^{-n}\|f\|_{\boldsymbol{T}}\mathfrak{p}(\rho,\varphi_2,\cdots,\varphi_n) \tag{29}$$

for some constant $C>0$, for we have then

$$\begin{aligned}&\mathfrak{p}(\rho,\varphi_1,\cdots,\varphi_n)/\mathfrak{p}(\rho,\varphi_2,\cdots,\varphi_n)\\ &\quad=P(\rho,e^{i\varphi_1})^n\prod_{1<j}P(\rho,e^{i\varphi_j})\prod_{1<k}|e^{i\varphi_1}-e^{i\varphi_k}|^2\\ &\quad\leq\left(\frac{1-\rho}{2\delta\rho}\right)^n\left(\frac{2}{1-\rho}\right)^{n-1}\cdot 2^{2(n-1)}=C(1-\rho)/(\rho\delta)^n\end{aligned}$$

using the following inequalities:

$$p(\rho, e^{i\varphi_j}) = \frac{1-\rho^2}{(1-\rho)^2 + 2\rho(1-\cos\varphi_j)} \leq \frac{1-\rho^2}{(1-\rho)^2} \leq \frac{2}{1-\rho}$$

$$p(\rho, e^{i\varphi_1}) = \frac{1-\rho^2}{(1-\rho)^2 + 2\rho(1-\cos\varphi_1)} \leq \frac{1-\rho^2}{2\rho(1-\cos\varphi_1)} \leq \frac{1-\rho}{\rho\delta}.$$

Thus we obtain, if one sets $1-\cos\delta' = \delta$ $0 < \delta' < \pi$,

$$\begin{aligned}
&(2\pi)^n n! \left|\int f\omega_\rho\right| \\
&\leq \int_{|\varphi_1|<\delta'} \cdots \int_{|\varphi_n|<\delta'} g(\varphi_1, \cdots, \varphi_n) \,|\, \mathfrak{p}(\rho, \varphi_1, \cdots, \varphi_m) d\varphi_1 \cdots d\varphi_n \\
&+ \int_{1-\cos\varphi_1 \leq \delta} d\varphi_1 \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |g(\varphi_1, \cdots, \varphi_n)| \,\mathfrak{p}(\rho, \varphi_1, \cdots, \varphi_n) d\varphi_2 \cdots d\varphi_n \\
&+ \cdots + \int_{1-\cos\varphi_n \leq \delta} d\varphi_n \int_{-\pi}^{\pi} \cdots \int_{-\pi}^{\pi} |g(\varphi, \cdots, \varphi_n)| \,\mathfrak{p}(\rho, \varphi_1, \cdots, \varphi_n) d\varphi_1 \cdots d\varphi_{n-1} \\
&\leq (2\pi)_n n! \sup_{T \in W_\delta} |f(T)| + (2\pi)^n n! \, C(1-\rho)(\rho\delta)^{-n} \|f\|_T
\end{aligned}$$

where we used (27), (28), (29) in the last step. Thus

$$\left|\int f\omega_\rho\right| \leq \sup_{T \in W_\delta} |f(T)| + c(\rho, \delta) \|f\|_T$$

where $c(\rho, \delta) = \dfrac{C(1-\rho)}{\rho^n \delta^n}$. Q.E.D.

Now we shall prove an approximation theorem for continuous functions on $\boldsymbol{K}$. First we prepare some notations. Given a continuous function f on $\boldsymbol{K}$, let f_ρ $(0<\rho<1)$ denote the continuous function f_ρ on $\boldsymbol{K}$ given by

$$(30) \qquad f_\rho(A) = \int_{[\boldsymbol{K}]} L_A^*(f) \mathfrak{p}_\rho v \qquad A \in \boldsymbol{K}$$

where L_A denotes the left multiplication by A. In view of the invariance of v, f_ρ can be also given by

$$(31) \qquad f_\rho(A) = \int_{[\boldsymbol{K}]} f L_{A^{-1}}^*(\mathfrak{p}_\rho) v \qquad A \in \boldsymbol{K}.$$

From (30) we have

$$(32) \qquad L_A^*(f_\rho) = (L_A^*(f))_\rho \qquad A \in \boldsymbol{K}$$

and from (31) and the fact that $L_B^*(\mathfrak{p}_\rho) = R_B^*(\mathfrak{p}_\rho)$ $B \in \boldsymbol{K}$ it follows

(33) $$R_A^*(f_\rho)=(R_A^*(f))_\rho \qquad A\in \boldsymbol{K}.$$

(31) shows also the real analyticity of f_ρ since $L_{A^{-1}}^*(\mathfrak{p}_\rho)$ is real analytic with respect to A. Now our second approximation theorem is the following

Theorem 3.1.2. $\|f_\rho-f\|_{\boldsymbol{K}}=\sup_{A\in\boldsymbol{K}}|f_\rho(A)-f(A)|\to 0$ *as* $\rho\to 1$.

Proof. Denote by V_δ the neighbourhood of the identity in $\boldsymbol{K}$ given by

$$V_\delta=\{A\in\boldsymbol{K}: \mathrm{Re}(1-\xi)<\delta \text{ for any } \xi \text{ such that } D(\xi I_n-A)=0\},$$

that is, $V_\delta=\varphi(\boldsymbol{K}/\boldsymbol{T}\times W_\delta)$ (or $\boldsymbol{K}/\boldsymbol{T}\times W_\delta=\varphi^{-1}(V_\delta)$). Then $\{V_\delta\}_{0<\delta<2}$ forms a complete system of neighbourhoods at the identity of $\boldsymbol{K}$. Since $\boldsymbol{K}$ is a compact group, f is equicontinuous on $\boldsymbol{K}$, i.e., for any $\varepsilon>0$ there exists δ $(0<\delta<2)$ such that

(34) $$\sup_{A\in V_\delta}|L_A^*(f)(B)-f(A)|\leq\varepsilon.$$

On the other hand we have by (24)

(35) $$f_\rho(A)-f(A)=\int_{\boldsymbol{K}}(L_A^*(f)-f(A))\mathfrak{p}_\rho v.$$

Therefore, if one sets $g_A=\varphi^*(L_A^*(f)-f(A))$, then by (34)

$$\sup_{(m,T)\in\boldsymbol{K}/\boldsymbol{T}\times W_\delta}|g_A(m,T)|\leq\varepsilon,$$

in particular

(36) $$\sup_{T\in W_\delta}|\hat{g}_A(T)|\leq\varepsilon$$

where we have set

$$\hat{g}_A(T)=\int_{[\boldsymbol{K}/\boldsymbol{T}]}g_A(m,T)v^1.$$

Using Fubini's theorem we deduce from (23), (35)

$$\begin{aligned} f_\rho(A)-f(A)&=\int_{[N]}g_A v^1\otimes\omega_\rho \\ &=\int_{[T]}\hat{g}_A\omega_\rho\,. \end{aligned}$$

Therefore by Theorem 3.1.1 and (36)

$$|f_\rho(A)-f(A)|\leq\varepsilon+c(\rho,\delta)\|\hat{g}_A\|_T$$
$$\leq\varepsilon+2c(\rho,\delta)\|f\|_K.$$

Since the right hand side is independent of A, we obtain

$$\|f_\rho-f\|_K\leq\varepsilon+2c(\rho,\delta)\|f\|_K.$$

Letting ρ tend to 1 from below,

$$\overline{\lim_{\rho\to1}}\|f_\rho-f\|_K\leq\varepsilon.$$

Since ε is arbitrary, we finally obtain

$$\overline{\lim_{\rho\to1}}\|f_\rho-f\|_K=0. \qquad \text{Q.E.D.}$$

Corollary 3.1.1. *For $f\in C(\boldsymbol{K})$ there exists a holomorphic function f_ρ^* on $\boldsymbol{G}_\rho$ such that $f_\rho^*|_K=f_\rho$, where $\boldsymbol{G}_\rho$ denotes the open set*

$$\{Z\in\boldsymbol{G};\ \rho^{-2}I_n>ZZ^*>\rho^2I_n\}.$$

To prove this we assume the following lemma.

Lemma $\boldsymbol{H}_1$ (Hua [4]). *For $Z\in\boldsymbol{G}_\rho$ and for $A\in\boldsymbol{K}$, the matrices*

$$I_n-\rho Z^{-1}A,\quad I_n-\rho A^{-1}Z$$

are non-singular.

Proof of Corollary 3.1.1. In view of this lemma define $\mathfrak{p}_\rho^Z\in C(\boldsymbol{K})$, setting

$$\mathfrak{p}_\rho^Z(A)=(1-\rho^2)^{n^2}/(D(I_n-\rho Z^{-1}A)D(I_n-\rho A^{-1}Z))^n\quad Z\in\boldsymbol{G}_\rho. \tag{37}$$

Note that $\mathfrak{p}_\rho^A=L_{A}^*(\mathfrak{p}_\rho)$ for $A\in\boldsymbol{K}$. Thus, if one sets for $Z\in\boldsymbol{G}_\rho$

$$f_\rho^*(Z)=\int f\mathfrak{p}_\rho^Z v, \tag{38}$$

it holds $f_\rho^*|_K=f_\rho$ and f_ρ^* is holomorphic since $\mathfrak{p}_\rho^Z(A)$ is holomorphic with respect to Z in $\boldsymbol{G}_\rho$. Q.E.D.

3.2. Application to Some Type of Vector Fields on Trivial $U(n)$-Bundle. Here we shall study how the map $f\to f_\rho$ (resp $f\to f_\rho^*$) relates left $\boldsymbol{K}$-invariant vector fields on the product of $\boldsymbol{K}$ (resp $\boldsymbol{G}_\rho$) with an open subset in $\boldsymbol{R}^N$.

Let $\mathfrak{p}_\rho^*$ denote the function on $\boldsymbol{K}\times\boldsymbol{G}_\rho$

$$(A, Z)\to\mathfrak{p}_\rho^Z(A).$$

Then $\mathfrak{p}_\rho^*$ satisfies the following relation

$$\mathfrak{p}_\rho^*(AB, Z)=\mathfrak{p}_\rho^*(A, ZB^{-1})=\mathfrak{p}_\rho^*(B, A^{-1}Z). \tag{39}$$

Choose a $\boldsymbol{K}$-invariant volume v^* on $\boldsymbol{G}$ (which certainly exists since $\boldsymbol{K}$ is compact) and for $\psi\in C_0^\infty(\boldsymbol{G}_\rho)$ define a function on $\boldsymbol{K}$ setting

$$\psi_{(\rho)}(A)=\int_{\boldsymbol{G}}\psi\mathfrak{p}_{(\rho)}^A v^* \qquad A\in K \tag{40}$$

where $\mathfrak{p}_{(\rho)}^A$ is the function on $\boldsymbol{G}_\rho$ defined by $\mathfrak{p}_{(\rho)}^A(Z)=\mathfrak{p}_\rho^*(A, Z)$. Here, for definiteness, the orientation of $\boldsymbol{G}$ in the integration should be the natural orientation of $\boldsymbol{G}$ as a complex manifold. From the invariance of v^* and from (39) it follows that

$$L_A(\psi_{(\rho)})=(L_A\psi)_{(\rho)} \tag{41}$$

$$R_A(\psi_{(\rho)})=(R_A\psi)_{(\rho)} \tag{42}$$

where $A\in\boldsymbol{K}$. From (33), (41) we obtain

Proposition 3.2.1. *For* $A\in\boldsymbol{K}$

$$\widetilde{A}(\psi_{(\rho)})=(\widetilde{A}\psi)_{(\rho)} \qquad \psi\in C_0^\infty(\boldsymbol{G}_\rho) \tag{43}$$

$$\widetilde{A}(f_\rho)=(\widetilde{A}f)_\rho \qquad f\in C^\infty(\boldsymbol{K}). \tag{43'}$$

In (43) $\widetilde{A}$ in the left hand side should be the left invariant vector field corresponding to A on $\boldsymbol{K}$ and $\widetilde{A}$ in the right should be that on $\boldsymbol{G}$.

Later we also need

Proposition 3.2.2. *For* $A\in\boldsymbol{K}$ *and for* $\varphi, \psi\in C_0^\infty(\boldsymbol{G})$

$$\int_{\boldsymbol{G}}(\widetilde{A}\varphi)\psi v^*+\int_{\boldsymbol{G}}\varphi(\widetilde{A}\psi)v^*=0. \tag{44}$$

Proof. Note that

$$\int_{\boldsymbol{G}}((\widetilde{A}\varphi)\psi+\varphi(\widetilde{A}\psi))v^*=\int_{\boldsymbol{G}}\widetilde{A}(\varphi\psi)v^*=\left[\frac{d}{dt}\int_{\boldsymbol{G}}R^*_{\exp tA}(\varphi\psi)v^*\right]_{t=0}.$$

But $\int_{\boldsymbol{G}}R^*_{\exp tA}(\psi)v^*$ is independent of t because of the $\boldsymbol{K}$-invariance of

v^*, therefore (44) is proved.

In the same way we obtain

Proposition 3.2.2′. *For $A\in\boldsymbol{K}$ and for f, $g\in C_0^\infty(\boldsymbol{K})$*

$$\int_{\boldsymbol{K}}(\widetilde{A}f)gv+\int_{\boldsymbol{K}}f(\widetilde{A}g)v=0. \tag{44}'$$

Now let Ω be a domain in $\boldsymbol{R}^N$ and denote by M, $\widetilde{M}$ the product manifolds $\boldsymbol{K}\times\Omega$, $\boldsymbol{G}\times\Omega$, respectively, then $T_{(A,x)}(M)$, $T_{(Z,x)}(\widetilde{M})$ can be canonically identified with $T_A(\boldsymbol{K})\oplus T_x(\Omega)$, $T_Z(\boldsymbol{G})\oplus T_x(\Omega)$ respectively and under this identification we call a vector field on M (resp $\widetilde{M}$) a *vertical vector field* if its value at each point (A, x) of M (resp (Z, x) of $\widetilde{M}$) lies in $T_A(\boldsymbol{K})$ (resp $T_Z(\boldsymbol{G})$), further a vector field on M (resp $\widetilde{M}$) is called a *horizontal vector field* if its value at each point (A, x) of M (resp (Z, x) of $\widetilde{M}$) lies in $T_x(\Omega)$. Then every vector field X on M (or on $\widetilde{M}$) can be uniquely written as the sum of a vertical vector field and a horizontal vector field. The former is called the *vertical part* of X and denoted by X^v and the latter is called the *horizontal part* of X and denoted by X^h. A vector field X on M (resp $\widetilde{M}$) is called *left invariant* if, for any A of $\boldsymbol{K}$ (resp Z of $\boldsymbol{G}$),

$$dL_A(X)=X \quad (\text{resp. } dL_Z(X)=X)$$

where L_A, L_Z are transformations given by

$$L_A(A', x)=(AA', x) \qquad (A', x)\in M,$$

$$L_Z(Z', x)=(ZZ', x) \qquad (Z', x)\in\widetilde{M}.$$

The horizontal part and the vertical part of a left invariant vector field X on M (resp. on $\widetilde{M}$) can be written as follows:

$$X^h=\sum_{i=1}^N c_i(x)\frac{\partial}{\partial x_i}$$

$$X^v=\widetilde{A(x)} \quad (\text{resp } \widetilde{Z(x)})$$

where $(x_1, \cdots, x_N)$ is the system of coordinates of $\boldsymbol{R}^N$ and $c_i(x)$ are C^∞ functions on Ω, and $A(x)$ (resp $Z(x)$) is a $\mathcal{K}$ (resp $\mathcal{G}$)-valued function on Ω. Here $\widetilde{A(x)}$ means the vector field which assigns to each point

(A, x) of M a vector $(\widetilde{A(x)})_A$ in $T_{(A,x)}(M)=T_A(\boldsymbol{K})\oplus T_x(\Omega)$: $\widetilde{Z(x)}$ should be interpreted in the same way. Note that also $A(x)$ can be regarded as a $\mathcal{G}$-valued function, therefore for a left invariant vector field X on M, there uniquely exists a left invariant vector field $\widehat{X}$ on $\widetilde{M}$ such that

$$\widehat{X}_p = X_p \quad \text{for } p \in M.$$

We call this $\widehat{X}$ the *extension* of X. A left invariant vector field is called *typical* if X^h has constant coefficients, i.e., it can be written as follows:

$$X^h = \sum_{i=1}^{N} c_i \frac{\partial}{\partial x_i} \quad (c_1, \cdots, c_N \text{ are constants}).$$

Now let $\boldsymbol{v}$, $\boldsymbol{v}^*$ be volumes on M, $\widetilde{M}$ defined by

$$\boldsymbol{v} = dx \otimes v, \quad \boldsymbol{v}^* = dx \otimes v^*$$

where $dx = dx^1 \cdots ex_N$. Then we have

Proposition 3. 2. 3. *The formal adjoint with respect to volume* $\boldsymbol{v}$ (resp $\boldsymbol{v}^*$) *of a typical vector field* X *on* M (resp $\widehat{X}$ *on* $\widetilde{M}$) *is* $-X$ (resp $-\widehat{X}$), *i.e.*,

$$(45) \qquad \int_M (Xf) g \boldsymbol{v} + \int_M f(Xg) \boldsymbol{v} = 0 \qquad f, g \in C_0^\infty(M)$$

$$\left(\text{resp} \int_{\widetilde{M}} (\widehat{X}\varphi) \psi \boldsymbol{v}^* + \int_{\widetilde{M}} \varphi(\widehat{X}\varphi) \boldsymbol{v}^* = 0 \qquad \varphi, \psi \in C_0^\infty(\widetilde{M})\right).$$

Proof. (45) for the horizontal parts of X, $\widehat{X}$ is obvious, and for the vertical parts it is an immediate consequence of Propositions 3. 2. 1, 3. 2. 2 and Fubini's theorem.

Now we shall extend mappings $f \to f_\rho$, $f \to f_\rho^*$ for $f \in C(\boldsymbol{K})$ and mapping $\psi \to \psi_{(\rho)}$ for $\psi \in C_0^\infty(\boldsymbol{G}_\rho)$ into mappings of $C(M)$ and of $C_0^\infty(\widetilde{M}_\rho)$ where $\widetilde{M}_\rho = \boldsymbol{G}_\rho \times \Omega$. For $f \in C(M)$ and $x \in \Omega$ let f^x denote the function on $\boldsymbol{K}$ given by $f^x(A) = f(A, x)$ $(A \in \boldsymbol{K})$, and define f_ρ by

$$f_\rho(A, x) = (f^x)_\rho(A)$$

and f_ρ^* setting

$$f_\rho^*(Z, x) = (f^x)_\rho^*(Z) \qquad Z \in \boldsymbol{G}_\rho.$$

In the same way we shall define $\psi_{(\rho)}$ for $\psi \in C_0^\infty(\widetilde{M}_\rho)$:

$$\psi_{(\rho)}(A, x) = (\psi^x)_{(\rho)}(A) \qquad A \in \boldsymbol{K},\ x \in \Omega$$

where ψ^x denotes the function on G_ρ given by $\psi^x(Z) = \psi(Z, x)$. Then we obtain an analogy of Theorem 3.1.2.

Theorem 3.1.2′. *For $f \in C(M)$, f_ρ tends to f in the topology of $C(M)$ as $\rho \nearrow 1$.*

Proof. Let K be a compact subset of Ω, then the set $\{f^x : x \in K\}$ is compact in $C(\boldsymbol{K})$ since the map $x \to f_x$ is continuous. But the convergence $g_\rho \to g$ in $C(\boldsymbol{K})$ is uniform on any compact subset of $C(\boldsymbol{K})$ since by definition

$$\|g_\rho\|_{\boldsymbol{K}} \leq \|g\|_{\boldsymbol{K}} \qquad g \in C(\boldsymbol{K}).$$

Thus

$$\sup_{(A,x) \in \boldsymbol{K} \times K} |f_\rho(A, x) - f(A, x)| \to 0 \quad \text{as } \rho \nearrow 1.$$

Q.E.D.

Now (43), (43)′ are generalized as follows.

Proposition 3.2.4. *If X is a typical vector field on M, then*

(46) $$(Xf)_\rho = X(f_\rho) \qquad f \in C^\infty(M)$$

(47) $$(\widehat{X}\Psi)_{(\rho)} = X(\psi_{(\rho)}) \qquad \psi \in C_0^\infty(\widetilde{M}_\rho).$$

These formulae are evident by Proposition 3.2.1 and the structure of typical vector fields on M.

Theorem 3.2.1. *Let X be a typical vector field on M and f be a continuous function on M such that $Xf = 0$ in the distribution sense. Then we have $Xf_\rho = 0$, $\widehat{X}f_\rho^* = 0$.*

Proof. Continuous mappings $f \to X(f_\rho)$, $f \to (Xf)_\rho$ of $C(M)$ into $\mathcal{D}'(M)$ coincide on dense subset $C^\infty(M)$ of $C(M)$, therefore there coincide on the whole space $C(M)$. $X(f_\rho) = 0$ is proved. In view of Proposition 3.2.2 $\widehat{X}(f_\rho^*) = 0$ is equivalent to $\int_{\widetilde{M}} f_\rho^* (\widehat{X}\Psi) v^* = 0$ $(\forall \psi \in C_0^\infty(\widetilde{M}_\rho))$. But by Fubini's theorem

$$\int_{\widetilde{M}} f_\rho^* (\widehat{X}\psi) v^* = \int_M f(\widehat{X}\psi)_{(\rho)} v = \int_M f X(\psi_{(\rho)}) v$$

where in the last step we have used Proposition 3.2.4. Since $\psi_{r(\rho)}$ has compact support, we have by the hypothesis

$$\int_{\tilde{M}} f_\rho^* (\hat{X}\psi_r) v^* = 0. \qquad \text{Q.E.D.}$$

Remark. Evidently Propositions 3.2.3 and 3.2.4 and Theorem 3.2.1 are still valid for a differential operator $X+iX'$, where X, X' are typical vector fields. Such an operator will be called also a typical vector field.

To end this section we add an interesting approximation theorem for solutions of certain overdetermined system of differential equations, but we do not use this in later discussion.

Theorem 3.2.2. *Let $X_1, \cdots, X_N$ be real analytic typical vector fields such that the differential operator with constant coefficients $X_1^h, \cdots, X_N^h$ are linearly independent. Then the set of real analytic solutions of the system of equations*

$$X_1 f = 0, \cdots, X_N f = 0$$

is dense in the space of all continuous solutions of it with respect to the relative topology induced from $C(M)$.

Proof. In view of Theorem 3.1.1′ it suffices to prove that f_ρ is real analytic for a solution f, which follows immediately from the real analyticity of f_ρ^*. By Theorem 3.2.1 we have

$$\hat{X}_1 f_\rho^* = 0, \cdots, \hat{X}_N f_\rho^* = 0. \tag{48}$$

Moreover since $f_\rho^*(Z, x)$ is holomorphic in variables Z_{jk} $(1 \leq j,\ k \leq n)$, we have

$$\frac{\partial}{\partial \bar{Z}_{jk}} f_\rho^* = 0 \quad (1 \leq j,\ k \leq n). \tag{49}$$

Since the system of differential equation (48), (49) is elliptic by the assumption of the theorem, f_ρ^* is real analytic.

Q.E.D.

§4. Problem H for D_0

4.1. A Local Parametrization of D_0. In this section we investigate Problem H for the Šilov boundary D_0 of the Siegel domain of second kind D in Section 2. For this purpose we shall give a more convenient parametrization of D_0 in order to apply the results of the preceding section to D_0.

Recall that D_0 was given by

$$D_0=\left\{(Z, x)\in\mathcal{G}\times V;\ \frac{1}{i}(Z-Z^*)-x\otimes x^*>0\right\}$$

where V is an n-dimensional complex Hilbert space and $\mathcal{G}$ the endomorphism ring of V. Fixing an orthonormal base $\{e_1, \cdots, e_n\}$ of V as before, we shall identify V with $\boldsymbol{C}^n$, $\mathcal{G}$ with $M_n(\boldsymbol{C})$, where $\boldsymbol{C}^n$ should be regarded as the space of column vectors with n components. Let $\boldsymbol{H}$ denote the subgroup of $\boldsymbol{G}$ given by

$$\boldsymbol{H}=\left\{\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix};\ h_1, \cdots, h_n\in\boldsymbol{C}\setminus\{0\}\right\}$$

and $\mathfrak{h}$ the Lie algebra of $\boldsymbol{H}$, i.e.,

$$\mathfrak{h}=\left\{\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix};\ h_1, \cdots, h_n\in\boldsymbol{C}\right\}.$$

We set further

$$\mathfrak{h}_r=\left\{\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix}\in\mathfrak{h};\ h_j\neq h_k \text{ if } j\neq k\right\}.$$

Now define a mapping ϕ of $\boldsymbol{G}\times\mathfrak{h}_r$ into $\mathcal{G}\times V$ by setting

$$\varphi(Z, H)=(ZHZ^{-1}, Z\cdot x_0);\quad Z\in\boldsymbol{G},\ H\in\mathfrak{h}_r$$

where $x_0=\begin{pmatrix}1\\ \cdots \\ 1\end{pmatrix}$. Then by simple calculation one can easily see that $(d\varphi)_p$ is non-singular at each point p of $\boldsymbol{G}\times\mathfrak{h}_r$. Let us determine $\phi(\boldsymbol{G}\times\mathfrak{h}_r)$. For this purpose let $\mathcal{G}_r$ be the set of elements in $\mathcal{G}$ whose characteristic polynomial has mutually distinct roots, i.e., $\mathcal{G}=\{Z\in\mathcal{G}$: the discriminant of equation in λ $D(\lambda I_n-Z)=0$ is non-zero$\}$ and for any permutation σ of n letters $\{1, 2, \cdots, n\}$ define $(\sigma) \in \boldsymbol{G}$ as follows:

$$(\sigma)_{jk} = \begin{cases} 1 & \text{if } \sigma(k) = j \\ 0 & \text{if } \sigma(k) \neq j. \end{cases}$$

Further we shall denote by $\boldsymbol{W}$ the set of all (σ). Then $\boldsymbol{W}$ is a finite subgroup of $\boldsymbol{G}$ and we have

$$(\sigma)\boldsymbol{H}(\sigma)^{-1} = \boldsymbol{H}.$$

Therefore $\boldsymbol{N} = \boldsymbol{W}\boldsymbol{H} = \boldsymbol{H}\boldsymbol{W}$ is a subgroup of $\boldsymbol{G}$. It is well-known that $\boldsymbol{N}$ is the normalizer of $\boldsymbol{H}$ in $\boldsymbol{G}$. Obviously

$$\boldsymbol{N}/\boldsymbol{H} \cong \boldsymbol{W}.$$

Suppose that $\zeta \in \mathcal{G}_r$, then, by Linear Algebra, there exists an element Z of $\boldsymbol{G}$ and an element H of $\mathfrak{h}_r$ such that

$$ZHZ^{-1} = \zeta.$$

Define a mapping S: $\mathcal{G}_r \to \boldsymbol{G}/\boldsymbol{N}$ setting

$$S(\zeta) = \pi(Z)$$

where π is the canonical projection of $\boldsymbol{G}$ onto $\boldsymbol{G}/\boldsymbol{N}$. The $S(\zeta)$ is independent of the choise of such Z, and S is well-defined. It is also evident that S is a *holomorphic* mapping. Define a function f_0 on V setting

$$f_0(x) = x_1^2 \cdots x_n^2 \qquad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

and set for $(Z, x) \in \boldsymbol{G} \times V$

$$f_1(Z, x) = f_0(Z^{-1}x).$$

Then $f_1(ZN, x) = (D(N))^{-2} f_1(Z, x)$ for $N \in \boldsymbol{N}$. Therefore, if one sets

$$f_2(Z, x) = D(Z)^2 f_1(Z, x),$$

we have $f_2(ZN, x) = f_2(Z, x)$. Hence there exists a function f_3 defined on $\boldsymbol{G}/\boldsymbol{N} \times V$ such that

$$f_3(\pi(Z), x) = f_2(Z, x).$$

Set $f(\zeta, x)=f_3(S(\zeta), x)$ for $(\zeta\times x)\in\mathcal{G}_r\times V$. Then f is a *holomorphic* function on $\mathcal{G}_r\times V$.

Let R be the subset of $\mathcal{G}_r\times V$ defined as follows

$$R=\{(\zeta, x)\in\mathcal{G}_r\times V;\ f(\zeta, x)\neq 0\}.$$

Since the complement of $\mathcal{G}_r\times V$ is an analytic variety and since f is holomprphic, $D_r=R\cap D_0$ is open, dense in generic submanifold D_0. We claim that

$$R=\varphi(\boldsymbol{G}\times\mathfrak{h}_r).$$

$R\supseteq\varphi(\boldsymbol{G}\times\mathfrak{h}_r)$ is obvious, in fact $ZHZ^{-1}\in\mathcal{G}^r$, $f(ZHZ^{-1}, Zx_0=D(Z)^2\times f_0(x_0)\neq 0$ for $(Z, H)\in\boldsymbol{G}\times\mathfrak{h}_r$. To show $R\subseteq\varphi(\boldsymbol{G}\times\mathfrak{h}_r)$ note that for any $(\zeta, x)\in R$ there exists $(Z, H)\in\boldsymbol{G}\times\mathfrak{h}_r$ such that

(1) $ZHZ^{-1}=\zeta$.

Then we have

$$f_0(Z^{-1}x)\neq 0$$

since $f(\zeta, x)=f_3(\pi(Z), x)=f_2(Z, x)=f_0(Z^{-1}x)D(Z)^2$. By the definition of f_0 there exists H_0 of $\boldsymbol{H}$ such that

(2) $H_0^{-1}Z^{-1}x=x_0$.

By (1), (2) $\varphi(ZH_0, H)=(\zeta, x)$. Thus $R\subseteq\varphi(\boldsymbol{G}\times\mathfrak{h}_r)$ is proved. As noted before, φ is holomorphic and regular. While by an easy computation we have

$$\varphi^{-1}(\varphi, H)=\{(Z\cdot(\sigma), H);\ (\sigma)\in\boldsymbol{W}\}.$$

Therefore φ is an $n!$-fold convering onto R. Set $\check{D}=\varphi^{-1}(D_r)$. Then $\check{D}$ is a real submanifold of $\boldsymbol{G}\times\mathfrak{h}_r$ which is in local holomorphically equivalent to D_r since φ is *holomorphic.* Thus we may consider our problem for $\check{D}$ instead of D_r.

Now let us determine $\check{D}$ explicitly. Suppose $(Z, H)\in\check{D}$. By Linear Algebra Z can be written uniquely as follows

$$Z=AP$$

where $A\in\boldsymbol{K}$, P is a positive definite hermitian matrix. Then by the difinition of D_0 we have

$$0=\frac{1}{i}(ZHZ^{-1}-{}^{t}\overline{(ZHZ^{-1})})-Zx_0{}^{t}\bar{x}_0{}^{t}\overline{Z}$$
$$=A\Big(\frac{1}{i}(PHP^{-1}-P^{-1}HP)-Px_0{}^{t}\bar{x}P\Big)A^{-1}.$$

Thus

$$PHP^{-1}-P^{-1}HP=iPx_0{}^{t}\bar{x}_0P,$$

or equivalently

$$HP^{-2}-P^{-2}H=ix_0{}^{t}\bar{x}_0.$$

Hence, if we set $Q=P^{-2}$, we have

$$(h_j-\overline{h}_k)Q_{jk}=i \qquad (j,k=1,2,\cdots,n)$$

where

$$H=\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix}, \qquad Q=(Q_{jk}).$$

In particular because of the positive definiteness we obtain

(3) $\operatorname{Im} h_j>0 \qquad (j=1,2,\cdots,n).$

Then automatically

$$h_j-\overline{h}_k\neq 0 \qquad (j,k=1,2,\cdots,n)$$

and we get

$$Q=(Q_{jk})=(i/(h_j-\overline{h}_k)).$$

Conversely for $H\in\mathfrak{h}_r$ which satisfies (3), the matrix $(i/(h_j-\overline{h}_k))$ is certainly *positive-definite*. For, determinants of its principal minors are all positive. This follows immediately from the following lemma

Lemma $\mathbf{H}_2$ (Hua [4]). *Set* $a_{jk}=\dfrac{1}{x_j+y_k}$, $a=(a_{jk})$.

Then

$$D(a)=\delta(x)\delta(y)\prod_{j,k=1}^{n}\frac{1}{x_j+y_k}$$

where $\delta(x)=\prod_{j<k}(x_j-x_k)$.

Now define $Q(H)$ by setting

$$Q(H)_{jk}=\frac{i}{h_j-\overline{h}_k}$$

where $H=\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \in \mathfrak{h}_r$ $\mathrm{Im}\, h_j > 0$ $(j=1, 2, \cdots, n)$, further set

$$P(H)=Q(H)^{-1/2}.$$

$P(H)$ is well-defined since $Q(H)$ is positive definite. Then we have from above argument

$$\check{D}=\{(AP(H), H):\ A\in K\ H\in\check{\mathfrak{h}}\}$$

where
$$\check{\mathfrak{h}}=\left\{H=\begin{pmatrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \in \mathfrak{h}_r;\ \mathrm{Im}\, h_j > 0\ \ (j=1, 2, \cdots, n)\right\}.$$

Define a real analytic diffeomorphism ψ of $\boldsymbol{G}\times\check{\mathfrak{h}}$ onto itself by setting

$$\psi(Z, H)=(ZP(H), H)\quad Z\in\boldsymbol{G},\ H\in\check{\mathfrak{h}}.$$

Further set $\bar{\psi}=\psi|_{\boldsymbol{K}\times\check{\mathfrak{h}}}$. Then $\bar{\psi}$ is a real analytic diffeomorphism of $\boldsymbol{K}\times\check{\mathfrak{h}}$ onto $\check{D}$.

4.2. Solution of Problem H for D_0. Let ω' denote the Maurer-Cartan form of $\mathfrak{h}$ when $\mathfrak{h}$ is regarded as an abelian Lie group. Define an endomorphism $f_{(Z,H)}$ of $\mathcal{G}+\mathfrak{h}$ for any $(Z, H)\in\boldsymbol{G}\times\mathfrak{h}$ so that the diagram

$$\begin{array}{ccc} T_{(Z,H)}(\boldsymbol{G}\times\check{\mathfrak{h}})=T_Z(\boldsymbol{G})\oplus T_H(\mathfrak{h}) & \xrightarrow{\ \delta Z_Z\oplus\omega'_H\ } & \mathcal{G}\oplus\mathfrak{h} \\ \Big\downarrow d\psi_{(Z,H)} & & \Big\downarrow f_{(Z,H)} \\ T_{\psi(Z,H)}(\boldsymbol{G}\times\check{\mathfrak{h}})=T_{ZP(H)}(\boldsymbol{G})\oplus T_H(\mathfrak{h}) & \xrightarrow{\ \delta Z_{ZP(H)}\oplus\omega'_H\ } & \mathcal{G}\oplus\mathfrak{h} \end{array}$$

is commutative. To determine $f_{(Z,H)}$ denote by f_H the linear mapping of $\mathfrak{h}$ into $\mathcal{G}$ so that the following diagram is commutative:

$$\begin{array}{ccc} T_H(\mathfrak{h}) & \xrightarrow{\ \omega'_H\ } & \mathfrak{h} \\ \Big\downarrow (dP)_H & & \Big\downarrow f_H \\ T_{P(H)}(\boldsymbol{G}) & \xrightarrow{\ \omega_{P(H)}\ } & \mathcal{G} \end{array}$$

and set $Z_1=ZP(H)$. Then

$$dZ_1=(dZ)P(H)+ZdP(H).$$

Therefore

$$\delta Z_1=Z_1^{-1}dZ_1=ad(P(H)^{-1})\delta Z+\delta P(H).$$

Thus

$$f_{(Z,H)}=ad(P(H)^{-1})+f_H+id_{\mathfrak{h}}.$$

From this it follows that for $A\in \boldsymbol{K}$, $H\in\mathfrak{h}$,

$$(4)\quad \omega_{AP(H)}\oplus\omega'_H(T_{\psi(A,H)}(\check{D}))$$
$$=\{(ad(p(H)^{-1})(B)+f_H(L),L);\ B\in\mathcal{K}\ L\in\mathfrak{h}\}.$$

Denote by $\mathcal{G}^{(0,1)}$ the subspace of the complexification $\mathcal{G}^C=\mathcal{G}+i\mathcal{G}$ of $\mathcal{G}$ given by

$$\mathcal{G}^{(0,1)}=\{Z+iIZ, Z\in\mathcal{G}\}$$

where I is the linear isomorphism of $\mathcal{G}$ which defines the complex structure of $\mathcal{G}$. Then we have

$$\mathcal{G}^{(0,1)}\cap\mathcal{K}^C=(0)$$
$$\mathcal{G}^{(0,1)}+\mathcal{K}^C=\mathcal{G}^C.$$

Thus by (4), for $H\in\mathfrak{h}$ and for $L\in\mathfrak{h}^{(0,1)}$ there exists a unique $B(L,H)$ of $\mathcal{K}^C$ such that

$$d\psi_{(A,H)}(\widetilde{B(L,H)}_A+\widetilde{L}_H)\in T^{(0,1)}_{(A,H)}(\boldsymbol{G}\times\mathfrak{h})\cap T^C_{\psi(A,H)}(\check{D})$$

where $\widetilde{L}$ denotes the vector field on $\mathfrak{h}$ such that $\omega'(\widetilde{L})=L$ and $\mathfrak{h}^{(0,1)}$ is the subspace similar to $\mathcal{G}^{(0,1)}$ for $\mathfrak{h}$ instead of $\mathcal{G}$. In other words we have

Theorem 4.2.1. *Let L be an element of $\mathfrak{h}^{(0,1)}$. Then there exists a unique typical vector field $X(L)$ on $M=K\times\check{\mathfrak{h}}$ such that $X(L)^h=\widetilde{L}$ and $d\Psi(X(L))$ is a section of the tangential Cauchy-Riemann bundle of $\check{D}$.*

Here we have used word 'typical' setting $\Omega=\check{\mathfrak{h}}$ in the preceding section. From this theorem we deduce

Corollary 4.2.1. *Under the notation of §3 the following system of differential equations is equivalent to the Cauchy-Riemann equation $\bar\partial u=0$ of $\boldsymbol{G}\times\mathfrak{h}$:*

$$d\psi\Big(\widehat{X}\Big(\frac{\partial}{\partial\bar h_1}\Big)\Big)u=0,\ \cdots,\ d\psi\Big(\widehat{X}\Big(\frac{\partial}{\partial\bar h_n}\Big)\Big)u=0\quad \frac{\partial}{\partial\overline{Z}_{jk}}u=0\quad(1\le j,\ k\le n).$$

Here $(h_1, \cdots, h_n)$ is a system of coordinates of $\mathfrak{h}$ and (Z_{jk}) is the system of coordinates of $\boldsymbol{G}$ as defined before.

Now let Ω be a domain of $\check{\mathfrak{h}}$, then we have an approximation theorem for $\mathcal{S}$-holomorphic functions on $\check{D}_\Omega = \psi(\boldsymbol{K}\times\Omega)$. For this purpose we have to extend mappings of $C(M_\Omega)$ $(M_\Omega = \boldsymbol{K}\times\Omega)$ $f\to f_\rho$, $f\to f_\rho^*$ to mappings of $C(\check{\boldsymbol{D}}_\Omega)$ by ψ and $\bar{\psi}$, i.e., we set for $f\in C(\check{D}_\Omega)$

$$f_\rho = (\bar{\psi}^{-1})^*(\bar{\psi}^* f)_\rho)$$

$$f_\rho^* = ((\psi|_{G_\rho\times\Omega})^{-1})^*((\bar{\psi}^* f)_\rho^*).$$

Here f_ρ^* is a function on $\widehat{M}_{\Omega,\rho} = \boldsymbol{G}_\rho\times\Omega$.

Lemma 4.2.1. *Let f be an S-holomorphic function on $\check{D}_\Omega$, then f_ρ^* is holomorphic in $\psi(\widetilde{M}_{\Omega,\rho})$.*

Proof. By Theorem 4.2.1, Corollary 4.2.1, Theorem 3.2.1 and the definition of $\mathcal{S}$-holomorphicity we have

$$\bar{\partial} f_\rho^* = 0$$

in the distribution sense. Since $\bar{\partial}$ is elliptic, f_ρ^* is holomorphic.

Q.E.D.

In view of this lemma Theorem 3.1.2′ implies

Theorem 4.2.2. *In the closed subspace of $\mathcal{S}$-holomorphic functions on $\check{D}_\Omega$ of $C(\check{D}_\Omega)$ the set of restrictions of holomorphic functions near $\check{D}_\Omega$ is dense with respect to the topology induced from $C(M)$.*

Now define an action of $\boldsymbol{K}$ on $\mathcal{G}\times V$ setting

$$\rho_0(A)(Z, x) = (AZA^*, Ax) \qquad A\in\boldsymbol{K},\ (Z, x)\in\mathcal{G}\times V.$$

Then D_0, D_r are $\boldsymbol{K}$-stable. Note that

$$\varphi\circ L_A = \rho_0(A)\circ\varphi.$$

Thus every $\boldsymbol{K}$-stable open subset S of D_r is given by $S=\varphi(\check{D}_\Omega)$ for some open subset Ω of $\check{\mathfrak{h}}$. Therefore, for any $\boldsymbol{K}$-stable open subset S of D_r the set of restrictions to S of holomorphic functions near S is dense in the space of $\mathcal{S}$-holomorphic functions on S, the same is true for $p\cdot S = \rho(p)S$ $(p\in D_0)$, ρ being defined in §2.2.

But the system of open subsets of D_0

$$\Pi=\{\rho(p)(D_r\cap B_\delta):\ p\in D_0,\ 0<\delta<+\infty\}$$

is a complete system of neighbourhoods. This can be seen as follows: Although $0\notin D_r$, we can choose a sequence (p_n) $n=1,2,\cdots$ of points of D_r such that $p_n\in B_{1/n}$ since D_r is open and dense in D_0. Then $\{\rho(p_n^{-1})(D_r\cap B_{1/n}):\ n=1,2,\cdots\}$ is certainly a complete system of neighbourhoods of 0. To see this note that for any $\delta>0$ there exists a positive integer n such that

$$(B_{1/n}\cap D_0)^{-1}(B_{1/n}\cap D_0)\subset B_\delta\cap D_0$$

because D_0 is a group manifold. Consequently for any $\delta>0$ we can choose $n\in N$ so that

$$\rho(p_n^{-1})(D_r\cap B_{1/n})\subset B_\delta\cap D_0\,.$$

Thus our assertion is proved, and hence even

$$\{\rho(pp_n^{-1})(D_r\cap B_{1/n}):\ p\in D_0,\ n=1,2,\cdots\}$$

is a complete system of neighbourhoods of D_0.

Therefore we conclude that there exists a complete system of neighbourhoods Π such that for $S\in\Pi$ $\Gamma_S(\mathcal{H}_{\mathcal{S}})=H(S)$ where $\mathcal{H}_{\mathcal{S}}$ denotes the sheaf of germs of $\mathcal{S}$-holomorphic functions, i.e., the solution sheaf of the tangential Cauchy Riemann equation. Thus we have proved that $\mathcal{H}_{\mathcal{S}}=\mathcal{H}$. Combining this with Theorem 2. 2. 1 we obtain

Theorem 4. 2. 2′.

$$\Gamma_S(\mathcal{H}_{\mathcal{S}})=\Gamma_S(\mathcal{H})=\underset{\leftarrow}{H}(S)=H_D(S)=H(U_1(S))\,|_S$$

for any open subset S of D_0.

§5. Application to Standard Real Submanifold of Second Kind

5. 1. Basic Notations and Definitions. Let V be an n-dimensional complex Hilbert space and $H(V)$ the space of hermitian forms on V^*. In §1, V^* was considered to be $\mathrm{Hom}_{\boldsymbol{R}}(V,\boldsymbol{R})$. However $\mathrm{Hom}_{\boldsymbol{R}}(V,\boldsymbol{R})$ can be canonically identified with $\mathrm{Hom}_{\boldsymbol{C}}(V,\boldsymbol{C})$ by the map

$$\mathrm{Hom}_{\boldsymbol{R}}(V,\boldsymbol{R})\ni f\to\tfrac{1}{2}(f-if\circ I)\in\mathrm{Hom}_{\boldsymbol{C}}(V,\boldsymbol{C}).$$

Here I denotes the automorphism of V defined by $I(x)=ix(x\in V)$.

Now we shall prove the equivalence of D_0 with M^0. Recall that standard real submanifold M^0 in $N^0=V\times H(V)^c$ is given by

$$M^0=\{(x,\alpha)\in V\times H(V)^c;\ \mathrm{Im}(\alpha)-\frac{1}{8}(x\otimes x+Ix\otimes Ix)=0\}.$$

Here $\mathrm{Im}(\alpha)$ is the element of $H(V)$ such that $\alpha-i\,\mathrm{Im}(\alpha)\in H(V)$, that is, $\mathrm{Im}(\alpha)=\frac{1}{2i}(\alpha-\alpha^*)$ where $*$ indicates the conjugation of $H(V)^c$ fixing $H(V)$. Consider the linear isomorphism π_0 of $\mathcal{G}_h$ onto $H(V)$ defined by

$$\pi_0(A)(\mathrm{Re}\,x^*, \mathrm{Re}\,y^*)\,(=\langle\pi_0(A), \mathrm{Re}\,x^*\otimes\mathrm{Re}\,y^*\rangle)=\tfrac{1}{4}\mathrm{Re}(Ax, y)\quad A\in\mathcal{G}_h$$

where we have used the notations in §2. Computing directly we obtain

$$\pi_0(x\otimes x^*)=\tfrac{1}{4}(x\otimes x+Ix\otimes Ix).$$

Let $\tilde{\pi}_0$ be the linear isomorphism of $\mathcal{G}\times V$ onto N^0 defined by $\tilde{\pi}_0(A, x)=(x, \pi_0^C(A))$. Then $\tilde{\pi}$ maps D_0 onto M^0. The image of the Siegel domain D by $\tilde{\pi}_0$ is the domain

$$\boldsymbol{D}=\left\{(x, \alpha):\ \mathrm{Im}(\alpha)-\frac{1}{8}(x\otimes x+Ix\otimes Ix)>0\right\}.$$

Note that $M^0, \boldsymbol{D}$ are obtained once the complex structure of V is given, i.e., they do not depend on the inner product on V.

Now let π be a linear mapping of $H(V)$ onto a real vector space W. Then we can construct the fundamental Lie algebra $\mathfrak{m}(\pi)$ and the corresponding standard real submanifold $M(\pi)$ as in §1. Further consider the canonical linear map $\tilde{\pi}$ of $N^0=V+H(V)^c$ onto $N(\pi)=V+W^C$. Then $\hat{\pi}=\tilde{\pi}|_{M^0}$ is the canonical homomorphism. In what follows we shall identify D, D_0 with $\boldsymbol{D}$, M^0 respectively by the map $\tilde{\pi}_0$, and use rather D, D_0 instead of $\boldsymbol{D}, M^0$ to avoid inessential complication of notaion. For example, $\hat{\pi}$, $\tilde{\pi}$ should be interpreted as the maps from D_0, $\mathcal{G}\times V$ onto $M(\pi)$, $N(\pi)$ respectively. Under this convention the image by $\tilde{\pi}$ of D is denoted by $D(\pi)$. $D(\pi)$ also does not depend on the inner product on V.

We shall now introduce function spaces and sheaves for $M(\pi)$

which are similar to those for D_0. For a subset S of $M(\pi)$, $H^\circ(S)$ is the set of restrictions to S of holomorphic functions defined near S. In case S is σ-compact, $H(S)$ is the closure of $H^\circ(S)$ in $C(S)$. For an open subset S of $M(\pi)$, $\overleftarrow{H}(S)$ is the set $\{f\in C(S); f|_{S'}\in H(S')$ for any relatively compact subset S' of $S\}$. $\mathcal{Q}$ is the sheaf induced from the presheaf given by the system $(H(S))_{S:\text{open}}$ with natural restriction maps. For a subset U of $\overline{D(\pi)}$ such that $U\cap D(\pi)$ is a non-empty open subset and such that $U\subset\overline{U\cap D(\pi)}$, $\widetilde{H}(U)$ is the set $\{u\in C(U);\ u|_{U\cap D(\pi)}$ is holomorphic$\}$. Here the second condition on U means that every continuous function on U is determined by its restriction on $U\cap D(\pi)$. Further we set for any open subset S of $M(\pi)$

$$U_\pi(S)=\tilde{\pi}(U(\hat{\pi}^{-1}(S)))$$

where $U(\hat{\pi}^{-1}(S))$ has appeared already in Lemma 2.2.10. $U_\pi(S)$ satisfies condition $U_\pi(S)\subset\overline{U_\pi(S)\cap D(\pi)}$ and $U_\pi(S)\cap D(\pi)$ is open. The first assertion follows immediately from

$$U(\hat{\pi}^{-1}(S))\subset\overline{U(\hat{\pi}^{-1}(S))\cap D}$$

which is obvious since $U(\hat{\pi}^{-1}(S))$ is an open subset of $\overline{D}$. The second follows from

Lemma 5.1.1. *If U is an open subset of $\overline{D}$, then $\tilde{\pi}(U)\cap D(\pi)$ $=\tilde{\pi}(U\cap D)$.*

Proof. Suppose $x\in\tilde{\pi}(U)\cap D(\pi)$. Then there are $p\in U$, $q\in D$ such that $\tilde{\pi}(p)=\tilde{\pi}(q)=x$. Set for $0\leq t\leq 1$

$$p(t)=(1-t)p+tq.$$

Then $\mathfrak{l}(p(t))=(1-t)\mathfrak{l}(p)+t\mathfrak{l}(q)>0$ when $0<t\leq 1$. For $\mathfrak{l}(p)\geq 0$ and $\mathfrak{l}(q)>0$. But U is open in $\overline{D}$, hence there exists $\delta>0$ such that $p(t)\in U$ for $0\leq t\leq\delta$. Thus $p(t)\in U\cap D$ for $0<t<\delta$. Therefore $x=\tilde{\pi}(p)=\tilde{\pi}(p(t))\in\tilde{\pi}(U\cap D)$. We have thus showed $\tilde{\pi}(U)\cap D(\pi)$ $\subset\tilde{\pi}(U\cap D)$. Since the opposite inclusion is obvious, the proof is complete.

We shall now define two important classes of standard real submanifolds which are the main objects in this paper. The standard real submanifold $M(\pi)$ is said to be *totally indefinite* or *$\forall$-indefinite* if every $0\neq\alpha\in W^*$ the symmetric bilinear form

$$\alpha(x, Iy) \qquad x, y\in V$$

is indefinite. If every non-zero element of $\mathrm{Ker}\,\pi$ is indefinite, then $M(\pi)$ is *dually* $\forall$*-indefinite.* This terminology derives its origin from the fact that we can assign its dual for every $M(\pi)$ so that $M(\pi)$ is $\forall$-indefinite if and only if the dual is dually $\forall$-indefinite. This is done as follows. Let K_d be the image of the transposed of π and π_d the canonical projection from $H(V^*)$ onto $W_d=H(V^*)/K_d$. Then π_d gives rise to a standard real submanifold $M(\pi_d)$ in complex vector space $N(\pi_d)=V^*\times(W_d)^c$. $M(\pi_d)$ is *the dual* of $M(\pi)$. Note here that the dual of the dual does not coincide with the original one in general. In fact $(\pi_d)_d$ is the projection of $H(V)$ onto $H(V)/\mathrm{Ker}\,\pi$. However we may consider that $M(\pi)$, $M((\pi_d)_d)$ are essentially the same. For, they are mutually equivalent by the canonical map $\tilde{\rho}$ induced from the commutative diagram

$$\begin{array}{ccc} H(V) & \xrightarrow{\ \pi\ } & W \\ \| & & \downarrow\rho \\ H(V) & \xrightarrow{(\pi_d)_d} & H(V)/\mathrm{Ker}\,\pi\ . \end{array}$$

Lemma 5.1.2. *$M(\pi)$ is $\forall$-indefinite if and only if $M(\pi_d)$ is dually $\forall$-indefinite.*

This follows immediately from

Lemma 5.1.3. *For a subspace L of $H(V)$ the following statements are mutually equivalent*:

(i) *L contains a (positive) definite element.*

(ii) *$L^\perp=\{\gamma\in H(V^*);\ \langle\gamma,\eta\rangle=0\ \forall\eta\in L\}$ contains no semidefinite element except for* 0.

Here $\langle\ ,\ \rangle$ is the bilinear form on $H(V)\times H(V^*)$ which gives the canonical identification of $H(V^*)$ with $H(V)^*$. For the proof of this see L. L. Dines [1] or Lemma 3 of I. Naruki [7]. Note that, in this notation, $\mathrm{Ker}\,\pi_d=(\mathrm{Ker}\,\pi)^\perp$, in view of which Lemma 5.1 2 is obvious.

If $M(\pi)$ is either $\forall$-indefinite, or dually $\forall$-indefinite, then $M(\pi)$ is said to be *stable.* The standard real submanifold which is not stable is called *unstable* one. The meaning of the word is the following: Let G_k denote the Grassmann manifold consisting of all k-dimensional subspaces of $H(V)$. Given an $H\in G_k$, M_H denotes the standard real submanifold induced from the canonical map $\pi_H: H(V)\to H(V)/H$. Then the set $\{H\in G_k: M_H$ is unstable$\}$ is closed in G_k and of the first category, i.e., without interior, while the sets $\{H\in G_k: M_H$ is $\forall$-indefinite$\}$, $\{H\in G_k: M_H$ is dually $\forall$-indefinite$\}$ are open in G_k. Thus one may say that *almost all* standard real submanifolds are stable.

In this paper we shall only concern ourselves with stable ones and the main objective is to show

$$\underleftarrow{H}(S)=\Gamma_S(\mathcal{H})=\Gamma_S(\mathcal{H}_{\mathcal{S}})=\widetilde{H}(U_\pi(S))|_S$$

when S is an open subset of a stable standard real submanifold. The proof is given first for $\forall$-indefinite case and next for dually $\forall$-indefinite case.

5.2. $\forall$-INDEFINITE CASE. In this case the study is based on the following

Lemma 5.2.1. *If $M(\pi)$ is $\forall$-indefinite, then $N(\pi)=\tilde{\pi}(D)=D(\pi)$.*

Proof. It suffices to show $D+(\mathrm{Ker}\,\pi)^C=N^0=V\times H(V)^C$. By assumption and Lemma 5.1.3, $\mathrm{Ker}\,\pi$ contains a positive definite element r_0 of $H(V)$. But, for any $p\in N^0$ $\mathfrak{l}(p)+tr_0$ is positive definite for sufficiently large $t>0$, i.e., $p+itr_0$ belongs to D. Hence $p\in D+i\,\mathrm{Ker}\,\pi$. Thus we have proved $N^0=D+i\,\mathrm{Ker}\,\pi=D+(\mathrm{Ker}\,\pi)^C$. Q.E.D.

Now we shall prove a stronger version of the statement announced at the end of 5.1 for totally indefinite $M(\pi)$.

Theorem 5.2.1. *If S is an open subset of an $\forall$-indefinite $M(\pi)$, then $U_\pi^1(S)$ is an open subset of $N(\pi)$ and*

$$\underleftarrow{H}(S)=\Gamma_S(\mathcal{H})=\Gamma_S(\mathcal{H}_{\mathcal{S}})=H(U_\pi^1(S))|_S=H^\circ(U_\pi^1(S))|_S.$$

Here we have put $U_\pi^1(S)=\tilde{\pi}(U_1(\hat{\pi}^{-1}(S)))$.

Proof. Suppose $f\in\Gamma_S(\mathcal{H}_{\mathcal{S}})$. Then $\hat{\pi}^*(f)\in\Gamma_{S'}(\mathcal{H}_{\mathcal{S}})$ where, for simplicity, we have set $S'=\hat{\pi}^{-1}(S)$. From Theorem 4.2.2′ it follows the existence of $u\in\widetilde{H}(U_1(S'))$ such that $u|_{S'}=\hat{\pi}^*(f)$. Note that both $\hat{\pi}^*(f)$ and S' are invariant under translations parallel to $\mathrm{Ker}\,\pi$. Thus Lemma 2.2.11 implies that there exists a function v on $U_\pi^1(S)=\tilde{\pi}(U_1(S'))$ such that

$$v(\tilde{\pi}(p))=u(p) \qquad p\in U_1(S').$$

Note that the combination of Lemmas 5.1.1 and 5.2.1 implies that $\tilde{\pi}|_D$ is open. Thus the continuity of v is obvious.

On the other hand we have

$$\tilde{\pi}(U_1(S'))=\tilde{\pi}(U_1(S))\cap D(\pi)=\tilde{\pi}(U_1(S)\cap D)$$

where we have used Lemma 5.2.1 in the first step and Lemma 5.1.1 in the last. This implies that $U_\pi^1(S)$ is open in $N(\pi)$. Furthermore, since $u|_{U_1(S)\cup D}$ is holomorphic, v is also holomorphic. Hence $f=v|_S \in H(U_\pi^1(S))|_S=H^\circ(U_\pi^1(S))|_S$. Thus we have proved $\Gamma_S(\mathcal{H}_{\mathcal{S}})\subset H(U^1(S))|_S$ which, together with obvious inclusions $\Gamma_S(\mathcal{H}_{\mathcal{S}})\supset\Gamma_S(\mathcal{H})\supset \underleftarrow{H}(S)\supset H(U_\pi^1(S))|_S$, implies

$$\underleftarrow{H}(S)=\Gamma_S(\mathcal{H})=\Gamma_S(\mathcal{H}_{\mathcal{S}})=H(U_\pi^1(S))|_S=H^\circ(U_\pi^1(S))|_S.$$

Q.E.D.

From this theorem it follows trivially

Corollary 5.2.1. *If $M(\pi)$ is $\forall$-indefinite, then every S-holomorphic function in an open set of $M(\pi)$ is real analytic.*

Note that $U_1(D_0)=\overline{D}$ by the construction of $U_1(S)$. Therefore $U_\pi^1(M(\pi))=N(\pi)$ by Lemma 5.2.1. Thus we obtain

Corollary 5.2.2. *Every $\mathcal{S}$-holomorphic function on $M(\pi)$ is the restriction of an entire holomorphic function of $N(\pi)$ if $M(\pi)$ is $\forall$-indefinite.*

By means of Theorem 5.2.1 we shall show that $\Gamma_S(\mathcal{H}_{\mathcal{S}})$ is a Montel space.

Corollary 5.2.3. *Under the hypothesis of Theorem* 5.2.1 *the map*

$$\Gamma_S(\mathcal{H}_S) \ni f \mapsto \varphi f \in C_0^\infty(S)$$

is a compact operator for $\varphi \in C_0^\infty(S)$.

Proof. Since the restriction map $r: H(U(S)) \to \Gamma_S(\mathcal{H}_S)$ is a continuous bijection and since $\widetilde{H}(U(S))$, $\Gamma_S(\mathcal{H}_S)$ are Fréchet spaces, the inverse η of r is also continuous (by Banach's theorem). Note that $\widetilde{H}(U(S)) = H(U(S))$ since $U(S)$ is an open set of $N(\pi)$. Therefore the map ζ

$$\widetilde{H}(U(S)) \ni u \mapsto \psi u \in C_0^\infty(U(S))$$

is compact where ψ is a function of $C_0^\infty(U(S))$ such that $\psi|_S = \varphi$. Therefore the map

$$\Gamma_S(\mathcal{H}_S) \ni f \mapsto \varphi f = \zeta \circ \eta(f)|_S \in C_0^\infty(S)$$

is compact. Q.E.D.

Now we shall extend this Corollary for a somewhat wider class of real submanifolds. Let M be a real submanifold of a complex manifold $\widetilde{M}$. M is called *a locally flat real submanifold of type* $M(\pi)$ if for any p of M there exist a neighbourhood U in $\widetilde{M}$ of p and a biholomorphic mapping φ of U into $N(\pi)$ such that $\varphi(U \cap M)$ is open in $M(\pi)$.

Corollary 5. 2. 3′. *Suppose that* $M(\pi)$ *is totally indefinite and* M *is a locally flat real submanifold of type* $M(\pi)$. *Then the map*

$$\Gamma_S(\mathcal{H}_S) \ni f \mapsto \varphi f \in C_0^\infty(S)$$

is a compact operator for an open set S *of* M *and* $\varphi \in C_0^\infty(S)$.

This corollary implies the finiteness theorem for $\mathcal{S}$-holomorphic vector bundles on a compact locally flat real submanifold. Let M be a real submanifold of a complex manifold such that the tangential Cauchy-Riemann bundle of M is well-defined. A complex vector bundle E on M with a subsheaf S of E is called an $\mathcal{S}$-holomorphic vector bundle on M if, for any point p of M, there exist a neighbourhood U and sections $s^1, s^2, \cdots, s^l$ (l = fiber dim E) over U of S such that

$$\sum_{k=1}^{l} f_k s^k \in \Gamma_U(S) \Longleftrightarrow f_1, \cdots, f_l \text{ are } \mathcal{S}\text{-holomorphic in } U.$$

By Corollary 5.2.3′, we obtain

Proposition 5.2.1. *Let M be a compact locally flat real submanifold of type $M(\pi)$ where $M(\pi)$ is totally indefinite and (E, S) an $\mathcal{S}$-holomorphic vector bundle over M. Then $\Gamma(S)$ is finite-dimensional.*

Proof. Obvious since $\Gamma(S)$ is a locally compact Banach space by Corollary 5.2.3′.

This proposition is a special case of more general theorem of Naruki [7] proved by using the method of Hörmander. But the reason why we have proved Proposition 5.2.1 using the holomorphic extension is that the method of this paper suggests us a conjecture on holomorphic extension. Let M be a real submanifold of a complex manifold for which Levi-Tanaka algebras are well-defined and let $\mathfrak{m}_p = \sum_{k=1}^{\mu} \mathfrak{g}_p^k$ be the Levi-Tanaka algebra of M at p. Then $\mathfrak{m}_p^2 = \mathfrak{g}_p^1 + \mathfrak{g}_p^2$ is also a fundamental Lie algebra under the convention that

$$[\mathfrak{g}_p^1, \mathfrak{g}_p^2] = [\mathfrak{g}_p^2, \mathfrak{g}_p^2] = (0).$$

M is called *totally indefinite* if for any $p \in M$ $\mathfrak{m}_p^2$ is totally indefinite (i.e., the corresponding standard real submanifold $M(\mathfrak{m}_p^2)$ is totally indefinite). Now our conjecture is the following.

Conjecture: *Let M be a totally indefinite real submanifold of a complex manifold M. Then there exists a neighbourhood $\widetilde{U}$ in $\widetilde{M}$ of M such that the restriction map*

$$H(\widetilde{U}) \to \Gamma_M(\mathcal{H}_{\mathcal{S}})$$

is onto.

If this conjecture is true, it will be almost obvious that the space of global sections of an $\mathcal{S}$-holomorphic vector bundle over a compact totally indefinite real submanifold is finite-dimensional, which is a main theorem of [7].

Remark. For any compact, locally flat and totally indefinite real submanifold M $\mathcal{S}$-holomorphic functions are constant on each component of M. This can be proved as follows. Let H be the vector space of

$\mathcal{S}$-holomorphic functions on M. Then by Proposition 1 this is a finite-dimensional algebra over $\boldsymbol{C}$. Therefore any element of H satisfies an algebraic equation

$$P(f)=0 \qquad P\in \boldsymbol{C}[x].$$

Suppose that $c_1, c_2, \cdots, c_k$ are the roots of P, then

$$(f-c_1)(f-c_2)\cdots(f-c_k)=0.$$

Therefore f takes only a finite number of values. Since f is continuous, f is constant on each component of M.

As an example, we shall now give a compact, $\forall$-indefinite and locally flat real submaifold, which is an orbit of a subgroup of the holomorphic transformation group on a complex Grassmann manifold.

Example (Tanaka). First we construct many linear groups:

$$G=GL(m, \boldsymbol{C}) \quad (m=n+2n' \ n=p+q \ p, q>0)$$

$$G^0=\{A\in G : AJ{}^t\bar{A}=J\} \quad J=\begin{pmatrix} & & & -iI_{n'} \\ & I_p & & \\ & & -I_q & \\ iI_{n'} & & & \end{pmatrix}$$

$$L=\{A=(a_{ij})\in G : a_{ij}=0 \text{ if } n'<i\leqq m,\ 1\leqq j\leqq n'\}$$

$$G'=G^0\cap L.$$

Then the Lie algebras $\mathcal{G}^0$, $\mathcal{G}'$, $\mathcal{L}$ of G^0, G', L are given by

$$\mathcal{G}^0=\{A\in M_m(\boldsymbol{C}) : AJ+J{}^t\bar{A}=0\}$$

$$\mathcal{L}=\{A=(a_{ij})\in M_m(\boldsymbol{C}) : a_{ij}=0 \text{ if } n'<i\leqq m,\ 1\leqq j<n'\}$$

$$\mathcal{G}'=\mathcal{G}^0\cap\mathcal{L}.$$

By direct calculation one sees that $\mathcal{G}^0$ is the set of matrices of the following form

$$\overbrace{}^{n'}\ \overbrace{}^{n}\ \overbrace{}^{n'}$$

$$\left(\begin{array}{c|c|c} r & \eta & \zeta \\ \hline \xi & E & iI'{}^t\bar{\eta} \\ \hline \sigma & -i\,{}^t\bar{\xi}I' & -r^* \end{array}\right)\begin{matrix}\}n' \\ \}n \\ \}n'\end{matrix} \qquad I'=\begin{pmatrix} I_p & \\ & -I_q \end{pmatrix}$$

where σ, ζ are hermitian, $EI'+I'{}^t\bar{E}=0$. We shall set

$$\mathfrak{g}_1=\left\{\left(\begin{array}{c|c|c} & & \\ \hline \xi & & \\ \hline & -i\,{}^t\bar{\xi}I' & \end{array}\right) : \xi\in M_{n,n'}(\boldsymbol{C})\right\}$$

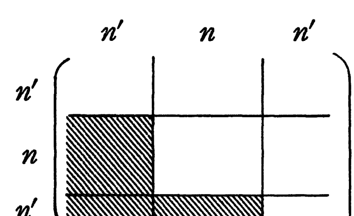
$$\mathfrak{g}_2=\left\{\left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline \sigma & & \end{array}\right) : {}^t\bar{\sigma}=\sigma\ \ \sigma\in M_{n'}(\boldsymbol{C})\right\}$$

$$\mathfrak{m}=\mathfrak{g}_1+\mathfrak{g}_2\,.$$

Then $\mathfrak{m}$ is a subalgebra of $\mathcal{G}^0$ such that $[\mathfrak{g}_1, \mathfrak{g}_1]\subseteq\mathfrak{g}_2$, $[\mathfrak{g}_2, \mathfrak{g}_2]$ = $=0$. We shall make $\mathfrak{g}_1$ a complex vector space so that th mapping τ

$$\xi \mapsto \left(\begin{array}{c|c|c} & & \\ \hline \xi & & \\ \hline & -i\,{}^t\bar{\xi}I' & \end{array}\right)$$

is a complex linear isomorphism of $M_{n,n'}(\boldsymbol{C})$ onto $\mathfrak{g}_1$. Thus fundamental Lie algebra and $\mathfrak{m}^{C}$ can be identified with the su of $M_n(\boldsymbol{C})$ consisting of matrices with the following form



by the mappings

$$\tau^{C} : \mathfrak{g}_1^{C} \ni \xi\otimes 1+\xi'\otimes i \mapsto \left(\begin{array}{c|c|c} & & \\ \hline \xi+i\xi' & & \\ \hline & ({}^t\bar{\xi}' - i\,{}^t\bar{\xi})I' & \end{array}\right)$$

and

$$\mathfrak{g}_2^{C} \ni (\sigma\otimes 1+\sigma'\otimes i) \mapsto \left(\begin{array}{c|c} & \\ \hline \sigma+i\sigma & \end{array}\right).$$

The connected Lie subgroup of G^0 with Lie algebra $\mathfrak{m}$ (resp $\mathfrak{m}^{C}$) shall be identified with M' (resp M^{C}) introduced for $\mathfrak{m}$ in §1. Under this identification we have

$$S=\left\{\left(\begin{array}{c|c|c} & & \\ \hline & & \\ \hline & \xi & \end{array}\right) : \xi\in M_{n,n'}(\boldsymbol{C})\right\}$$

where S is the abelian subalgebra of $\mathfrak{m}$ introduced in §1. Denote by H the Lie subgroup of G with Lie algebra S.

$$S=L\cap\mathfrak{m}^{c},\quad H=L\cap M^{c}.$$

Then the canonical map

$$M^{c}/H\rightarrow G/L$$

is one to one and regular (not onto).

On the other hand the canonical map

$$M'\rightarrow G^0/G'$$

is also one to one and regular. In fact $G'\cap M'=(e)$ and $\mathfrak{m}+\mathcal{G}'=\mathcal{G}^0$. Moreover the following diagram is commutative

$$\begin{array}{ccc} M' & \longrightarrow & G^0/G' \\ \downarrow & & \downarrow \\ M^{c}/H & \longrightarrow & G/L \end{array}$$

where all arrows are canonical. Thus G^0/G' contains a locally flat open subset. But the canonical left operation of G^0 on G/L is holomorphic. Therefore G^0/G' itself is locally flat.

Now we take $K=U(m)\cap G^0$ as a maximal compact subgroup of G^0 and denote by $\mathcal{K}$ the Lie algebra of K. Then

$$\mathcal{G}^0=\mathcal{K}+\mathcal{G}'.$$

Therefore orbits of K in G^0/G' are open, and it is obviously compact. Since G^0/G' is connected, K operates transitively on G^0/G', and G^0/G' itself is compact. Thus G^0/G' is a compact locally flat real submanifold of G/L. (Note that G/L is a Grassmann manifold.)

In order to show that G^0/G' is totally indefinite one may investigate Lie algebra $\mathfrak{m}$. Recall that $\mathfrak{g}_1$ is isomorphic to $M_{n,n'}(C)$ by τ. Note that

$$[\tau(\xi), I\tau(\eta)]={}^t\bar{\xi}I'\eta+{}^t\bar{\eta}I'\xi\,.$$

We shall prove that for any non-zero hermitian (n', n')-matrix A, the symmetric form

$$[A](\xi,\eta)=\mathrm{Sp}(A[\tau(\xi), I\tau(\eta)])$$

is indefinite. Denote by S_k the subspace of $M_{n,n'}(C)$ of k-th column vectors and set $A=(a_{ij})_{1\leq i,j\leq n'}$.

1) Suppose that $a_{ij}\neq 0$ for some $i\neq j$. For any $u\in S_i$, $v\in S_j$

$$[A](u+v, u+v)=2\,\mathrm{Re}(a(\sum_{k=1}^{p}u_k\bar{v}_k-\sum_{k=p+1}^{n}u_k\bar{v}_k))$$

where we set $a_{ij}=a$. Choosing a suitable $e^{i\varphi}$ so that $e^{i\varphi}$ a is real, we have

$$[A](e^{i\varphi}u+v, e^{i\varphi}u+v)=2e^{i\varphi}a\,\mathrm{Re}(\sum_{k-1}^{p}u_k\bar{v}_k-\sum_{k=p+1}^{n}u_k\bar{v}_k)$$

which can obviously be made positive as well as negative by taking suitable u, v.

2) Suppose $a=a_{ii}\neq 0$ for some i. Then, for $u\in S_i$

$$[A](u,u)=2a(\sum_{k=1}^{p}u_k\bar{u}_k-\sum_{k=p+1}^{n}u_k\bar{u}_k).$$

Thus the restriction of $[A]$ to S_i is indefinite.

To sum up we conclude that G^0/G' is a *compact, locally flat* and *totally indefinite* real submanifold of G/L.

5.3. DUALLY $\forall$-INDEFINITE CASE. As announced before the objective of this paragraph is to prove the following

Theorem 5.3.1. *If $M(\pi)$ is dually $\forall$-indefinite, then we have*

$$\underleftarrow{H}(S)=\Gamma_S(\mathcal{H})=\Gamma_S(\mathcal{H}_{\mathcal{S}})=\widetilde{H}(U_\pi(S))|_S$$

for any open subset S of $M(\pi)$.

To prove this we need a lemma concerning closed cones in real vector spaces.

Lemma 5.3.1. *Let K be a closed cone in a real vector space W, and H be a subspace of W such that $H\cap K=(0)$. Then the set $(B+H)\cap K$ is bounded if B is a bounded set of W.*

Proof. Choose a fixed norm $\| \ \|$ on W. Suppose that $(B+H)\cap K$ is not bounded while B is bounded. Then there exist sequences $\{x_\nu\}_{\nu=1}^\infty\subseteq B$ and $\{h_\nu\}_{\nu=1}^\infty\subset H\backslash\{0\}$ such that $\{x_\nu+h_\nu\}_{\nu=1}^\infty\subset K$ and such that $\|h_\nu\|\to\infty$ $(\nu\to\infty)$. Set $\|h'_\nu\|=\|h_\nu\|^{-1}h_\nu$. Since $\|h'_\nu\|=1$, we may assume that h'_ν converges to some $h_0\in H$ when $\nu\to\infty$. Then $\|h_0\|=1$, and further $\|h_\nu\|^{-1}(x_\nu+h_\nu)\to h_0$ $(\nu\to\infty)$. On the other hand, since $\|h_\nu\|^{-1}(x_\nu+h_\nu)\in K$, h_0 also belongs to $\overline{K}=K$. But then $H\cap K=(0)$ implies $h_0=0$, which, however, contradicts $\|h_0\|=1$. Thus $(B+H)\cap K$ is bounded. Q.E.D.

Proof of Theorem 5.3.1. Suppose $f\in\Gamma_S(\mathcal{H}_{\mathcal{S}})$. Set for simplicity $H=\mathrm{Ker}\,\pi$, $S'=\hat{\pi}^{-1}(S)$. Then $\hat{\pi}^*(f)\in\Gamma_{S'}(\mathcal{H}_{\mathcal{S}})$. Moreover $\hat{\pi}^*(f)$ together with S' is invariant by translation parallel to H. Hence by Lemma 2.2.11 and Theorem 4.2.2$'$ there exists a function u on $U_1(S')+H^C$, such that $u|_{S'}=\hat{\pi}^*(f)$, $u|_{U_1(S')}\in\widetilde{H}(U_1(S'))$, and $\tau_h^*(u)=u$ $(h\in H^C)$. Here τ_h is the translation $p\to p+h$ $(p\in N^0)$.

Now we shall prove the continuity of $v=u|_{U(S')+H^C}$. First recall that $U(S')+H=U(S')$, so that

$$U(S')+H^C=U(S')+iH.$$

Observe that the following statement is sufficient for v to be continuous: *From every sequence* $\{x_\nu\}_{\nu=1}^{\infty}\subset U(S')+iH$ *converging to* $x_0\in U(S')+iH$, *it is possible to choose a subsequence* $\{x_{\nu_k}\}_{k=1}^{\infty}$ *such that* $u(x_{\nu_k})\to u(x_0)$ $(k\to\infty)$. Let us prove this statement. Write each x_ν $(\nu=0, 1, 2, \cdots)$ in the form $x_\nu=p_\nu+ih_\nu$ $(p_\nu\in U(S'),\ h_\nu\in H)$. Then $\mathfrak{l}(x_\nu)=\mathfrak{l}(p_\nu)+h_\nu$ and $\mathfrak{l}(p_\nu)\geq 0$, that is, $\mathfrak{l}(p_\nu)$ lies in the closure $\overline{K}_0$ of the cone K_0 consisting of positive definite elements of $H(V)$. Here we have set as in §2, $\mathfrak{l}(x, \alpha)=\mathrm{Im}(\alpha)-\frac{1}{8}(x\otimes x+Ix\otimes Ix)$ for $(x, \alpha)\in V\times H(V)^C$. By the assumption $H\cap\overline{K}_0=\mathrm{Ker}\,\pi\cap\overline{K}_0=(0)$. Hence Lemma 5.3.1 implies that $\{h_\nu\}_{\nu=1}^{\infty}$ is bounded. Thus there certainly exists a subsequence $\{h_{\nu_k}\}_{k=1}^{\infty}$ which converges to some h_0' of H. But then $\{p_{\nu_k}\}_{k=1}^{\infty}$ also converges to $p_0'=x_0-ih_0'\in\overline{U(S')}$. However $p_0'\in\partial S'$ implies that $0=\mathfrak{l}(p_0')=\mathfrak{l}(x_0)-h_0'=\mathfrak{l}(p_0)+h_0-h_0'$, from which $\mathfrak{l}(p_0)=h_0'-h_0\in\overline{K}_0\cap H=(0)$, that is, $p_0=p_0'\in\partial S'$, contradicting $p_0\in U(S')$. (Recall $S'=U(S')\cup D_0$.) Thus $p_0'\in\overline{U(S')}\backslash\partial S$, which, according to Lemma 2.2.10, implies $p_0'\in U_1(S)$. Since u is continuous in $U_1(S)$ it follows

$$u(x_\nu)=u(p_\nu)\to u(p_0')=u(x_0)$$

when $\nu\to\infty$. The required statement is thus proved, hence v is continuous. Recall now that u, hence also v are invariant by τ_h $(h\in H^C)$. Thus there exists a continuous function w on $U_\pi(S)=\tilde{\pi}(U(S'))$ such that $w(\tilde{\pi}(p))=v(p)$ $p\in U(S')$. The function w is certainly holomorphic in $U_\pi(S)\cap D(\pi)$, for $U_\pi(S)\cap D(\pi)=\tilde{\pi}(U(S')\cap D)$ by Lemma 5.1.1 and v is holomorphic in $U(S')\cap D$. Thus $f=w|_S\in H(U_\pi(S))|_S$. And thus we have proved

$$(1)\qquad \Gamma_S(\mathcal{H}_S)\subseteq\widetilde{H}(U(S))|_S.$$

Now let γ_0 be a positive definite form in $H(V)$ and S' a relatively compact subset of S. Then there exists a relatively compact subset S'' of $\hat{\pi}^{-1}(S)$ such that $\hat{\pi}(S'')=S'$. Since $U(\hat{\pi}^{-1}(S))$ is open in $\overline{D}$, $S''+it\gamma_0$ is contained in $U(\hat{\pi}^{-1}(S))\cap D$ for sufficiently small $t>0$. Thus, for a sufficiently small $t>0$,

$$S'+it\pi(\gamma_0)\subseteq U(S)\cap D(\pi).$$

Therefore, arguing as in the proof of Lemma 2.2.9, we obtain

(2) $\widetilde{H}(U(S))|_S \subseteq \underset{\leftarrow}{H}(S).$

In view of obvious inclusions $\underset{\leftarrow}{H}(S) \subseteq \Gamma_S(\mathcal{H}) \subseteq \Gamma(\mathcal{H}_S)$, (1) and (2) imply

$$\underset{\leftarrow}{H}(S) = \Gamma_S = (\mathcal{H}_S) = \widetilde{H}(U(S))|_S.$$

Q.E.D.

Here it should be remarked that $\tilde{\pi}(\overline{D}) = \overline{D(\pi)}$ when $M(\pi)$ is dually $\forall$-indefinite. This follows from

$$\overline{D} = \{(x, \alpha) : \mathfrak{l}(x, \alpha) \in \overline{K}_0\}$$
$$\overline{D(\pi)} = \{\tilde{\pi}(x, \alpha); \pi(\mathfrak{l}(x, \alpha)) \in \overline{\pi(K_0)}\}$$
$$\pi(K_0) = \overline{\pi(K_0)}.$$

The last relation is an immediate consequence of Lemma 5. 3. 1.

Note that $U_\pi(S)$ never swells up to $\overline{D(\pi)}$ even if $S = M(\pi)$. However we can prove the result parallel to Corollary 5. 2. 2 also in the present case.

Theorem 5. 2. 2. *If $M(\pi)$ is dually $\forall$-indefinite, then the restriction map $\widetilde{H}(\overline{D(\pi)}) \to \Gamma(\mathcal{H}_S)$ is a topological isomorphism.*

For the proof, first note that $U_1(D_0) = \overline{D}$, next use the relation $\overline{U_1(D_0)} = U_1(D_0)$ instead of $\overline{U(S)} \backslash \partial S \subset U_1(S)$. Then the proof of Theorem 5. 3. 1 can be applied without further change.

Now we shall show that the maximal open subset of $\overline{D(\pi)}$ contained in $U_\pi(S)$ always contains S if $M(\pi)$ is dually $\forall$-indefinite. The proof of this will require some preparations. Let H be a subspace such that $H \cap \overline{K}_0 = (0)$. For $p \in \overline{D} + iH$ we set

$$d_H(p) = \sup\{\|h\|; \mathfrak{l}(p) + h \geq 0 \text{ (or } p + ih \in \overline{D}_0)\}.$$

Then certainly $0 \leq d_H(p) < +\infty$ according to Lemma 5. 3. 1, and further $d_H(p) = 0$ if $p \in D_0$.

Lemma 5. 3. 2. *The function d_H is continuous at each point of D_0.*

Proof. Suppose that d_H is not continuous at $p_0 \in D_0$, that is, there exists $\delta>0$ and a sequence $\{p_\nu\}_{\nu=1}^{\infty}\subset\overline{D}+iH$ converging to p_0 such that $d_H(p_\nu)\geq\delta$. For $\nu=1, 2, \cdots$ choose $h_\nu\in H$ such that $\mathfrak{l}(p_\nu)+h_\nu\in\overline{K}_0$, $\|h_\nu\|>\delta/2$. Then according to Lemma 5. 3. 1 $\{h_\nu\}_{\nu=1}^{\infty}$ is bounded. Hence we may assume that h_ν converges to some $h_0\in H$. The closedness of $\overline{K}_0$ implies that $\mathfrak{l}(p_0)+h_0\in\overline{K}_0$, from which it follows $h_0=0$. But this contradicts $\|h_\nu\|\geq\delta/2>0$. Q.E.D.

Lemma 5. 3. 3. *Suppose that $M(\pi)$ is dually $\forall$-indefinite. If U is an open subset of $\overline{D}$, then the maximal open subset of $\overline{D(\pi)}$ contained in $\tilde{\pi}(U)$ always contains $\pi(U\cap D_0)$.*

Proof. Instead of proving the conclusion we shall prove the following equivalent statement: *If $p\in D_0\cap U$, then p lies in some open subset of $\overline{D}+H^C$ contained in $U+H^C$.*

Suppose $p\in D_0\cap U$. Set $H=\operatorname{Ker}\pi$. For $q\in\overline{D}+iH$ the set

$$S(q)=q+i\{h\in H: \|h\|\leq d_H(q)\}$$

certainly intersects $\overline{D}$ by the definition of d_H. Now choose an open subset $\widetilde{U}$ of N^0 such that $U=\overline{D}\cap\widetilde{U}$. Since $p\in D_0$, $S(p)=\{p\}\subset\widetilde{U}$. Therefore, according to the continuity of d_H at p, there exists a neighbourhood $\widetilde{V}\ni p$ in N^0 such that $S(q)\subset\widetilde{U}$ when $q\in(\overline{D}+iH)\cap\widetilde{V}$. But then $S(q)\cap U=S(q)\cap\widetilde{U}\cap\overline{D}=S(q)\cap\overline{D}\neq\phi$. Thus $U\cap(q+iH)\neq\phi$, that is, $q\in U+iH$. Thus $\widetilde{V}\cap(\overline{D}+iH)\subset U+iH$ and this implies the required statement.

In view of this lemma S lies in the maximal open subset of $\overline{D(\pi)}$ contained in $U_\pi(S)$, which we shall denote by $V_\pi(S)$. Then it follows from Theorem 5. 3. 1 that the restriction map

$$\widetilde{H}(V_\pi(S))\to\Gamma_S(\mathcal{H}_S)$$

is a topological linear isomorphism. This and Theorem 5. 2. 1 imply the following stronger form of the solution of Problem H mentioned in the introduction for stable standard real submanifolds.

Theorem A. *If $M(\pi)$ is a stable standard real submanifold, then, for any open subset S of $M(\pi)$, there exists an open subset U*

of the closure of the domain $D(\pi)$ *such that* $S\subset U$, *and the restriction map*

$$\widetilde{H}(U)\rightarrow\Gamma_s(\mathcal{H}_S)$$

is a topological linear isomorphism (onto $\Gamma_s(\mathcal{H}_S)$*).*

Now we shall devote the rest for the study of nondegenerate Siegel domains of the second kind. Each of them is $D(\pi)$ for some map $\pi: H(V)\rightarrow W$, taking suitable V and W. $M(\pi)$ is the Šilov boundary of $D(\pi)$ then. First of all we begin with the definition. Let W be a real vector space and K an open convex cone in W which does not contain any line. Further let V be a complex vector space. A W^C-valued sesqui-linear form $F(x, y)$ $(x, y\in V)$ is called K-hermitian form on V if

(1) $\qquad F(x, y)=F(y, x)^*$

(2) $\qquad F(x, x)\in\overline{K}$

(3) $\qquad F(x, x)=0\Rightarrow x=0$

where the map $\alpha\rightarrow\alpha^*$ is the conjugation of W^C fixing W. Denote by $D(F, K)$ the domain in $V\times W^C$

$$\left\{(x, \alpha)\,;\ \frac{1}{2i}(\alpha-\alpha^*)-F(x, x)\in K\right\}.$$

If $D(F, K)$ is affine homogeneous, that is, the group of affine transformation leaving $D(F, K)$ invariant operates transitively on $D(F, K)$, $D(F, K)$ is called *a Siegel domain of second kind.* If moreover the set $\left\{\frac{1}{i}(F(x, y)-F(y, x))\right\}$ generates W, we call $D(F, K)$ *non-degenerate.* For a non-degenerate Siegel domain we shall assign a fundamental Lie algebra $\mathfrak{m}(F)$ setting

$$\mathfrak{g}_1=V,\quad \mathfrak{g}_1=W,\quad \mathfrak{m}=\mathfrak{g}_1+\mathfrak{g}_2\,,$$

$$[\mathfrak{g}_2, \mathfrak{g}_1]=[\mathfrak{g}_2, \mathfrak{g}_2]=(0),$$

$$[x, y]=\frac{1}{i}(F(x, y)-F(y, x))\quad x, y\in\mathfrak{g}_1\,.$$

Then as proved in §1 we obtain a (unique) linear mapping π of $H(V)$ onto W such that $\mathfrak{m}(\pi)=\mathfrak{m}(F)$. $M(\pi)$ is then the Šilov boundary of

$D(F, K)$. (See [7].)

Lemma 2. *The map π being as above, $D(\pi)=D(F, K)$.*

Proof. Let F_0 denote the K_0-hermitian form on V given by

$$F_0(x, y)=\tfrac{1}{2}([Ix, y]+i[x, y]) \qquad x, y\in V$$

where [,] denotes the bracket operation of $\mathfrak{m}^0=V+H(V)$. Then

$$D=D(F_0, K_0)$$

and D is certainly non-degenerate. Since $\overline{K}_0$ is the convex closures of the set of positive semi-definite hermitian forms of rank 1, condition (2) implies that $\pi(\overline{K}_0)\subseteq\overline{K}$. Hence $\pi(K_0)\subseteq K$. Thus $D(\pi)\subseteq D(F, K)$.

In order to prove $D(\pi)\supseteq D(F, K)$ we need the following theorem of Piatetski-Šapiro [8].

Theorem P. *Every affine transformation of $D(F, K)$ is of the following form*

$$\alpha\to A\alpha+2iF(Bx, b)+iF(b, b)$$
$$x\to Bx+b$$

where $A\in\mathrm{End}(W)$ $B\in\mathrm{End}_C(V)$ *and* $AF(x, y)=F(Bx, By)$ $(x, y\in V)$.

Suppose that g is the affine automorphism of $D(F, K)$ indicated in the above theorem. If we denote by $S^2(B)$ the restriction to $H(V)$ of $B\otimes B$, then the map g_0 of $N^0=V\oplus H(V)^C$ given by

$$\alpha\to S^2(B)\alpha+2iF_0(Bx, b)+iF_0(b, b)$$
$$x\to Bx+b$$

is an affine automorphism of $D=D(F_0, K_0)$. Thus we obtain a group-isomorphism $g\to g_0$ of the affine automorphism group of $D(F, K)$ into that of D such that $\tilde{\pi}\circ g_0=g\circ\tilde{\pi}$. Therefore any orbit of the affine automorphism group of $D(F, K)$ is contained in the image by $\tilde{\pi}$ of an orbit of the affine automorphism group of D. In particular $D(\pi)\supseteq D(F, K)$. Q.E.D.

Now we shall prove that the Šilov boundary of a non-generate

Siegel domain $D(F, K)$ is dually totally indefinite. Since K does not contain any line, there exists a hyperplane P of W such that $P\cap\overline{K}=(0)$. Suppose that P is given by $P=\{\alpha\in W: \mathfrak{l}(\alpha)=0\}$ where $\mathfrak{l}\in W^*$. Without loss of generality we may assume that $x\in K\Rightarrow\mathfrak{l}(x)>0$. Then the hermitian form

$$\mathfrak{l}(F(x, y)) \qquad x, y\in V$$

is certainly positive definite. In fact if

$$\mathfrak{l}(F(x, x))=0,$$

then $F(x, x)\in P\cap\overline{K}=(0)$, i.e., $F(x, x)=0$ and by condition (3) we obtain $x=0$. Thus we conclude that the Šilov boundary of $D(F, K)$ is dually totally indefinite.

References

[1] Dines, L. L., On linear combinations of quadratic forms, Bull. Amer. Math. Soc. **49** (1943), 383-393.

[2] Greenfield, S. J., Cauchy-Riemann equations in several variables, Ann. Scuola Norm. Sup. Pisa **22** (1968), 175-314.

[3] Hörmander, L., Hypoelliptic second order differential equations, Acta Math. **119** (1967), 147-171.

[4] Hua, L. K., Harmonic analysis of functions of several complex variables in the classical domains, Transl. Math. Monogr. **6**, Amer. Math. Soc., 1963.

[5] Lewy, H., On the local character of the solutions of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, Ann. of Math. **64** (1956), 514-522.

[6] ――――, On hulls of holomorphy, Comm. Pure Appl. Math. **13** (1960), 587-591.

[7] Naruki, I., An analytic study of a pseudo-complex-structure, to appear in Proc. Intern. Conf. Func. Anal., Tokyo, 1969.

[8] Piatetski-Šapiro, Géometrie des domaines classiques et théorie des fonctions automorphes, Travaux et Recherches Mathématiques, Dunod, Paris, 1966.

[9] Tanaka, N., Graded Lie algebras and geometric structures, Proc. U. S.-Japan Seminar Diff. Geometry (1965), 147-150.

[10] ――――, On generalized graded Lie algebras and geometric structures I, J. Math. Soc. Japan **19** (1967), 215-254.

[11] ――――, On differential systems, graded Lie algebras and pseudo-groups, to appear.

[12] Wells, R. O., On the holomorphic hull of a real submanifold in several complex variables, Comm. Pure Appl. Math. **19** (1966), 145-165.

[13] ――――, Holomorphic hulls and holomorphic convexity of differentiable submanifolds, Trans. Amer. Math. Soc. **132** (1968), 245-262.