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Finiteness of the Number of Discrete Eigenvalues of the Schrödinger Operator for a Three Particle System II*

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§4. Introduction

The present paper is the continuation of [2] with the same title. In the previous papers [1] [2], we have studied the Schrödinger operator of the form

$$H=-\Delta_1-\Delta_2-\frac{Z_1}{r_1}-\frac{Z_2}{r_2}+\frac{Z_3}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}, \tag{4.1}$$

where $Z_1 \geq Z_2$ and Z_3 are positive constants. There we have shown the results:

i) If $Z_2 > Z_3$, H has an infinite number of discrete eigenvalues in $(-\infty, -Z_1^2/4)$.

ii) If Z_1, $Z_2 < Z_3$, H has at most a finite number of discrete eigenvalues in $(-\infty, -Z_1^2/4)$.

In this article we shall study the case $Z_1 \geq Z_3 > Z_2$. In this case, the conditions in [1] or [2] are not satisfied, but we have the same results as (ii) by modifying slightly the proof of Theorem 1 in [2].

The theorems proved in this paper assert that the number of discrete eigenvalues of the operator H of the form

$$H=-\Delta_1-\Delta_2+q_1(\boldsymbol{r}_1)+q_2(\boldsymbol{r}_2)+P(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{4.2}$$

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depends essentially on the behavior of $q_2(\boldsymbol{r}_2)$ in the region $r_2 \geq R$, and $P(\boldsymbol{r}_1, \boldsymbol{r}_2)$ in $|\boldsymbol{r}_1 - \boldsymbol{r}_2| > R$, if

$$(4.3) \qquad \mu_1 < \mu_2,$$

where

$$(4.4) \qquad \mu_i = \inf_{\varphi \in \mathcal{D}^2_{L^2}(R^3)} \frac{(H_i\varphi, \varphi)_{R^3}}{\|\varphi\|^2} \quad (i=1, 2),$$

$$H_i = -\Delta_i + q_i(\boldsymbol{r}_i) \qquad (i=1, 2).$$

On the other hand the condition (4.3) depends on the behavior of $q_1(\boldsymbol{r}_1)$ and $q_2(\boldsymbol{r}_2)$ not only at infinity, but also in the whole space R^3. Then the structure of the spectrum of the operator of the form (4.2) is complicated.

Since the proofs of the theorems are essentially the same as the one applied in [1] and [2], we shall only sketch the outline. For the convenience, we shall use the same notation as the one introduced in [2].

§5. Some Theorems and Proofs

Let H of the form (4.2) satisfy the conditions:

$$(5.1) \qquad q_i(\boldsymbol{r}_i) \in L^2_{loc}(R^3) \quad (i=1, 2) \quad \text{and} \quad P(\boldsymbol{r}_1, \boldsymbol{r}_2) \in Q_\alpha(R^6)$$

(for some $\alpha > 0$) are real-valued functions,

$$(5.2) \qquad q_i(\boldsymbol{r}_i) \ (i=1, 2) \text{ converge uniformly to zero as } r_i \to \infty,$$

$$(5.3) \qquad P(\boldsymbol{r}_1, \boldsymbol{r}_2) \geq 0 \text{ in } R^6,$$

$$(5.4) \qquad P(\boldsymbol{r}_1, \boldsymbol{r}_2) \text{ converges uniformly to zero as } r_1 \to \infty$$

whenever $\boldsymbol{r}_2$ is fixed, and as $r_2 \to \infty$ whenever $\boldsymbol{r}_1$ is fixed (see (2.2)—(2.5) in [2]). Then it is known that

i) if the domain $D(H)$ of H is $\mathcal{D}^2_{L^2}(R^6)$, H is a lower semi-bounded selfadjoint operator in $L^2(R^6)$,

ii) if $\mu_1 \leq \mu_2$, where $\mu_i (i=1, 2)$ are defined by (4.3) and (4.4), $\sigma_e(H) = [\mu_1, \infty)$ (see Theorem 1 in [2]).

Moreover we remark the fact that if we assume conditions (5.1) and (5.2), and if $D(H_i) = \mathcal{D}^1_{L^2}(R^3)$, then $H_i (i=1, 2)$ are a lower semi-bounded selfadjoint operator in $L^2(R^3)$, $\sigma_e(H_i)=[0,\infty)$ and $\mu_i\leq 0$. Now we have

Theorem 3. *If we assume for the operator H of the form* (4.2) *the conditions* (4.3), (5.1)–(5.4) and

$$(5.5)\qquad P(\boldsymbol{r}_1,\boldsymbol{r}_2)+q_2(\boldsymbol{r}_2)\begin{cases}\geq \dfrac{C}{r_2^{2\beta-\varepsilon}} & \text{for } k\leq \dfrac{r_2^\beta}{r_1}\leq k' \text{ and } r_2\geq R\\[2ex] \geq 0 & \text{for } k'<\dfrac{r_2^\beta}{r_1} \text{ and } r_2\geq R\end{cases}$$

for some constants k, $k'(1<k<k'<+\infty)$, $\beta(0<\beta\leq 1)$, $\varepsilon>0$, $R>0$ and $C>0$, then H has at most a finite number of discrete eigenvalues in $(-\infty,\mu_1)$.

Proof. Let $g(t)$ be a function having the following properties: $g(t)\in C^\infty(0,\infty)$, $g(t)\equiv 1$ for $t\geq k'$, $g(t)\equiv 0$ for $0<t<k$ and $0\leq g(t)\leq 1$ for $0<t<+\infty$. By the conditions (4.3) and (5.2), we can choose $R>1$ large enough to satisfy the following inequalities:

$$(5.6)\qquad q_1(\boldsymbol{r}_1)>\frac{\mu_1-\mu_2}{3}\quad\text{for}\quad r_1>\frac{R^\beta}{k'}$$

$$q_2(r_2)>\frac{\mu_1-\mu_2}{3}\quad\text{for}\quad r_2>\frac{R^\beta}{k'},$$

$$(5.7)\qquad t^4g(t)g''(t)+CR^\varepsilon\geq 0\quad\text{for}\quad k\leq t\leq k',$$

$$(5.8)\qquad \frac{\mu_2-\mu_1}{3}-\frac{1}{R^{2\beta}}|g(t)g''(t)t^4|\geq 0\quad\text{for}\quad k\leq t\leq k'.$$

Then we define domains $\{\Omega_i\}_{i=1,\dots,4}$ in the same way as in [2], namely

$$\Omega_1=\{r_1<R \text{ and } r_2<R\},\quad \Omega_2=\left\{r_1\geq R \text{ and } r_2\leq\frac{r_1^\beta}{k}\right\},$$

$$\Omega_3=\left\{r_2\geq R \text{ and } r_1\leq\frac{r_2^\beta}{k}\right\}\quad\text{and}\quad \Omega_4=R^6-\bigcup_{i=1}^{3}\Omega_i,$$

and for $\psi\in D(H)=D^2_{L^2}(R^6)$

$$(5.9)\qquad L[\psi]\equiv(H\psi,\psi)_{R^6}=\sum_{i=1}^{4}\{\|\,|\nabla_1\psi|\,\|^2_{\Omega_i}+\|\,|\nabla_2\psi|\,\|^2_{\Omega_i}$$

$$+(q_1\psi,\psi)_{\Omega_i}+(q_2\psi,\psi)_{\Omega_i}+(p\psi,\psi)_{\Omega_i}\}\equiv\sum_{i=1}^{4}L_i[\psi].$$

Then we have in the same way as in [2] (see Lemma 1, Lemma 3 and Lemma 4 in [2])

Lemma 7.

(i) *For any* $\psi\in D(H)$, $L_3[\psi]\geq\mu_1\|\psi\|^2_{\Omega_3}$.

(ii) *For any* $\psi\in D(H)$, $L_4[\psi]\geq\mu_1\|\psi\|^2_{\Omega_4}$.

(iii) *There exists some finite dimensional subspace* $\mathfrak{M}$ *in* $L^2(R^6)$ *such that for any* $\psi\in D(H)\cap\mathfrak{M}^\perp$, $L_1[\psi]\geq\mu_1\|\psi\|^2_{\Omega_1}$.

In fact we have only to take into account for (i) that we have for any $\varphi\in\mathcal{D}^2_{L^2}(R^3)$ $(H_1\varphi,\varphi)_{R^3}\geq\mu_1\|\varphi\|^2_{R^3}$, and for (ii) $\frac{\mu_1-\mu_2}{3}>\frac{\mu_1}{2}$ because of $\mu_1<\mu_2\leq 0$.

Now we shall show by modifying the proof of Lemma 1 in [2].

Lemma 8. *For any* $\psi\in D(H)$, $L_2[\psi]\geq\mu_1\|\psi\|^2_{\Omega_2}$.

Proof. Making use of the relation $(H_2\varphi,\varphi)_{R^3}\geq\mu_2\|\varphi\|^2_{R^3}$ for any $\varphi\in\mathcal{D}^2_{L^2}(R^3)$, we have in the same way as the proof of Lemma 1 in [2].

$$L_2[\psi]\geq\int_{\Omega_2}\left\{\mu_2 g\left(\frac{r_1^\beta}{r_2}\right)^2+g\left(\frac{r_1^\beta}{r_2}\right)g''\left(\frac{r_1^\beta}{r_2}\right)\cdot\frac{r_1^{2\beta}}{r_2^4}+q_2(r_2)\left(1-g\left(\frac{r_1^\beta}{r_2}\right)^2\right)+P+q_1\right\}|\psi|^2dx \tag{5.10}$$

for any $\psi\in D(H)$. Let $\frac{r_1^\beta}{r_2}=t$, and we have by (5.6)–(5.8)

$$(1-g(t)^2)(-\mu_2+q_2(r_2))+(\mu_2-\mu_1)+\frac{1}{r_1^{2\beta}}g(t)g''(t)t^4+q_1(r_1)+P(r_1,r_2)\geq 0 \tag{5.11}$$

for $t\geq k$ and $r_1\geq R$. In fact $g(t)\equiv 1$ and $g''(t)\equiv 0$ for $t\geq k'$, and $(1-g(t)^2)(-\mu_2+q_2(r_2))\geq\frac{\mu_1-\mu_2}{3}$ for $k\leq t<k'$ i. e. $r_2>\frac{r_1^\beta}{k'}>\frac{R^\beta}{k'}$, $P(r_1, r_2)\geq 0$, and $q_1(r_1)\geq\frac{\mu_1-\mu_2}{3}$ for $r_1>R>\frac{R^\beta}{k'}$. Therefore for any $\psi\in D(H)$, we have $L_2[\psi]\geq\mu_1\|\psi\|^2_{\Omega_2}$ by (5.10) and (5.11).

Making use of Lemma 7 and Lemma 8 in the same way as applied in the proof of Theorem 1 in [2], we have the assertion of Theorem 3.

Remark 6. The operator of the form (4.1) has at most a finite number of discrete eigenvalues in $(-\infty, \mu_1)$, if $Z_1 \geq Z_3 > Z_2$. In fact $\mu_1 = -\frac{Z_1^2}{4} < \mu_2 = -\frac{Z_2^2}{4}$ and the condition (5.5) is satisfied (see, Remark 1 in [2]).

If $q_2(\boldsymbol{r}_2)$ tends to zero more rapidly than the conditions given in Remark 1 in [2] which satisfies (5.5), we have only to assume (5.3) in place of (5.3) and (5.5) as for $P(\boldsymbol{r}_1, \boldsymbol{r}_2)$. Namely we have

Theorem 4. *If we assume* (4.3), (5.1)-(5.4) *and the condition*

$$q_2(\boldsymbol{r}_2) \geq -\frac{1}{4}\frac{1}{r_2^2} \quad \text{for} \quad r_2 \geq R, \tag{5·12}$$

the Schrödinger operator H of the form (4.2) *has at most a finite number of discrete eigenvalues in* $(-\infty, \mu_1)$.

Proof. Let

$$T = -\Delta_1 - \Delta_2 + q_1(\boldsymbol{r}_1) + q_2(\boldsymbol{r}_2). \tag{5.13}$$

If $D(T) = D(H) = \mathscr{D}_{L^2}^2(R^6)$, T is a selfadjoint operator in $L^2(R^6)$ and $\sigma_e(T) = \sigma_e(H) = [\mu_1, \infty)$. By (5.12) H_2 has at most a finite number of discrete eigenvalues in $(-\infty, 0)$. Let the discrete eigenvalues of H_2 be $\lambda_1^{(2)} = \mu_2 \leq \lambda_2^{(2)} \leq \cdots \leq \lambda_m^{(2)} < 0$ (m is finite), if they exist, and let those of H_1 be $\lambda_1^{(1)} = \mu_1 \leq \lambda_2^{(1)} \leq \cdots \leq \lambda_n^{(1)} < 0$ (n may be infinite). Then by the method of the separation of variables, we have, by the method applied to the proof of Lemma 6 in [2],

Lemma 9. *If* λ *is an eigenvalue of* T *smaller than* μ_1, *then* $\lambda \in \{\lambda_k^{(1)} + \lambda_l^{(2)}\}_{k=1,\cdots,n,\, l=1,\cdots,m}$.

Taking into consideration that $\{\lambda_k^{(1)} + \lambda_l^{(2)}\}_{k=1,\cdots,n,\, l=1,\cdots,m}$ concentrates at most $\{\lambda_l^{(2)}\}_{l=1,\cdots,m}$, T has at most a finite number of discrete eigenvalues in $(-\infty, \mu_1)$ by Lemma 9 and the condition (4.3). Let $\mathfrak{N}$ be the finite dimensional subspace in $L^2(R^6)$ spanned by the eigenfunctions belonging to the eigenvalues of T in $(-\infty, \mu_1)$. Then by (5.3) and $\sigma_e(T) = [\mu_1, \infty)$ we have for any $\psi \in D(H) \cap \mathfrak{N}^\perp$,

$$(H\psi, \psi)_{R^6} \geq (T\psi, \psi)_{R^6} \geq \mu_1 \|\psi\|_{R^6}^2, \tag{5.14}$$

which asserts that Theorem 2 holds.

In case $\mu_1 \leq \mu_2$, in order to obtain the result that H has at most a finite number of discrete eigenvalues in $(-\infty, \mu_1)$, we must impose some conditions on the behavior of $q_1(\boldsymbol{r}_1)$ for $r_1 \geq R$, and $P(\boldsymbol{r}_1, \boldsymbol{r}_2)$ for $|\boldsymbol{r}_1 - \boldsymbol{r}_2| \geq R$ in addition to (5.5) or (5.12) (see, for example, Theorem 1 or Theorem 2 in [2]). Otherwise there exists the case that H has an infinite number of discrete eigenvalues in $(-\infty, \mu_1)$. Namely we have

Theorem 5. *If we assume* (5.1)-(5.4) *and*

(5.15) $$\mu_1 \leq \mu_2,$$

(5.16) $$q_2(\boldsymbol{r}_2) \leq -\frac{c}{r_2^\beta} \quad \text{for} \quad r_2 \geq R_0,$$

(5.17) $$0 \leq P(r_1, r_2) \begin{cases} \leq \dfrac{dR_1^{\beta'-\beta} R_2^{\gamma-\beta}}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^\gamma} & \text{for} \quad |\boldsymbol{r}_1 - \boldsymbol{r}_2| \leq R_2, \\ \leq \dfrac{dR_1^{\beta'-\beta}}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^{\beta'}} & \text{for} \quad R_2 \leq |\boldsymbol{r}_1 - \boldsymbol{r}_2| \leq R_1, \\ \leq \dfrac{d}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|^\beta} & \text{for} \quad R_1 \leq |\boldsymbol{r}_1 - \boldsymbol{r}_2|, \end{cases}$$

for some constants $\beta (0 < \beta \leq 2)$, $\gamma (0 < \gamma < 3/2)$, $\beta' (\max(\beta, \gamma) < \beta' < 3)$, $c > 0$, $d > 0$, $R_2 (0 < R_2 < 1)$ *and sufficiently large* $R_0 > 0$, $R_1 > 0$, and

(5.18) $$c - d \begin{cases} > 0 & \text{for} \quad 0 < \beta < 2, \\ > \dfrac{1}{4} & \text{for} \quad \beta = 2, \end{cases}$$

then there exist an infinite number of discrete eigenvalues in $(-\infty, \mu_1)$.

Proof. We can prove the above theorem in a manner similar to [1].

By (5.16) and (5.18) ((5.18) is necessary for $\beta = 2$), H_2 has an infinite number of discrete eigenvalues in $(-\infty, 0)$. Then taking account of (5.15) and $\sigma_e(H_1) = \sigma_e(H_2) = [0, \infty)$, μ_1 is a discrete eigenvalue of H_1. Let a normalized eigenfunction belonging to μ_1 be $\varphi_0(\boldsymbol{r}_1) \in D(H_1) = \mathcal{D}_{L^2}^2(R^3)$. Moreover we can choose the function having the following property (see Lemma 5 in [1]); for $\varepsilon>0$ satisfying the relation $c-d-\frac{1}{4-\varepsilon}>0$ in case $c-d>\frac{1}{4}$,

$$\left\{\begin{array}{l} g_1(\boldsymbol{r}_2)\in C_0^\infty(R^3),\ \|g_1\|_{R^3}=1,\ g_1(\boldsymbol{r}_2)\equiv 0 \quad \text{for} \quad r_2\leq R_0 \\ \int_{R^3}\frac{|g_1(\boldsymbol{r}_2)|^2}{r_2^2}d\boldsymbol{r}_2\geq(4-\varepsilon)\int_{R^3}|\nabla_1 g_1|^2 d\boldsymbol{r}_2. \end{array}\right. \tag{5.19}$$

Let $g_\alpha(\boldsymbol{r}_2)=\alpha^{3/2}g_1(\alpha\boldsymbol{r}_2)$ and $\psi_\alpha(x)=\varphi_0(\boldsymbol{r}_1)g_\alpha(\boldsymbol{r}_2)$, where α is a positive parameter, we have $\|\psi_\alpha\|_{R^6}=1$ and $\psi_\alpha(x)\in D(H)=\mathcal{D}_{L^2}^{2}(R^6)$. Then taking account of the relation

$$\lim_{\alpha\to 0}\int_{R^6}\frac{|\varphi_0(\boldsymbol{r}_1)g_1(\boldsymbol{r}_2)|^2}{|\alpha\boldsymbol{r}_1-\boldsymbol{r}_2|^\delta}dx=\int_{R^3}\frac{|g_1|^2}{r_2^\delta}d\boldsymbol{r}_2 \quad \text{for any} \quad \delta(0<\delta<3), \tag{5.20}$$

(see Lemma 4 in [1]), there exists some constant $\alpha_0'(0<\alpha_0'<1)$ such that for any $\alpha(0<\alpha<\alpha_0')$ we have

$$(H\psi_\alpha,\psi_\alpha)_{R^6}\leq\mu_1+M\alpha^{\beta'}-\left\{\begin{array}{ll} \frac{1}{2}(c-d)\alpha^\beta\int_{R^3}\frac{|g_1|^2}{r_2^\beta}d\boldsymbol{r}_2+M'\alpha^2 & \\ & (\text{for } 0<\beta<2) \\ \frac{1}{4}\left(c-d-\frac{1}{4-\varepsilon}\right)\alpha^2\int_{R^3}\frac{|g_1|^2}{r_2^2}d\boldsymbol{r}_2 & \\ & (\text{for } \beta=2), \end{array}\right. \tag{5.21}$$

where M and M' are constants independent of α (see (4.9) or (4.9′) in [1]). Then by (5.18) and (5.21) there exists some constant $\alpha_0(0<\alpha_0<\alpha_0')$ such that for any $\alpha(0<\alpha<\alpha_0)$ we have

$$(H\psi_\alpha,\psi_\alpha)_{R^6}<\mu_1. \tag{5.22}$$

Now we assume that H has at most a finite number of discrete eigenvalues in $(-\infty,\mu_1)$. Let their number be p, and the subspace in $L^2(R^6)$ spanned by their eigenfunctions be $\mathfrak{M}$. Then we can choose $\{\alpha_i\}_{i=1,\cdots,p+1}$ such that $0<\alpha_{p+1}<\alpha_p<\cdots<\alpha_1<\alpha_0$ and the support of $g_{\alpha_i}(\boldsymbol{r}_2)$ and that of $g_{\alpha_j}(\boldsymbol{r}_2)$ are disjoint in R^3 for $i\neq j$. Since the dimension of the subspace spanned by $\{\varphi_0(\boldsymbol{r}_1)g_{\alpha_i}(\boldsymbol{r}_2)\}_{i=1,\cdots,p+1}$ is $p+1$, we can choose constants $\{c_i\}_{i=1,\cdots,p+1}$ such that $\sum_{i=1}^{p+1}c_i\psi_{\alpha_i}(x)\in\mathfrak{M}^\perp\cap D(H)$ and $\sum_{i=1}^{p+1}|c_i|^2=1$. Let $f(x)=\sum_{i=1}^{p+1}c_i\psi_{\alpha_i}(x)$. Then by (5.22) and the condition that the supports of g_{α_i} and g_{α_j} are disjoint for each $i \neq j$, we have $\|f\|_{R^6}=1$ and

$$(5.23) \qquad (Hf, f)_{R^6} = \sum_{i=1}^{p+1} |c_i|^2 (H\psi_{\alpha_i}, \psi_{\alpha_i})_{R^6} < \mu_1 \sum_{i=1}^{p+1} |c_i|^2 = \mu_1.$$

On the other hand by $f \in \mathfrak{M}^{\perp} \cap D(H)$ and $\|f\|_{R^6}=1$, we have $(Hf, f)_{R^6} \geq \mu_1$, which contradicts (5.23). Thus the assertion of Theorem 5 is proved.

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