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Corrections to "Finiteness of the Number of Discrete Eigenvalues of the Schrödinger Operator for a Three Particle System"

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Remark 4, p. 59, was an error due to the negligence of the fact that R depends on Z. The correct assertion is the following:

There exists some constant $Z_0\left(\frac{Z_3}{2}\geq Z_0>0\right)$ depending only on Z_3 such that for any positive constants Z_1, $Z_2(Z_0\geq Z_1\geq Z_2>0)$ the operator of the form

$$H=-\Delta_1-\Delta_2-\frac{Z_1}{r_1}-\frac{Z_2}{r_2}+\frac{Z_3}{|\boldsymbol{r}_1-\boldsymbol{r}_2|} \tag{1}$$

has no discrete eigenvalues.

In fact let $\frac{Z_3}{2}\geq Z_1\geq Z_2>0$. Then taking into consideration Remark 1 and the fact that μ given to (1) by (2.7) and (2.8) equals $-\frac{Z_1^2}{4}$, (3.1) and (3.2) are satisfied by $R=\frac{c_1}{Z_1}$, where c_1 is sufficiently large constant depending only on Z_3. On the other hand, we can show the following fact which is more precise than Lemma 5.

There exist an "extension operator" Φ_1 which maps $\mathcal{E}^1_{L^2}(\Omega_1)$ to $\mathcal{D}^1_{L^2}(R^6)$ and some constant c_2 depending only on Z_3 such that for any $\varphi\in\mathcal{E}^1_{L^2}(\Omega_1)$

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$$(\Phi_1\varphi)(x)=\varphi(x) \quad \text{for} \quad x\in\Omega_1 \tag{2}$$

and

$$\begin{cases} \| |\nabla(\Phi_1\varphi)| \|^2_{R^6} \leq c_2(\| |\nabla\varphi| \|^2_{\Omega_1}+Z_1^2\|\varphi\|^2_{\Omega_1}), \\ \|\Phi_1\varphi\|^2_{R^6} \leq c_2\Delta\|\varphi\|^2_{\Omega_1}. \end{cases} \tag{3}$$

Indeed, for $\varphi\in\mathcal{E}^1_{L^2}(\Omega_1)$ we define $f\in\mathcal{E}^1_{L^2}(\Omega_0)$ by

$$f(x)=\varphi(Rx) \quad \text{for} \quad x\in\Omega_0, \tag{4}$$

where $\Omega_0=\{x\in R^6;\ r_1<1,\ r_2<1\}$. Then we have

$$\begin{cases} \|f\|^2_{\Omega_0}=\dfrac{1}{R^6}\|\varphi\|^2_{\Omega_1}, \\ \| |\nabla f| \|^2_{\Omega_0}=\dfrac{R^2}{R^6}\| |\nabla\varphi| \|^2_{\Omega_1}. \end{cases} \tag{5}$$

Let Φ_2 and $c_3=c_3(\Omega_0,\Phi_2)$ be the Φ and $\tilde{c}$ satisfying the relations given in Lemma 5 with Ω_1 replaced by Ω_0. Now we define $\Phi_1\varphi\in\mathcal{D}^1_{L^2}(R^6)$ by

$$(\Phi_1\varphi)(x)=(\Phi_2 f)\left(\frac{x}{R}\right) \quad \text{for} \quad x\in R^6. \tag{6}$$

Then by Lemma 5 Φ_1 satisfies (2), and

$$\frac{1}{R^6}\|\Phi_1\varphi\|^2_{R^6}=\|\Phi_2 f\|_{R^6}\leq c_3\|f\|^2_{\Omega_0}=\frac{c_3}{R^6}\|\varphi\|^2_{\Omega_1} \tag{7}$$

and

$$\begin{aligned} \frac{R^2}{R^6}\| |\nabla(\Phi_1\varphi)| \|^2_{R^6} &= \| |\nabla(\Phi_2 f)| \|^2_{R^6}\leq c_3(\| |\nabla f| \|^2_{\Omega_0}+\|f\|^2_{\Omega_0}) \\ &= \frac{c_3R^2}{R^6}\left(\| |\nabla\varphi| \|^2_{\Omega_1}+\frac{1}{R^2}\|\varphi\|^2_{\Omega_1}\right). \end{aligned} \tag{8}$$

By (7) and (8) we have (3).

Then using the well-known inequality

$$\int_{R^3}\frac{|\varphi|^2}{r^2}dx\leq 4\int_{R^3}|\nabla\varphi|^2dx \quad \text{for} \quad \varphi\in\mathcal{D}^1_{L^2}(R^3), \tag{9}$$

and Schwartz's inequality, we have by (2) and (3) for any $\psi\in\mathcal{D}^2_{L^2}(R^6)$

$$\int_{\Omega_1}\frac{|\psi|^2}{r_1}dx\leq\int_{R^6}\frac{|\Phi_1\psi|^2}{r_1}dx\leq\left(\int_{R^6}\frac{|\Phi_1\psi|^2}{r_1^2}dx\right)^{1/2}\cdot\|\Phi_1\psi\|_{R^6} \tag{10}$$

$$=\left(\int_{R^3}d\boldsymbol{r}_2\int_{R^3}\frac{|\Phi_1\psi|^2}{r_1^2}d\boldsymbol{r}_1\right)^{1/2}\cdot\|\Phi_1\psi\|_{R^6}\leq 2\left(\int_{R^3}d\boldsymbol{r}_2\int_{R^3}|\nabla_1(\Phi_1\psi)|^2d\boldsymbol{r}_1\right)^{1/2}\cdot\|\Phi_1\psi\|_{R^6}$$

$$\leq\frac{\eta}{Z_1}\||\nabla_1(\Phi_1\psi)\|_{R^6}^2+\frac{Z_1}{\eta}\|\Phi_1\psi\|_{R^6}^2$$

$$\leq\frac{c_2\eta}{Z_1}\||\nabla\psi|\|_{\Omega_1}^2+c_2\left(\eta+\frac{1}{\eta}\right)Z_1\|\psi\|_{\Omega_1}^2$$

and similarly

$$\int_{\Omega_1}\frac{|\psi|^2}{r_2}dx\leq\frac{c_2\eta}{Z_1}\||\nabla\psi|\|_{\Omega_1}^2+c_2\left(\eta+\frac{1}{\eta}\right)Z_1\|\psi\|_{\Omega_1}^2, \tag{11}$$

where η is an arbitrary positive constant. Let $\eta=\frac{1}{2c_2}$. Then if we take into consideration $\frac{Z_3}{|\boldsymbol{r}_1-\boldsymbol{r}_2|}>\frac{Z_3}{2R}=\frac{Z_1Z_3}{2c_1}$ in Ω_1, we have by (10) and (11) for any $\psi\in\mathcal{D}_{L^2}^2(R^6)$

$$\begin{aligned}L_1[\psi]&\geq\left(1-c_2\eta-\frac{Z_2}{Z_1}c_2\eta\right)\||\nabla\psi|\|_{\Omega_1}^2+\\&\quad+\left\{\frac{Z_1Z_3}{2c_1}-c_2\left(\eta+\frac{1}{\eta}\right)Z_1(Z_1+Z_2)\right\}\|\psi\|_{\Omega_1}^2\\&\geq N_1\left\{\frac{Z_3}{2c_1}-2c_2\left(\eta+\frac{1}{\eta}\right)Z_1\right\}\|\psi\|_{\Omega_1}^2.\end{aligned}\tag{12}$$

Then there exists some constant Z_0 such that for any Z_1 and $Z_2 (Z_0\geq Z_1\geq Z_2>0)$ we have

$$L_1[\psi]\geq\mu\|\psi\|_{\Omega_1}^2=-\frac{Z_1^2}{4}\|\psi\|_{\Omega_1}^2 \tag{13}$$

for any $\psi\in\mathcal{D}_{L^2}^2(R^6)$. By Lemma 1, Lemma 3 and (13), we have the assertion.

Remark. There exists some constant Z_0' depending on $Z_1 (Z_0'>Z_1\geq Z_2>0)$ such that for any $Z_3>Z_0'$ the operator of the form (1) has no discrete eigenvalues.

In fact R satisfying (3.1) and (3.2) is independent of Z_3 ($Z_3\geq 2Z_1$). Then by the same calculation as (3.16) and (3.17) we have

$$L_1[\psi]\geq(1-2\tilde{c}\eta Z_1)\||\nabla\psi|\|_{\Omega_1}^2+\left(\frac{Z_3}{2R}-2Z_1(\eta+c(\eta))\tilde{c}\right)\|\psi\|_{\Omega_1}^2. \tag{14}$$

Take $\eta=\frac{1}{2\tilde{c}Z_1}$ and Z_3 sufficiently large. We have for any $\psi\in\mathcal{D}^2_{L^2}(R^6)$

$$(15) \qquad L_1[\psi]\geq-\frac{Z_1^2}{4}\|\psi\|^2_{\Omega_1}=\mu\|\psi\|^2_{\Omega_1}.$$

Then by Lemma 1, Lemma 3 and (15), we have the assertion.