Publ. RIMS, Kyoto Univ.
Vol. 6 (1970), 205-236

On Asymptotic Solutions of the Functional Difference Equations Associated with Some Nonlinear Difference Equations

By

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0. Introduction

Let

$$y(x+1)=g(x, y) \tag{0.1}$$

be a system of m nonlinear difference equations for the vector-valued function $y(x)=(y_1(x), y_2(x), \cdots, y_m(x))$ of a complex variable x. $g(x, y)$ is an analytic function of $m+1$ complex variables (x, y) defined in the region $X_0 \times Y_0$, where

$$X_0 : |x| > R,$$
$$Y_0 : \|y\| < r \qquad \|y\| = \max_j |y_j|,$$

R, r being positive constants. When we consider the expansion

$$g(x, y)=g_0(x)+A(x)y+\sum_{|p|\geqq 2} g_p(x)y^p, \tag{0.2}$$

the case in which $g_0(x)\not\equiv 0$ was discussed in [1], and the case in which $g_0(x)\equiv 0$, $A(x)\not\equiv 0$ was discussed in [2]. We shall deal with some cases in which $g_0(x)\equiv 0$ and $A(x)\equiv 0$.

In Part II we shall discuss about a system of nonlinear difference equations of the form:

$$y_i(x+1)=\prod_{j=1}^{m} y_j^{\lambda_{ij}} f_i(x, y) \qquad (i=1, 2, \cdots, m) \tag{0.3}$$

Received February 6, 1970.
Communicated by M. Urabe.
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where λ_{ij}'s are nonnegative integers such that

$$\sum_{j=1}^{m} \lambda_{ij} \geqq 2 \qquad (i=1, 2, \cdots, m)$$

and $f_i(x, y)$ are analytic functions of x and y in the region $X_0 \times Y_0$ such that $f_i(\infty, 0) \neq 0$ $(i=1, 2, \cdots, m)$. Under the further assumption that each eigenvalue λ_i of the $m \times m$ matrix $\Lambda = (\lambda_{ij})$ is absolutely greater than one it may be shown that without loss of generality we may assume $f_i(\infty, 0)=1$.

In §6 we shall prove the existence of a transformation of the form:

$$(0.4) \qquad y_i = u_i\left(1 + \sum_{|k| \geqq 1} p_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}\right) \qquad (i=1, 2, \cdots, m)$$

by which the equation (0.3) are transformed into the most simple equations

$$(0.5) \qquad u_i(x+1) = \prod_{j=1}^{m} u_j^{\lambda_{ij}} \qquad (i=1, 2, \cdots, m),$$

so that we can conclude that the equation (0.3) has a formal solution of the form (0.4) in which u_i is substituted by any solution $u_i(x)$ of the equation (0.5).

On the basis of this situation we find it effective to regard the solution (0.4) as a function $y_i(x, u)$ of $m+1$ independent variables x, u_1, $u_2, \cdots, u_m$ satisfying the following equations:

$$(0.6) \qquad y_i\left(x+1, \prod_{j=1}^{m} u_j^{\lambda_{1j}}, \prod_{j=1}^{m} u_j^{\lambda_{2j}}, \cdots, \prod_{j=1}^{m} u_j^{\lambda_{mj}}\right)$$

$$= \prod_{j=1}^{m} y_j^{\lambda_{ij}}(x, u) f_i(x, y(x, u)) \qquad (i=1, 2, \cdots, m)$$

which we shall call a system of functional difference equations associated with the original system of difference equations (0.3). In §7 we shall give a detailed discussion on this matter.

In §8 and §9 we shall prove the existence and the uniqueness of an analytic solution of (0.6) of the form

$$(0.7) \qquad y_i(x, u)=u_i(1+\sum_{|k|=1}^{N} p_{k_0k_1\cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m} + z_{iN}(x, u)) \qquad (i=1, 2, \cdots, m)$$

such that the inequality

$$\|z_N\| \leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

holds with some arbitrary constant M in a certain region of (x, u) space. In §9 we shall show the existence of an asymptotic analytic solution of (0.6).

In preparation for the discussions in Part II, we shall deal with a following particular case of equation (0.3) in Part I:

$$(0.8) \qquad y_i(x+1)=y_i^{\lambda_i} f_i(x, y) \qquad (i=1, 2, \cdots, m),$$

where λ_i's are integers $\geqq 2$.

Equations of the form (0.8) are important in themselves since they appear in the study of some important nonlinear difference equations. For example, consider a nonlinear difference equation of the form:

$$(0.9) \qquad z(x+1)=a_0(x)+a_1(x)z(x)+\cdots\cdots+a_m(x)z^m(x) \qquad (m\geqq 2),$$

where $a_j(x)$ $(j=0, 1, \cdots, m)$ are analytic functions of x in the region $|x|>R$ such that $a_m(\infty)\neq 0$. By the transformation $z=1/y$, equation (0.9) is transformed to the following equation of the form (0.8):

$$(0.10) \qquad y(x+1)=y^m f(x, y),$$

where

$$f(x, y)=(a_m(x)+a_{m-1}(x)y+\cdots\cdots+a_0(x)y^m)^{-1}.$$

Any solution of (0.9) that approaches to ∞ as x tends to ∞ is then obtained from a solution of (0.10) that approaches to 0 as x tends to ∞.

In conclusion the author expresses his hearty thanks to Professor HUKUHARA, Professor SIBAGAKI and Professor URABE who had through the preparation of this paper incessant interest on our problem and gave many criticisms and improvements.

Part I. On the System of Nonlinear Difference Equations of the Form $y_i(x+1)=y_i^{\lambda_i}f_i(x, y)$ $(i=1, 2, \cdots, m)$

1. Formal solutions. We consider the system of nonlinear difference equations of the form

$$(1.1) \qquad y_i(x+1)=y_i^{\lambda_i}f_i(x, y_1, y_2, \cdots, y_m) \qquad (i=1, 2, \cdots, m)$$

where λ_i's are integers greater than one and f_i's are analytic functions of x and $y=(y_1, y_2, \cdots, y_m)$ which are defined in the domain $X_0\times Y_0=\{x \mid |x|>R\}\times\{y \mid \|y\|<r\}$ $(\|y\|=\max_i |y_i|)$ and satisfy the conditions

$$(1.2) \qquad f_i(\infty, 0, 0, \cdots, 0)=\mu_i\neq 0 \qquad (i=1, 2, \cdots, m).$$

We remark that we may assume $\mu_i=1$ $(i=1, 2, \cdots, m)$ without loss of generality. Indeed, setting $\xi_i=\mu_i^{1/(\lambda_i-1)}$ and introducing the unknown $z_i=\xi_i y_i$, we have for z_i the equations

$$z_i(x+1)=z_i^{\lambda_i}g_i(x, z_1, z_2, \cdots, z_m),$$

where $g_i(x, z_1, z_2, \cdots, z_m)=\mu_i^{-1}f_i(x, \xi_1^{-1}z_1, \xi_2^{-1}z_2, \cdots, \xi_m^{-1}z_m)$ for which it holds

$$g_i(\infty, 0, 0, \cdots, 0)=1 \qquad (i=1, 2, \cdots, m).$$

Owing to the assumptions just made we have the expansions

$$f_i(x, y_1, y_2, \cdots, y_m)=1+\sum_{|k|\geqq 1} a_{k_0k_1\cdots k_m}^{(i)}x^{-k_0}y_1^{k_1}y_2^{k_2}\cdots y_m^{k_m}$$

where $|k|=k_0+k_1+\cdots+k_m$.

In finding a formal solution of the system of equations (1.1) we will show that a transformation of the type

$$(1.3) \qquad y_i=u_i(1+\sum_{|k|\geqq 1} p_{k_0k_1\cdots k_m}^{(i)}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m})$$

is effective, the coefficients $p_{k_0k_1\cdots k_m}^{(i)}$ being suitably chosen.

First, we consider the transformation of the first step

$$(1.4) \qquad y_i=u_i(1+q_0^{(i)}x^{-1}+\sum_{j=1}^{m} q_j^{(i)}u_j),$$

which has the inverse transformation

$$u_i = y_i(1 - q_0^{(i)} x^{-1} - \sum_{j=1}^{m} q_j^{(i)} y_j + \cdots), \tag{1.5}$$

where $\cdots$ represents the terms which are of higher degree than one in x^{-1} and y_j. Setting $x+1$ for x in (1.5) and using (1.1) and (1.4) it is immediately shown that the new unknown u_i satisfies the following equation

$$\begin{aligned} u_i(x+1) = u_i^{\lambda_i}(1 + q_0^{(i)} x^{-1} &+ \sum_{j=1}^{m} q_j^{(i)} u_j)^{\lambda_i} \\ &\times (1 + \sum_{|k|=1} a_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m} + \cdots) \\ &\times (1 - q_0^{(i)} x^{-1} + \cdots), \end{aligned}$$

i.e.,

$$\begin{aligned} u_i(x+1) = u_i^{\lambda_i}\{1 + ((\lambda_i - 1) q_0^{(i)} + a_{10\cdots 0}^{(i)}) x^{-1} \\ + \sum_{j=1}^{m} (\lambda_i q_j^{(i)} + a_{00\cdots 10\cdots 0}^{(i)}) u_j + \cdots\}, \end{aligned} \tag{1.6}$$

where $\cdots$ represents higher terms in x^{-1} and u_j. Now by the assumption $\lambda_i \geqq 2$ we can choose the coefficients $q_0^{(i)}$, $q_j^{(i)}$ $(j = 1, 2, \cdots, m)$ so that the coefficients of x^{-1} and u_j in the right-hand side of (1.6) may all vanish, in other words, equation (1.6) may be of the form

$$u_i(x+1) = u_i^{\lambda_i}(1 + \sum_{|k| \geqq 2} b_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}).$$

This is of the form similar to (1.1) but the terms of degree one in x^{-1} and u_j are lacking in the right-hand expansion. Now, to this equation we perform the transformation of the second step

$$u_i = v_i(1 + \sum_{|k|=2} r_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} v_1^{k_1} v_2^{k_2} \cdots v_m^{k_m}),$$

which has the inverse transformation

$$v_i = u_i(1 - \sum_{|k|=2} r_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m} + \cdots).$$

Then, it can be shown as before that the new unknown v_i satisfies the equation of the form

$$v_i(x+1)=v_i^{\lambda_i}\{1+((\lambda_i-1)r^{(i)}_{200\cdots0}+b^{(i)}_{200\cdots0})x^{-2}$$
$$+\sum_{|k|=2}{}' (\lambda_i r^{(i)}_{k_0k_1\cdots k_m}+b^{(i)}_{k_0k_1\cdots k_m})x^{-k_0}\, v_1^{k_1}v_2^{k_2}\cdots v_m^{k_m}+\cdots\}.$$

Here it is observed that the expansion $\{\quad\}$ has no terms of degree one x^{-1} and v_i, and $\cdots$ means terms of degree higher than two in x^{-1} and v_i. Now by $\lambda_i\geqq 2$ we can choose the coefficients $r^{(i)}_{k_0k_1\cdots k_m}$ for $|k|=2$ so that the coefficients of x^{-2} and $x^{-k_0}v_1^{k_1}v_2^{k_2}\cdots v_m^{k_m}$ for $|k|=2$ may all vanish, in other words, the resulted equation for the unknown v may be of the form

$$v_i(x+1)=v_i^{\lambda_i}(1+\sum_{|k|\geqq 3} c^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}v_1^{k_1}v_2^{k_2}\cdots v_m^{k_m}).$$

It has a form similar to (1.1) but the terms of degree one and two in x^{-1} and v's are lacking in the right-hand expansion.

Repeating similar processes we have after N steps N transformations

$$y_i=u_i(1+q_0^{(i)}x^{-1}+\sum_{j=1}^{m} q_j^{(i)}\, u_j),$$
$$u_i=v_i(1+\sum_{|k|=2} r^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}v_1^{k_1}v_2^{k_2}\cdots v_m^{k_m}),$$
$$\cdots\cdots\cdots\cdots$$
$$u_{i,N-1}=u_{i,N}(1+\sum_{|k|=N} t^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m}).$$

Here u_i, v_i means, respectively, $u_{i,1}$ and $u_{i,2}$. As composite transformation from $\{y_i\}$ to $\{u_{i,N}\}$ we have

$$y_i=u_{i,N}(1+\sum_{|k|\geqq 1} p^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m}) \tag{1.7}$$

and the equation satisfied by u_{iN} has the form

$$u_{i,N}(x+1)=u_{i,N}^{\lambda_i}\{1+\sum_{|k|\geqq N+1} b^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m}\}. \tag{1.8}$$

It is observed from the nature of the process that in all transformations (1.7) the coefficient of $x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}$ preserves the value $p^{(i)}_{k_0k_1\cdots k_m,|k|}$ for $N=|k|$, $|k|+1$, $|k|+2$, $\cdots\cdots$. Hence we may define the coefficients in (1.3) by setting

$$p^{(i)}_{k_0k_1\cdots k_m}=p^{(i)}_{k_0k_1\cdots k_m,|k|}. \tag{1.9}$$

Thus we have defined the transformation (1.3) from $\{y_i\}$ to $\{u_i\}$. Now, two transformations (1.3) and (1.7) give a relation between $\{u_i\}$ and $\{u_{i,N}\}$ defined by

$$u_i(1+\sum_{|k|\geqq 1}p^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m})$$
$$=u_{i,N}(1+\sum_{|k|\geqq 1}p^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m})$$

which can be solved formally in the form

$$u_i=u_{i,N}(1+\sum_{|k|\geqq 1}\bar{p}^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m}), \tag{1.10}$$

and as its inverse we have

$$u_{i,N}=u_i(1+\sum_{|k|\geqq 1}\bar{\bar{p}}^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}). \tag{1.11}$$

By the definition (1.9) we have in (1.10) and (1.11) the relations

$$\bar{p}^{(i)}_{k_0k_1\cdots k_m,N}=\bar{\bar{p}}^{(i)}_{k_0k_1\cdots k_m,N}=0 \qquad (1\leqq|k|\leqq N),$$

so that we have

$$u_i=u_{i,N}(1+\sum_{|k|\geqq N+1}\bar{p}^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m})$$

and

$$u_{i,N}=u_i(1+\sum_{|k|\geqq N+1}\bar{\bar{p}}^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}).$$

When these expressions are substituted into the equation (1.8) we find equation satisfied by $\{u_i\}$ are of the form:

$$u_i(x+1)=u_i^{\lambda_i}(1+\sum_{|k|\geqq N+1}c^{(i)}_{k_0k_1\cdots k_m,N}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}).$$

Here N has been an arbitrary integer, so that the equation satisfied by $\{u_i\}$ is really

$$u_i(x+1)=u_i^{\lambda_i}. \tag{1.12}$$

Thus we have proved that *the equation* (1.1) *is reduced to* (1.12) *when the transformation given by* (1.3) *is performed on it. Consequently we can conclude that the equation* (1.1) *has a formal solution of the form*

$$y_i(x, u)=u_i(1+\sum_{|k|\geqq 1} p^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}), \tag{1.13}$$

in which u_i *is substituted by a solution* $u_i(x)$ *of the equation* (1.12).

2. Associated functional difference equations. Making use of the formal solution (1.13) of the equation (1.1) we shall conveniently deal with a solution of (1.1) of the form

$$y_i(x)=y_i(x, u(x)) \qquad (i=1, 2, \cdots, m) \tag{2.1}$$

depending on the general solution $u(x)=(u_1(x), u_2(x), \cdots, u_m(x))$ of the system of equations: $u(x+1)=u^{\lambda}(x)$, i.e.,

$$u_i(x+1)=u_i^{\lambda_i}(x) \qquad (i=1, 2, \cdots, m). \tag{2.2}$$

The general solution $u(x)$ of (2.2) is

$$u_i(x)=\exp(\pi_i(x)\lambda_i^x), \tag{2.3}$$

containing an arbitrary periodic function $\pi(x)=(\pi_1(x), \pi_2(x), \cdots, \pi_m(x))$ with period 1, so that the function (2.1) corresponding to (2.3) is a general solution of (1.1).

Now substitute (2.1) and (2.2) into (1.1), then we have

$$y_i(x+1, u^{\lambda})=y_i^{\lambda_i}(x, u)f_i[x, y(x, u)] \qquad (i=1, 2, \cdots, m). \tag{2.4}$$

This equality holds formally in x and u, because $y=y(x, u)$ is a transformation reducing (1.1) to (1.12) formally in x and u. Let us write (2.4) briefly as

$$y_i(x+1, u^{\lambda})=y_i^{\lambda_i}f_i(x, y) \qquad (i=1, 2, \cdots, m) \tag{2.5}$$

and call this equation an associated functional difference equation of (1.1).

Clearly $y(x, u)$ given by (1.13) is a formal solution of (2.5). In order to prove the asymptoticity of this formal solution, let us put

$$y_i(x, u)=u_i(P_{iN}(x, u)+z_{iN}(x, u)),$$

where

$$P_{iN}(x, u)=1+\sum_{|k|=1}^{N} p_{k_0k_1\cdots k_m}^{(i)} x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m} \qquad (i=1, 2, \cdots, m),$$

thus introducing new unknown function $z_N(x, u)$.

Substituting them in (2.5), the equation for z_N becomes

$$z_{iN}(x+1, u^{\lambda})$$
$$=(P_{iN}(x, u)+z_{iN}(x, u))^{\lambda_i}f_i(x, u(P_N+z_N))-P_{iN}(x+1, u^{\lambda}),$$

where for simplicity we use the notation $u(P_N+z_N)$ for the vector $\{u_1(P_{1N}+z_{1N}), u_2(P_{2N}+z_{2N}), \cdots, u_m(P_{mN}+z_{mN})\}$. Let $\varphi_{iN}(x, u, z)$ and $c_{iN}(x, u)$ be defined by

$$\varphi_{iN}(x, u, z)=(P_{iN}+z_{iN})^{\lambda_i}f_i(x, u(P_N+z_N))-P_{iN}^{\lambda_i}f_i(x, uP_N),$$
$$c_{iN}(x, u)=P_{iN}^{\lambda_i}f_i(x, uP_N)-P_{iN}(x+1, u^{\lambda}),$$

then we get

$$(2.6) \qquad z_{iN}(x+1, u^{\lambda})=\varphi_{iN}(x, u, z)+c_{iN}(x, u) \qquad (i=1, 2, \cdots, m).$$

Writing z_{iN}, φ_{iN}, c_{iN}, respectively, as z_i, φ_i, c_i in (2.6) for simplicity, we have

$$(2.7) \qquad z_i(x+1, u^{\lambda})=\varphi_i(x, u, z)+c_i(x, u) \qquad (i=1, 2, \cdots, m).$$

Owing to the assumption made on the functions f_i in (1.1), and to the fact that P_{iN} in (2.1) are polynomials in x^{-1} and u_i, we see that there exist positive constants R_2, r_2 such that $c_i(x, u)$ are defined and analytic in

$$(28) \qquad |x|>R_2, \qquad \|u\|<r_2.$$

Referring to the fact that (1.3) is a formal solution of (1.1) we can conclude that there exists a positive constant L such that the inequality

$$(2.9) \qquad \|c(x, u)\|\leqq L\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

holds in the region (2.8). Choosing positive constants R_3, r_3 suitably we see that $\varphi_i(x, u, z)$ are analytic in the region

$$(2.10) \qquad |x|>R_3, \quad \|u\|<r_3, \quad \|z\|<r_3,$$

and moreover we have

$$\varphi_i(x, u, 0)=0 \qquad (i=1, 2, \cdots, m).$$

Therefore we may write $\varphi(x, u, z)$ in the form:

$$\varphi(x, u, z)=B(x, u)z+\psi(x, u, z).$$

Here $B(x, u)$ is an m by m matrix $B(x, u)=(b_{ik}(x, u))$, whose elements are analytic functions in the region $|x|>R_3$, $\|u\|<r_3$ such that

$$b_{ii}(x, u)=\frac{\partial}{\partial z_i}\varphi_i(x, u, 0)$$

$$=\lambda_i P_i^{\lambda_i-1}f_i(x, uP)+u_i P_i^{\lambda_i}\frac{\partial}{\partial y_i}f_i(x, uP),$$

$$b_{ik}(x, u)=u_k P_i^{\lambda_i}\frac{\partial}{\partial y_k}f_i(x, uP) \qquad (i\neq k),$$

so that it holds

$$B(\infty, 0)=\operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m).$$

$\psi(x, u, z)$ is an analytic function in the region (2.10), and there exists a positive quantity K such that the inequality

$$\|\psi(x, u, z)\|\leqq K\|z\|^2$$

holds in (2.10). Thus (2.6) may be written as

$$(2.11) \qquad z_i(x+1, u^{\lambda})=c_i(x, u)+\sum_{k=1}^{m} b_{ik}(x, u)z_k+\psi_i(x, u, z).$$

3. Existence theorem. In this section, we shall prove that the functional difference equation (2.11) has an analytic solution such that

$$(3.1) \qquad \|z(x, u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

in a certain region $\Gamma \times U$ such as

$$\Gamma: |x| > \rho_1, \qquad |\arg x| < \delta,$$
$$U: ||u|| < \rho_2 (<1),$$

where M and δ are arbitrary but fixed constants such that $M > L$ and $\delta < \frac{\pi}{2}$, and the constants ρ_1 and ρ_2 will be determined in the course of proof. We shall employ the Fixed-Point Theorem to prove this.

Let F be a family of functions $z(x, u)$ that are analytic and satisfying the inequality (3.1) for x and u in the region $\Gamma \times U$. F is a convex set, because if $f(x, u)$ and $g(x, u)$ belong to F the inequality

$$\begin{aligned} &||\lambda f(x, u) + (1-\lambda) g(x, u)|| \\ &\quad \leqq \lambda ||f(x, u)|| + (1-\lambda) ||g(x, u)|| \\ &\quad \leqq M\{|x|^{-(N+1)} + ||u||^{N+1}\} \end{aligned}$$

holds in $\Gamma \times U$ for any λ such as $0 < \lambda < 1$. F is a closed set, because if $\{z_i(x, u)\}$ is a sequence of functions belonging to F, and converging uniformly in any compact set in $\Gamma \times U$, then the limiting function $z(x, u)$ satisfies (3.1) and is analytic in $\Gamma \times U$, which means that $z(x, u)$ belongs to F.

Now, referring to the equation (2.11) consider the mapping T defined as

$$\begin{aligned} (3.2) \qquad \bar{z}(x, u) &= T(z(x, u)) \\ &= \Lambda^{-1}\{z(x+1, u^{\lambda}) - c(x, u) - H(x, u)z - \psi(x, u, z)\} \end{aligned}$$

where $H(x, u) = B(x, u) - \Lambda$, $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \cdots, \lambda_m)$. We may choose positive constants L, B, K, R', r' (<1) and r'' (<1) such that the inequalities

$$||c(x, u)|| \leqq L\{|x|^{-(N+1)} + ||u||^{N+1}\},$$
$$||H(x, u)|| \leqq B\{|x|^{-1} + ||u||\},$$
$$||\psi(x, u, z)|| \leqq K||z||^2$$

hold in the region $|x|>R'$, $\|u\|<r'$ and $\|z\|<r''$. We notice that if $z(x, u)\in F$, then $z(x+1, u^{\lambda})\in F$.

Now, $\|A^{-1}\|=\max_i \lambda_i^{-1}\leqq\frac{1}{2}$ and $M>L$, we have from (3.2)

$$\|\bar{z}(x, u)\|$$

$$\leqq\frac{1}{2}\{\|z(x+1, u^{\lambda})\|+\|c(x, u)\|+\|H(x, u)\|\,\|z\|+\|\psi(x, u, z)\|\}$$

$$\leqq\frac{M}{2}\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

$$\times\left[1+\frac{L}{M}+B(|x|^{-1}+\|u\|)+KM(|x|^{-(N+1)}+\|u\|^{N+1})\right],$$

so that choosing ρ_1 sufficiently large and $\rho_2<1$ sufficiently small we have

$$\|\bar{z}(x, u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

in the region $\Gamma\times U$. Thus the mapped function $\bar{z}(x, u)$ which is clearly analytic in $\Gamma\times U$ belongs also to F.

If $\{z_n(x, u)\}$ is a sequence of functions that belong to F and converges uniformly in any compact set in $\Gamma\times U$, then the sequence $\{\bar{z}_n(x, u)\}$ of the mapped functions clearly converges uniformly to a function $\bar{z}(x, u)$ in $\Gamma\times U$ which is the image of the limiting function $z(x, u)$.

Lastly we notice that the family of the mapped functions forms a normal family in $\Gamma\times U$. This is clear from the fact that $\bar{z}_n(x, u)$ are analytic and equibounded by $\|\bar{z}_n(x, u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$.

Hence all the necessary assumptions of the Fixed-Point Theorem were shown to be satisfied in our case, so that we have the following existence theorem.

Theorem 1. *Let $f_i(x, y)$ be analytic in the region $X_0\times Y_0$ such as*

$$X_0: |x|>R,$$

$$Y_0: \|y\|<r \qquad \textit{where} \quad \|y\|=\max_i |y_i|$$

and let us consider the functional difference equation

$$y_i(x+1, u^{\lambda}) = y_i^{\lambda_i} f_i(x, y), \tag{2.5}$$

where $y = y(x, u)$ *is a function of independent variables* x *and* $u = (u_1, u_2, \cdots, u_m)$. *Then in accordance with the following formal solution*

$$y_i = u_i\{1 + \sum_{|k| \geqq 1} p_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}\}$$

of (2.5) *there exists correspondingly to an arbitrary but fixed quantity* M (*which is greater than the quantity* L *introduced in* (2.9)) *an analytic solution*

$$y_i = u_i(1 + \sum_{|k|=1}^{N} p_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u^{k_1} u^{k_2} \cdots u^{k_m} + z_{iN}) \tag{S}$$

such that the inequality

$$\|z_N\| \leqq M\{|x|^{-(N+1)} + \|u\|^{N+1}\}$$

holds in a certain region $\Gamma_N \times U_N$ *which is defined for a sufficiently large quantity* ρ_{1N} *and a sufficiently small quantity* $\rho_{2N} < 1$ *as*

$$\Gamma_N : |x| > \rho_{1N}, \qquad |\arg x| < \delta\left(< \frac{\pi}{2}\right),$$

$$U_N : \|u\| < \rho_{2N}.$$

4. Uniqueness theorem. We shall consider the uniqueness of the solution of the functional difference equation

$$z(x+1, u^{\lambda}) = c(x, u) + B(x, u)z(x, u) + \psi(x, u, z). \tag{2.11}$$

We have proved that it has an analytic and bounded solution in a region $\Gamma \times U$. We shall show that such a solution is unique.

Supposing two solutions $z_1(x, u)$ and $z_2(x, u)$ we set

$$v(x, u) = z_2(x, u) - z_1(x, u).$$

Then v satisfies the equation

$$v(x+1, u^{\lambda}) = B(x, u)v + \psi(x, u, z_1 + v) - \psi(x, u, z_1). \tag{4.1}$$

Expanding the right-hand side in power of v, we can write the above equation in the form

$$v(x+1, u^{\lambda})=A(x, u, z_1)v+\Psi(x, u, z_1, v) \tag{4.2}$$

where Ψ denotes the part containing terms of degrees higher than one in v. By assumption $A(x, u, z_1)$ is analytic in the region $X_1\times U_1\times Z_1$ and $\Psi(x, u, z_1, v)$ is analytic in the region $X_1\times U_1\times Z_1\times V_1$ where

$$D_1: |x|>R_1, \qquad |\arg x|<\delta,$$

$$U_1: ||u||<r_1,$$

$$Z_1: ||z_1||<\bar{r}_1,$$

$$V_1: ||v||<\bar{\bar{r}}_1$$

for some suitable quantities $R_1, r_1, \bar{r}_1, \bar{\bar{r}}_1$, and it holds $A(\infty, 0, 0)=\Lambda$. Set

$$H(x, u, z_1)=A(x, u, z_1)-\Lambda.$$

Since $\lambda_0=\min_i \lambda_i\geqq 2$, we can select a number ρ such that $\lambda_0>\rho>1$. Then for the positive quantity $\lambda_0-\rho$ we choose $R_2>0$ sufficiently large, $r_2<0$ sufficiently small so that in the region $V=D_2\times U_2\times Z_2\times V_2$ such as

$$D_2: |x|>R_2, \qquad |\arg x|<\delta,$$

$$U_2: ||u||<r_2,$$

$$Z_2: ||z_1||<M(R_2^{-(N+1)}+r_2^{N+1}),$$

$$V_2: ||v||<2M(R_2^{-(N+1)}+r_2^{N+1})$$

the inequality

$$||H(x, u, z_1)v+\Psi(x, u, z_1, v)||\leqq(\lambda_0-\rho)||v||$$

holds. It follows that

$$||\Lambda v||-||A(x, u, z)v+\Psi(x, u, z, v)||\leqq(\lambda_0-\rho)||v||.$$

Since $\lambda_0||v||\leqq||\Lambda v||$, we have ultimately

$$||A(x, u, z_1)+\Psi(x, u, z_1, v)||\geqq\rho||v||,$$

so that by (4.2) we have

(4.3) $$||v(x+1, u^{\lambda})|| \geqq \rho ||v(x, u)||.$$

Since $v(x, u)$ is bounded in the region $D_2 \times U_2$, putting

$$S = \sup_{D_2 \times U_2} ||v(x, u)||$$

we get $S \geqq \rho S$ where $\rho > 1$. So we must have $S=0$. Hence $z_1(x, u) = z_2(x, u)$ in $D_2 \times U_2$. Since the solution is analytic $\Gamma \times U$, such a solution is unique in $\Gamma \times U$. Thus we get the following theorem.

Theorem 2. *Under the conditions of Theorem* 1 *the solution stated in it is unique.*

5. Existence of an asymptotic solution. We have proved that for the equation

(2.5) $$y_i(x+1, u^{\lambda}) = y_i^{\lambda_i} f_i(x, y) \qquad (i=1, 2, \cdots, m)$$

there exists an analytic solution

$$y_i = u_i(P_{iN}(x, u) + z_{iN}(x, u)),$$

where

$$P_{iN}(x, u) = 1 + \sum_{|k|=1}^{N} p^{(i)}_{k_0 k_1 \cdots k_m} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}$$

such as $||z_N|| \leqq M\{|x|^{-(N+1)} + ||u||^{N+1}\}$, and that such a solution is unique in $\Gamma_N \times U_N$. The solution $y_i = y_i(x, u)$ might depend on N, so that temporarity let us denote it by $y_i[N]$. Then

$$y_i[N] = u_i(P_{iN}(x, u) + z_{iN}(x, u)),$$

where $||z_N|| \leqq M\{|x|^{-(N+1)} + ||u||^{N+1}\}$ in $\Gamma_N \times U_N$. Now this can be rewritten as

$$y_i[N] = u_i(P_{i,N-1}(x, u) + v_{i,N-1}(x, u))$$

where

$$v_{i,N-1}(x, u) = \sum_{|k|=N} p^{(i)}_{k_0 k_1 \cdots k_m} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m} + z_{iN}(x, u).$$

Since $||z_N||\leqq M\{|x|^{-(N+1)}+||u||^{N+1}\}$ in $\Gamma_N\times U_N$, it follows that

$$||v_{N-1}||\leqq M'_{N-1}\{|x|^{-N}+||u||^N\},$$

where M'_{N-1} is some positive quantity. On the other hand, according to uniqueness theorem the solution such as

$$y_i[N-1]=u_i(P_{i,N-1}(x, u)+z_{i,N-1}(x, u))$$

where $||z_{N-1}||\leqq M_{N-1}\{|x|^{-N}+||u||^N\}$ in $\Gamma_{N-1}\times U_{N-1}$ is unique, so that it must hold

$$y_i[N]=y_i[N-1]$$

and consequently

$$z_{i,N-1}(x, u)=\sum_{|k|=N} p^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}+z_{iN}(x, u)$$

$$\text{in}\quad (\Gamma_{N-1}\cap\Gamma_N)\times(U_{N-1}\cap U_N).$$

Therefore we can conclude that there exists an analytic solution which does not depend on N in some region of the form previously denoted by $\Gamma\times U$, having the property that it holds

$$y_i(x, u)=u_i(P_{iN}(x, u)+z_{iN}(x, u)),$$

$$||z_N(x, u)||\leqq M_N\{|x|^{-(N+1)}+||u||^{N+1}\}\quad \text{in}\quad \Gamma\times U,$$

which means that we have asymptotically

$$y_i(x, u)\sim u_i(1+\sum_{|k|\geqq 1} p^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m})$$

in $\Gamma\times U$. Thus we have

Theorem 3. *Under the conditions of Theorem* 1 *there exists an analytic asymptotic solution of the equation* (2.5) *such as*

$$(5.1)\qquad y_i(x, u)\sim u_i(1+\sum_{|k|\geqq 1} p^{(i)}_{k_0k_1\cdots k_m}x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m})$$

in the region $\Gamma\times U$.

Part II. On the System of Nonlinear Difference Equations of the Form $y_i(x+1)=\prod_{j=1}^{m} y_j^{\lambda_{ij}} f_i(x, y)$ $(i=1, 2, \cdots, m)$

6. Formal solutions. We consider the system of nonlinear difference equations of the form:

$$y_i(x+1)=\prod_{j=1}^{m} y_j^{\lambda_{ij}} f_i(x, y) \qquad (i=1, 2, \cdots, m) \tag{6.1}$$

where λ_{ij}'s are nonnegative integers such that $\sum_{j=1}^{m} \lambda_{ij} \geqq 2$ $(i=1, 2, \cdots, m)$. We assume that $f_i(x, y)=f_i(x, y_1, y_2, \cdots, y_m)$ are analytic functions of x and $y=(y_1, y_2, \cdots, y_m)$ defined in the region $X_0\times Y_0=\{x \mid |x|>R\} \times \{y \mid \|y\|<r\}$ $(\|y\|=\max_i |y_i|)$. Clearly functions $f_i(x, y)$ then can be expanded in the following form:

$$f_i(x, y)=a_{00\cdots 0}^{(i)}+\sum_{|k|\geqq 1} a_{k_0k_1\cdots k_m}^{(i)} x^{-k_0} y_1^{k_1} y_2^{k_2} \cdots y_m^{k_m}, \tag{6.2}$$

where $|k|$ means $k_0+k_1+\cdots+k_m$. We assume that $f_i(\infty, 0)=a_{00\cdots 0}^{(i)}\equiv\mu_i \neq 0$ $(i=1, 2, \cdots, m)$. Furthermore we assume that each eigenvalue λ_i of the m by m matrix $\Lambda=(\lambda_{ij})$ is absolutely greater than one.

We may assume $f_i(\infty, 0)\equiv\mu_i=1$ $(i=1, 2, \cdots, m)$ without loss of generality. Indeed, putting

$$z_i=\xi_i y_i \qquad (i=1, 2, \cdots, m) \tag{6.3}$$

with undetermined constants ξ_i and substituting (6.3) in (6.1), we get the following system of difference equations for $z=(z_1, z_2, \cdots, z_m)$:

$$\prod_{j=1}^{m} \xi_j^{\lambda_{ij}} z_i(x+1)=\xi_i\mu_i \prod_{j=1}^{m} z_j^{\lambda_{ij}} g_i(x, z) \qquad (i=1, 2, \cdots, m) \tag{6.4}$$

where

$$\begin{aligned} g_i(x, z)&=g_i(x, z_1, z_2, \cdots, z_m) \\ &=\mu_i^{-1} f_i(x, \xi_1^{-1}z_1, \xi_2^{-1}z_2, \cdots, \xi_m^{-1}z_m). \end{aligned} \tag{6.5}$$

By our assumption one is not an eigenvalue of the matrix $\Lambda=(\lambda_{ij})$ so that the constants $\xi_i (i=1, 2, \cdots, m)$ can be so chosen that the equalities

(6.6) $$\prod_{j=1}^{m} \xi_j^{\lambda_{ij}} = \xi_i \mu_i \qquad (i=1, 2, \cdots, m)$$

hold. By such choice of the constant $\xi=(\xi_1, \xi_2, \cdots, \xi_m)$, (6.4) has the required form:

$$z_i(x+1)=\prod_{j=1}^{m} z_j^{\lambda_{ij}} g_i(x, z) \qquad (i=1, 2, \cdots, m).$$

$g_i(x, z)$ are analytic functions in $X_0 \times Z_0$ where

$$X_0 : |x|>R,$$

$$Z_0 : \|z\|<r' \qquad (r'=r \min_i |\xi_i|),$$

and they can be expanded in the same form as (6.2) but with $a_{00\cdots 0}^{(i)}=1$.

In this section, we shall show that we can find a formal solution of (6.1) by means of a transformation of the form:

(6.7) $$y_i=u_i(1+\sum_{|k|\geqq 1} p_{k_0 k_1 \cdots k_m}^{(i)} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m}).$$

Adopting the same idea used in Part I, we shall decompose the transformation (6.7) in a series of steps.

First we take $u_0(x)=y(x)$. Then we consider the transformation from $y(x)$ to $u_1(x)$ defined by the relation of the form

(6.8) $$y_i=u_{i1}(1+q_0^{(i)} x^{-1}+\sum_{n=1}^{m} q_n^{(i)} u_{n1}),$$

which has the inverse transformation

(6.9) $$u_{i1}=y_i(1-q_0^{(i)} x^{-1}-\sum_{n=1}^{m} q_n^{(i)} y_n+\cdots),$$

where $\cdots$ represents terms which are of degree higher than one in x^{-1}, $y_1, y_2, \cdots, y_m$. Setting $x+1$ for x in (6.9) and using (6.1) and (6.8), it is immediately shown that the new unknown u_1 satisfies the following equation

$$(6.10)\qquad u_{i1}(x+1)=\prod_{j=1}^{m} u_{j1}^{\lambda_{ij}}(1+q_0^{(j)}x^{-1}+\sum_{n=1}^{m} q_n^{(j)}u_{n1})^{\lambda_{ij}}$$
$$\times(1+\sum_{|k|\geqq 1} a_{k_0k_1\cdots k_m}^{(i)}x^{-k_0}u_{11}^{k_1}u_{21}^{k_2}\cdots u_{m1}^{k_m})$$
$$\times(1-q_0^{(i)}x^{-1}-\cdots).$$

Let E be the unit matrix and $M=(\mu_{ij})$ be the m by m matrix defined by $M=\Lambda-E$. Then (6.10) becomes

$$(6.11)\qquad u_{i1}(x+1)=\prod_{j=1}^{m} u_{j1}^{\lambda_{ij}}\{1+(\sum_{j=1}^{m}\mu_{ij}q_0^{(j)}+a_{10\cdots0}^{(i)})x^{-1}$$
$$+\sum_{n=1}^{m}(\sum_{j=1}^{m}\lambda_{ij}q_n^{(j)}+a_{00\cdots010\cdots0}^{(i)})u_{n1}+\cdots\}.$$

By the assumption one and zero are not the eigenvalue of Λ, so that Λ and M are regular matrices. Hence we can determine the constants $q_0^{(i)}$ and $q_n^{(i)}$ appearing in (6.8) so that the coefficients of x^{-1} and u_{n1} in the right-hand side of (6.11) may all vanish.

The next step which transforms $u_1(x)$ to $u_2(x)$ is similar. Inductively let us assume that we already have the system of difference equations for u_{N-1} of the form:

$$(6.12)\quad u_{i,N-1}(x+1)=\prod_{j=1}^{m} u_{j,N-1}^{\lambda_{ij}}(1+\sum_{|k|\geqq N}\beta_{k_0k_1\cdots k_m,N-1}^{(i)}x^{-k_0}u_{1,N-1}^{k_1}\cdots u_{m,N-1}^{k_m}).$$

We shall show that a transformation from u_{N-1} to u_N of the form

$$(6.13)\qquad u_{i,N-1}(x+1)=u_{i,N}(1+\sum_{|k|=N}\gamma_{k_0k_1\cdots k_m,N}^{(i)}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m})$$

can be determined in such a way that we may have the system of difference equations for u_N of the form:

$$(6.14)\qquad u_{i,N}(x+1)=\prod_{j=1}^{m} u_{j,n}^{\lambda_{ij}}(1+\sum_{|k|\geqq N+1}\beta_{k_0k_1\cdots k_m,N}^{(i)}x^{-k_0}u_{1,N}^{k_1}u_{2,N}^{k_2}\cdots u_{m,N}^{k_m}).$$

Now (6.13) has the inverse transformation

$$(6.15)\qquad u_{i,N}=u_{i,N-1}(1-\sum_{|k|=N}\gamma_{k_0k_1\cdots k_m,N}^{(i)}x^{-k_0}u_{1,N-1}^{k_1}\cdots u_{m,N-1}^{k_m}+\cdots),$$

where $\cdots$ shows the terms which are of degree higher than N in x^{-1}, $u_{1,N-1}, u_{2,N-1}, \cdots, u_{m,N-1}$. Setting $x+1$ for x in (6.15), (6.12) and (6.13), it is immediately shown that the new unknown $u_{i,N}$ satisfy the following equations

$$u_{iN}(x+1)=\prod_{j=1}^{m} u_{jN}^{\lambda_{ij}}(1+\sum_{|k|=N}\gamma_{k_0k_1\cdots k_m,N}^{(j)}x^{-k_0}u_{1N}^{k_1}\cdots u_{mN}^{k_m})^{\lambda_{ij}}$$
$$\times(1+\sum_{|k|=N}\beta_{k_0k_1\cdots k_m,N-1}^{(i)}x^{-k_0}u_{1N}^{k_1}\cdots u_{mN}^{k_m}+\cdots)$$
$$\times(1-\gamma_{N0\cdots0,N}^{(i)}x^{-N}+\cdots),$$

i.e.,

$$(6.16)\quad u_{iN}(x+1)=\prod_{j=1}^{m} u_{jN}^{\lambda_{ij}}\{1+(\sum_{j=1}^{m}\mu_{ij}\gamma_{N0\cdots0,N}^{(j)}+\beta_{N0\cdots0,N-1}^{(i)})x^{-N}$$
$$+\sum_{\substack{|k|=N\\ k_0\neq N}}(\sum_{j=1}^{m}\lambda_{ij}\gamma_{k_0k_1\cdots k_m,N}^{(j)}+\beta_{k_0k_1\cdots k_m,N-1}^{(i)})x^{-k_0}u_{1N}^{k_1}\cdots u_{mN}^{k_m}$$
$$+\cdots\}.$$

Since the matrices $M=(\mu_{ij})$ and $\Lambda=(\lambda_{ij})$ are regular, we can determine the coefficients $\gamma_{k_0k_1\cdots k_m,N}^{(i)}$ so that in the right-hand side of (6.16) all the coefficients of the terms of degree N in $x^{-1}, u_{1N}, \cdots, u_{mN}$ may vanish, that is, (6.6) may be of the form (6.14).

Now, by the composition of the mappings

$$y_i(x)=u_{i0}(x),$$
$$u_{i0}=u_{i1}(1+q_0^{(i)}x^{-1}+\sum_{n=1}^{m}q_n^{(i)}u_{n,1}),$$
$$u_{i1}=u_{i2}(1+\sum_{|k|=2}\gamma_{k_0k_1\cdots k_m,2}^{(i)}x^{-k_0}u_{12}^{k_1}u_{22}^{k_2}\cdots u_{m2}^{k_m}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$u_{i,N-1}=u_{i,N}(1+\sum_{|k|=N}\gamma_{k_0k_1\cdots k_m,N}^{(i)}x^{-k_0}u_{1N}^{k_1}u_{2N}^{k_2}\cdots u_{mN}^{k_m})$$
$$(i=1, 2, \cdots, m),$$

we get the following transformation from y to u_N:

(6.17) $$y_i = u_{iN}(1 + \sum_{|k| \geqq 1} p^{(i)}_{k_0 k_1 \cdots k_m, N} x^{-k_0} u^{k_1}_{1N} u^{k_2}_{2N} \cdots u^{k_m}_{mN}).$$

It is observed from the nature of the process that in all transformations (6.17) the coefficient of $x^{-k_0} u^{k_1}_{1N} u^{k_2}_{2N} \cdots u^{k_m}_{mN}$ preserves the value $p^{(i)}_{k_0 k_1 \cdots k_m, |k|}$ for $N = |k|, |k|+1, |k|+2, \cdots$. Hence we may define the coefficients in (6.7) by setting

(6.18) $$p^{(i)}_{k_0 k_1 \cdots k_m} = p^{(i)}_{k_0 k_1 \cdots k_m, |k|}.$$

Thus we have defined the transformation (6.7) from y to u. Now, the two transformations (6.7) and (6.17) give a relation between u and u_N defined as

(6.19) $$u_i(1 + \sum_{|k| \geqq 1} p^{(i)}_{k_0 k_1 \cdots k_m} x^{-k_0} u^{k_1}_1 u^{k_2}_2 \cdots u^{k_m}_m)$$
$$= u_{iN}(1 + \sum_{|k| \geqq 1} p^{(i)}_{k_0 k_1 \cdots k_m, N} x^{-k_0} u^{k_1}_{1N} u^{k_2}_{2N} \cdots u^{k_m}_{mN})$$

which can be solved formally in the form

(6.20) $$u_i = u_{iN}(1 + \sum_{|k| \geqq 1} \bar{p}^{(i)}_{k_0 k_1 \cdots k_m\ N} x^{-k_0} u^{k_1}_{1N} u^{k_2}_{2N} \cdots u^{k_m}_{mN})$$

and as its inverse we have

(6.21) $$u_{iN} = u_i(1 + \sum_{|k| \geqq 1} \bar{\bar{p}}^{(i)}_{k_0 k_1 \cdots k_m, N} x^{-k_0} u^{k_1}_1 u^{k_2}_2 \cdots u^{k_m}_N).$$

By the definition (6.18) we have in (6.20) and (6.21) the relations

$$\bar{p}^{(i)}_{k_0 k_1 \cdots k_m, N} = \bar{\bar{p}}^{(i)}_{k_0 k_1 \cdots k_m, N} = 0$$

for k such as $1 \leqq |k| \leqq N$, so that we have

(6.22) $$u_i = u_{iN}(1 + \sum_{|k| \geqq N+1} \bar{p}^{(i)}_{k_0 k_1 \cdots k_m, N} x^{-k_0} u^{k_1}_{1N} u^{k_2}_{2N} \cdots u^{k_m}_{mN}),$$

and

(6.23) $$u_{iN} = u_i(1 + \sum_{|k| \geqq N+1} \bar{\bar{p}}^{(i)}_{k_0 k_1 \cdots k_m, N} x^{-k_0} u^{k_1}_1 u^{k_2}_2 \cdots u^{k_m}_m).$$

When these expressions are substituted into the equation (6.14) we find that the equations satisfied by $\{u_i\}$ are of the form

$$(6.24) \qquad u_i(x+1)=\prod_{j=1}^{m} u_j^{\lambda_{ij}}(1+\sum_{|k|\geqq N+1} c_{k_0k_1\cdots k_m,N}^{(i)} x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}).$$

Here N has been an arbitrary integer, so that the equation satisfied by $\{u_i\}$ is really

$$(6.25) \qquad u_i(x+1)=\prod_{j=1}^{m} u_j^{\lambda_{ij}}.$$

Thus we have proved that *the equation* (6.1) *is reduced to* (6.25) *when the transformation given by* (6.7) *is performed on it.* Consequently we can conclude that *the equation* (6.1) *has a formal solution of the form*

$$(6.26) \qquad y_i(x, u)=u_i(1+\sum_{|k|\geqq 1} p_{k_0k_1\cdots k_m}^{(i)} x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m})$$

in which u_i *is substituted by a solution* $u_i(x)$ *of the equation* (6.25).

7. Associated functional difference equation. Making use the formal solution (6.7) of the equation (6.1) we shall conveniently deal with a solution of (6.1) of the form

$$(7.1) \qquad y_i(x, u(x)) \qquad (i=1, 2,\cdots, m)$$

depending on the general solution $u(x)=(u_1(x), u_2(x), \cdots, u_m(x))$ of the system of equations

$$(7.2) \qquad u_i(x+1)=\prod_{j=1}^{m} u_j(x)^{\lambda_{ij}} \qquad (i=1, 2, \cdots, m).$$

The general solution $u(x)$ of (7.2) may be found easily by putting $v_i(x)=\log u_i(x)$. $u(x)$ contains an arbitrary periodic function $\pi(x)=(\pi_1(x), \pi_2(x), \cdots, \pi_m(x))$ with period one, so that the function (7.1) corresponding to the general solution $u(x)$ of (7.2) is a general solution of (7.1).

Substitute (7.1) and (7.2) into (6.1), then we have

$$(7.3) \qquad y(x+1, \pi u^{\Lambda})=\prod_{j=1}^{m} y_j^{\lambda_{ij}}(x, u) f_i[x, y(x, u)]$$

$$(i=1, 2, \cdots, m),$$

where the left-hand side denotes the rather complicated quantity

$$y_i(x+1, \prod_{j=1}^{m} u_j^{\lambda_{1j}}, \prod_{j=1}^{m} u_j^{\lambda_{2j}}, \cdots, \prod_{j=1}^{m} u_j^{\lambda_{mj}}).$$

Similarly to (2.4), equality (7.3) holds formally in x and u for $y(x, u)$ given by (6.7). In what follows, we shall call (7.3) an *associated functional difference equation* of (6.1).

Corresponding to the formal solution (6.7) of the associated functional difference equation (7.3), let us put

$$y_i(x, u)=u_iP^{(i)}(x, u), \tag{7.4}$$

$$y_i(x, u)=u_i(P_{iN}(x, u)+z_{iN}(x, u)) \tag{7.5}$$

where

$$\begin{aligned} P^{(i)}(x, u)&=1+\sum_{|k|\geqq 1} p_{k_0k_1\cdots k_m}^{(i)} x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}, \\ P_{iN}(x, u)&=1+\sum_{|k|=1}^{N} p_{k_0k_1\cdots k_m}^{(i)} x^{-k_0}u_1^{k_1}u_2^{k_2}\cdots u_m^{k_m}. \end{aligned} \tag{7.6}$$

Substituting (7.5) in (7.3) the equation for z_N becomes

$$\begin{aligned} &z_{iN}(x+1, \pi u^A) \\ &\quad=\prod_{j=1}^{m} (P_{jN}(x, u)+z_{jN}(x, u))^{\lambda_{ij}} f_i(x, u(P_N+z_N)) \\ &\quad\quad -P_{iN}(x+1, \pi u^A), \end{aligned}$$

where

$$\begin{aligned} &f_i(x, u(P_N+z_N)) \\ &\quad=f_i(x, u_1(P_{1N}+z_{1N}), u_2(P_{2N}+z_{2N}), \cdots, u_m(P_{mN}+z_{mN})), \end{aligned}$$

and

$$\begin{aligned} &P_{iN}(x+1, \pi u^A) \\ &\quad\equiv P_{iN}(x+1, \prod_{j=1}^{m} u_j^{\lambda_{1j}}, \prod_{j=1}^{m} u_j^{\lambda_{2j}}, \cdots, \prod_{j=1}^{m} u_j^{\lambda_{mj}}). \end{aligned}$$

Let $\varphi_{iN}(x, u, z)$ and $c_{iN}(x, u)$ be defined by

$$\begin{aligned}(7.7)\qquad \varphi_{iN}(x, u, z)=\prod_{j=1}^{m}(P_{jN}+z_{jN})^{\lambda_{ij}}f_i(x, u(P_N+z_N))\\ -\prod_{j=1}^{m}(P_{jN})^{\lambda_{ij}}f_i(x, uP_N),\end{aligned}$$

$$(7.8)\qquad c_{iN}(x, u)=\prod_{j=1}^{m}(P_{jN}(x, u))^{\lambda_{ij}}f_i(x, uP_N)-P_{iN}(x+1, \pi u^{\Lambda}).$$

Then we get

$$z_{iN}(x+1, \pi u^{\Lambda})=\varphi_{iN}(x, u, z)+c_{iN}(x, u)\qquad (i=1, 2, \cdots, m).$$

Writing for simplicity z_{iN}, φ_{iN}, c_{iN}, respectively, as z_i, φ_i, c_i in this equation we shall study the following functional difference equation:

$$(7.9)\qquad z_i(x+1, \pi u^{\Lambda})=\varphi_i(x, u, z)+c_i(x, u)\qquad (i=1, 2, \cdots, m).$$

Owing to the assumption on the functions f_i in (7.8) and to the fact that P_{iN} in (7.6) are polynomials in x^{-1} and u_i, we see that there exist positive constants R_2, r_2 such that $c_i(x, u)$ defined by (7.8) are analytic in the region

$$(7.10)\qquad |x|>R_2,\qquad ||u||<r_2.$$

Referring to the fact that (6.7) is a formal solution of (7.3), we get the formal equalities

$$\prod_{j=1}^{m}u_j^{\lambda_{ij}}P^{(i)}(x+1, \pi u^{\Lambda})=\prod_{j=1}^{m}(u_jP^{(j)}(x, u))^{\lambda_{ij}}f_i(x, uP(x, u)),$$

that is,

$$\begin{aligned}(7.11)\qquad \prod_{j=1}^{m}(P^{(j)}(x, u))^{\lambda_{ij}}f_i(x, uP(x, u))-P^{(i)}(x+1, \pi u^{\Lambda})=0\\ (i=1, 2, \cdots, m).\end{aligned}$$

By (7.6), (7.11) and the definition (7.8) of $c_i(x, u)$, we get the estimation

$$c_i(x, u)=O(|x|^{-(N+1)}+||u||^{N+1})\qquad (i=1, 2, \cdots, m).$$

Therefore we can conclude that there exists a positive constant L such that

$$\|c(x, u)\| \leqq L\{|x|^{-(N+1)} + \|u\|^{N+1}\}$$

holds in (7.10). Choosing positive constants R_3, r_3 suitably we see that $\varphi_i(x, u, z)$ are analytic in the region

$$(7.12) \qquad |x| > R_3, \qquad \|u\| < r_3, \qquad \|z\| < r_3,$$

and moreover we have $\varphi_i(x, u, 0) = 0$ $(i = 1, 2, \cdots, m)$. Therefore we may write $\varphi = (\varphi_1, \varphi_2, \cdots, \varphi_m)$ in the form

$$\varphi(x, u, z) = B(x, u)z + \psi(x, u, z).$$

Here $B(x, u)$ is an m by m matrix $B(x, u) = (b_{ik}(x, u))$ whose elements are analytic in $|x| > R_3$, $\|u\| < r_3$ such that

$$b_{ik}(x, u) = \frac{\lambda_{ik}}{P_{kN}} \prod_{j=1}^{m} (P_{jN})^{\lambda_{ij}} f_i(x, uP_N) + u_k \prod_{j=1}^{m} (P_{jN})^{\lambda_{ij}} \frac{\partial f_i}{\partial y_k}(x, uP_N).$$

Hence we have

$$B(\infty, 0) = (\lambda_{ik}) = \Lambda.$$

$\psi(x, u, z)$ is an analytic function in (7.12), and there exists a positive quantity K such that the inequality

$$\|\psi(x, u, z)\| \leqq K \|z\|^2$$

holds in (7.12). Therefore (7.9) may be written as

$$(7.13) \qquad z_i(x+1, \pi u^{\Lambda}) = c_i(x, u) + \sum_{j=1}^{m} b_{ij}(x, u) z_j + \psi_i(x, u, z) \qquad (i = 1, 2, \cdots, m).$$

Let us put $\lambda_0 = \min_i |\lambda_i|$. By assumption we have $\lambda_0 > 1$. For an arbitrary but fixed constant δ such as

(7.14) $$\lambda_0 - 1 > \delta > 0$$

there exists a regular matrix S such that

(7.15) $$\Lambda_J = S^{-1}\Lambda S \qquad (\Lambda = (\lambda_{ij})),$$

where Λ_J is a Jordan's canonical form of Λ:

(7.16) $$\Lambda_J = \begin{pmatrix} \Lambda_1 & & & 0 \\ & \Lambda_2 & & \\ 0 & & \ddots & \\ & & & \Lambda_l \end{pmatrix}, \quad \Lambda_j = \begin{pmatrix} \lambda_j & \delta_{j1} & & & \\ & \lambda_j & \delta_{j2} & & \\ & & \ddots & & \\ & & & \lambda_j & \delta_{j,m_j-1} \\ & & & & \lambda_j \end{pmatrix}$$

and each δ_{jk} satisfies the following inequality

$$|\delta_{jk}| \leqq \delta.$$

Denoting the diagonal matrix $\operatorname{diag}(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots, \lambda_l, \cdots, \lambda_l)$ by Λ_D, we have

(7.17) $$\|\Lambda_J - \Lambda_D\| \leqq \delta.$$

Now, putting

(7.18) $$z(x, u) = Sw(x, u),$$

and substituting this in (7.13), we have the equation for w

(7.19) $$w(x+1, \pi u^\Lambda) = S^{-1}c(x, u) + S^{-1}B(x, u)Sw + S^{-1}\psi(x, u, Sw).$$

For simplicity, we put

$$\mathscr{C}(x, u) = S^{-1}c(x, u),$$
$$\mathscr{B}(x, u) = S^{-1}B(x, u)S,$$
$$\Psi(x, u, w) = S^{-1}\psi(x, u, Sw).$$

Then (7.19) becomes

(7.20) $$w(x+1, \pi u^\Lambda) = \mathscr{C}(x, u) + \mathscr{B}(x, u)w + \Psi(x, u, w).$$

We may choose R^* sufficiently large and r^* (<1) sufficiently small so that the following conditions (A.I), (A.II) and (A.III) may be satisfied.

(A.I) The components of $\mathscr{C}(x, u)$ are analytic in $X\times U$ where

$$X: |x|>R^*, \qquad U: \|u\|<r^*,$$

and the inequality

$$\|\mathscr{C}(x, u)\|\leqq L_1\{|x|^{-(N+1)}+\|u\|^{N+1}\} \qquad (L_1=\|S^{-1}\|L) \tag{7.21}$$

holds in $X\times U$.

(A.II) The inequality

$$\|\mathscr{B}(x, u)-\Lambda_D\|\leqq\delta+B\{|x|^{-1}+\|u\|\} \tag{7.22}$$

holds in $X\times U$.

(A.III) The components of $\Psi(x, u, w)$ are analytic in $X\times U\times W$ such as

$$X: |x|>R^*, \qquad U: \|u\|<r^*, \qquad W: \|w\|<r^*,$$

and the inequality

$$\|\Psi(x, u, w)\|\leqq K_1\|w\|^2 \qquad (K_1=K\|S^{-1}\|\|S\|^2) \tag{7.23}$$

holds in $X\times U\times W$.

8. Existence theorem. In this section, we shall prove that the system of the functional difference equation (7.20) has an analytic solution such that

$$\|w(x, u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\} \tag{8.1}$$

in a certain region $\Gamma\times U_0$ such as

$$\begin{aligned}&\Gamma: |x|>\rho_1, \qquad |\arg x|<\alpha,\\ &U_0: \|u\|<\rho_2 \qquad (<1)\end{aligned} \tag{8.2}$$

where M and α are arbitrary but fixed constants such that $M>L/(\lambda_0-1-\delta)$ and $0<\alpha<\frac{\pi}{2}$, and ρ_1 and ρ_2 are constants to be determined in the course of the proof. We shall employ the Fixed-Point Theorem to prove this.

Let F be a family of vector-valued functions $w(x, u)$ whose elements are analytic and satisfy the inequality (8.1) for (x, u) in the region $\Gamma\times U_0$.

Now, rewriting the equation (7.20), we have

$$w(x, u)=\Lambda_D^{-1}\{w(x+1, \pi u^{\Lambda})-\mathscr{C}(x, u)-(\mathscr{B}(x, u)-\Lambda_D)w-\Psi(x, u, w)\}$$

where Λ_D is the diagonal matrix diag $(\lambda_1, \cdots, \lambda_1, \lambda_2, \cdots, \lambda_2, \cdots\cdots, \lambda_l, \cdots, \lambda_l)$.

Consider the mapping $T: w(x, u)\to\bar{w}(x, u)$ defined by

$$\begin{aligned}\bar{w}(x, u)&=T(w(x, u))\\&=\Lambda_D^{-1}\{w(x+1, \pi u^{\Lambda})-\mathscr{C}(x, u)\\&\quad-(\mathscr{B}(x, u)-\Lambda_D)w-\Psi(x, u, w)\}.\end{aligned}\tag{8.3}$$

The most important fact to be proved is that if $w(x, u)\in F$ then also $\bar{w}(x, u)\in F$. We shall prove this only, since the rests may be proved similarly as in Part I.

We shall notice that if $(x, u)\in\Gamma\times U_0$ then $(x+1, \pi u^{\Lambda})\in\Gamma\times U_0$, i.e., if $w(x, u)\in F$, then also $w(x+1, \pi u^{\Lambda})\in F$. On the other hand, by (A.I), (A.II) and (A.III) the inequalities

$$||\mathscr{C}(x, u)||\leqq L_1\{|x|^{-(N+1)}+||u||^{N+1}\}\tag{7.21}$$

$$||\mathscr{B}(x, u)-\Lambda_D||\leqq\delta+B\{|x|^{-1}+||u||\}\tag{7.22}$$

hold in the region $X\times U$ and the inequality

$$||\Psi(x, u, w)||\leqq K_1||w||^2\tag{7.23}$$

holds in the region $X\times U\times W$. By these facts we may get the inequality

$$\begin{aligned}||\bar{w}(x, u)||\leqq||\Lambda_D^{-1}||[(M+L_1)(|x|^{-(N+1)}+||u||^{N+1})\\+(\delta+B(|x|^{-1}+||u||))M(|x|^{-(N+1)}+||u||^{N+1})\\+K_1M^2(|x|^{-(N+1)}+||u||^{N+1})^2].\end{aligned}$$

By $||\Lambda_D^{-1}||=1/\lambda_0$, the inequality

$$(8.4)\qquad \|\bar{w}(x,\ u)\| \leqq \frac{1}{\lambda_0}\Big[1+\frac{L_1}{M}+\delta+B(|x|^{-1}+\|u\|)$$

$$+K_1M(|x|^{-(N+1)}+\|u\|^{N+1})\Big]M(|x|^{-(N+1)}+\|u\|^{N+1})$$

holds in the region $X\times U$.

Let us choose M so that $M>L_1/(\lambda_0-1-\delta)$, and let us take $\rho_1(>R^*)$ sufficiently large and $\rho_2(<r^*)$ sufficiently small. Then we have

$$M=L/\{\lambda_0-1-\delta-B(|x|^{-1}+\|u\|)-K_1M(\|x\|^{-(N+1)}+\|u\|^{N+1})\}$$

in the region $\Gamma\times U_0$ which is contained in $X\times U$. Hence we get the inequality

$$\frac{1}{\lambda_0}\Big\{1+\frac{L_1}{M}+\delta+B(|x|^{-1}+\|u\|)+K_1M(|x|^{-(N+1)}+\|u\|^{N+1})\Big\}\leqq 1$$

in $\Gamma\times U_0$. Therefore by (8.4), we have

$$\|\bar{w}(x,\ u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

for any $w(x,\ u)\in F$. Hence the mapped function $\bar{w}(x,\ u)$ which is clearly analytic in $\Gamma\times U_0$ belongs also to F. Thus we get the following

Theorem 4. *Consider the system of functional difference equations*

$$(7.20)\qquad w(x+1,\ \pi u^{\Lambda})=\mathscr{C}(x,\ u)+\mathscr{B}(x,\ u)w(x,\ u)+\Psi(x,\ u,\ w(x,\ u))$$

where $\mathscr{C}(x,\ u)$, $\mathscr{B}(x,\ u)$ and $\Psi(x,\ u,\ w)$ satisfy the conditions (A.I), (A.II) and (A.III) in §7. *Equation* (7.20) *then has an analytic solution such that*

$$\|w(x,\ u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$$

in a certain region $\Gamma\times U_0=\{x\,|\,|x|>\rho_1,\ |\arg x|<\alpha\}\times\{u\,|\,\|u\|<\rho_2<1\}$, where M and α are arbitrary but fixed constants such that $M>L_1/(\lambda_0-1-\delta)$ and $0<\alpha<\frac{\pi}{2}$, and ρ_1, ρ_2 are suitable constants.

9. Uniqueness theorem. In this section, we shall consider the uniqueness of the solution of the functional difference equation

(7.20) $$w(x+1, \pi u^{\Lambda})=\mathscr{C}(x, u)+\mathscr{B}(x, u)w(x, u)+\Psi(x, u, w(x, u))$$

under the conditions (A.I), (A.II) and (A.III) in §7. We shall prove that if (7.20) has an analytic solution satisfying $\|w(x, u)\|\leqq M\{|x|^{-(N+1)}+\|u\|^{N+1}\}$ in the region $\Gamma\times U_0$, then such a solution is unique.

Let us assume that there exist two solutions $w_1(x, u)$ and $w_2(x, u)$. Put

(9.1) $$v(x, u)=w_2(x, u)-w_1(x, u),$$

then v satisfies the following equation

(9.2) $$v(x+1, \pi u^{\Lambda})=\mathscr{B}(x, u)v+\Psi(x, u, w_1+v)-\Psi(x, u, v).$$

Expanding the right-hand side in power of v, we can write the above equation in the form

(9.3) $$v(x+1, \pi u^{\Lambda})=A(x, u, w_1)v+\Psi_1(x, u, w_1, v),$$

where Ψ_1 denotes the part containing terms of degrees higher than one in v.

Let us choose constants $\bar{R}_1$ ($\geqq \max(\rho_1, R^*)$) sufficiently large, and $\bar{r}_1$ ($=\min(\rho_2, r^*)$) sufficiently small, and consider the regions D_1, U_1, W_1 and V_1 defined by

$$D_1:|x|\geqq\bar{R}_1, \qquad |\arg x|<\alpha,$$

$$U_1:\|u\|\leqq\bar{r}_1,$$

$$W_1:\|w_1\|\leqq M(\bar{R}_1^{-(N+1)}+\bar{r}_1^{N+1}),$$

$$V_1:\|v\|\leqq 2M(\bar{R}_1^{-(N+1)}+\bar{r}_1^{N+1}).$$

Then we may assume that $A(x, u, w_1)$ is analytic in the region $D_1\times U_1\times W_1$ and the equality

(9.4) $$A(\infty, 0, 0)=\Lambda_J$$

holds, where Λ_J is the Jordan's canonical form (7.16) of $\Lambda=(\lambda_{ij})$. On the other hand, $\Psi_1(x, u, w_1, v)$ is analytic in the region $D_1\times U_1\times W_1\times V_1$ and the inequality

$$\|\psi_1(x, u, w_1, v)\| \leqq K_2 \|v\|^2$$

holds in $D_1 \times U_1 \times W_1 \times V_1$, where K_2 is a constant suitably chosen.

Set $H(x, u, w_1) = A(x, u, w_1) - \Lambda_J$, then (9.3) becomes

(9.5) $$v(x+1, \pi u^{\Lambda}) = (\Lambda_J + H(x, u, w_1))v + \Psi_1(x, u, w_1, v).$$

As is easily seen, the inequality

(9.6) $$\|\Lambda_J v\| \geqq (\lambda_0 - \delta)\|v\|$$

holds for any v, where $\lambda_0 = \min |\lambda_j|$ and δ is a constant satisfying

(7.14) $$\lambda_0 - 1 > \delta > 0.$$

Now let us choose an arbitrary but fixed positive constant δ_1 such that

(9.7) $$\lambda_0 - 1 - \delta > \delta_1 > 0.$$

Then, choosing suitably the constants $\bar{R}_2$, $\bar{r}_2$, $\bar{\bar{r}}_2$ and $\tilde{r}_2$ the inequality

(9.8) $$\|H(x, u, w_1)v + \Psi_1(x, u, w_1, v)\| \leqq \delta_1 \|v\|$$

holds in the region $V = D_2 \times U_2 \times W_2 \times V_2$ such as

$$D_2 : |x| \geqq \bar{R}_2 \quad (\geqq \bar{R}_1), \qquad |\arg x| < \alpha,$$

$$U_2 : \|u\| \leqq \bar{r}_2 \quad (\leqq \bar{r}_1),$$

$$W_2 : \|w_1\| \leqq M(\bar{R}_2^{-(N+1)} + \bar{r}_2^{N+1}) \equiv \bar{\bar{r}}_2,$$

$$V_2 : \|v\| \leqq 2M(\bar{R}_2^{-(N+1)} + \bar{r}_2^{N+1}) \equiv \tilde{r}_2.$$

By (9.5), (9.6) and (9.8), we have

(9.9)
$$\begin{aligned}
&\|v(x+1, \pi u^{\Lambda})\| \\
&= \|\Lambda_J v + H(x, u, w_1)v + \Psi_1(x, u, w_1, v)\| \\
&\geqq \|\Lambda_J v\| - \|H(x, u, w_1)v + \Psi_1(x, u, w_1, v)\| \\
&\geqq (\lambda_0 - \delta)\|v\| - \delta_1 \|v\| \\
&= (\lambda_0 - \delta - \delta_1)\|v\|
\end{aligned}$$

in the region V. Putting $\rho = \lambda_0 - \delta - \delta_1$ we get $\rho > 1$ by (9.7). Then the inequality

$$\|v(x+1, \pi u^{\Lambda})\| \geqq \rho\|v\| \tag{9.10}$$

holds in the region V. Since v is bounded in the region $D_2 \times U_2$, putting $S=\sup_{D_2\times U_2}\|v(x, u)\|$ we have $S \geqq \rho S$ where $\rho > 1$. Hence $S=0$, i.e., $w_1(x, u)=w_2(x, u)$. The solution being analytic $\Gamma \times U$, such a solution is unique in $\Gamma \times U$. Thus we get the following

Theorem 5. *Under the conditions of Theorem* 4 *the solution stated in it is unique.*

The existence of an analytic asymptotic solution can be proved in a similar manner as in Part I by using Theorem 4, Theorem 5 and the inequality $\|z\| \leqq \|S\|\,\|w\|$ in (7.18). Thus we have

Theorem 6. *Under the conditions of Theorem* 4 *there exists an analytic solution of* (7.3) *for which we have the asymptotic expansion*

$$y_i(x, u) \sim u_i(1+\sum_{|k|\geqq 1} p^{(i)}_{k_0k_1\cdots k_m} x^{-k_0} u_1^{k_1} u_2^{k_2} \cdots u_m^{k_m})$$

in the region $\Gamma \times U$.

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