Publ. RIMS, Kyoto Univ.
Vol. 6 (1970), 307-321

Spectral Representation for Branching Processes with Immigration on the Real Half Line

By

Yukio OGURA*

§0. Introduction

In the previous paper [6], we have obtained the spectral representation for the semigroups of continuous state branching processes (CB-processes). In this paper, we shall obtain the analogous results for continuous state branching processes with immigration (CBI-processes). CBI-processes were introduced by Kawazu and Watanabe [3] as a continuous version of Galton-Watson processes with immigration and our results are similar to those of Karlin and McGregor [2] for Galton-Watson processes with immigration. When CBI-processes are diffusions, our representations are some concrete examples of the general representation theory for one-dimensional diffusions (cf. [5] e.g.).

Generally speaking, the spectrum appearing in the representation for a CBI-process is a constant multiple of that for the corresponding CB-process. The eigenmeasures of a CBI-process are given by convolutions of those of the CB-process and so called an α-stationary measure (cf. (2.1) below and [6] (2.1)). The right eigenfunctions are given by a similar way (cf. (2.4) below and [6] (2.6)).

In §1, we shall define an α-stationary measure, and represent the semigroup by it ((1.15) below). In §2, we prepare some general lemmas. §3 is devoted to obtain the spectral representation for sub- and supercritical cases. In these cases, only the discrete spectrum appears. In §4, we deal with the critical case, in which the spectrum is continuous.

Received May 4, 1970.
Communicated by K. Yosida.
* Department of Mathematics, Faculty of Science and Engineering, Saga University.

Some examples are given in §3 and §4. They contain all diffusion CBI-processes.

The author wishes to express his thanks to Professor N. Ikeda and Professor S. Watanabe for their valuable discussions.

§1. α-stationary Measure

A CBI-process is a Markov process (x_t, P_x) on the real half line $[0, \infty]$ with ∞ as a trap, satisfying for each $t\geq 0$, $\lambda\geq 0$, and $x\geq 0$,

$$E_x[e^{-\lambda x_t};\ t<e_\infty]=\varphi_t(\lambda)e^{-x\psi_t(\lambda)}.\text{[1]} \tag{1.1}$$

Here, $\varphi_t(\lambda)$ and $\psi_t(\lambda)$ are nonnegative functions of t and λ, and they satisfy

$$\psi_{t+s}(\lambda)=\psi_t(\psi_s(\lambda)), \qquad \psi_0(\lambda)=\lambda, \tag{1.2}$$

$$\varphi_{t+s}(\lambda)=\varphi_t(\lambda)\varphi_s(\psi_t(\lambda)), \qquad \varphi_0(\lambda)=1. \tag{1.3}$$

Now we shall assume that the process is stochastically continuous. Then $h(\lambda)\equiv\partial\psi_t(\lambda)/\partial t|_{t=0}$ and $g(\lambda)\equiv-\partial\varphi_t(\lambda)/\partial t|_{t=0}$ exist and are given by

$$h(\lambda)=-a\lambda^2+b\lambda+c-\int_0^\infty(e^{-\lambda y}-1+\lambda(y\wedge 1))\,n_1(dy), \tag{1.4}$$

$$g(\lambda)=d\lambda+e-\int_0^\infty(e^{-\lambda y}-1)\,n_2(dy), \tag{1.5}$$

with real constants $a\geq 0$, b, $c\geq 0$, $d\geq 0$, $e\geq 0$ and nonnegative measures n_1 and n_2 on $(0, \infty)$ such that $\int(y^2\wedge 1)\,n_1(dy)+\int(y\wedge 1)\,n_2(dy)<\infty$ (cf. [3]). From (1.2) and (1.3), we have

$$\frac{\partial}{\partial t}\psi_t(\lambda)=h(\psi_t(\lambda)), \qquad \psi_0(\lambda)=\lambda, \tag{1.6}$$

$$\varphi_t(\lambda)=\exp\left(-\int_0^t g(\psi_s(\lambda))ds\right). \tag{1.7}$$

Conversely, for given $h(\lambda)$ and $g(\lambda)$ with above properties, $\psi_t(\lambda)$ and $\varphi_t(\lambda)$ are uniquely determined by (1.6) and (1.7). Furthermore,

---

1) $e_\infty=\inf\{t;\ x_t=\infty\}$, $(\inf\phi=+\infty)$.

$\{\psi_t(\lambda)\}_{t\in[0,\infty)}$ is a Ψ-semigroup[2)] and $\varphi_t(\lambda)$ is completely monotone, and hence there corresponds a unique CBI-process.

Definition 1.1. The function $h(\lambda)$ is called *supercritical* (*subcritical*, *critical*) if $h'(0)>0$ or $c>0$ (resp. $h'(0)<0$ and $c=0$, resp. $h'(0)=0$ and $c=0$).

In the sequel, we shall assume

$$\int_\lambda^\infty \frac{d\tau}{h(\tau)} \text{ converges for a large } \lambda.^{3)} \tag{1.8}$$

Since $h(\lambda)$ is concave and $h(0)\geq 0$, we can find its largest zero point γ. By (1.8), $\gamma>0$ if it is supercritical, and $\gamma=0$ otherwise.

Definition 1.2. Let $\alpha=g(\gamma)$. A nonnegative measure $\pi_0(dx)$ on $[0,\infty)$ is called an *α-stationary measure of the CBI-process*, if it satisfies

$$\int_0^\infty \pi_0(dx)P_t(x,E)=e^{-\alpha t}\pi_0(E), \qquad E\in\mathscr{K}[0,\infty),^{4)} \tag{1.9}$$

$$0<\hat{\pi}_0(\lambda)\equiv\int_0^\infty e^{-\lambda x}\pi_0(dx)<\infty, \qquad \text{for a large } \lambda. \tag{1.10}$$

Lemma 1.1. *A nonnegative measure $\pi_0(dx)$ on $[0,\infty)$ is an α-stationary measure, if and only if $\hat{\pi}_0(\lambda)<\infty$ for all $\lambda>\gamma$ and*

$$\hat{\pi}_0(\psi_t(\lambda))\varphi_t(\lambda)=e^{-\alpha t}\hat{\pi}_0(\lambda), \qquad \lambda>\gamma,\ t>0. \tag{1.11}$$

Proof. Since all the measures in (1.9) are nonnegative, (1.11) follows from (1.9) by taking the Laplace transforms, (it is allowed that both sides are $+\infty$ at this step). If $\pi_0(dx)$ is an α-stationary measure, $\hat{\pi}_0(L)<\infty$ for some $L>0$. Since $\psi_t(L)\to\gamma$ as $t\to\infty$ by [6] Lemma 1.1, $\hat{\pi}_0(\lambda)<\infty$ for each $\lambda>\gamma$ by (1.11). The converse is obvious.

---

2) $\{\psi_t(\lambda)\}_{t\in[0,\infty)}$ is called Ψ-semigroup, if it satisfies (1.2) and each $\psi_t(\lambda)$ is nonnegative, having the completely monotone derivative in λ.

3) This condition is same as [6] (1.5). However, this is not always necessary for the spectral representation in [6] or in this paper. Indeed, the parallel arguments are available if we use $-\int_\lambda^{\lambda_0}\frac{d\tau}{h(\tau)}$ instead of $\hat{\pi}(\lambda)=-\int_\lambda^\infty\frac{d\tau}{h(\tau)}$, (cf. [6] (1.8)). But there is no more stationary measure or α-stationary measure if this fails.

4) $\mathscr{K}(0,\infty)$ ($\mathscr{K}[0,\infty)$) is the class of all Borel measurable sets in $(0,\infty)$ (resp. $[0,\infty)$) with compact closures.

Proposition 1.1. *There is a unique α-stationary measure up to a constant multiple. It is given by*

$$\hat{\pi}_0(\lambda)=\exp\left(\int_{\lambda_0}^{\lambda}\frac{g(\tau)-\alpha}{h(\tau)}\,d\tau\right),\qquad \lambda>\gamma, \tag{1.12}$$

where λ_0 is a constant with $\lambda_0>\gamma$.

Proof. First, we shall show that $\hat{\pi}_0(\lambda)$ of (1.12) is completely monotone on $\lambda>\lambda_0$, by showing that $f(\lambda)\equiv-\int_{\lambda_0}^{\lambda}((g(\tau)-\alpha)/h(\tau))\,d\tau$ has the completely monotone derivative (cf. [1] p. 417). Since $g(\lambda)$ is non-decreasing and $h(\lambda)<0$ on $\lambda>\gamma$, $f(\lambda)\geq 0$ on $\lambda>\lambda_0$. Similarly,

$$f'(\lambda)=\frac{g(\lambda)-\alpha}{\lambda-\gamma}\bigg/-\frac{h(\lambda)}{\lambda-\gamma}\geq 0,\qquad \lambda>\gamma. \tag{1.13}$$

$(g(\lambda)-\alpha)/(\lambda-\gamma)$ is completely monotone, since

$$\frac{d^n}{d\lambda^n}\frac{g(\lambda)-\alpha}{\lambda-\gamma}=(-1)^n\frac{n!}{(\lambda-\gamma)^{n+1}}\sum_{\nu=0}^{n}\frac{[-(\lambda-\gamma)]^\nu}{\nu!}\frac{d^\nu}{d\lambda^\nu}(g(\lambda)-\alpha)$$

$$=(-1)^n\frac{n!}{(\lambda-\gamma)^{n+1}}\int\left\{1-e^{-(\lambda-\gamma)y}\sum_{\nu=0}^{n}\frac{[(\lambda-\gamma)y]^\nu}{\nu!}\right\}e^{-\gamma y}n_2(dy).$$

Similarly, $-h(\lambda)/(\lambda-\gamma)$ has the completely monotone derivative. Hence $f'(\lambda)$ in (1.13) is completely monotone on $\lambda>\gamma$.

(1.11) follows from (1.12), (1.6) and (1.7).

By differentiating the both sides of (1.11) at $t=0$, we have

$$\hat{\pi}_0'(\lambda)h(\lambda)-\hat{\pi}_0(\lambda)g(\lambda)=-\alpha\hat{\pi}_0(\lambda).$$

This equation has the unique solution (1.12) up to a constant multiple.

q.e.d.

Let

$$\hat{\pi}(\lambda)=-\int_{\lambda}^{\infty}\frac{d\tau}{h(\tau)},\qquad \lambda>\gamma, \tag{1.14}$$

and $\hat{\theta}(w)$ be its inverse function (cf. [6] p. 426). Since $\psi_t(\lambda)=\hat{\theta}(\hat{\pi}(\lambda)+t)$, $\lambda>\gamma$, $t\geq 0$ ([6] (1.10)), (1.11) implies

Proposition 1.2. *The CBI-process has the representation*

$$(1.15)\qquad E_x[e^{-\lambda x_t}; t<e_\infty]=\frac{e^{-\alpha t}\hat{\pi}_0(\lambda)}{\hat{\pi}_0(\hat{\theta}(\hat{\pi}(\lambda)+t))}\, e^{-x\hat{\theta}(\hat{\pi}(\lambda)+t)},\qquad \lambda>\gamma,\ t\geq 0.$$

Corollary 1.1. From this, it is clear that the sample paths hit the origin with positive probability, if and only if (1.8) holds and

$$(1.16)\qquad \int_\lambda^\infty \frac{g(\tau)}{h(\tau)}\, d\tau \text{ converges for a large } \lambda.$$

(Cf. [6] Remark 1.1).

§2. General Lemmas of the Representations

In this section, we show that the representation (1.15) in §1 signifies the spectral representation of the semigroup. We start with

Lemma 2.1. *For each* $u\in[0, \infty)$, *there exists a signed measure* $\zeta(u; dx)$ *on* $\mathscr{K}[0, \infty)$ *such that*

$$(2.1)\qquad \int_0^\infty e^{-\lambda x}\zeta(u; dx)=\hat{\pi}_0(\lambda)e^{-u\hat{\pi}(\lambda)},\qquad \lambda>\gamma,$$

$$(2.2)\qquad \int_0^\infty e^{-\lambda x}|\zeta|(u; dx)\leq\hat{\pi}_0(\lambda)\, e^{u\hat{\pi}(\lambda)},\qquad \lambda>\gamma,$$

where $|\zeta|(u; E)$ *is the total variation of* $\zeta(u; \cdot)$ *on* E.

Moreover $\zeta(u; dx)$ *is an eigenmeasure of* $P_t(x, E)$ *in the sense that*

$$(2.3)\qquad \int_0^\infty \zeta(u; dx)P_t(x, E)=e^{-(u+\alpha)t}\zeta(u; E),\qquad t\geq 0,\ E\in\mathscr{K}[0, \infty).$$

The proof is similar to those of Lemmas 2.1 and 2.2 [6].

As in [6], we shall put some assumptions:

Condition I. *For each* $x\in[0, \infty)$, *there exists a signed measure* $\eta(x; du)$ *on* $\mathscr{K}[0, \infty)$ *such that for some* $t_0\geq 0$,

$$(2.4)\qquad \int_0^\infty e^{-wu}\eta(x; du)=\frac{e^{-x\hat{\theta}(w)}}{\hat{\pi}_0(\hat{\theta}(w))},\qquad w>t_0,$$

$$(2.5)\qquad H_w(x)\equiv\int_0^\infty e^{-wu}|\eta|(x; du)<\infty,\qquad w>t_0.$$

Condition II. *Condition* I *is satisfied, and*

$$\int_0^\infty P_t(x,\ dy) H_w(y) < \infty, \qquad t>0,\ w>t_1, \tag{2.6}$$

for some $t_1 \geq t_0$.

Lemma 2.2. *If Condition* II *is satisfied,* $\eta(x;\ du)$ *is an eigenfunction of* $P_t(x, E)$ *in the sense that*

$$\int_0^\infty P_t(x,\ dy)\eta(y;\ U) = \int_U e^{-(u+\alpha)t}\eta(x;\ du), \qquad U \in \mathscr{K}[0,\ \infty). \tag{2.7}$$

Lemma 2.3. *If Condition* I *is satisfied, then* $P_t(x, E)$ *has the spectral representation*

$$P_t(x,\ E) = \int_0^\infty \eta(x;\ du)\, e^{-(u+\alpha)t}\zeta(u;\ E), \qquad t>t_0,\ x\in[0,\ \infty),\ E\in\mathscr{K}[0,\ \infty). \tag{2.8}$$

The proofs are similar to those of [6] Lemmas 2.3 and 2.4, and will be omitted. (Use (1.11) and (1.15).)

§3. Discrete Spectrum Case

In this section, we deal with CBI-processes, whose $h(\lambda)$ and $g(\lambda)$ satisfy

$$h'(\gamma)<0, \text{ and } h(\lambda),\ g(\lambda) \text{ are analytic at } \gamma. \tag{3.1}$$

When a CBI-process is supercritical, (3.1) is always satisfied: when it is subcritical, since $h'(\gamma)=h'(0)<0$, the only assumption is that $h(\lambda)$ and $g(\lambda)$ are analytic at 0.

Let $\mu=-h'(\gamma)$. Then $A(\lambda)\equiv e^{-\mu\hat{\pi}(\lambda)}$ is analytic at γ and $A'(\gamma)>0$ (cf. [6]). Hence the inverse function $B(v)$ of $A(\lambda)$ is analytic on a neighbourhood $V(0)$ of $v=0$.[5] Moreover $B(v)$ and $\hat{\theta}(w)$ have the relation

$$\hat{\theta}(w) = B(e^{-\mu w}), \qquad e^{-\mu w}\in V(0). \tag{3.2}$$

5) Note that $A(\gamma)=0$.

Now, by (1.12) and (3.1), $1/\hat{\pi}_0(\lambda)$ is analytic at γ. Since $B(v)$ is analytic at 0 and $B(0)=\gamma$, $e^{-xB(v)}/\hat{\pi}_0(B(v))$ is analytic at 0. Therefore we can define the functions $\{\eta(x;k\mu)\}_{k=0}^{\infty}$ uniquely by

$$\sum_{k=0}^{\infty}\eta(x;k\mu)v^k=e^{-xB(v)}/\hat{\pi}_0(B(v)). \tag{3.3}$$

Theorem 3.1. *Let* (3.1) *be satisfied. Then* $\eta(x;k\mu)$ *is an eigenfunction of* $P_t(x,E)$ *corresponding to the eigenvalue* $e^{-(\alpha+k\mu)t}$. *Furthermore* $P_t(x,E)$ *has the spectral representation*

$$P_t(x,E)=\sum_{k=0}^{\infty}\eta(x;k\mu)e^{-(\alpha+k\mu)t}\zeta(k\mu;E), \tag{3.4}$$

$$t>t_0,\quad x\in[0,\infty),\quad E\in\mathscr{K}[0,\infty),$$

for some $t_0\geq 0$.

The proof is similar to that of [6] Theorem 3.1.

Remark 3.1. If $h(\lambda)$ is supercritical with $c=0$, t_0 in (3.4) can be taken to be 0. Indeed, in this case, (3.3) is satisfied on $(-1,1)$, since $B(v)$ is analytic there and $\hat{\pi}_0(\lambda)$ is analytic on $\lambda>0$ (cf. [6] Remark 3.1).

Remark 3.2. $\eta(x;k\mu)=e^{-\gamma x}\times$(*a polynomial in* x *with degree* k). This follows from (3.3) (cf. [6] Remark 3.2).

Note that in this case

$$\psi_t(\lambda)=B(e^{-\mu t}A(\lambda)),\qquad \text{for large } t, \tag{3.5}$$

(cf. [6] (3.6)). An asymptotic property is also obtained:

Proposition 3.1. *If* (3.1) *is satisfied,*

$$E_x[e^{-\lambda x_t};t<e_\infty]=c_1(\lambda)e^{-\gamma x}e^{-\alpha t}[1-e^{-\mu t}c_2(\lambda)(c_3+x)+0(e^{-2\mu t})],\qquad t\to\infty, \tag{3.6}$$

where $c_1(\lambda)=\exp\int_\gamma^\lambda((g(\tau)-\alpha)/h(\tau))d\tau$, $c_2(\lambda)=A(\lambda)/A'(\gamma)$ *and* $c_3=g'(\gamma)/h'(\gamma)$. $0(\ )$ *is uniform on* $\lambda\geq\gamma-\varepsilon$ *and* $x\leq L$ *for some* $\varepsilon>0$ *and each* $L>0$.

Proof. Since $\hat{\pi}_0(B(v))$ is analytic at 0 and $\hat{\pi}_0(B(0))=\hat{\pi}_0(\gamma)\neq 0$,

$$\frac{1}{\hat{\pi}_0(B(v))}=\frac{1}{\hat{\pi}_0(\gamma)}\left[1-\frac{\hat{\pi}_0'(\gamma)}{\hat{\pi}_0(\gamma)}B'(0)v+0(v^2)\right],\qquad v\to 0.$$

Hence, by (3.5) and (1.11)

$$(3.7)\quad \varphi_t(\lambda)=e^{-\alpha t}\hat{\pi}_0(\lambda)\frac{1}{\hat{\pi}_0(B(e^{-\mu t}A(\lambda)))}$$

$$=e^{-\alpha t}\frac{\hat{\pi}_0(\lambda)}{\hat{\pi}_0(\gamma)}\left[1-\frac{\hat{\pi}_0'(\gamma)}{\hat{\pi}_0(\gamma)}B'(0)e^{-\mu t}A(\lambda)+0(e^{-2\mu t})\right],\qquad t\to\infty.$$

On the other hand, by [6] (3.8)

$$(3.8)\qquad e^{-x\psi t(\lambda)}=e^{-\gamma x}\left[1-xe^{-\mu t}\frac{A(\lambda)}{A'(\gamma)}+0(e^{-2\mu t})\right],\qquad t\to\infty.$$

(3.7) and (3.8) imply (3.6).

Now we shall give a few examples (cf. [6] Examples 1 and 2).

Example 3.1. Let

$$\begin{cases} h(\lambda)=-a\lambda^{1+p}+b\lambda, \\ g(\lambda)=c\lambda^p+d, \qquad a,\ b>0,\ c,\ d\geq 0,\ 0<p\leq 1. \end{cases}$$

$h(\lambda)$ is supercritical, and the largest zero point γ is $(b/a)^q$, where $q=1/p$. As [6], $\mu=pb$ and

$$A(\lambda)=1-\frac{b}{a\lambda^p},\qquad \lambda>0,$$

$$B(v)=\gamma\left(\frac{1}{1-v}\right)^q,\qquad |v|<1.$$

Since $(g(\lambda)-g(\gamma))/h(\lambda)=-c/a\lambda$,

$$\hat{\pi}_0(\lambda)=\Lambda_0\frac{1}{\lambda^\kappa},\qquad \kappa=\frac{c}{a},\ \Lambda_0=\lambda_0^\kappa,$$

(λ_0 is a constant larger than γ). $\alpha=bc/a+d$ and the α-stationary measure is

$$\pi_0(dx)=\frac{\Lambda_0}{\Gamma(\kappa)}x^{\kappa-1}dx.$$

The eigenmeasures $\zeta(k\mu;\,dx)$, $k=0,1,2,\cdots$, are given by

$$\zeta(k\mu;\ dx)=A_0\sum_{l=0}^{k}\binom{k}{l}\left(-\frac{b}{a}\right)^l\frac{x^{pl+\kappa-1}}{\Gamma(pl+\kappa)}\,dx,$$

and the eigenfunctions $\eta(x;k\mu)$, $k=0, 1, 2, \cdots$, are

$$\eta(x;k\mu)=\frac{\gamma^\kappa}{A_0}\sum_{l=0}^{\infty}\frac{(-\gamma x)^l}{l!}\binom{q(l+\kappa)+k-1}{k}.$$

When $p=1$, the CBI-process is a diffusion with the generator

$$\mathscr{G}u=axu''+bxu'+cu'-du.$$

In this case, (3.4) can be written in the symmetric form;

$$P_t(x,\ dy)=\sum_{k=0}^{\infty}\eta_0(x;k\mu)\,e^{-(k\mu+\alpha)t}\,\eta_0(y;k\mu)\,m(dy),\qquad dy\in\mathscr{K}(0,\infty),$$

where

$$\eta_0(x;k\mu)=\sqrt{\frac{\gamma^\kappa\Gamma(k+\kappa)}{k!}}\;e^{-\gamma x}\sum_{l=0}^{k}\binom{k}{l}\frac{(-\gamma x)^l}{\Gamma(l+\kappa)},$$

and $m(dx)\equiv x^{\kappa-1}e^{\gamma x}dx$ is the canonical measure of the diffusion. This is an example of the general spectral representation theory of one-dimensional diffusions (cf. [5] e.g.).

Example 3.2. Let

$$\begin{cases} h(\lambda)=-a(\lambda+\gamma_0)^{1+p}+b(\lambda+\gamma_0),\\ g(\lambda)=c(\lambda+\gamma_0)^p+d, \qquad a,\ b>0,\ c,\ d\geq 0,\ 0<p\leq 1, \end{cases}$$

where $\gamma_0=(b/a)^q$, $q=1/p$. In this case $h(\lambda)$ is subcritical. $\alpha=bc/a+d$ and the α-stationary measure is

$$\pi_0(dx)=\frac{A_1}{\Gamma(\kappa)}\,e^{-\gamma_0 x}x^{\kappa-1}dx,\qquad \kappa=\frac{c}{a},\ A_1=(\lambda_0+\gamma_0)^\kappa.$$

$\mu=pb$ and the eigen-measures and -functions are

$$\zeta(k\mu;\ dx)=A_1\sum_{l=0}^{k}\binom{k}{l}\left(-\frac{b}{a}\right)^l\frac{x^{pl+\kappa-1}}{\Gamma(pl+\kappa)}\,e^{-\gamma_0 x}dx,$$

$$\eta(x;k\mu)=\frac{\gamma_0^\kappa e^{\gamma_0 x}}{A_1}\sum_{l=0}^{\infty}\frac{(-\gamma_0 x)^l}{l!}\binom{q(l+\kappa)+k-1}{k}.$$

When $p=1$, the CBI-process is a diffusion with the generator

$$\mathscr{G}u=axu''-bxu'+cu'-du.$$

(3.4) has the symmetric form

$$P_t(x, dy)=\sum_{k=0}^{\infty}\eta_0(x; k\mu)\, e^{-(k\mu+\alpha)t}\eta_0(y; k\mu)\, m(dy), \qquad dy\in\mathscr{K}(0, \infty),$$

where

$$\eta_0(x; k\mu)=\sqrt{\frac{\gamma_0^{\kappa}\Gamma(k+\kappa)}{k!}}\sum_{l=0}^{k}\binom{k}{l}\frac{(-\gamma_0 x)^l}{\Gamma(l+\kappa)},$$

and $m(dx)\equiv x^{\kappa-1}e^{-\gamma_0 x}\, dx$ is the canonical measure of the diffusion. This is also an example of the general theory.

§4. Continuous Spectrum Case

In this section, we discuss the spectral representation of CBI-processes with continuous spectra. First, we shall deal with the case

$$\text{(4.1)}\quad \begin{cases} h(\lambda)=\lambda^{1+p}h_1(\lambda) \text{ and } g(\lambda)=\lambda^p g_1(\lambda)+d, \text{ where } 0<p\leq 1,\ d\geq 0 \\ \text{and } h_1(\lambda),\ g_1(\lambda) \text{ are analytic at } 0 \text{ with } h_1(0)g_1(0)\neq 0. \end{cases}$$

This is satisfied of course only when $h(\lambda)$ is critical. Since $h(\lambda)$ is concave and $g(\lambda)$ nondecreasing, $h_1(0)<0$ and $g_1(0)>0$. We set

$$\kappa=-\frac{g_1(0)}{h_1(0)}.$$

Lemma 4.1. *Let* (4.1) *be satisfied. Then* $\hat{\theta}(w)$ *is analytically continued to a domain containing a right half-plane* Re $w\geq t_1$, *and*

$$\text{(4.2)}\qquad \left|\frac{1}{\hat{\pi}_0(\hat{\theta}(w))}\right|\leq\frac{M}{|w|^{\kappa/p}}, \qquad \text{Re } w\geq t_1.$$

Proof. By (1.12) and (4.1), for some $\rho>0$,

$$\text{(4.3)}\qquad \frac{1}{\hat{\pi}_0(\lambda)}=\lambda^{\kappa}f(\lambda), \qquad |\lambda|<\rho,$$

where $f(\lambda)$ is an analytic function with $f(0)\neq 0$. On the other hand, by the same reason as in the proofs of [6] Lemmas 4.1 and 4.2,

$$\sup_{\operatorname{Re} w>t_1}\left|\frac{w^{\kappa/p}}{\hat{\pi}_0(\hat{\theta}(w))}\right| \leq \begin{cases} \sup\limits_{\substack{0<r<\rho \\ |\theta|<\pi/2+\varepsilon_0}} \left|\dfrac{(1+c_0+re^{i\theta}k_0(re^{i\theta}))^{\kappa/p}}{\hat{\pi}_0(re^{i\theta})(\alpha pr^p e^{ip\theta})^{\kappa/p}}\right|, & 0<p<1, \\ \sup\limits_{\substack{0<|\lambda|<\rho \\ |\arg\lambda|<\pi/2+\varepsilon_1}} \left|\dfrac{(1+c_1\lambda\log\lambda+\lambda k_1(\lambda))^{\kappa}}{\hat{\pi}_0(\lambda)(\alpha\lambda)^{\kappa}}\right|, & p=1, \end{cases} \tag{4.4}$$

where $\varepsilon_0, \varepsilon_1>0$, c_0, c_1 are some constants, and $k_0(\lambda)$, $k_1(\lambda)$ analytic functions. Now (4.2) follows from (4.3) and (4.4).

Moreover, it holds that

$$|1-e^{-x\hat{\theta}(w)}| \leq \frac{Me^{\rho x}}{|w|}, \qquad \operatorname{Re} w \geq t_2, \text{ (cf. [6] (4.3)).} \tag{4.5}$$

Let $t_0=t_1\vee t_2$. Then we have:

Theorem 4.1. *Suppose that* (4.1) *is satisfied and* $\kappa>p/2$. *Then the transition function* $P_t(x, E)$ *admits the spectral representation*

$$P_t(x, E)=\int_0^\infty \eta(x, u)\, e^{-(u+\alpha)t}\zeta(u; E)\, e^{t_0 u} du, \qquad t>t_0,\ x\geq 0,\ E\in\mathscr{K}(0, \infty), \tag{4.6}$$

where $\eta(x, u)$ *is in* $L^2(0, \infty)$ *as a function in* u. *Furthermore, if* $p=1$, $\eta(x, u)$ *is an eigenfunction of* $P_t(x, E)$ *for each continuity point* u *of* $\eta(x, u)$.

Proof. By (4.2), (4.5) and the Paley-Wiener theorem ([4] p. 131),

$$\frac{1-e^{-x\hat{\theta}(w)}}{\hat{\pi}_0(\hat{\theta}(w))}=\int_0^\infty e^{-wu}\beta(x, u)\, e^{t_0 u} du, \qquad \operatorname{Re} w\geq t_0, \tag{4.7}$$

$$\frac{1}{\hat{\pi}_0(\hat{\theta}(w))}=\int_0^\infty e^{-wu}\beta(u)\, e^{t_0 u} du, \qquad \operatorname{Re} w\geq t_0, \tag{4.8}$$

where $\beta(x, u)$ and $\beta(u)$ are in $L^2(0, \infty)$ as functions in u. Hence (2.4) follows with $\eta(x; du)/e^{t_0 u}du \equiv \eta(x, u)=\beta(u)-\beta(x, u)$. (2.5) is obvious since $\eta(x, u)$ is in $L^2(0, \infty)$. Thus Condition I is satisfied and hence (4.6) follows by Lemma 2.3.

For the latter assertion, we shall only note that (4.7) and (4.8) imply

$$\int_0^\infty e^{-wu}|\eta|(x;\, du)\leq M_\rho e^{\rho x}, \qquad w>t_0,$$

(cf. the proof of [6] Theorem 4.1).

Proposition 4.1. *If* (4.1) *is satisfied, we have*

(4.9) $\quad E_x[e^{-\lambda x_t};\, t<e_\infty]$

$$=\begin{cases} \dfrac{e^{-\alpha t}\hat{\pi}_0(\lambda)f(0)}{(\alpha pt)^{q\kappa}}\left\{1-\dfrac{\kappa(\alpha\hat{\pi}(\lambda)-qc)}{\alpha pt}+0\left(\dfrac{1}{t^{q+2}}\vee\dfrac{1}{t^{2q}}\right)\right\}, \\ \hfill 0<p<1, \\ \dfrac{e^{-\alpha t}\hat{\pi}_0(\lambda)f(0)}{(\alpha t)^{\kappa}}\left\{1+\dfrac{\kappa\beta\log t}{\alpha^2 t}+\dfrac{f'(0)/f(0)-\kappa(\hat{\pi}(\lambda)+c_1)-x}{\alpha t}\right. \\ \left. \quad +0\left(\left(\dfrac{\log t}{t}\right)^2\right)\right\}, \qquad p=1,\ t\to\infty \end{cases}$$

where $f(\lambda)$ *is that of* (4.3) *and* α, c, q, c_1 *are those of* [6] §4. $0(\ )$ *are uniform on* $\lambda\geq K$ *and* $x\leq L$ *for each* K, $L>0$.

Proof. Note that (1.1) and (1.11) imply

$$E_x[e^{-\lambda x_t};\, t<e_\infty]=\frac{e^{-\alpha t}\hat{\pi}_0(\lambda)}{\hat{\pi}_0(\psi_t(\lambda))}\, e^{-x\psi_t(\lambda)}. \tag{4.10}$$

Since $\psi_t(\lambda)\to 0$ as $t\to\infty$, we have by (4.3)

$$\frac{1}{\hat{\pi}_0(\psi_t(\lambda))}=f(0)\,\psi_t(\lambda)^{\kappa}\left(1+\frac{f'(0)}{f(0)}\,\psi_t(\lambda)+0(\psi_t(\lambda)^2)\right), \qquad t\to\infty. \tag{4.11}$$

(4.9) follows from (4.10), (4.11) and [6] (4.12).

Remark 4.1. The higher approximations may be obtained by the same methods.

Now we shall give an example of (4.1).

Example 4.1. Let

$$\begin{cases} h(\lambda)=-a\lambda^{1+p}, \\ g(\lambda)=b\lambda^p+c, \qquad 0<p\leq 1,\ a>0,\ b\geq 0,\ c\geq 0. \end{cases}$$

By a simple calculation,

$$\hat{\pi}(\lambda)=\frac{1}{pa\lambda^{p}}, \qquad \hat{\theta}(w)=\frac{1}{(paw)^{q}},$$

with $q=1/p$. $\kappa=b/a$ and

$$\hat{\pi}_0(\lambda)=\frac{\lambda_0^{\kappa}}{\lambda^{\kappa}},$$

for $\lambda_0>0$. Hence the α-stationary measure $(\alpha=c)$ is

$$\pi_0(dx)=\frac{\lambda_0^{\kappa}}{\Gamma(\kappa)}x^{\kappa-1}dx.$$

The eigenmeasures $\zeta(u; dx)$ are

$$\zeta(u; dx)=\lambda_0^{\kappa}\sum_{l=0}^{\infty}\frac{1}{l!}\left(-\frac{u}{pa}\right)^{l}\frac{x^{pl+\kappa-1}}{\Gamma(pl+\kappa)}dx.$$

In this case, $\kappa>p/2$ is not necessary for the spectral representation (4.6).[6] Indeed, in our case, Condition I is satisfied with

$$\eta(x; du)=\frac{1}{\lambda_0^{\kappa}(pa)^{q\kappa}}\sum_{l=0}^{\infty}\frac{1}{l!}\frac{(-x)^{l}u^{q(l+\kappa)-1}}{(pa)^{ql}\Gamma(q(l+\kappa))}du.$$

When $p=1$, the corresponding CBI-process is a diffusion with the generator

$$\mathscr{G}u=axu''+bu'-cu, \tag{4.12}$$

and (2.8) has a symmetric form

$$P_t(x, dy)=\int_0^{\infty}\eta^0(x; u)\,e^{-(u+c)t}\eta^0(y; u)\,du\,m(dy), \qquad dy\in\mathscr{K}(0, \infty),$$

where

$$\eta^0(x; u)=\sqrt{\frac{u^{\kappa-1}}{a^{\kappa}}}\sum_{l=0}^{\infty}\frac{1}{l!}\left(-\frac{ux}{a}\right)^{l}\frac{1}{\Gamma(l+\kappa)},$$

and $m(dx)\equiv x^{\kappa-1}dx$ is the canonical measure of the diffusion.

Finally, we will give an example of the spectral representation with continuous spectra which does not satisfy (4.1).

---

6) But $\eta(x, u)$ does not belong to $L^2(0, \infty)$, when $\kappa\leq p/2$.

Example 4.2. Let

$$\begin{cases} h(\lambda) = -a\lambda^{1+p}, \\ g(\lambda) = b\lambda^r + c, \qquad 0<p,\ r\leq 1,\ p\neq r,\ a>0,\ b,\ c\geq 0. \end{cases}$$

The functions $\hat{\pi}(\lambda)$ and $\hat{\theta}(w)$ are as same as those of Example 4.1.

Suppose first that $0<p<r\leq 1$, and put $\rho = r-p$. Then

$$\hat{\pi}_0(\lambda) = \Lambda_0 \exp\left(-\frac{b\lambda^\rho}{a\rho}\right), \qquad \Lambda_0 = \exp\left(\frac{b\lambda_0^\rho}{a\rho}\right),$$

for $\lambda_0 > 0$. Hence the α-stationary measure $\pi_0(dx)$ is

$$\pi_0(dx) = \Lambda_0 P^{(\rho)}((a\rho/b)^{1/p} dx),$$

where $P^{(\rho)}(dx)$ is the one-sided stable distribution of exponent ρ.[7] Let $\xi_u(dx)$ be that of [6] Example 3. Then the eigenmeasure $\zeta(u;dx)$ is the convolution of $\pi_0(dx)$ and $\xi_u(dx)$. Further,

$$\frac{1}{\hat{\pi}_0(\hat{\theta}(w))} = \int_0^\infty e^{-wu}\beta_0(du),$$

$$\beta_0(du) = \frac{1}{\Lambda_0}\left\{\sum_{l=1}^\infty \frac{b^l u^{q\rho l-1} du}{l!(a\rho)^l (ap)^{q\rho l}\Gamma(q\rho l)} + \delta(du)\right\}.$$

Hence Condition I is satisfied with $\eta(x;du) \equiv (\beta_0 * \phi_\cdot(x))(du)$, where $\phi_{du}(x)$ is that of [6] Example 3.

Next, let $0<r<p\leq 1$, and put $\sigma = p-r$. Then

$$\hat{\pi}_0(\lambda) = \Lambda^0 \exp\left(\frac{b}{a\sigma\lambda^\sigma}\right), \qquad \Lambda^0 = \exp\left(-\frac{b}{a\sigma\lambda_0^\sigma}\right),$$

for $\lambda_0 > 0$. The α-stationary measure $\pi_0(dx)$ is

$$\pi_0(dx) = \Lambda^0\left\{\sum_{l=1}^\infty \frac{1}{l!}\left(\frac{b}{a\sigma}\right)^l \frac{x^{l\sigma-1}}{\Gamma(\sigma l)}\, dx + \delta(dx)\right\},$$

and the eigenmeasure $\zeta(u;dx)$ is the convolution of $\pi_0(dx)$ and $\xi_u(dx)$.

$$\frac{1}{\hat{\pi}_0(\hat{\theta}(w))} = \int_0^\infty e^{-wu}\beta^0(du),$$

---

7) That is $P^{(\rho)}$ is a probability measure given by

$$\int_0^\infty e^{-\lambda x} P^{(\rho)}(dx) = e^{-\lambda^\rho}, \qquad \lambda \geq 0.$$

$$\beta^0(du)=\frac{1}{\Lambda^0}P^{(q\sigma)}\left(\frac{1}{pa}\left(\frac{q\sigma}{b}\right)^{1/q\sigma}du\right),$$

so that Condition I is satisfied with $\eta(x;du)=(\beta^0*\phi.(x))(du)$. Hence (2.8) follows by Lemma 2.3.

References

[1] Feller, W., An Introduction to Probability Theory and Its Applications, II, Wiley, New York, 1966.

[2] Karlin, S. and J. McGregor, Spectral theory of branching processes, I, II, Z. Wahrsheinlichkeitstheorie und verw. Geb. **5** (1966), 1-33; 34-54.

[3] Kawazu, K. and S. Watanabe, Branching processes with immigration and related limit theorems, to appear in Z. Wahrscheinlichkeitstheorie und verw. Gebiete.

[4] Hoffman, K., Banach spaces of Analytic Functions, Prentice-Hall, New Jersey, 1962.

[5] McKean, Jr., H. P., Elementary solutions for certain parabolic partial differential equations, Trans. Amer. Math. Soc. **82** (1956), 519-548.

[6] Ogura, Y., Spectral representation for branching processes on the real half line, Publ RIMS, Kyoto Univ. **5** (1970), 423-441.

[7] Widder, D. V., The Laplace transform, Princeton University Press, 1946.