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On Quasifree States of CAR and Bogoliubov Automorphisms

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Abstract

A necessary and sufficient condition for two quasifree states of CAR to be quasiequivalent is obtained. Quasifree states is characterized as the unique KMS state of a Bogoliubov automorphism of CAR. The structure of the group of all inner Bogoliubov automorphisms of CAR is clarified.

§1. Introduction

A classification of gauge invariant quasifree states of the canonical anticommutation relations (CAR) up to quasi and unitary equivalence is recently obtained by Powers and Størmer [12]. We shall generalize their result to arbitrary quasifree states.

We use the formalism developped earlier [2] and study quasifree state φ_S of a selfdual CAR algebra. It is then shown that φ_{S_1} and φ_{S_2} are quasiequivalent if and only if $S_1^{1/2}-S_2^{1/2}$ is in the Hilbert Schmidt class. For a gauge invariant quasifree state φ_A in the paper of Powers and Størmer, $S=A\oplus(1-A)$ and hence our result is a direct generalization of Powers and Størmer.

The quasifree primary state φ_S for which S does not have eigenvalue 1 is shown to be the unique KMS state for the one parameter group $\tau(U(\lambda))$ of Bogoliubov $*$ automorphisms of CAR, where $\tau(U(\lambda))$ corresponds to a unitary transformation $U(\lambda)=\exp i\lambda H$ of the direct sum of testing function spaces of creation and annihilation operators and H is related to S by $S=(1+e^{-H})^{-1}$. This is used to simplify some of arguments. A quasifree state φ_S is primary unless $1/2$ is an isolated point spectrum of S, has an odd multiplicity and $S(1-S)$ is in the

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Hilbert Schmidt class.

It is shown that a Bogoliubov automorphism $\tau(V)$ is inner if and only if $V-1$ is in the trace class and $\det V>0$ or $V+1$ is in the trace class and $\det V<0$. It is a $*$ automorphism if and only if V is unitary. A double valued representation of the identity component (i.e. $\det V>0$) of the group of inner Bogoliubov automorphisms of a CAR algebra by elements of CAR algebra (such that it implements the automorphism) is obtained with a help of bilinear hamiltonians. It is a generalization of the observable algebra introduced by Araki and Wyss [4].

A necessary and sufficient condition for the unitary implementability of a Bogoliubov transformation in a Fock representation is obtained.

In an appendix, a general structure of two projections is presented and an angle operator is inroduced. Some of the discussions in the main text can be carried out by introducing a specific basis, although we have avoided this in the present paper. For such a purpose, this general analysis of two projections is useful.

The CAR algebra has been extensively studied by many authors ([4~7, 10, 12~17]) and some of our results such as Theorem 6 and 7 are in these earlier references.

§2. Basic Notations

We quote a few notions concerning a self dual CAR algebra from an earlier paper [2].

Let K be a complex Hilbert space and Γ be an antiunitary involution (a complex conjugation, $\Gamma^2=1$, $(\Gamma f, \Gamma g)=(g,f)$) on K. A *self dual CAR algebra* $\mathfrak{A}_{SDC}(K,\Gamma)$ over (K,Γ) is a $*$ algebra generated by $B(f)$, $f\in K$, its conjugate $B(f)^*$, $f\in K$ and an identity which satisfy the following relations: (1) $B(f)$ is (complex) linear in f, (2) $B(f)B(g)^*+B(g)^*B(f)=(g,f)\,1$, and (3) $B(f)^*=B(\Gamma f)$.

If K has a finite dimension, $\mathfrak{A}_{SDC}(K,\Gamma)$ has a finite dimension. Irrespective of the dimension of K, $\mathfrak{A}_{SCD}(K,\Gamma)$ has a unique C^* norm and $\overline{\mathfrak{A}}_{SDC}(K,\Gamma)$ denotes its C^* completion.

Any unitary operator U on K commuting with Γ preserves the above relations (1) ~ (3) and hence defines a $*$ automorphism $\tau(U)$ of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ by $\tau(U)\mathrm{B}(f)=\mathrm{B}(Uf)$. U and $\tau(U)$ shall be called a *Bogoliubov transformation* and a *Bogoliubov $*$ automorphism.*

The antilinear transformation

$$\tau(\Gamma)\sum_{n=1}^{N}c_n\mathrm{B}(f_1^{(n)})\cdots\mathrm{B}(f_{k_n}^{(n)})=\sum_{n=1}^{N}c_n^*\mathrm{B}(\Gamma f_1^{(n)})\cdots\mathrm{B}(\Gamma f_{k_n}^{(n)})$$

also leaves relations (1) ~ (3) invariant and hence can be extended to a conjugate $*$ automorphism (i.e. antilinear $*$ isomorphism onto itself) which will be denoted by $\tau(\Gamma)$.

Any projection operator P on K satisfying $\Gamma P\Gamma=1-P$ is called a *basis projection.* There exists a basis projection P if and only if the dimension of K is even or infinite. Any two basis projections P_1 and P_2 can be transformed to each other by a Bogoliubov transformation $U: P_1=UP_2U^*$.

Any projection P on K such that $P\perp\Gamma P\Gamma$ is called a *partial basis projection.* $\dim(1-P-\Gamma P\Gamma)$ is called the *Γ codimension* of P.

By identifying $\mathrm{B}(f)$ and $B(\Gamma f)$, $f\in PK$ with creation and annihilation operators on a CAR algebra $\mathfrak{A}_{\mathrm{CAR}}(K_1)$ over $K_1=PK$, we have a $*$ isomorphism of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ with $\overline{\mathfrak{A}}_{\mathrm{CAR}}(K_1)$, where P is any basis projection.

Here $\mathfrak{A}_{\mathrm{CAR}}(K_1)$ is the $*$ algebra generated by creation operators $(\mathrm{a}^\dagger,f)$, $f\in K_1$, their conjugates $(\mathrm{a}^\dagger,f)^*\equiv(f,\mathrm{a})$ (annihilation operators) and an identity, satisfying the following relations: (1) $(\mathrm{a}^\dagger,f)$ is (complex) linear in f, (2) $(\mathrm{a}^\dagger,f)(\mathrm{a}^\dagger,g)+(\mathrm{a}^\dagger,g)(\mathrm{a}^\dagger,f)=(f,\mathrm{a})(g,\mathrm{a})+(g,\mathrm{a})(f,\mathrm{a})=0$, $(\mathrm{a}^\dagger,f)(g,\mathrm{a})+(g,\mathrm{a})(\mathrm{a}^\dagger,f)=(g,f)1$. $\overline{\mathfrak{A}}_{\mathrm{CAR}}(K_1)$ is the completion of $\mathfrak{A}_{\mathrm{CAR}}(K_1)$ with respect to its unique C^* norm.

(A more precise notation will be something like $\mathrm{B}_{K,\Gamma}(f)$, $(\mathrm{a}_{K_1}^\dagger,f)$ and (f,a_{K_1}), which is useful whenever elements of more than one algebras with different K, Γ, and K_1 appear at the same time. We shall meet in later sections a case where elements of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ and $\overline{\mathfrak{A}}_{\mathrm{SDC}}(\hat{K},\hat{\Gamma})$, $\hat{K}=K\oplus K$, $\hat{\Gamma}=\Gamma\oplus(-\Gamma)$, appear at the same time. In this case, $\mathrm{B}_{K,\Gamma}(f)$, $f\in K$ is identified with $\mathrm{B}_{\hat{K},\hat{\Gamma}}(f\oplus 0)$ and will be denoted simply as $\mathrm{B}(f)$.)

§3. Quasiequivalence of Quasifree States

Definition 3.1. A state φ on $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ satisfying the following relation is called a *quasifree state*:

(3.1) $\varphi(B(f_1)\cdots B(f_{2n+1}))=0,$

(3.2) $\varphi(B(f_1)\cdots B(f_{2n}))=(-1)^{n(n-1)/2}\sum\varepsilon(s)\prod_{j=1}^{n}\varphi(B(f_{s(j)})B(f_{s(j+n)})),$

where $n=1, 2, \cdots$, the sum is over all permutations s satisfying

$$s(1)<s(2)<\cdots<s(n),$$
$$s(j)<s(j+n), \qquad j=1, \cdots, n,$$

and $\varepsilon(s)$ is the signature of s.

Lemma 3.2. *For any state φ over $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$, there exists a bounded operator S on K satisfying*

(3.3) $\varphi(B(f)^*B(g))=(f, Sg),$

(3.4) $1\geqq S^*=S\geqq 0,$

(3.5) $S+\Gamma S\Gamma=1.$

Proof. We have

(3.6) $B(f)^*B(f)\leqq B(f)^*B(f)+B(f)B(f)^*=\|f\|^2,$

(3.7) $\|B(f)\|=\|B(f)^*B(f)\|^{1/2}\leqq\|f\|.$

Hence (3.3) defins a bounded linear operator S on K.

From the positivity of φ, it follows that $S^*=S\geqq 0$. From the anticommutation relations, we have

$$\begin{aligned}\varphi(B(f)^*B(g))&=(f, g)-\varphi(B(g)B(f)^*)\\&=(f, g)-\varphi(B(\Gamma g)^*B(\Gamma f))\\&=(f, g)-(\Gamma g, S\Gamma f).\end{aligned}$$

Since

(3.8) $(h, \Gamma f)=(\Gamma(\Gamma h), \Gamma f)=(f, \Gamma h),$

we have $(\Gamma g, S\Gamma f)=(S\Gamma g, \Gamma f)=(f, \Gamma S\Gamma g)$. Hence (3.5) follows.

From $S\geqq 0$ and $1-S=\Gamma S\Gamma$, it follows that $1-S\geqq 0$. Q.E.D.

Lemma 3.3. *For any bounded linear operator S satisfying (3.4) and (3.5), there exists a unique quasifree state φ satisfying (3.3).*

The uniqueness is immediate from (3.1) and (3.2). The existence follows from Lemma 4.6.

Definition 3.4. The unique quasifree state of Lemma 3.3 is denoted φ_S.

From Lemmas 3.2 and 3.3, φ_S exhausts all quasifree states of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$.

Theorem 1. *Two quasifree states φ_S and $\varphi_{S'}$ give rise to mutually quasiequivalent representations of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ if and only if $S^{1/2}-(S')^{1/2}$ is in the Hilbert Schmidt class.*

The proof will be presented in section 5.

§4. Fock Representation Induced by Quasifree States

Definition 4.1. $\mathfrak{H}_S$, π_S, and Ω_S denote the Hilbert space, the representation and the cyclic unit vector canonically associated with the quasifree state φ_S through the relation

$$\varphi_S(A)=(\Omega_S, \pi_S(A)\Omega_S), \quad A\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma).$$

Lemma 4.2. *Let φ_S be a quasifree state. If a Bogoliubov transformation U commutes with S, then there exists a unitary operator $\mathrm{T}_S(U)$ on $\mathfrak{H}_S$ such that*

$$\mathrm{T}_S(U)\Omega_S=\Omega_S \tag{4.1}$$

and

$$\mathrm{T}_S(U)\pi_S(A)\mathrm{T}_S(U)^*=\pi_S(\tau(U)A) \tag{4.2}$$

for all $A\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$.

Proof. If $[U, S]=0$, then $\varphi_S(\tau(U)A)=\varphi_S(A)$. Hence

$$\mathrm{T}_S(U)\sum c_i\pi_S(A_i)\Omega_S=\sum c_i\pi_S(\tau(U)A_i)\Omega_S,$$

and

$$\mathrm{T}_S(U^*)\sum c_i\pi_S(A_i)\Omega_S=\sum c_i\pi_S(\tau(U^*)A_i)\Omega_S$$

define isometric linear mappings from a dense subset of $\mathfrak{H}_S$ into $\mathfrak{H}_S$ satisfying

$$\mathrm{T}_S(U)\mathrm{T}_S(U^*)=\mathrm{T}_S(U^*)\mathrm{T}_S(U)\subset 1,$$
$$\mathrm{T}_S(U)\subset \mathrm{T}_S(U^*)^*.$$

Therefore, the closure of this $\mathrm{T}_S(U)$ is unitary and satisfies (4.1) and (4.2).

Note that $\mathrm{T}_S(-1)$ is defined for all S.

Lemma 4.3. *Let P be a basis projection. If a state φ of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ satisfies*

(4.3) $$\varphi(\mathrm{B}(f)\mathrm{B}(f)^*)=0, \qquad f\in PK,$$

then $\varphi=\varphi_P$. The representation π_P is irreducible.

Proof. By splitting every $\mathrm{B}(f)$ as $\mathrm{B}(Pf)++\mathrm{B}(P\Gamma f)^*$ and using commutation relations to bring $\mathrm{B}(Pf)$ to the left and $\mathrm{B}(Pg)^*$ to the right, any element A in $\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$ can be written as

$$A=\sum_j \mathscr{P}_j\mathrm{B}(f_j)^*+\sum_j \mathrm{B}(g_j)\mathscr{P}'_j+\lambda$$

where $f_j, g_j\in PK$, $\mathscr{P}_j$ and $\mathscr{P}'_j$ are polynomials. The condition (4.3) implies $\varphi(A)=\lambda$ and hence state φ satisfying (4.3) is unique.

From (3.3), φ_P satisfies (4.3).

The condition (4.3) may be stated as $\varphi(A^*A)=0$ whenever A belongs to the closed left ideal $\mathfrak{L}$ generated by $\mathrm{B}(f)^*$, $f\in PK$. The uniqueness of such state implies that $\mathfrak{L}$ is maximal and the unique state φ is pure [9]. Q.E.D.

The state φ_P is called a *Fock state* and π_P is called a *Fock representation.* Under the identification of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ with $\overline{\mathfrak{A}}_{\mathrm{CAR}}(PK)$, this coincides with an ordinary definition of the Fock vaccuum of CAR and the existence of such state φ_P is known. A different choice of the basis projection P produces a different identification α_P of the selfdual CAR algebra with a CAR algebra and correspondingly different Fock state φ_P. All of them are mutually related by Bogoliubov automorphisms.

Definition 4.4. Let S be an operator on K. Then P_S donotes the operator on $K\oplus K$ given by a matrix

$$P_S=\begin{pmatrix} S & S^{1/2}(1-S)^{1/2} \\ S^{1/2}(1-S)^{1/2} & 1-S \end{pmatrix}. \tag{4.4}$$

Lemma 4.5. *If S satisfies* (3.4) *and* (3.5), *then P_S is a basis projection on $(\widehat{K},\widehat{\Gamma})$ where $\widehat{K}=K\oplus K$, $\widehat{\Gamma}=\Gamma\oplus(-\Gamma)$.*

Proof. A direct computation shows $P_S^2=P_S=P_S^*$, $\Gamma P_S\Gamma=1-P_S$.

Lemma 4.6. *Let S, P_S, $\widehat{K}$, $\widehat{\Gamma}$ be as in Lemma* 4.5. *Then the restriction of the Fock state φ_{P_S} of $\overline{\mathfrak{A}}_{\rm SDC}(\widehat{K},\widehat{\Gamma})$ to $\overline{\mathfrak{A}}_{\rm SDC}(K,\Gamma)$ is the quasifree state φ_S.*

Since φ_{P_S} is quasifree, its restriction is also quasifree. $\varphi_{P_S}(\mathrm{B}(f)^*\mathrm{B}(g))=(f,P_Sg)=(\widehat{f},S\widehat{g})$ if $f=\widehat{f}\oplus 0$, $g=\widehat{g}\oplus 0$. Q.E.D.

Lemma 4.7. *Let P be a basis projection and*

$$\pi_P^{\rm t}(\mathrm{B}(f))=\pi_P[\mathrm{B}([2P-1]f)]\mathrm{T}_P(-1), \qquad f\in K. \tag{4.5}$$

Then there exists a representation $\pi_P^{\rm t}$ of $\overline{\mathfrak{A}}_{\rm SDC}(K,\Gamma)$ on $\mathfrak{H}_P$ which is uniquely determined by (4.5). *Ω_P is cyclic for $\pi_P^{\rm t}$ and the corresponding vector state is φ_P.*

Proof. It follows from (4.5) that $\pi_P^{\rm t}(\mathrm{B}(f))$ satisfies relations (1), (2) and (3) for $\overline{\mathfrak{A}}_{\rm SDC}(K,\Gamma)$ in section 2. Hence the existence of a representation $\pi_P^{\rm t}$ of $\overline{\mathfrak{A}}_{\rm SDC}(K,\Gamma)$ satisfying (4.5) follows. Since $\mathrm{B}(f)$ generates $\overline{\mathfrak{A}}_{\rm SDC}(K,\Gamma)$, $\pi_P^{\rm t}$ is unique. By applying $\pi_P^{\rm t}(\mathrm{B}(f_i))$, $i=1,\cdots,n$ successively on Ω_P, one can reproduce $\pi_P(\mathrm{B}(f_1))\cdots\pi_P(\mathrm{B}(f_n))\Omega_P$ up to $\pm$ sign and hence Ω_P is cyclic. From the same computation, it is seen that the vector state given by Ω_P is φ_P.

Lemma 4.8. *Let K_0 be a Γ invariant subset of K, $\mathrm{E}(K_0)$ be the projection operator for the smallest closed subspace of K containing K_0 and*

$$\mathrm{R}_P(K_0)=\{\pi_P(\mathrm{B}(f))\,;\ f\in K_0\}''. \tag{4.6}$$

The following conditions are equivalent.

(1) *Ω_P is cyclic for* $\mathrm{R}_P(K_0)$,

(2) $(1-\mathrm{E}(K_0))\wedge(1-P)=0$,

(3) $(1-\mathrm{E}(K_0))\wedge P=0$.

Here $P\wedge P'$ denotes the projection for $PK\cap P'K$. The following conditions are also equivalent.

(1)′ *Ω_P is separating for* $\mathrm{R}_P(K_0)$,

(2)′ $\mathrm{E}(K_0)\wedge(1-P)=0$,

(3)′ $\mathrm{E}(K_0)\wedge P=0$.

Proof. (3)→(1): As is known, $\mathfrak{H}_P$ is a direct sum of subspaces $\mathfrak{H}_P^{(n)}$, $n=0,1,\cdots$, such that the set of vectors $\prod_{i=1}^{n}(\mathrm{B}(f_i))\Omega_P$, $f_i\in PK$, is total in $\mathfrak{H}_P^{(n)}$. (3) implies that Pf, $f\in K_0$ is total in PK. [If $(g,Pf)=0$ for all $f\in K_0$ and $g\in PK$, then $(g,f)=(g,Pf)=0$ and hence $g\in K_0^{\perp}\cap PK=\{0\}$.] Assume that $\mathfrak{H}_P^{(k)}\subset\overline{\mathrm{R}_P(K_0)\Omega_P}$ for $k<n$. (This is true for $n=1$.) Then

$$\pi_P[\mathrm{B}(Pf)]\mathfrak{H}_P^{(k)}\subset\pi_P[\mathrm{B}(f)]\mathfrak{H}_P^{(k)}-\pi_P[\mathrm{B}(\{1-P\}f)]\mathfrak{H}_P^{(k)},$$
$$\pi_P[\mathrm{B}(\{1-P\}f)]\mathfrak{H}_P^{(k)}\subset\mathfrak{H}_P^{(k-1)}.$$

Therefore, $\mathrm{B}(Pf)\mathfrak{H}_P^{(k)}$ and hence $\mathfrak{H}_P^{(k+1)}$ are in $\overline{\mathrm{R}_P(K_0)\Omega_P}$.

(1)→(2): Assume that (2) does not hold and $(1-\mathrm{E}(K_0))g=(1-P)g=g\neq0$. Then $\pi_P[\mathrm{B}(g)]$ anticommutes with all $\pi_P[B(f)]$, $f\in K_0$ and hence $\pi_P[B(g)]\mathrm{R}_P(K_0)\Omega_P=0$. Therefore $\pi_P[\mathrm{B}(g)]^*\Omega_P\neq0$ is orthogonal to $\mathrm{R}_P(K_0)\Omega_P$ and (1) is false.

(2)→(3): Immediate from $\Gamma\mathrm{E}(K_0)\Gamma=\mathrm{E}(K_0)$ and $\Gamma(1-P)\Gamma=P$.

To prove the rest, let

$$\mathrm{R}_P^{\mathrm{t}}(K_0)=\{\pi_P^{\mathrm{t}}(\mathrm{B}(f));\ f\in K_0\}''. \tag{4.7}$$

Then

$$\mathrm{R}_P^{\mathrm{t}}((2P-1)K_0^{\perp})\subset\mathrm{R}_P(K_0)'. \tag{4.8}$$

(3)′→(1)′: We have $\mathrm{E}(K_0)=1-\mathrm{E}(K_0^{\perp})$. Hence (3)′ implies that Ω_P is cyclic for $\mathrm{R}_P(K_0^{\perp})$ by (3)→(1) and so for $\mathrm{R}_P^{\mathrm{t}}((2P-1)K_0^{\perp})$. Due to (4.8), this implies (1)′.

(1)′→(2)′: Assume that (2)′ does not hold and $\mathrm{E}(K_0)g=(1-P)g$

$=g\neq 0$. Then $Q\equiv\pi_P(\mathrm{B}(g))$ is in $\mathrm{R}_P(K_0)$ and $Q\Omega_P=0$. Hence $(1)'$ is false.

$(2)'\rightarrow(3)'$: Same as $(2)\rightarrow(3)$. Q.E.D.

Remark 4.9. It is known that an equality holds in (4.8).

Corollary 4.10. *Let*

$$R_S\equiv\pi_{P_S}(\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma))''. \tag{4.9}$$

Then the following conditions are equivalent.

(1) Ω_{P_S} *is cyclic for* R_S.

(2) Ω_{P_S} *is separating for* R_S.

(3) S *does not have an eigenvalue* 1.

(4) S *does not have an eigenvalue* 0.

Proof. Let Q be the projection $1\oplus 0$ acting on $\widehat{K}=K\oplus K$. Then, by Lemma 4.8, (1) is equivalent to $0=(1-Q)\wedge P_S=$ the eigenprojection of $(1-Q)P_S(1-Q)=0\oplus(1-S)$ for an eigenvalue 1 and hence is equivalent to (4). Similarly (2) is equivalent to $0=Q\wedge P_S=$ the eigenprojection of $QP_SQ=S\oplus 0$ for an eigenvalue 1 and hence is equivalent to (3). Since $\Gamma S\Gamma=1-S$, $(3)\Leftrightarrow(4)$. Q.E.D.

If any of the conditions $(1)\sim(4)$ is satisfied, we can identify $\mathfrak{H}_S$ and Ω_S with $\mathfrak{H}_{P_S}$ and Ω_{P_S}. In general, $\mathfrak{H}_S$ is identified with a subspace of $\mathfrak{H}_{P_S}$.

Lemma 4.11. *If* 1/2 *is an eigenvalue of* S *with an even or* 0 *or infinite multiplicity, then* R_S *is a factor.*

Proof. First we consider the case where S does not have an eigenvalue 1/2 (i.e. its multiplicity is 0). We show that $R=\{R_S\cup R_S'\}''$ is irreducible. For this purpose, it is enough to show that Ω_{P_S} is cyclic for R and that there exists a subset $\mathfrak{L}\subset R$ such that $Q\Psi=0$ for all $Q\in\mathfrak{L}$ is equivalent to $\Psi=c\Omega_{P_S}$ for some complex number c.

Vectors $\pi_{P_S}(\prod_{i=1}^{n}\mathrm{B}(f_i\oplus g_i))\Omega_{P_S}$ are total in $\mathfrak{H}_{P_S}$. Since $\pi_{P_S}(\prod_{i=1}^{n}\mathrm{B}(f_i\oplus 0))\in R_S$ and $\pi_{P_S}(\prod_{i=1}^{n}\mathrm{B}(0\oplus g_i))\mathrm{T}_{P_S}(-1)\in R_S'$, Ω_{P_S} is cyclic for R.

We now take the set of all

$$\mathrm{A}(f)\equiv\pi_{P_S}(\mathrm{B}[(1-S)^{1/2}f\oplus 0])-\pi_{P_S}(\mathrm{B}[0\oplus S^{1/2}f])\mathrm{T}_{P_S}(-1)$$

to be $\mathfrak{L}$. The first term is in R_S and the second term is in R_S' by (4.8). $A(f)\Omega_{P_S}=0$. We shall show that $\mathrm{A}(f)\Psi=0$ for all $f\in K$ implies $\Psi=c\Omega_{P_S}$.

Let

$$\Psi_{\pm}=[1\pm\mathrm{T}_{P_S}(-1)]\Psi.$$

Since $\mathrm{T}_{P_S}(-1)$ anticommutes with $\mathrm{A}(f)$, we have $\mathrm{A}(f)\Psi_{\pm}=0$.

On Ψ_+, $\mathrm{T}_{P_S}(-1)=1$ and hence $\mathrm{A}(f)\Psi_+=\pi_{P_S}(\mathrm{B}(f'))\Psi_+$, $f'=(1-S)^{1/2}f\oplus(-S^{1/2}f)$. Obviously $P_Sf'=0$. Since $[(1-S)^{1/2}f]'+\widehat{\Gamma}[\Gamma S^{1/2}f]'=f\oplus 0$ and $-[S^{1/2}f]'+\widehat{\Gamma}[\Gamma(1-S)^{1/2}f]'=0\oplus f$, the set $\{f';f\in K\}$ coincides with $(1-P_S)\widehat{K}$. Therefore $\Psi_+=c\Omega_{P_S}$ by Lemma 4.3.

On Ψ_-, $\mathrm{A}(f)\Psi_-=\pi_{P_S}(\mathrm{B}(f''))\Psi_-$, $f''=(1-S)^{1/2}f\oplus S^{1/2}f$. $\mathrm{A}(f)\Psi_-=0$ for all $f\in K$ implies that the vector state of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(\widehat{K},\widehat{\Gamma})$ induced by $\|\Psi_-\|^{-1}\Psi_-$ is a Fock state for the basis projection $P_S'\equiv 2(S\oplus(1-S))-P_S$ provided that $\Psi_-\neq 0$. Here $(\Psi_-,\Omega_{P_S})=0$ while, from the equation (9.27) and Theorem 6, (Ψ_-,Ω_{P_S}) can vanish only when $P_S'(1-P_S)P_S'$ has an eigenvalue 1. From $P_S'f=f$ and $P_Sf=0$ for $f=f_1\oplus f_2$, we have $(Sf_1)=(1/2)f_1$, $Sf_2=(1/2)f_2$. Hence, if S does not have an eigenvalue 1/2, then $\Psi_-=0$.

We now consider the general case where the eigenvalue 1/2 of S has a nonvanishing multiplicity. We shall reduce it to the previous case by Lemma 5.3. Let $E_{1/2}$ be the eigenprojection of S for an eigenvalue 1/2. By Lemma 3.3 of [2], there exists a subprojection E of $E_{1/2}$ such that $E+\Gamma E\Gamma=E_{1/2}$. Let T be a Hilbert Schmidt class operator such that $0\leqq T\leqq 2^{-1}$ and $(1-E)$ is the eigenprojection of T for an eigenvalue 0. Let

$$\widehat{S}=S-T+\Gamma T\Gamma.$$

Then $\widehat{S}=\widehat{S}^*$, $\Gamma\widehat{S}\Gamma=1-\widehat{S}$, $\widehat{S}$ does not have an eigenvalue 1/2 and

$$\widehat{S}^{1/2}-S^{1/2}=[(1/2-T)^{1/2}-(1/2)^{1/2}]+\Gamma[(1/2+T)^{1/2}-(1/2)^{1/2}]\Gamma.$$

Since $(1/2\pm T)^{1/2}-(1/2)^{1/2}=\pm[(1/2\pm T)^{1/2}+(1/2)^{1/2}]^{-1}T$ is in the

Hilbert Schmidt class, R_S and $R_{\hat{S}}$ are quasiequi valent by Lemma 5. 3. We already know that $R_{\hat{S}}$ is a factor. Therefore R_S is a factor.

Q.E.D.

A full characterization of the case where R_S becomes a factor is given in Theorem 9.

Remark 4. 12. From the beginning part of the preceding proof, it follows that Ω_{P_S} is cyclic for $(R_S \cup R'_S)''$ for any S and hence is separating for the center of R_S.

§5. Proof of Theorem 1

The following is Lemma 4. 5 of [12], for which we give a different proof.

Lemma 5. 1. *$S^{1/2}-(S')^{1/2}$ is in the Hilbert Schmidt class if and only if $P_S-P_{S'}$ is in the Hilbert Schmidt class.*

Proof. Let $\rho=S^{1/2}$, $\rho'=(S')^{1/2}$. If $\rho-\rho'$ is HS (a Hilbert Schmidt class operator), then all of

(5. 1) $$S-S'=\{(\rho-\rho')(\rho+\rho')+(\rho+\rho')(\rho-\rho')\}/2,$$

(5. 2) $$(1-S)^{1/2}-(1-S')=\Gamma(\rho-\rho')\Gamma,$$

(5. 3) $$\rho(1-S)^{1/2}-\rho'(1-S')^{1/2} = (\rho-\rho')(1-S)^{1/2}+\rho'((1-S)^{1/2}-(1-S')^{1/2}),$$

are HS. Hence $P_S-P_{S'}$ is HS.

Conversely, assume $P_S-P'_S$ is HS. Then, by Lemma 5. 2,

(5. 4) $$\| |P_S-Q'|-|P_{S'}-Q'| \|_{\text{H.S.}} \leqq \|P_S-P_{S'}\|_{\text{H.S.}}$$

where $Q'=0\oplus 1$ on $K\oplus K$. Since $|P_S-Q'|^2=S\oplus S$, $|P_{S'}-Q'|^2=S'\oplus S'$, we have $S^{1/2}-(S')^{1/2}$ in the HS class. Q.E.D.

Lemma 5. 2. *Let A and B be bounded selfadjoint operators, then*

(5. 5) $$\|A-B\|_{\text{H.S.}} \geqq \| |A|-|B| \|_{\text{H.S.}}.$$

Proof. (5. 5) is equivalent to

(5. 6) $$\operatorname{tr}\{A^2+B^2-AB-BA\} \geqq \operatorname{tr}\{A^2+B^2-|A||B|-|B||A|\}.$$

First consider the case, where A has purely discrete spectrum. Let Ψ_α be a complete orthonormal set of eigenvectors of A with eigenvalues λ_α. Then

$$\text{tr}\{A^2+B^2-AB-BA\}=\sum_\alpha\{\lambda_\alpha^2+(\Psi_\alpha, B^2\Psi_\alpha)-2\lambda_\alpha(\Psi_\alpha, B\Psi_\alpha)\}, \tag{5.7}$$

$$\begin{aligned}\text{tr}\{A^2+B^2-|A||B|-|B||A|\}\\ =\sum_\alpha\{\lambda_\alpha^2+(\Psi_\alpha, B^2\Psi_\alpha)-2|\lambda_\alpha|(\Psi_\alpha, |B|\psi_\alpha)\}.\end{aligned} \tag{5.8}$$

Since $|B|\geqq B\geqq-|B|$, $(\Psi_\alpha, |B|\Psi_\alpha)\geqq|(\Psi_\alpha, B\Psi_\alpha)|$. Therefore we have (5.6), where $+\infty$ is allowed.

For any selfadjoint operator A and $\varepsilon>0$, there exists a selfadjoint operator A_ε with purely discrete spectrum such that $\|A-A_\varepsilon\|_{\text{H.S.}}<\varepsilon$. Hence [5]. From (5.5), we have $\||A|-|A_\varepsilon|\|_{\text{H.S.}}\leqq\|A-A_\varepsilon\|_{\text{H.S.}}<\varepsilon$.

$$\begin{aligned}\|A-B\|_{\text{H.S.}}&\geqq\|A_\varepsilon-B\|_{\text{H.S.}}-\varepsilon\\ &\geqq\||A_\varepsilon|-|B|\|_{\text{H.S.}}-\varepsilon\\ &\geqq\||A|-|B|\|_{\text{H.S.}}-2\varepsilon.\end{aligned} \tag{5.9}$$

Since ε is arbitrary, we have (5.5) for general A and B. Q.E.D.

Lemma 5.3. *If $S^{1/2}-(S')^{1/2}$ is in the Hilbert Schmidt class, then φ_S and $\varphi_{S'}$ are quasiequivalent. If S and S' satisfy any of conditions* (1)~(4) *of Corollary* 4.10, *in addition, then π_S and $\pi_{S'}$ are unitarily equivalent.*

Proof. If $S^{1/2}-(S')^{1/2}$ is HS, then $P_S-P_{S'}$ is HS. Hence by the first half of Theorem 6 (essentially Lemma 9.4), there exists a vector Ω' in $\mathfrak{H}_{P_S}$ such that the vector state of Ω' on $\overline{\mathfrak{A}}(\widehat{K}, \widehat{\Gamma})$ is $\varphi_{P_S}{}'$, where $\widehat{K}=K\oplus K$, $\widehat{\Gamma}=\Gamma\oplus(-\Gamma)$. Hence φ_S and $\varphi_{S'}$ are given as vector states of $\pi_{P_S}(\overline{\mathfrak{A}}(K, \Gamma))$ by Ω_{P_S} and Ω' which are separating for the center of $\pi_{P_S}(\overline{\mathfrak{A}}(K, \Gamma))''$ due to Remark 4.12. Therefore φ_S and $\varphi_{S'}$ are quasiequivalent. If both S and S' satisfy conditions in Corollary 4.10, then $\mathfrak{H}_S$ and $\mathfrak{H}_{S'}$ can be both identified with $\mathfrak{H}_{P_S}$ and hence π_S and $\pi_{S'}$ are unitarily equivalent.

Lemma 5.4. *Let A_n be a sequence of bounded linear operators on a Hilbert space with a strong limit A. Then*

(5.10) $$\|A\|_{\text{H.S.}} \leqq \underline{\lim}\|A_n\|_{\text{H.S.}}.$$

(Here $\|C\|_{\text{H.S.}}=\{\sum\|C\psi_i\|^2\}^{1/2}$ for a complete orthonormal basis $\{\psi_i\}$ and we allow $+\infty$. It is independent of the basis.)

Proof. We have

$$\underline{\lim}\|A_n\|^2_{\text{H.S.}} \geqq \underline{\lim}\sum_{i=1}^{N}\|A_n\psi_i\|^2=\sum_{i=1}^{N}\|A\psi_i\|^2.$$

Since N is arbitrary, we obtain (5.10). Q.E.D.

Lemma 5.5. *If S and S' satisfy any of conditions* (1)~(4) *of Corollary* 4.10, *and if $P_S-P_{S'}$ is not in the Hilbert Schmidt class, then π_S and $\pi_{S'}$ are not quasiequivalent.*

Proof. Let Q_n be an increasing sequence of finite even dimensional projections commuting with Γ and tending to 1 on K. From Lemma 5.4, we have

$$\lim_{n\to\infty}\|(Q_nSQ_n)^{1/2}-(Q_nS'Q_n)^{1/2}\|_{\text{H.S.}}=\infty.$$

From (5.4), we have

$$\lim_{n\to\infty}\|P_{Q_nSQ_n}-P_{Q_nS'Q_n}\|_{\text{H.S.}}=\infty.$$

From Lemma 6.6, we obtain

$$\lim_{n\to\infty}\|(\varphi_S-\varphi_{S'})\big|\mathfrak{A}_{\text{SDC}}(Q_nK,\Gamma)\|=2.$$

Therefore, we have

(5.11) $$\|\varphi_S-\varphi_{S'}\|=2.$$

Since S and S' both satisfy the condition of Corollary 4.10, the representations π_S and $\pi_{S'}$ have cyclic and separating vectors Ω_S and $\Omega_{S'}$. If π_S and $\pi_{S'}$ are quasiequivalent, then they are unitarily equivalent. Therefore there exists a separating vector Ω' in $\mathfrak{H}_S$ such that $(\Omega',\pi_S(A)\Omega')=\varphi_{S'}(A)$. Since Ω' is cyclic for the commutant, there exists a unitary operator W in $\pi_S(\mathfrak{A}_{\text{SDC}}(K,\Gamma))'$ such that $(W\Omega',\Omega_S)\neq 0$. Then the vector state for $\Omega''=W\Omega'$ is again $\varphi_{S'}$ and we have

(5.12) $$\begin{aligned}&\|\varphi_S-\varphi_{S'}\|\\ &\leqq \text{tr}\,|P(\Omega_S)-P(\Omega'')|=2\{1-|(\Omega_S,\Omega'')|^2\}^{1/2}\\ &<2,\end{aligned}$$

where $P(\Psi)$ denote the projection operator on the one dimensional space spanned by Ψ. The contradiction of (5.11) and (5.12) proves the Lemma. Q.E.D.

Proof of Theorem 1. If $S^{1/2}-(S')^{1/2}$ is in the Hilbert Schmidt class, then φ_S and $\varphi_{S'}$ are quasiequivalent by Lemma 5.3.

Now assume that $S^{1/2}-(S')^{1/2}$ is not in the Hilbert Schmidt class. Let E_1 and E_1' be eigenprojections of S and S' for an eigenvalue 1. Let T and T' be Hilbert Schmidt class operators such that $0\leqq T<1$, $0\leqq T'<1$ and the eigenprojection of T and T' for an eigenvalue 0 are $1-E_1$ and $1-E_1'$. Let

$$\hat{S}=S-T^2+\Gamma T^2\Gamma$$
$$\hat{S}'=S'-(T')^2+\Gamma(T')^2\Gamma.$$

Then $\hat{S}$ and $\hat{S}'$ have the properties (3.4) and (3.5) and satisfy the condition (3) of Corollary 4.10. Further,

$$\hat{S}^{1/2}-S^{1/2}=\Gamma T\Gamma-[(1-T^2)^{1/2}+1]^{-1}T^2$$
$$(\hat{S}')^{1/2}-(S')^{1/2}=\Gamma T'\Gamma-[(1-(T')^2)^{1/2}+1]^{-1}(T')^2$$

are both in the Hilbert Schmidt class. This implies by Lemma 5.3 that $\varphi_{\hat{S}}$ is quasiequivalent to φ_S and $\varphi_{\hat{S}'}$ is quasiequivalent to $\varphi_{S'}$. It also implies that $(\hat{S}')^{1/2}-(\hat{S})^{1/2}$ is not in the Hilbert Schmidt class.

We can now apply Lemma 5.5 and conclude that $\varphi_{\hat{S}'}$ is not quasiequivalent to $\varphi_{\hat{S}}$ and hence that $\varphi_{S'}$ is not quasiequivalent to φ_S.

Q.E.D.

In the present section, we have assumed Lemma 9.4 and Lemma 6.6. We shall prove Lemma 9.4 in the course of our discussion on the unitary implementability of Bogoliubov transformations, although a more direct and hence shorter proof of this Lemma is also possible. We shall prove Lemma 6.6 by using a known structure of *KMS* states.

§6. Uniqueness Theorems

Let $\tau(\lambda)$ be a continuous one parameter group of automorphisms of a C^*-algebra $\mathfrak{A}$. A state φ of $\mathfrak{A}$ is said to be a *state of finite* $\tau(\lambda)$-

energy if there exists a such that

(6.1) $$\int\varphi(B\tau(\lambda)A)f(\lambda)\mathrm{d}\lambda=0, \qquad A, B\in\mathfrak{A}$$

whenever $f\in\mathcal{S}$ and

(6.2) $$\tilde{f}(p)=\int f(\lambda)e^{i\lambda p}\mathrm{d}\lambda=0$$

for $p\geq a$. When a can be chosen to be 0, φ is called *$\tau(\lambda)$-vacuum.*

A state φ is called a *KMS state of $\tau(\lambda)$ with inverse temperature* β, if

(6.3) $$\int\varphi(B\tau(\lambda)A)f(\lambda)\mathrm{d}\lambda=\int\varphi((\tau(\lambda)A)B)f(\lambda+i\beta)\mathrm{d}\lambda$$

for $A, B\in\mathfrak{A}$ and $\tilde{f}\in\mathcal{D}$ such that

(6.4) $$f(\lambda)=\frac{1}{2\pi}\int e^{-i\lambda p}\tilde{f}(p)\mathrm{d}p.$$

(6.3) is referred to as the *KMS* condition.

Theorem 2. *Let* $\mathrm{U}(\lambda)$ *be a continuous one parameter group of Bogoliubov transformations. Let* $\mathrm{E}(p)$ *be the spectral projections*:

(6.5) $$\mathrm{U}(\lambda)=\int e^{i\lambda p}\mathrm{E}(\mathrm{d}p)=e^{i\lambda H},$$

$$H=\int p\mathrm{E}(\mathrm{d}p).$$

Let $E_+=\mathrm{E}((0,\infty))$, $E_0=\mathrm{E}(\{0\})$. *Then φ is a $\tau(\lambda)$-vacuum if and only if*

(6.6) $$\varphi(AB)=\varphi_{E_+}(A)\varphi'(B),$$

$$A\in\overline{\mathfrak{A}}_{\mathrm{SDC}}((1-E_0)K,\Gamma), \quad B\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(E_0K,\Gamma),$$

where φ_{E_+} is a Fock state and φ' is an arbitrary state on $\overline{\mathfrak{A}}_{\mathrm{SDC}}(E_0K,\Gamma)$.

Proof. Since $\mathrm{U}(\lambda)$ is a Bogoliubov transformation, $\Gamma E_0\Gamma=E_0$ and $\Gamma E_+\Gamma=1-E_0-E_+$. Namely E_+ is a basis projection for $(1-E_0)K$. Let φ_1 be the restriction of φ to $\overline{\mathfrak{A}}_{\mathrm{SDC}}((1-E_0)K,\Gamma)\equiv\mathfrak{A}$.

Next we have

$$\int\tau(\mathrm{U}(\lambda))\mathrm{B}(g)f(\lambda)\mathrm{d}\lambda=\mathrm{B}(\tilde{f}(H)g).$$

If $\tilde{f}$ runs over all $\tilde{f}\in S$ such that $\tilde{f}(p)=0$ for $p\geq 0$, then the set of $\tilde{f}(H)g$, $g\in K$ is a dense subset of $E_{-}K$. Hence (6.1) requires

$$\varphi_1(\mathrm{B}(f)^*\mathrm{B}(f))=0$$

for all $f\in E_{-}K$. By Lemma 4.3, this implies $\varphi_1=\varphi_{E_+}$.

Let π_φ be the representation of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ and Ω_φ be a cyclic vector associated with φ. If $h_j\in E_0K$, $\|h_j\|^2=2$ and $\Gamma h_j=h_j$, $j=1,\cdots$, then the vector states of $\mathfrak{A}$ by $\Psi=\pi_\varphi(\prod_{j=1}^{n}\mathrm{B}(h_j))\Omega_\varphi$ are the same Fock states φ_{E_+}. Since the union of $\pi_\varphi(\mathfrak{A})\Psi$ for all such Ψ is total in $\mathfrak{H}_\varphi$, $\pi_\varphi|\mathfrak{A}$ is quasiequivalent to the Fock representation π_{E_+}. Hence, by Lemma 4.2 for $U=-1$ and $S=E_+$ and by the irreducibility of π_{E_+}, there exists $T\in\pi_\varphi(\mathfrak{A})''$ (corresponding to $\mathrm{T}_{E_+}(-1)$) such that $T\Omega_\varphi=\Omega_\varphi$, $T^*=T$, $T^2=1$ and $T\pi_\varphi(A)T^*=\pi_\varphi(\tau(-1)A)$ for $A\in\mathfrak{A}$. Let $\pi'_\varphi(\mathrm{B}(h))=\pi_\varphi(\mathrm{B}(h))T$ for $h\in E_0K$. We have $\pi'_\varphi(\mathrm{B}(h))\in\pi_\varphi(\mathfrak{A})'$. Hence $\pi_\varphi(\mathrm{B}(h))=\pi'_\varphi(\mathrm{B}(h))T$ commutes with T. Therefore $\pi'_\varphi(\mathrm{B}(h))$ generates a representation of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(E_0K,\Gamma)$, which we denote by π'_φ. More explicitly, $\pi'_\varphi(C)=\pi_\varphi(C)(1+T)/2+\pi_\varphi(\tau(-1)C)(1-T)/2$. Let φ_2 be the restriction of φ to $\overline{\mathfrak{A}}_{\mathrm{SDC}}(E_0K,\Gamma)$. Since $T\Omega_\varphi=\Omega_\varphi$, φ_2 is the vector state given by $\varphi_2(C)=(\Omega_\varphi,\pi'_\varphi(C)\Omega_\varphi)$. Since Ω_φ gives rise to a pure state of $\mathfrak{A}$, we have $\varphi(AC)=(\Omega_\varphi,\pi_\varphi(A)\pi'_\varphi(C)\Omega_\varphi)=\varphi_{E_+}(A)\varphi_2(C)$ for $A\in\mathfrak{A}$ and $C\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(E_0K,\Gamma)$.

Conversely, if φ_2 is a state on $\overline{\mathfrak{A}}_{\mathrm{SDC}}(E_0K,\Gamma)$ and $\mathfrak{H}_2$, Ω_2, π_2 is canonically associated with it, then

$$\begin{aligned}\pi(AB)=&\pi_{E_+}(A)\otimes(\pi_2(B)+\pi_2(\tau(-1)B))/2\\&+\pi_{E_+}(A)\mathrm{T}_{E_+}(-1)\otimes(\pi_2(B)-\pi_2(\tau(-1)B))/2\end{aligned}$$

on $\mathfrak{H}_{E_+}\otimes H_2$ uniquely extends to a representation of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ and $\Omega=\Omega_{E_+}\otimes\Omega_2$ satisfies $(\Omega,\pi(AB)\Omega)=\varphi_{E_+}(A)\varphi_2(B)$. Further, $\tau(\mathrm{U}(\lambda))$ leaves the vector state by Ω invariant, and is unitarily implementable by an operator $T_{E_+}(\mathrm{U}(\lambda))\otimes 1$, whose generator is known to be positive semidefinite for $HE_+\geq 0$. Hence (6.1) is satisfied. Q.E.D.

Lemma 6.1. *If the dimension of K is finite and even or infinite, $\varphi_{1/2}$ is the unique state of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ satisfying*

(6.7) $\varphi_{1/2}(AB)=\varphi_{1/2}(BA)$.

Proof. $\varphi_{1/2}$ satisfies (6.7) due to (3.1), (3.2) and

(6.8) $$\varphi_{1/2}(B(f)^* B(g)) = \varphi_{1/2}(B(g), B(f)^*) = (f, g)/2.$$

Let $\{f_\alpha\}$ be a Γ invariant orthonormal basis of K. (Such basis exists). Any element in $\mathfrak{A}_{SDC}(K, \Gamma)$ is a polynomial of $B(f_\alpha)$. Since $B(f_\alpha)^2 = 1/2$ and $B(f_\alpha)$ anticommutes with other $B(f_\beta)$, it is enough to deduce the value $\varphi(B(f_{\alpha_1})\cdots B(f_{\alpha_n}))$ uniquely from (6.7) when $\alpha_1 \cdots \alpha_n$ are distinct. If $n \neq 0$ is even, then $B(f_{\alpha_1})\cdots B(f_{\alpha_n}) = -B(f_{\alpha_n})B(f_{\alpha_1})\cdots$ implies that $\varphi(\prod_k B(f_{\alpha_k})) = 0$. If n is odd and if there is β distinct from all α_k, then

$$\begin{aligned} B(f_{\alpha_1})\cdots B(f_{\alpha_n}) &= 2B(f_{\alpha_1})\cdots B(f_{\alpha_n})B(f_\beta)^2 \\ &= -2B(f_\beta)B(f_{\alpha_1})\cdots B(f_{\alpha_n})B(f_\beta) \end{aligned}$$

implies again that $\varphi(\prod_k B(f_{\alpha_k})) = 0$. If $\dim K$ is even or infinite, this shows the uniqueness. Q.E.D.

$\varphi_{1/2}$ is called the *central state*. Existence of such $\varphi_{1/2}$ follows from Lemma 3.3. If $\dim K = 2n$, $\varphi_{1/2}$ is the trace of a full matrix algebra divided by 2^n.

Corollary 6.2. *For any $*$ automorphism τ of $\mathfrak{A}_{SDC}(K, \Gamma)$, $\varphi_{1/2}$ is invariant and there exists a unitary operator $T_{1/2}(\tau)$ on $H_{1/2}$ such that*

$$T_{1/2}(\tau)\Omega_{1/2} = \Omega_{1/2}, \quad T_{1/2}(\tau)\pi_{1/2}(A)T_{1/2}(\tau)^* = \pi_{1/2}(\tau A),$$
$$T_{1/2}(\tau_1)T_{1/2}(\tau_2) = T_{1/2}(\tau_1\tau_2).$$

Theorem 3. *Let $U(\lambda)$ be as in the previous theorem. Then a KMS state of $\tau(U(\lambda))$ with inverse temperature β is unique and is given by a quasifree state φ_S with*

(6.9) $$S = (1 + e^{-\beta H})^{-1},$$

provided that R_S is a factor.

Proof. It is known that any *KMS* state has a central decomposition as an integral over primary *KMS* states. Hence it is enough to prove the uniqueness of primary *KMS* state.

Let φ_1 be a primary *KMS* state and $\varphi(A) = (\varphi_1(A) + \varphi_1(\tau(-1)A))/2$. Then φ is again a *KMS* state and has the property that $\varphi(Q) = 0$ for any odd polynomial Q of $B(f)$.

Let $\mathfrak{H}_\varphi$, π_φ, Ω_φ be canonically associated with φ, $R=\pi_\varphi(\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma))''$. Since $\varphi(\tau(-1)A)=\varphi(A)$ by construction, there exists a unitary operator $\mathrm{T}_\varphi(-1)$ such that $\mathrm{T}_\varphi(-1)\pi_\varphi(A)\Omega_\varphi=\pi_\varphi(\tau(-1)A)\Omega_\varphi$.

A $\tau(\lambda)$ *KMS* state is known to be $\tau(\lambda)$ invariant. Let $\mathrm{T}_\varphi(\mathrm{U}(\lambda))$ be the unitary operator determined by $\mathrm{T}_\varphi(\mathrm{U}(\lambda))\pi_\varphi(A)\Omega_\varphi=\pi_\varphi(\tau(\mathrm{U}(\lambda))A)\Omega_\varphi$. Let $\mathrm{T}_\varphi(\mathrm{U}(\lambda))=e^{i\lambda\theta}$, $\Delta=e^{-\beta\theta/2}$.

Since $\tau(-1)$ commutes with $\tau(\mathrm{U}(\lambda))$, $\mathrm{T}_\varphi(-1)$ commutes with $\mathrm{T}_\varphi(\mathrm{U}(\lambda))$ and Δ. Ω_φ is cyclic for R by construction.

The *KMS* condition implies that Ω_φ is separating. Further, there exists an antiunitary involution J (a complex conjugation) on $\mathfrak{H}_\varphi$ such that

$$J\Omega_\varphi=\Omega_\varphi,\ JRJ=R',\ [J,e^{i\lambda\theta}]=0, \tag{6.10}$$

$$JA\Omega_\varphi=\Delta A^*\Omega_\varphi,\qquad A\in\mathfrak{A}, \tag{6.11}$$

where $\mathfrak{A}$ is a dense $*$ subalgebra of R consisting of all $\int\pi_\varphi(\tau(\mathrm{U}(\lambda)))\cdot Af(\lambda)\mathrm{d}\lambda$ with $A\in\pi_\varphi(\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma))$ and $f(\lambda)=(2\pi)^{-1}\int\tilde{f}(p)\exp-ip\lambda\mathrm{d}p$, $\tilde{f}\in\mathscr{D}$. From the commutativity of Δ and $T_\varphi(-1)$ we have

$$\begin{aligned}&T_\varphi(-1)^*JT_\varphi(-1)A\Omega_\varphi\\&\quad=T_\varphi(-1)^*\Delta(T_\varphi(-1)AT_\varphi(-1))^*\Omega_\varphi=\Delta A^*\Omega_\varphi=JA\Omega_\varphi.\end{aligned}$$

Hence J commutes with $T_\varphi(-1)$.

Let

$$\pi'_\varphi(A)=J\pi_\varphi(\tau(\Gamma)A)J. \tag{6.12}$$

It is another representation of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ such that the closure of $\pi'_\varphi(\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma))$ is R'.

Let $\hat{K}=K\oplus K$, $\hat{\Gamma}=\Gamma\oplus(-\Gamma)$ and consider the representation $\hat{\pi}$ of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(\hat{K},\hat{\Gamma})$ generated by

$$\hat{\pi}(\mathrm{B}(f\oplus g))=\pi_\varphi(\mathrm{B}(f))+\pi'_\varphi(\mathrm{B}(g))\mathrm{T}_\varphi(-1). \tag{6.13}$$

It is easily verified that $\hat{\pi}(\mathrm{B}(h))$ satisfies the relations (1), (2), (3) for selfdual CAR algebra and hence determines a representation of $\mathfrak{A}_{\mathrm{SDC}}(\hat{K},\hat{\Gamma})$.

Let $\mathrm{E}(\cdot)$ be the spectral projection of $\mathrm{U}(\lambda)$ in (6.5) and $g\in K$ be such that $\|\mathrm{E}(\mathrm{d}p)g\|^2$ has a compact support. Then $\pi_\varphi(\mathrm{B}(g))\in\mathfrak{A}$.

By an analytic continuation of $\mathrm{T}_{\varphi}(\mathrm{U}(\lambda))\pi_{\varphi}(\mathrm{B}(g))\Omega_{\varphi}=\pi_{\varphi}(\mathrm{B}[\mathrm{U}(\lambda)g])\Omega_{\varphi}$, we obtain

(6.14) $$\Delta\pi_{\varphi}(\mathrm{B}(g))\Omega_{\varphi}=\pi_{\varphi}(\mathrm{B}(e^{-\beta H/2}g))\Omega_{\varphi}.$$

Hence,

(6.15) $$\hat{\pi}(\mathrm{B}(f\oplus g))\Omega_{\varphi}=\pi_{\varphi}(B(f+e^{-\beta H/2}g))\Omega_{\varphi}.$$

Let P be the projection on the subspace of $\hat{K}$ spanned by elements of the form

(6.16) $$\mathrm{h}_1(f)=e^{-\beta H_-/2}f\oplus e^{-\beta H_+/2}f, \qquad f\in K,$$

where $H_+=HE_+$ and $H_-=H_+-H$. Then $\hat{\Gamma}P\hat{K}$ is spanned by

(6.17) $$\mathrm{h}_2(f)=e^{-\beta H_+/2}f\oplus(-e^{-\beta H_-/2}f)$$

which is orthogonal to (6.16). Further,

(6.18) $$\begin{aligned}&\mathrm{h}_1(e^{-\beta H_-/2}f)+\mathrm{h}_2(e^{-\beta H_+/2}f)\\ &=(e^{-\beta H_-}+e^{-\beta H_+})f\oplus 0.\end{aligned}$$

Since $e^{-\beta H_\pm}$ are mutually commuting positive selfadjoint operators, their sum has a dense range and hence (6.18) is dense in $K\oplus 0$. Similarly, $\mathrm{h}_1(e^{-\beta H_+/2}f)-\mathrm{h}_2(e^{-\beta H_-/2})$ is dense in $0\oplus K$. Hence the sum $\mathrm{h}_1(f_1)+\mathrm{h}_2(f_2)$ is dense in $\hat{K}$ and we have $\hat{\Gamma}P\hat{\Gamma}=1-P$. Therefore, P is a basis projection.

(6.15) shows that

(6.15) $$\hat{\pi}(\mathrm{B}(h))\Omega_{\varphi}=0$$

for $h=\mathrm{h}_2(f)$ in a dense subset of $(1-P)\hat{K}$ and hence for all h in $(1-P)\hat{K}$. Hence the vector state of $\mathfrak{A}_{\mathrm{SDC}}(\hat{K},\hat{\Gamma})$ given by Ω_{φ} is unique (a Fock state $\hat{\varphi}_P$) by Lemma 4.3. Then its restriction to $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ is also unique.

Since

(6.20) $$P=\begin{pmatrix}(1+e^{-\beta H})^{-1} & (1+e^{-\beta H})^{-1}e^{-\beta H/2}\\ (1+e^{-\beta H})^{-1}e^{-\beta H/2} & (1+e^{-\beta H})^{-1}e^{-\beta H}\end{pmatrix}.$$

We have

(6.21) $$\varphi=\varphi_S, \qquad S=(1+e^{-\beta H})^{-1}$$

Since $R=R_S$ is a factor by assumption and since a primary KMS state is an extremal KMS state, we have $\varphi_1=\varphi=\varphi_S$. Therefore the uniqueness is proved.

It remains to show that φ_S given by (6.21) is actually a KMS state. The KMS condition amounts to $((\Delta A^*\Delta^{-1})\Omega_\varphi, (\Delta B^*\Delta^{-1})\Omega_\varphi) = (B\Omega_\varphi, A\Omega_\varphi)$, for A, B in $\mathfrak{A}$. Hence we only have to prove the antiunitarity of J defined by (6.11).

Let ε be the Bogoliubov transformation on $(\hat{K}=K\oplus K, \hat{\Gamma}=\Gamma\oplus-\Gamma)$ given by the matrix $\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$. Then $\varepsilon P\varepsilon=1-P$ and hence the continuous extension of $J_0\hat{\pi}(C)\Omega_\varphi=\hat{\pi}(\tau(\hat{\Gamma})\tau(\varepsilon)C)\Omega_\varphi$, $C\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(\hat{K},\hat{\Gamma})$ defines obviously an antiunitary operator J_0. If we restrict C to $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ $(K\oplus 0\subset\hat{K})$, we have

$$(6.22)\qquad \begin{aligned} &J_0\pi_\varphi(\mathrm{B}(f_1))\cdots\pi_\varphi(\mathrm{B}(f_n))\Omega_\varphi \\ &=(-1)^{n(n-1)/2}(-i)^n\pi'_\varphi(\mathrm{B}(\Gamma f_1))\cdots\pi'_\varphi(\mathrm{B}(\Gamma f_n))\Omega_\varphi \\ &=(-i)^{n^2}(\Delta\pi_\varphi(\mathrm{B}(f_n))^*\Delta^{-1})\cdots(\Delta\pi_\varphi(\mathrm{B}(f_1))^*\Delta^{-1})\Omega_\varphi, \end{aligned}$$

where f_i is any element in K such that $\|\mathrm{E}(d\lambda)f_i\|$ has a compact support. Hence

$$(6.23)\qquad J=\alpha J_0,$$

where α is a function of $\mathrm{T}_\varphi(-1)$, being $=1$ if $\mathrm{T}_\varphi(-1)=1$ and $=i$ if $\mathrm{T}_\varphi(-1)=-1$. Since α is unitary, J is antiunitary. Q.E.D.

Corollary 6.3. *Assume that* $\dim K$ *is not odd. If* $S-1/2$ *is of finite rank and* S *does not have an eigenvalue* 1, *we have for* $A\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$

$$(6.24)\qquad \begin{aligned} \varphi_S(A) &= \varphi_{1/2}(e^{-(\mathrm{B},H\mathrm{B})/2}A)/\varphi_{1/2}(e^{-(\mathrm{B},H\mathrm{B})/2}) \\ &=(\det 2S)^{1/2}\varphi_{1/2}(e^{-(\mathrm{B},H\mathrm{B})/2}A) \end{aligned}$$

where $(\mathrm{B}, H\mathrm{B})$ *is defined in Lemma* 7.3 *and* $H=\log S(1-S)^{-1}$

Proof. The left hand side is a unique KMS state for $\mathrm{U}(\lambda)=e^{iH\lambda}$ with $\beta=1$. Since $S-1/2$ is of finite rank and $\Gamma S\Gamma=1-S$, an eigenvalue $1/2$ of S has an infinite or even multiplicity (according as K has an infinite or finite even dimension). Hence we have only to show that the right hand side is a *KMS* state. Since $S(1-S)^{-1}-1$ is of finite rank due to the assumption on S, H is of finite rank. It is hermitian and satisfies $\Gamma H\Gamma=-H$. Hence, there exists $Q\equiv(\mathrm{B}, H|\mathrm{B}|)/2\in\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$. We then have from (6.7)

$$\varphi_{1/2}(e^{-Q}AB)=\varphi_{1/2}(e^{-Q}B'A), \qquad B'=e^{Q}Be^{-Q}.$$

Since $e^{Q}\mathrm{B}(f)e^{-Q}=\mathrm{B}(e^{H}f)$, B' is an analytic continuation of $\tau(\mathrm{U}(\lambda))B$ to $\lambda=-i$. Hence the right hand side of (6.24) is a *KMS* state for $U(\lambda)$ with $\beta=1$.

The normalization factor is computed by (8.25). Q.E.D.

Lemma 6.4. *Let* $\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2$ *and* $R=\mathscr{B}(\mathfrak{H}_1)\otimes 1$. *Let* Ω *be a unit vector, cyclic and separating for* R, *and* J *be an antiunitary involution satisfying* $JRJ=R'$ *and* $J\Omega=\Omega$. *Assume that* $(\Omega, A\mathrm{j}(A)\Omega)\geqq 0$ *for all* $A\in R$ *where* $\mathrm{j}(A)=JAJ$. *Then there exists a standard diagonal expansion* ([3], *Definition* 2.2)

$$\Omega=\sum\lambda_i\Phi_{1i}\otimes\Phi_{2i} \tag{6.25}$$

such that $\lambda_i>0$ *and*

$$J(\Phi_{1i}\otimes\Phi_{2j})=(\Phi_{1j}\otimes\Phi_{2i}). \tag{6.26}$$

Proof. Let

$$\Omega=\sum\lambda_i'\Psi_{1i}\otimes\Psi_{2i}, \qquad \lambda_i'>0 \tag{6.27}$$

be a standard diagonal expansion of Ω and let J_0 be an antiunitary involution defined by

$$J_0\sum c_{ij}\Psi_{1i}\otimes\Psi_{2j}=\sum c_{ij}^*\Psi_{1j}\otimes\Psi_{2i}. \tag{6.28}$$

Let $W=J_0J$.

Then W is unitary and satisfies $W\Omega=\Omega$, $WRW^*=R$. Hence there exists a unitary U_1 in $\mathscr{B}(\mathfrak{H}_1)$ such that $WAW^*=(U_1\otimes 1)A(U_1^*\otimes 1)$ for all $A\in R$. Since $(U_1^*\otimes 1)W$ is in R', it can be written as $1\otimes U_2$. Then $W=U_1\otimes U_2$.

Let ρ_1 and ρ_2 be the unique trace class operators on $\mathfrak{H}_1$ and $\mathfrak{H}_2$ satisfying

$$\mathrm{tr}\,\rho_1A_1=(\Omega, (A_1\otimes 1)\Omega), \tag{6.29a}$$

(6. 29b) $$\mathrm{tr}\,\rho_2 A_2 = (\Omega, (1\otimes A_2)\Omega).$$

From $W\Omega=\Omega$ and $W=U_1\otimes U_2$, we have $[\rho_\nu, U_\nu]=0$, $\nu=1, 2$.

Let $\rho_\nu=\sum x\mathrm{P}_\nu(x)$ be the spectral resolution of ρ_ν. Then $\mathrm{P}_\nu(x) =\sum \mathrm{P}(\Psi_{\nu i})$ where $\mathrm{P}(\Psi_{\nu i})$ denotes the minimal projection of $\mathcal{B}(\mathfrak{H}_\nu)$ corresponding to $\Psi_{\nu i}$ and the sum extends over those i such that $(\lambda_i')^2=x$.

Let

(6. 30) $$\Phi_{1k}=\sum_i u_{ki}\Psi_{1i}$$

be a complete orthonormal set of eigenvectors of U_1 belonging to eigenvalues $e^{i\theta_k}$. Since $[\mathrm{P}_1(x), U_1]=0$ and each $\mathrm{P}_1(x)$ has a finite dimension, U_1 has a purely discrete spectrum and we can chose u_{k1} such that $u_{ki}u_{kj}\neq 0$ only if $\lambda_i'=\lambda_j'$.

Let

(6. 31) $$\Phi_{2k}=\sum_i (u_{ki})^*\Psi_{2i}\,.$$

Since (u_{ki}) is unitary, we have (6. 25) where $\lambda_k=\lambda_i'$ if $u_{ki}\neq 0$.

From $W\Omega=\Omega$ and (6. 25), we have

(6. 32) $$U_2\Phi_{2k}=e^{-i\theta_k}\Phi_{2k}\,.$$

Since $J=J_0W$, we have

(6. 33) $$J(\Phi_{1k}\otimes\Phi_{2l})=\varepsilon_{lk}(\Phi_{1l}\otimes\Phi_{2k}),$$

where $\varepsilon_{kl}=e^{i(\theta_k-\theta_l)}$. Since $J^2=1$, we have $(\varepsilon_{kl})^2=1$. Therefore $\varepsilon_{kl}=\varepsilon_{lk} =\pm 1$.

Let $A_{kl}\in\mathcal{B}(\mathfrak{H}_1)$ be defined by $A_{kl}\sum c_j\Phi_{1j}=c_l\Phi_{1k}+c_k\Phi_{1l}$. From $(\Omega, A\mathrm{j}(A)\Omega)\geqq 0$ with $A=A_{kl}\otimes 1$, we have $A\Omega=\lambda_k\Phi_{1l}\otimes\Phi_{2k}+\lambda_l\Phi_{1k}\otimes\Phi_{2l}$, $JAJ\Omega=JA\Omega=\varepsilon_{lk}(\lambda_k\Phi_{1k}\otimes\Phi_{2l}+\lambda_l\Phi_{1l}\otimes\Phi_{2k})$, and hence $2\lambda_k\lambda_l\varepsilon_{lk}\geqq 0$. From this we have $\varepsilon_{lk}\geqq 0$ and hence (6. 26) holds.

Lemma 6. 5. *Let R be a type I factor, Ω and Ω' be cyclic and separating unit vectors and J be an antiunitary involution such that $JRJ=R'$, $J\Omega=\Omega$, $J\Omega'=\Omega'$, $(\Omega, A\mathrm{j}(A)\Omega)\geqq 0$ and $(\Omega', A\mathrm{j}(A)\Omega')\geqq 0$ for all $A\in R$ where $\mathrm{j}(A)=JAJ$. Let φ and φ' be the vector states of R given by Ω and Ω'. Then*

(6. 34) $$\|\varphi-\varphi'\|\geqq 2(1-|(\Omega, \Omega')|).$$

Proof. Since R is a type I factor, we can identify the Hilbert space and R as follows:

$$\mathfrak{H}=\mathfrak{H}_1\otimes\mathfrak{H}_2,$$
$$R=\mathscr{B}(\mathfrak{H}_1)\otimes 1.$$

Let

$$\Omega=\sum\lambda_i\Phi_{1i}\otimes\Phi_{2i},$$
$$\Omega'=\sum\lambda_i'\Phi_{1i}'\otimes\Phi_{2i}',$$

be the standard diagonal expansions of Ω and Ω' given by the previous Lemma.

From (6.26) and antiunitarity of J, we have

$$(\Phi_{1i},\Phi_{1k}')^*(\Phi_{2j},\Phi_{2l}')^*=(\Phi_{1j},\Phi_{1l}')(\Phi_{2i},\Phi_{2k}'). \tag{6.35}$$

Since the matrices $u_{ij}=(\Phi_{1i},\Phi_{1j}')$ and $v_{ij}=(\Phi_{2i},\Phi_{2j})$ are unitary, there exists $u_{ik}\neq 0$. Setting $\varepsilon=v_{ik}/u_{ik}^*$, we have $v_{jl}^*=\varepsilon u_{jl}$, where ε is common for all j, l. From the unitarity, we have $|\varepsilon|=1$. From (6.35), we have $\varepsilon=\varepsilon^*$. Hence $\varepsilon=\pm 1$.

We now have

$$(\Omega,\Omega')=\sum\lambda_i\lambda_j'(\Phi_{1i},\Phi_{1j}')(\Phi_{2i},\Phi_{2j}') \tag{6.36}$$
$$=\varepsilon\sum\lambda_i\lambda_j'|(\Phi_{1i},\Phi_{1j}')|^2.$$

Let

$$\rho=\sum\lambda_i^2\mathrm{P}(\Phi_{1i}), \tag{6.37}$$
$$\rho'=\sum(\lambda_i')^2\mathrm{P}(\Phi_{1i}'). \tag{3.38}$$

Then $\varphi(A)=\mathrm{tr}\,\rho A$ and $\varphi'(A)=\mathrm{tr}(\rho' A)$. We now have

$$\|\varphi-\varphi'\|=\sup_{\|A\|\leqq 1}|\varphi(A)-\varphi'(A)|=\mathrm{tr}|\rho-\rho'| \tag{6.39}$$
$$\geqq\mathrm{tr}(\rho^{1/2}-(\rho')^{1/2})^2=2(1-\mathrm{tr}\,\rho^{1/2}(\rho')^{1/2})$$

where the inequality is due to Lemma 4.1 of [12]. From (6.36), we have

$$\mathrm{tr}\rho^{1/2}(\rho')^{1/2}=|(\Omega,\Omega')|. \tag{6.40}$$

From (6.39) and (6.40), we have (6.34). Q.E.D.

Lemma 6.6. *Assume that* $\dim K$ *is finite and even. Let* φ_S *and*

$\varphi_{S'}$ *be two quasifree states of* $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$. *Then*

$$\|\varphi_S-\varphi_{S'}\|\geqq 2[1-\det(1-(P_S-P_{S'})^2)^{1/8}]. \tag{6.41}$$

Proof. Let $\hat{K}=K\oplus K$, $\hat{\Gamma}=\Gamma\oplus-\Gamma$.

First consider the case where S and S' do not have an eigenvalue 1. In this case we can show

$$P_S\wedge(1-P_{S'})=0, \tag{6.42}$$

as follows:

Let $g=g_1\oplus g_2$, $P_S g=g$, $P_{S'}g=0$. Then $S^{1/2}g_2=(1-S)^{1/2}g_1$ and $(S')^{1/2}g_1=-(1-S')^{1/2}g_2$. Since S and S' do not have an eigenvalue 1, the same holds for $1-S=\Gamma S\Gamma$ and for $1-S'=\Gamma S'\Gamma$. Therefore S, S', $(1-S)$ and $(1-S')$ have their inverses. We have $\{S^{-1/2}(1-S)^{1/2}+(1-S')^{-1/2}(S')^{1/2}\}g_1=0$. From $S^{-1/2}(1-S)^{1/2}>0$ and $(1-S')^{-1/2}(S')^{1/2}>0$, we have $g_1=0$. Similarly we have $\{(1-S)^{-1/2}S^{1/2}+(S')^{-1/2}(1-s')^{1/2}\}g_2=0$ and hence $g_2=0$. This proves (6.42).

Let $\varepsilon=\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix}$. Then $\varepsilon^*=\varepsilon^{-1}=\varepsilon$, $[\hat{\Gamma},\varepsilon]=0$ and $\varepsilon P_S\varepsilon=1-P_S$ for any S. Consequently, $\hat{\Gamma}\varepsilon$ commutes with P_S and $P_{S'}$, and hence anti-commutes with $\mathrm{H}(P_{S'}/P_S)$ defined as in (9.2).

Let J_0 be an antiunitary involution on $\mathfrak{H}_{P_S}$ defined by

$$J_0\pi_{P_S}(C)\Omega_{P_S}=\pi_{P_S}(\tau(\hat{\Gamma})\tau(\varepsilon)C)\Omega_{P_S} \tag{6.43}$$

for $C\in\mathfrak{A}_{\mathrm{SDC}}(\hat{K},\hat{\Gamma})$. Let $J=\alpha J_0$ where α is a function of $\mathrm{T}_{P_S}(-1)=\mathrm{T}_S(-1)$ being $=1$ if $\mathrm{T}_S(-1)=1$ and $=i$ if $\mathrm{T}_S(-1)=-1$. For a finite even dimensional K, $\pi_S(\mathfrak{A}(K,\Gamma))$ is always a factor and hence the proof of Theorem 3 is applicable where $H=\log\{S(1-S)^{-1}\}$ and $\beta=1$. From (6.11) we have

$$(\Omega_{P_S},A(JAJ)\Omega_{P_S})=(A^*\Omega_{P_S},\Delta A^*\Omega_{P_S})\geqq 0 \tag{6.44}$$

for $A\in\pi_{P_S}(\mathfrak{A}_{\mathrm{SDC}}(K\oplus 0,\hat{\Gamma}))$, where $\Delta=e^{-\theta/2}>0$.

Let Q be defined as in (9.4) where $n=0$. Then $\pi_\varphi(Q)$ commutes with J_0 and $\mathrm{T}_{P_S}(-1)$. Hence $\Omega'=\pi_\varphi(Q)^*\Omega_{P_S}$ is invariant under J_0 and $\mathrm{T}_{P_S}(-1)$. Furthermore, we have

$$\varphi_{P_{S'}}(C)=(\Omega',\pi_{P_S}(C)\Omega'), \tag{6.45}$$

(6.46) $$J_0\pi_{P_S}(C)\Omega'=\pi_{P_S}(\tau(\widehat{\Gamma})\tau(\varepsilon)C)\Omega',$$

(6.47) $$\mathrm{T}_{P_S}(-1)\pi_{P_S}(C)\Omega'=\pi_{P_S}(\tau(-1)C)\Omega',$$

for $C\in\mathfrak{A}_{\mathrm{SDC}}(\widehat{K},\widehat{\Gamma})$. Therefore

(6.48) $$(\Omega', A(JAJ)\Omega')=(A^*\Omega', \Delta'A^*\Omega')\geqq 0$$

for $A\in\pi_{P_S}(\mathfrak{A}_{\mathrm{SDC}}(K\oplus 0,\widehat{\Gamma}))$, where $\Delta'=e^{-\theta'/2}>0$ denotes the Δ in the proof of Theorem 3 corresponding to S'.

We can now apply the previous Lemma and obtain

(6.49) $$\|\varphi_{S_1}-\varphi_{S_2}\|\geqq 2(1-|(\Omega_{P_S}, \Omega')|).$$

From (9.8), we obtain (6.41).

The general case, where one or both of S and S' have an eigenvalue 1, can be obtained by taking a limit. Q.E.D.

§7. Bilinear Hamiltonian

Lemma 7.1. *There exists a derivation* $\delta(H)$ *on* $\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$ *satisfying*

(7.1) $$\delta(H)\mathrm{B}(f)=\mathrm{B}(Hf)$$

if and only if H *is a bounded linear operator on* K *satisfying*

(7.2) $$H^*=-\Gamma H\Gamma.$$

If (7.2) *holds,* (7.1) *uniquely defines* $\delta(H)$. *It is a* $*$ *derivation of* $\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$ *if and only if*

(7.3) $$H^*=-H.$$

Proof. For the first part, we have to check the condition that $\delta(H)$ is consistent with the relations (1) and (2) for the definition of $\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$. For the condition (1), it is necessary and sufficient that H is linear. For the condition (2), it is necessary and sufficient that (7.2) holds. From (7.2) it follows that H^* is defined on all K and hence H must be bounded.

For the second part, the uniqueness of $\delta(H)$ is immediate. The relation (3) for $\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$ implies that $(\delta(H)\mathrm{B}(f))^*=\delta(H)\mathrm{B}(f)^*$

if and only if $\Gamma H = H\Gamma$. Under (7.2), this is equivalent to (7.3).

Lemma 7.2. *The * derivation $\delta(H)$ is the infinitesimal generator of the Bogoliubov automorphism $\tau(e^{\lambda H})$.*

Proof. From (7.3) it follows that $e^{\lambda H}$ is unitary. From (7.2) and (7.3), it follows that $[H, \Gamma]=0$ and hence $[e^{\lambda H}, \Gamma]=0$. Hence $e^{\lambda H}$ is a Bogoliubov transformation. The rest is immediate.

Lemma 7.3. *Let H be a finite rank operator on K and*

$$Hh=\sum_{i=1}^{n} f_i(g_i, h), \qquad h \in K. \tag{7.4}$$

Let

$$(\mathrm{B}, H\mathrm{B})=\sum_{i=1}^{n} \mathrm{B}(f_i)\mathrm{B}(g_i)^*. \tag{7.5}$$

$(\mathrm{B}, H\mathrm{B})$ *does not depend on the choice of f_i and g_i for a given H, is linear in H and satisfies*

$$(\mathrm{B}, H\mathrm{B})^*=(\mathrm{B}, H^*\mathrm{B}). \tag{7.6}$$

In addition, it satisfies

$$(\mathrm{B}, H\mathrm{B})=(\mathrm{B}, \alpha(H)\mathrm{B})+\frac{1}{2}\operatorname{tr} H, \tag{7.7}$$

$$\alpha(H)=\frac{1}{2}(H-\Gamma H^*\Gamma). \tag{7.8}$$

$H=\alpha(H')$ satisfies (7.2) for any H'.

If H satisfies (7.2), then $\alpha(H)=H$, $\operatorname{tr} H=0$ and

$$[(\mathrm{B}, H\mathrm{B}), A]=2\delta(H)A, \quad A \in \mathfrak{A}_{\mathrm{SDC}}(K, \Gamma), \tag{7.9}$$

$$\varphi_S((\mathrm{B}, H\mathrm{B}))=-\operatorname{tr} SH, \tag{7.10}$$

$$(1/4)\|H\|_{\mathrm{tr}} \leqq \|(\mathrm{B}, HB)\| \leqq \|H\|_{\mathrm{tr}}, \tag{7.11}$$

$$\tau(U)(\mathrm{B}, H\mathrm{B})=(B, UHU^{-1}\mathrm{B}), \tag{7.12}$$

$$\delta(H_1)(\mathrm{B}, H\mathrm{B})=(\mathrm{B}, [H_1, H]\mathrm{B}). \tag{7.13}$$

Here φ_S is a quasifree state and $\|H\|_{\mathrm{tr}}=\operatorname{tr}[(H^*H)^{1/2}]$.

(The formulae (7.12) and (7.13) hold for a general H not satisfying (7.2).)

Proof. For $(\mathrm{B}, H\mathrm{B})$ defined by (7.5), we have (cf. [2])

(7.14) $$[(\mathrm{B}, H\mathrm{B}), \mathrm{B}(f)] = \mathrm{B}(2\alpha(H)f),$$

(7.15) $$\varphi_S((\mathrm{B}, H\mathrm{B})) = \mathrm{tr}\, H - \mathrm{tr}\, SH.$$

For the central state, $S=1/2$ and

(7.16) $$\varphi_{1/2}((\mathrm{B}, H\mathrm{B})) = \mathrm{tr}\, H/2.$$

If K has an infinite or even dimension, $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ is known to have the trivial center. Hence (7.14) determines (B, HB) up to a constant and (7.16) fixes that constant. Even if the dimension of K is odd, we can make this argument by imbedding $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ in $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K', \Gamma)$ with a bigger K' with an even dimension.

This argument shows that (B, HB) is independent of the way in which H is expressed in (7.4) and also that (7.7) holds because $\alpha(H)$ has trace 0 and both sides of (7.7) satisfy (7.14) and (7.16). Note that $\mathrm{tr}\,\Gamma H^* \Gamma = \sum_i (\Gamma^2 e_i, \Gamma H^* \Gamma e_i) = \sum_i (H^* \Gamma e_i, \Gamma e_i) = \sum_i (\Gamma e_i, H\Gamma e_i) = \mathrm{tr}\, H.$

The linearity of (B, HB) in H, (7.6), (7.12) and (7.13) follow from the definition (7.5). (7.9) and (7.10) follow from (7.14) and (7.15).

If H satisfies (7.2) and is selfadjoint, it has the spectral decomposition

(7.17) $$H = \sum_\lambda \lambda E_\lambda,$$

where

(7.18) $$\Gamma E_\lambda \Gamma = E_{-\lambda}.$$

Hence we have a partial basis projection $\sum_{\lambda>0} E_\lambda \equiv E_+$ and an orthonormal basis f_i in $E_+ K$ such that

(7.19) $$(\mathrm{B}, H\mathrm{B}) = \sum_i \lambda_i (\mathrm{B}(f_i)\mathrm{B}(f_i)^* - B(f_i)^* B(f_i)).$$

Since $\|\mathrm{B}(f)\mathrm{B}(f)^* - \mathrm{B}(f)^*\mathrm{B}(f)\| \leqq \mathrm{B}(f)\mathrm{B}(f)^* + \mathrm{B}(f)^*\mathrm{B}(f)\| = \|f\|^2$, we have

(7.20) $$\|(\mathrm{B}, H\mathrm{B})\| \leqq \sum_i \lambda_i = \frac{1}{2}\|H\|_{\mathrm{tr}}.$$

On the other hand, $\varphi_S[(\mathrm{B}, H\mathrm{B})] = -\sum \lambda_i = -\frac{1}{2}\|H\|_{\mathrm{tr}}$ for $S = E_+ + (1/2)E_0$. Hence, for a selfadjoint H satisfying (7.2), we have

(7.21) $$\|(\mathrm{B}, H\mathrm{B})\|=\|H\|_{\mathrm{tr}}/2.$$

If $H=H_1+iH_2$, $H_1^*=H_1$, $H_2^*=H_2$, then consider the polar decomposition $H_1=|H_1|U_1$ where $U_1^*=U_1$, $U_1^2=1$. Then $(\mathrm{tr}\, H_2U_1)^*=\mathrm{tr}\, U_1H_2=\mathrm{tr}\, H_2U_1$ is real and we have

(7.22) $$\|H\|_{\mathrm{tr}}=\sup_{\|Q\|\leqq 1}|\mathrm{tr}\, HQ|\geqq|\mathrm{tr}\, HU_1|\geqq \mathrm{tr}|H_1|=\|H_1\|_{\mathrm{tr}}.$$

Hence

(7.23) $$\|H_1\|_{\mathrm{tr}}+\|H_2\|_{\mathrm{tr}}\geqq\|H\|_{\mathrm{tr}}\geqq\max(\|H_1\|_{\mathrm{tr}}, \|H_2\|_{\mathrm{tr}}).$$

On the other hand, for any operator $A=A_1+iA_2$, $A_1^*=A_1$, $A_2^*=A_2$, we have $\|A\|=\sup_{\|\varphi\|\|\psi\|\leqq 1}|(\varphi, A\psi)|\geqq\sup|(\varphi, A\varphi)|\geqq\sup|(\varphi, A_1\varphi)|=\|A_1\|$. Hence

(7.24) $$\|A_1\|+\|A_2\|\geqq\|A\|\geqq\max(\|A_1\|, \|A_2\|).$$

By combining (7.21), (7.23) and (7.24), we have (7.11). Q.E.D.

Lemma 7.4. *Let H be a trace class operator and H_n be a sequence of finite rank operators such that $\|H-H_n\|_{\mathrm{tr}}\to 0$ as $n\to\infty$. Then* $(\mathrm{B}, H_n\mathrm{B})$ *has a limit* $(\mathrm{B}, H\mathrm{B})$ *in* $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ *independent of the sequence for a given H. It is linear in H, and satisfies* (7.6) *and* (7.7). *If H satisfies* (7.2), $(\mathrm{B}, H\mathrm{B})$ *satisfies* (7.9), (7.10), (7.11), (7.12) *and* (7.13).

Proof. From (7.11) and (7.7), the convergence and the uniqueness follow. The rest follows from the corresponding properties for H_n.

Theorem 4. *The derivation $\delta(H)$ can be extended to an inner derivation of* $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ *if and only if H is in the trace class.*

Proof. "If" part follows from Lemma 7.4 and (7.9). $\delta(H)$ can be extended to an inner derivation if and only if $\delta(i(H^*+H))$ and $\delta(H^*-H)$ can be extended to inner $*$ derivations. For an inner $*$ derivation $\delta(H)$, $\tau(e^{\lambda H})$ for all real λ must be an inner automorphism by Lemma 7.2. From later result in Theorem 5 this implies that either $e^{\lambda H}-1$ is in the trace class or $e^{\lambda H}+1$ is in the trace class. In either case, the selfadjoint operator iH must have purely discrete spectrum.

If $e^{\lambda H}+1$ is in the trace class, then the eigenvalues x_j of iH can have an accumulation point only at $\frac{\pi}{\lambda}(2n+1)$, $n=0, \pm 1, \pm 2, \cdots$ which can happen only for a countable number of λ. For other values of λ, $e^{\lambda H}-1$ must be in the trace class and hence iS can have an accumulation point only at $2\pi n\lambda^{-1}$. This, first of all, excludes the other possibility and further implies that i can have an accumulation point of eigenvalues only at 0. From the condition $\|e^H-1\|_{tr}=\sum_i\{2(1-\cos x_i)\}^{1/2}<\infty$, and the inequality $1-\cos x\geqq x^2/3$ for $|x|\leqq 1$, we obtain $\|H\|_{tr}<\infty$. Q.E.D.

§8. Inner Bogoliubov Transformations

Definition 8.1. *$\mathcal{T}_\pm$ denotes the set of invertible bounded linear operators V on K such that $V-1$ is in the trace class,* $\det V=\pm 1$, *respectively, and*

$$\Gamma V^*\Gamma=V^{-1}. \tag{8.1}$$

*$\mathcal{T}_\pm$ is equipped with an operator multiplication, an adjoint operation * and a topology induced by spheres $\{V':\|V'-V\|_{tr}\leqq\varepsilon\}$.*

$\mathcal{T}_+$ and $\mathcal{T}_+\cup\mathcal{T}_-$ are topological groups and $\mathcal{T}_+$ is connected.

Since $V-1$ is compact, it has a (Jordan) expansion:

$$V-1=V_\Delta+\sum_{\lambda\neq\Delta}E_\lambda(\lambda-1+N_\lambda) \tag{8.2}$$

where Δ is a bounded open set containing 1, $(V_\Delta-\lambda)^{-1}$ is holomorphic for $\lambda\notin\Delta$,

$$E_\lambda E_{\lambda'}=\delta_{\lambda\lambda'}E_{\lambda'}, \tag{8.3}$$

$$E_{\lambda'}N_\lambda=N_\lambda E_{\lambda'}=\delta_{\lambda\lambda'}N_\lambda, \tag{8.4}$$

$$N_\lambda^{\dim E_\lambda}=0,\quad \dim E_\lambda<\infty, \tag{8.5}$$

$$V_\Delta E_\lambda=E_\lambda V_\Delta=0\quad(\lambda\notin\Delta), \tag{8.6}$$

$$\lim_{n\to\infty}(V_\Delta/r)^n=0,\quad r=\sup_{\lambda\in\Delta-1}|\lambda|. \tag{8.7}$$

V_Δ, E_λ and N_λ are uniquely determined by these properties and are given by

$$E_\lambda=\lim_{\rho\to 0}(2\pi)^{-1}\int_0^{2\pi}\rho e^{i\theta}(\lambda+\rho e^{i\theta}-V)^{-1}d\theta, \tag{8.8}$$

(8·9) $$N_\lambda = E_\lambda(V-\lambda)$$

and (8. 2). The $\det V$ may be computed by

(8. 10) $$\det V = \exp \operatorname{tr} \log V$$

(8. 11) $$\log V = (1-\sum_{\lambda\notin\Delta} E_\lambda)\log(1+V_\Delta) + \sum_{\lambda\notin\Delta} E_\lambda[\log\lambda + \log(1+\lambda^{-1}N_\lambda)],$$

where $\log(1+V_\Delta)$ and $\log(1+\lambda^{-1}N_\lambda)$ are defined by power series, which converge due to (8. 7) and (8. 5), and we take Δ such that $r<1$. Since V is invertible, $\lambda\neq 0$.

For V satisfying (8. 1), the uniqueness implies

(8. 12) $$\Gamma E_\lambda^* \Gamma = E_{(\lambda^{-1})},$$

(8. 13) $$(\lambda+\Gamma N_\lambda^* \Gamma)(\lambda^{-1}+N_{(\lambda^{-1})})=1,$$

(8. 14) $$(1+\Gamma V_\Delta^* \Gamma)(1+V_\Delta)=1,$$

where (8. 14) holds if Δ is invariant under $\lambda\to\lambda^{-1}$. If we choose branches of $\log\lambda$ in (8. 11) such that $\log\lambda+\log\lambda^{-1}=0$ for $\lambda\neq -1$ and $\log(-1) = i\pi$, then we have

(8. 15) $$\log V + \Gamma(\log V)^*\Gamma = 2\pi i E_{-1}.$$

Hence we have

(8. 16) $$\det V = (-1)^{\dim E_{-1}}.$$

Thus the condition $\det V = \pm 1$ can be replaced by $\dim E_{-1}$ = even or odd.

Lemma 8. 2. *If H is an operator in the trace class satisfying (7. 2), then $e^H \in \mathcal{T}_+$. If V is a normal operator in $\mathcal{T}_+$ or $V\in\mathcal{T}_+$ does not have an eigenvalue -1, then there exists a trace class operator H satisfying (7. 2) such that $V=e^H$. If $V>0$, H can be chosen hermitian and if V is unitary, iH can be chosen hermitian.*

Proof. The first part is immediate. Let $V\in\mathcal{T}_+$ be normal. We then have $N_{-1}=0$, $\dim E_{-1}$ = even, $E_{-1}^* = E_{-1}$, $\Gamma E_{-1}\Gamma = E_{-1}$. Hence there exists a subprojection F of E_{-1}, which satisfies $\Gamma F\Gamma + F = E_{-1}$, $F^* = F = F^2$. $H = \log V - 2\pi i F$ satisfies $e^H = Ve^{-2\pi i F} = V$ and (7. 2) due to (8. 15).

H is obviously in the trace class. If $V \in \mathcal{I}_+$ does not have an eigenvalue -1, then $H = \log V$ has the required property.

If $V > 0$, then we can choose $\log \lambda$ to be real and for this choice of the branch of log, H is hermitian. If V is unitary, we may take $|\mathrm{Im} \log \lambda| < \pi$ for $\lambda \neq -1$, and for this choice of the branch of log, iH is hermitian.

Lemma 8.3. *There exists a covering group $\mathcal{I}_+^*$ of $\mathcal{I}_+$ equipped with adjoint operation and a $*$ homomorphic homeomorphism π of $\mathcal{I}_+^*$ onto $\mathcal{I}_+$ such that π is 2 to 1 and the loop $\{\exp 2\pi i\lambda E;\ 0 \leqq \lambda \leqq 1\}$ for an odd dimensional partial basis projection E gives an element of $\mathcal{I}_+^*$ different from 1. There exists a homeomorphic $*$ isomorphism Q from $\mathcal{I}_+^*$ into $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ such that*

(8.17) $$Q(g)\mathrm{B}(f)Q(g)^{-1} = \mathrm{B}(\pi(g)f),$$

(8.18) $$\varphi_{1/2}(Q(g)) = \exp \frac{1}{2} \operatorname{tr} \log [(1 + \pi(g))/2],$$

(8.19) $$\|Q(g)\| = \exp \frac{1}{4} \operatorname{tr} |\log |\pi(g)||,$$

(8.20) $$\min(\|Q(g) - 1\|, \|Q(g) + 1\|) \leqq \|Q(g)\| - 1 + \frac{1}{2} \operatorname{tr} |(\pi(g)|\pi(g)|^{-1})^{1/2} - 1|.$$

Here the branch in (8.18) *is to be determined by the analytic continuation of* $(\det[(1 + e^{zH})/2])^{1/2}$ *from* $z = 0$ *to* 1 *if* $\pi(g) = e^H$ *and if the continuous inverse image of the path* $\{e^{\lambda H};\ 0 \leqq \lambda \leqq 1\}$ *ends at* g. *It is to be determined by the continuity for other* g.

Proof. Let $\sum$ be the set of trace class operators satisfying (7.2), equipped with a topology induced by spheres $\{H';\ \|H' - H\|_{\mathrm{tr}} < \varepsilon\}$. Let $\mathcal{I}_0$ be the set of $V \in \mathcal{I}_+$ such that -1 is not an eigenvalue of V.

For any $V \in \mathcal{I}_0$, we see from the Jordan expansion (8.2) that the following $\mathrm{H}_\gamma(V)$ satisfies (7.2) and $V = e^{\mathrm{H}_\gamma(V)}$ for each path γ from $z = 0$ to $z = 1$ avoiding zeroes of $\det(1 + z(V - 1))$.

(8.21) $$\mathrm{H}_\gamma(V) = \frac{1}{2}\{H_{0\gamma}(V) - \Gamma \mathrm{H}_{0\gamma}(V)^* \Gamma\},$$

(8.22) $$\mathrm{H}_{0\gamma}(V) = \int_\gamma (1 + z(V - 1))^{-1}(V - 1)\,\mathrm{d}z.$$

For V in the sphere $|V-1|<1$, we can take γ to be the interval $[0, 1]$ on real axis. Then $\mathrm{H}_\gamma(V)\to V$ is a one-to-one homeomorphism of an open neighbourhood of 0 in $\sum$ onto an open neighbourhood of 1 in $\mathcal{T}_+$.

Let $V\in\mathcal{T}_+$. Let $V^*V=e^{H_1}$, $H_1\in\sum$, $H_1^*=H_1$. Let $|V|=\exp(1/2)H_1$, $U=V|V|^{-1}=e^{H_2}$, $H_2\in\sum$, $H_2^*=-H_2$. (Since V is invertible, U is unitary.) Let $\mathrm{V}(z)=e^{zH_2}|V|$. $\mathrm{V}(z)\in\mathcal{T}_+$ for all complex number z. Since $\mathrm{V}(z)$ is an entire function of z and $\mathrm{V}(0)$ does not have an eigenvalue -1, $\det(\mathrm{V}(z)+1)=0$ has isolated roots. Hence $\mathcal{T}_0$ and $\exp\sum$, which contains $\mathcal{T}_0$, are dense in $\mathcal{T}_+$.

For $H\in\sum$, define

$$\widehat{\mathrm{Q}}(H)=\exp\frac{1}{2}(\mathrm{B}, H\mathrm{B}). \tag{8.23}$$

From Lemma 4.3, we have

$$\widehat{\mathrm{Q}}(H)\mathrm{B}(f)\widehat{\mathrm{Q}}(H)^{-1}=\mathrm{B}(e^H f). \tag{8.24}$$

Obviously $\widehat{\mathrm{Q}}(H)^*=\widehat{\mathrm{Q}}(H^*)$.

Let $H\in\sum$ be selfadjoint. Then, in the Jordan expansion $H=H_\Delta+\sum_{\lambda\neq\Delta}E_\lambda\lambda$, $E_+=\sum_{\lambda>0}E_\lambda$ is a partial basis projection and $\widehat{\mathrm{Q}}(H)$ belongs to $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K', \Gamma)$ for $K'=(E_++\Gamma E_+\Gamma)K$. By identifying $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K', \Gamma)$ with $\overline{\mathfrak{A}}_{\mathrm{CAR}}(E_+K)$, $(\mathrm{B}, H\mathrm{B})=2(\mathrm{a}^\dagger, H_+\mathrm{a})-\mathrm{tr}\,H_+$ where $H_+=HE_+$ and $(\mathrm{B}, H_+\mathrm{B})=(\mathrm{a}^\dagger, H_+\mathrm{a})$ is $Q_\psi(H_+)$ in the notation of [4]. By using the formula (12.3) of [4] with $\rho=1/2$, we have

$$\begin{aligned}\varphi_{1/2}(\mathrm{Q}(H)) &=\exp\left\{\mathrm{tr}\log[(1+e^{H_+})/2]-\frac{1}{2}\,\mathrm{tr}\,H_+\right\}\\ &=\exp\mathrm{tr}\log\cosh(H_+/2)\\ &=\exp\frac{1}{2}\,\mathrm{tr}(\log[(1+e^{H_-})/2]+\log[(1+e^{-H_+})/2])\\ &=\exp\frac{1}{2}\,\mathrm{tr}\log[(1+e^{H})/2].\end{aligned} \tag{8.25}$$

Note that the central state of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K', \Gamma)$ is the same as the restriction of the central state of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ to $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K', \Gamma)$.

Let $\mathrm{V}(z)$ be holomorphic in z and $\mathrm{V}(z)\in\mathcal{T}_+$. Then

(8.26) $$\det[(1+V)/2]=\exp \operatorname{tr}\log[(1+V)/2]$$

for $V=\mathrm{V}(z)$ is holomorphic in z and have zeros of an even order unless it is identically 0. Hence its square root is locally holomorphic at every z. We define

(8.27) $$\mathrm{f}(H)=\exp\frac{1}{2}\operatorname{tr}\log[(1+e^{zH})/2]|_{z=1},$$

where the value is the analytic continuation from $\mathrm{f}(0)=1$ and does not depend on the path of the analytic continuation.

By setting $H=H_1+zH_2$ in (8.25) and making an analytic continuation from real z to $z=i$, we have

(8.28) $$\varphi_{1/2}(\widehat{\mathrm{Q}}(H))=\mathrm{f}(H)$$

for all $H\in\sum$.

If $H\in\sum$ is selfadjoint, we have from (7.21)

(8.29) $$\|\widehat{\mathrm{Q}}(H)\|\leqq\exp\frac{1}{2}\|(\mathrm{B}, H\mathrm{B})\|=\exp\frac{1}{4}\operatorname{tr}|H|.$$

On the other hand, we can consider the Fock state $\varphi_{\Gamma E,\Gamma}$ of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K',\Gamma)$ and use (12.3) of [4] with $\rho=1$, we have

(8.30) $$\varphi_{\Gamma E,\Gamma}(\widehat{\mathrm{Q}}(H))=\exp\frac{1}{2}\operatorname{tr}H_+=\exp\frac{1}{4}\operatorname{tr}|H|.$$

Therefore

(8.31) $$\|\widehat{\mathrm{Q}}(H)\|=\exp\frac{1}{4}\operatorname{tr}|H|. \quad (H^*=H)$$

Since $\widehat{\mathrm{Q}}(H)^{-1}=\widehat{\mathrm{Q}}(-H)$ and $\widehat{\mathrm{Q}}(H)>0$, we have

(8.32) $$\exp\frac{1}{4}\operatorname{tr}|H|\geqq\widehat{\mathrm{Q}}(H)\geqq\exp-\frac{1}{4}\operatorname{tr}|H|. \quad (H^*=H)$$

Let $iH\in\sum$ be selfadjoint. Then $\widehat{\mathrm{Q}}(H)$ as well as e^H are unitary. If E is a one dimensional partial basis projection and $H=i\lambda(E-\Gamma E\Gamma)$, then $(\mathrm{B}, H\mathrm{B})$ has the spectrum $\{i\lambda, -i\lambda\}$ and hence $\|\widehat{\mathrm{Q}}(H)-1\|=\frac{1}{2}\operatorname{tr}|e^{H/2}-1|$. If $H=\sum H_i$, $H_i^*=-H_i$, $H_iH_j=H_jH_i=0$, then $\widehat{\mathrm{Q}}(H)=\Pi\widehat{\mathrm{Q}}(H_i)$, each $\widehat{\mathrm{Q}}(H_i)$ is unitary and $\|\widehat{\mathrm{Q}}(H)-1\|\leqq\sum\|\widehat{\mathrm{Q}}(H_i)-1\|$.

(Here we have used $\widehat{\mathrm{Q}}(H)-1=\sum_j(\prod_{k>j}\widehat{\mathrm{Q}}(H_k))(\widehat{\mathrm{Q}}(H_j)-1)$.) Hence we have

(8.33) $$\|\widehat{\mathrm{Q}}(H)-1\|\leqq\frac{1}{2}\operatorname{tr}|e^{H/2}-1|. \quad (H^*=-H)$$

Let $V=e^H=e^{H'}\in\mathcal{T}_+$, $H, H'\in\sum$. We show that $\widehat{\mathrm{Q}}(H)=\pm\widehat{\mathrm{Q}}(H')$. Let $r>0$ be sufficiently small such that $\lambda+2n\pi i\neq x$ for any n, $0<|x|\leqq r$ and $|\lambda|>r$ in the following Jordan expansion of H:

(8.34) $$H=\sum_{|\lambda|>r}\mathrm{E}_\lambda(H)(\lambda+\mathrm{N}_\lambda(H))+H_r.$$

Define

(8.35) $$\mathrm{E}_\lambda(H_0)=\sum_n\mathrm{E}_{\lambda+2n\pi i}(H) \qquad (\lambda\neq\pi i),$$

(8.36) $$\mathrm{N}_\lambda(H_0)=\sum_n\mathrm{N}_{\lambda+2n\pi i}(H) \qquad (\lambda\neq\pi i),$$

(8.37) $$\mathrm{E}_{\pm\pi i}(H_0)=\sum_{n\geqq0}\mathrm{E}_{\pm(1+2n)\pi i}(H),$$

(8.38) $$\mathrm{N}_{\pm\pi i}(H_0)=\sum_{n\geqq0}\mathrm{N}_{\pm(1+2n)\pi i}(H),$$

(8.39) $$H_0=\sum{}'\mathrm{E}_\lambda(H_0)(\lambda+\mathrm{N}_\lambda(H_0))+H_r+\sum_{n\neq0}\mathrm{N}_{2n\pi i}(H)$$

where the sum $\sum'$ is over λ such that $|\mathrm{Im}\lambda|\leqq\pi$ and $\lambda\neq\pm(\pi i-\rho)$, $\rho>0$. Similar definitions are made for H'. $H_0, H_0'\in\sum$.

If E is of a finite rank, $E^*\Gamma E=E\Gamma E^*=0$ and $E^2=E$, then we have

(8.40) $$\mathrm{f}[2\pi zi(E-\Gamma E^*\Gamma)]=(\cos\pi z)^{\dim E}.$$

Hence $\widehat{\mathrm{Q}}(2\pi i(E-\Gamma E^*\Gamma))=(-1)^{\dim E}$. If H_1 and H_2 commute with each other, we have

(8.41) $$\widehat{\mathrm{Q}}(H_1)\widehat{\mathrm{Q}}(H_2)=\widehat{\mathrm{Q}}(H_1+H_2).$$

(8.40) and (8.41) implies

(8.42) $$\widehat{\mathrm{Q}}(H)=\pm\widehat{\mathrm{Q}}(H_0), \quad \widehat{\mathrm{Q}}(H')=\pm\widehat{\mathrm{Q}}(H_0').$$

From $e^H=e^{H'}$, we have $e^{H_0}=e^{H_0'}$ and hence

(8.43) $$H_0-H_{00}=H_0'-H_{00}',$$

(8.44) $$H_{00}\equiv\pi i[\mathrm{E}_{\pi i}(H_0)-\mathrm{E}_{-\pi i}(H_0)],$$

(8.45) $$H_{00}'\equiv\pi i[\mathrm{E}_{\pi i}(H_0')-\mathrm{E}_{-\pi i}(H_0')],$$

(8.46) $$\mathrm{E}_{\pi i}(H_0)+\mathrm{E}_{-\pi i}(H_0)=\mathrm{E}_{\pi i}(H_0')+\mathrm{E}_{-\pi i}(H_0').$$

Since $[H_0, H_{00}]=[H_0', H_{00}']=0$, we would obtain $\widehat{\mathrm{Q}}(H)=\pm\widehat{\mathrm{Q}}(H')$ by (8.41), (8.42) and (8.43) if we can prove $\widehat{\mathrm{Q}}(H_{00})=\pm\widehat{\mathrm{Q}}(H_{00}')$.

From (8.24), $\widehat{\mathrm{Q}}(H_{00})\widehat{\mathrm{Q}}(H_{00}')^{-1}$ commutes with all $\mathrm{B}(f)$. The algebra $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$, which is isomorphic to CAR C^* algebra, is known to be simple [8] and in particular has a trivial center. Therefore $\widehat{\mathrm{Q}}(H_{00})\widehat{\mathrm{Q}}(H_{00}')^{-1}=c\mathbb{1}$ for some complex number c.

On the other hand

$$\widehat{\mathrm{Q}}(H_{00})^2=\widehat{\mathrm{Q}}[2\pi i(\mathrm{E}_{\pi i}(H_0)-\mathrm{E}_{-\pi i}(H_0))]=(-1)^{\dim \mathrm{E}_{\pi i}(H_0)}$$

and $\widehat{\mathrm{Q}}(H_{00}')^2=(-1)^{\dim \mathrm{F}_{\pi i}(H_0')}$. From (8.46), $\dim\mathrm{E}_{\pi i}(H_0)=\dim\mathrm{E}_{\pi i}(H_0')$ and hence $\widehat{\mathrm{Q}}(H_{00})^2=\widehat{\mathrm{Q}}(H_{00}')^2$. Hence $c^2\mathbb{1}=\widehat{\mathrm{Q}}(H_{00})^2\widehat{\mathrm{Q}}(H_{00}')^{-2}=1$. Therefore $c=\pm 1$ and $\widehat{\mathrm{Q}}(H_{00})=\pm\widehat{\mathrm{Q}}(H_{00}')$. This completes the proof of $\widehat{\mathrm{Q}}(H)=\pm\widehat{\mathrm{Q}}(H')$.

From the above argument, $\widehat{\mathrm{Q}}(\mathrm{H}_\gamma(V))=\pm\widehat{\mathrm{Q}}(\mathrm{H}_{\gamma'}(V))$ for any γ and γ' where $\mathrm{H}_\gamma(V)$ is defined by (8.21).

Let $e^{H_1}e^{H_2}\in\mathcal{T}_0$ and $\mathrm{V}(z)=e^{zH_1}e^{zH_2}$. Let $z(t)$, $0\leqq t\leqq 1$ be a path between $z(0)=0$ and $z(1)=1$ avoiding zeroes of $\det(\mathrm{V}(z)+1)$. For each $0\leqq t\leqq 1$, there exists an open interval I_t containing t and a fixed path γ_t such that $\det(1+z'(\mathrm{V}[z(t')]-1))\neq 0$ for $z'\in\gamma_t$ and $t'\in I_t$. $\mathrm{H}_{\gamma_t}(\mathrm{V}(z))$ defined by (8.21) is holomorphic in z at $z(t')$, $t'\in I_t$. The equality between two analytic functions of z

(8.47) $$\widehat{\mathrm{Q}}(\mathrm{H}_{\gamma_t}(\mathrm{V}(z)))=\pm\widehat{\mathrm{Q}}(zH_1)\widehat{\mathrm{Q}}(zH_2)$$

hold for all $z=z(t')$, $t'\in I_t$ if it holds for $z=z(t)$, t in some dense subset of an open interval in I_t, by the continuity and an analytic continuation.

The formula (8.47) holds with the plus sign if $|z|$ is sufficiently small by the Baker Haussdorf formula and Lemma 7.3. Since $[0,1]$ is covered by a finite number of I_+, (8.47) holds for all $z=z(t)$, $0\leqq t\leqq 1$. Hence, if $e^{H_1}e^{H_2}=e^H\in\mathcal{T}_0$, we have

(8.48) $$\widehat{\mathrm{Q}}(H_1)\widehat{\mathrm{Q}}(H_2)=\pm\widehat{\mathrm{Q}}(H).$$

Let $V=e^H\in\mathcal{T}_0$. Then

(8.49) $$\widehat{\mathrm{Q}}(H)=\pm\widehat{\mathrm{Q}}(H_2)\widehat{\mathrm{Q}}(H_1)$$

where $|V|=e^{H_1}$, $V|V|^{-1}=e^{H_2}$. Since $\widehat{\mathrm{Q}}(H_2)$ is unitary, we obtain from (8.31)

(8.50) $$\|\widehat{\mathrm{Q}}(H)\|=\exp\frac{1}{4}\operatorname{tr}|\log|V||$$

where $\log|V|$ is hermitian.

We also have

$$\min(\|\widehat{\mathrm{Q}}(H)-1\|,\ \|\widehat{\mathrm{Q}}(H)+1\|)$$
$$\leqq\|\pm\widehat{\mathrm{Q}}(H_2)\widehat{\mathrm{Q}}(H_1)-1\|\leqq\|\widehat{\mathrm{Q}}(H_1)-1\|+\|\pm\widehat{\mathrm{Q}}(H_2)-1\|.$$

Hence we obtain from (8.32) and (8.33)

(8.51) $$\min(\|\widehat{\mathrm{Q}}(H)-1\|,\ \|\widehat{\mathrm{Q}}(H)+1\|)$$
$$\leqq\|\widehat{\mathrm{Q}}(H)\|-1+\frac{1}{2}\operatorname{tr}|e^{(H_2/2)}-1|.$$

Let $V_n=e^{H_n}\in\mathcal{T}_0$, $H_n\in\sum$ and $\lim\|V_n-V\|_{\mathrm{tr}}=0$, $V\in\mathcal{T}_+$. Then $\operatorname{tr}|\log|V_n||$ is bounded. Since $\|V_nV_m^{-1}-1\|_{\mathrm{tr}}\to 0$ as $n, m\to\infty$, there exists $\varepsilon_n=\pm 1$ such that

(8.52) $$[\varepsilon_n\widehat{\mathrm{Q}}(H_n)]\,[\varepsilon_m\widehat{\mathrm{Q}}(H_m)]^{-1}\to 1$$

as $n, m\to\infty$ due to (8.51). Since $\|\widehat{\mathrm{Q}}(H_n)\|$ is bounded due to (8.50), we have

(8.53) $$\|\varepsilon_n\widehat{\mathrm{Q}}(H_n)-\varepsilon_m\widehat{\mathrm{Q}}(H_m)\|\to 0,$$

as $n, m\to\infty$. Hence there exists a limit of $\varepsilon_n\widehat{\mathrm{Q}}(H_n)$ as $n\to\infty$. The limit does not depend on the choice of H_n and ε_n except for a factor ± 1. We shall write the limit as $\mathrm{Q}(V,\varepsilon)$ where $\varepsilon=\pm 1$ and $\mathrm{Q}(V,1)=-\mathrm{Q}(V,-1)$. The properties $\widehat{\mathrm{Q}}(S)^*=\widehat{\mathrm{Q}}(S^*)$, (8.24), (8.28), (8.48), (8.50) and (8.51) extends to $\mathrm{Q}(V,\varepsilon)$ by the continuity.

Let $\mathcal{T}_+^*$ be the abstract group with an involution $*$, which is $*$ isomorphic to the group of operators $\mathrm{Q}(V,\varepsilon)$, Q be the $*$ isomorphism from $g\in\mathcal{T}_+^*$ to the corresponding $\mathrm{Q}(V,\varepsilon)$ and $\pi(g)=V$ if $Q(g)=\mathrm{Q}(V,\varepsilon)$. From (8.24), $\mathrm{Q}(V_1,\varepsilon_1)=\mathrm{Q}(V_2,\varepsilon_2)$ only if $V_1=V_2$. Hence π is well defined. From $\mathrm{Q}(V,\varepsilon)^*=\pm\mathrm{Q}(V^*,\varepsilon')$ and (8.48) for $\mathrm{Q}(V,\varepsilon)$, π is a $*$ homomorphism. From (8.51), it is a homeomorphism. Since

$Q(V, \varepsilon)=-1$ exists by (8.40), the mapping π is two to one. (8.17)～(8.20) follows from the corresponding properties for $Q(V, \varepsilon)$. Q.E.D.

Theorem 5. *Let K be an infinite dimensional Hilbert space. A Bogoliubov transformation $\tau(U)$ of $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ is inner if and only if $U \in \mathcal{I}_+$ or $-U \in \mathcal{I}_-$.*

Proof of "if" part.

If $U \in \mathcal{I}_+$, then $Q(\pi^{-1}U)$ is unitary and induces the desired automorphism $\tau(U)$ on $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$.

If $U_1 \in \mathcal{I}_{\sigma_1}$, $U_2 \in \mathcal{I}_{\sigma_2}$, then $U_1U_2 \in \mathcal{I}_{\sigma_1\sigma_2}$. Hence it is enough to show that $\tau(U)$ is inner for at least one unitary U in $\mathcal{I}_-$.

Let $\{e_\alpha\}$ be a Γ invariant orthonormal basis of K. (A complete orthogonal family of Γ invariant vectors f_α can be obtained inductively by picking up f in $\{f_\beta;\ \beta<\alpha_0\}^\perp$ and defining $f_{\alpha_0}=f+\Gamma f$, $f_{\alpha_0+1}=i(f-\Gamma f)$.) Let U be defined by the requirement of linearity, boundedness and $Ue_0=e_0$, $Ue_\alpha=-e_\alpha$ for all $\alpha\neq 0$. Then $-U\in\mathcal{I}_-$ and $\tau(U)$ is implementable by $\sqrt{2}\,B(e_0)$. Q.E.D.

To prove the "only if" part, we need some preparations.

Let K_n be a Γ invariant finite even dimensional subspace of K, $\mathfrak{A}_n=\overline{\mathfrak{A}}_{SDC}(K_n, \Gamma)$ and $\mathfrak{A}_n^c$ be the set of elements in $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ commuting with every element of $\mathfrak{A}_n$. We know the following properties.

(a) $\bigcup_n \mathfrak{A}_n$ is dense in $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$.

(b) Let φ_1 and φ_2 be states of $\mathfrak{A}_n$ and $\mathfrak{A}_n^c$.

Then there exists a state φ of $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ such that $\varphi\big|\mathfrak{A}_n=\varphi_1$, $\varphi\big|\mathfrak{A}_n^c=\varphi_2$.

Property (b) follows from the fact that $\mathfrak{A}_n$ is a full matrix algebra.

Lemma 8.4. *Let U be a unitary element of $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$. Then there exists a unitary V_n in $\mathfrak{A}_n$ such that* $\lim\|V_n-U\|=0$.

Proof. From (a), there exists $A_n\in\mathfrak{A}_n$ such that $\lim_n\|A_n-U\|=0$. Let $\|A_n-U\|=\varepsilon_n$. Then $\|A_n\|\leqq\|U\|+\varepsilon_n=1+\varepsilon_n$ and

$$\|A_n^*A_n-1\|\leqq\|A_n^*-U^*\|\|A_n\|+\|U^*\|\|A_n-U\|\leqq\varepsilon_n(2+\varepsilon_n).$$

Hence $\||A_n|^{-1}\|\leqq[1-\varepsilon_n(2+\varepsilon_n)]^{-1/2}$ and $\||A_n|^{-1}-1\|\leqq[1-\varepsilon_n(2+\varepsilon_n)]^{-1/2}-1$ provided that $\varepsilon_n(2+\varepsilon_n)\leqq 1$. ($|A_n|=(A_n^*A_n)^{1/2}$.)

Let $V_n = A_n|A_n|^{-1}$. Then V_n is isometric and

$$\|V_n - U\| \leqq \|A_n - U\| \| |A_n|^{-1}\| + \| |A_n|^{-1} - 1\|$$
$$\leqq \varepsilon_n [1 - \varepsilon_n(2+\varepsilon_n)]^{-1/2} + [1 - \varepsilon_n(2+\varepsilon_n)]^{-1/2} - 1 \to 0.$$

Therefore $\lim \|V_n - U\| = 0$.

We now have $\lim_n \|V_n V_n^* - 1\| = 0$ due to $UU^* = 1$. Since $V_n V_n^*$ is a projection, $\|V_n V_n^* - 1\| = 1$ unless $V_n V_n^* = 1$. Hence V_n is unitary for sufficiently large n. Q.E.D.

Lemma 8.5. *Let U be a unitary element of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ and U_n be unitary element in $\mathfrak{A}_n$ such that $UAU^{-1} = U_n A U_n^{-1}$ for all $A \in \mathfrak{A}_n$. Then there exists a complex number λ_n such that $|\lambda_n| = 1$ and* $\lim \lambda_n U_n = U$.

Proof. Let $U_n^c = U_n^{-1} U$. We have $U_n^c \in \mathfrak{A}_n^c$. Let V_n be as given by Lemma 8.4. We then have

$$\lim \|U_n^c - U_n^{-1} V_n\| = 0.$$

Let φ_{1n} and φ_{2n} be states of $\mathfrak{A}_n$ and $\mathfrak{A}_n^c$ and let φ_n be a state of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$ such that $\varphi_n \big| \mathfrak{A}_n = \varphi_{1n}$, $\varphi_n \big| \mathfrak{A}_n^c = \varphi_{2n}$. We have

$$\sup_{\varphi_{2n}} |\varphi_{2n}(U_n^c) - \varphi_{1n}(U_n^{-1} V_n)|$$
$$\leqq \sup_{\varphi} |\varphi(U_n^c - U_n^{-1} V_n)| = \|U_n^c - U_n^{-1} V_n\| \to 0.$$

Let $\lambda_n = \varphi_{1n}(U_n^{-1} V_n)$ for a fixed sequence φ_{1n}. Then

$$\|U_n^c - \lambda_n\| = \sup_{\varphi_{2n}} |\varphi_{2n}(U_n^c - \lambda_n)| \to 0.$$

Therefore $\lim \|U - \lambda_n U_n\| = 0$. Q.E.D.

Proof of "only if" part of Theorem 5. Let U be a Bogoliubov transformation which can be implemented by a unitary W in $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K, \Gamma)$: $WAW^* = \tau(U)A$. Any inner $*$automorphism is unitarily implementable in any representation. From Theorem 8, we see that $U-1$ or $U+1$ is in the Hilbert Schmidt class. In either case, U has a purely discrete spectrum.

First consider the case where multiplicities of eigenvalues 1 and -1 of U are not odd. Then there exists Γ invariant finite even dimensional spectral projections E_n of U such that $\lim E_n=1$. Let $U_n=E_nU$. Let W_n be a unitary element of $\overline{\mathfrak{A}}_{SDC}(E_nK,\Gamma)$ such that $W_nAW_n^*=\tau(U_n)A$ for $A\in\overline{\mathfrak{A}}_{SDC}(E_nK,\Gamma)$. By Lemma 8. 5, there exists complex numbers λ_n such that $\lim\lambda_nW_n=W$.

Let $U=e^{iH}$, $\|H\|\leqq\pi$, $H^*=H$, $\Gamma H\Gamma=-H$, and E_+ be a basis projection such that $[E_+, H]=0$, $E_+H\leqq 0$. If multiplicities of eigenvalues 1 and -1 of U are not odd, such H and E_+ exists. Let $E_-=1-E_+$, $H_\pm=E_\pm H$.

By Lemma 8. 3, W_n is proportional to $\widehat{Q}(iE_nH)$ and by Lemma 9. 2,

$$\varphi_{E\pm}(W_n)=c\exp(i/2)\operatorname{tr}(E_nH_\pm),$$

where $|c|=1$ is common for two equations. Since $[U, E_\pm]=0$, $W\Omega_{E\pm}$ must be a multiple of $\Omega_{E\pm}$ by Lemma 4. 3. Therefore

$$\lim_{n\to\infty}\lambda_n\exp(i/2)\operatorname{tr}(E_nH_\pm)=c'$$

where c' is common for $\pm$. This implies

$$\lim_n\exp(i/2)\operatorname{tr}E_n(H_+-H_-)=1.$$

From $\Gamma H_+\Gamma=-H_-$, $[\Gamma, E_n]=0$, we have $\operatorname{tr}E_nH_-=-\operatorname{tr}E_nH_+$. Therefore

$$\lim_n\exp i\operatorname{tr}E_nH_+=1.$$

Since $0\leqq H_+\leqq\pi$ and E_n can be chosen to pick up (an increasing sequence of) any finite number of eigenvectors of H_+, this implies that H_+ must be in the trace class. Therefore $U\in\mathcal{I}_+$ in this case.

In order to consider a general case, we again use Theorem 8. If both $\dim E_1$ and $\dim E_{-1}$ are finite, then either 1 or -1 is an accumulation point of the spectrum of U. Then there exists a Bogoliubov transformation $U'\in\mathcal{I}_+$ which commutes with U such that UU' has an infinite multiplicity for an eigenvalue 1 or -1. Since we know already that U' is inner, it is sufficient to consider the case where either $\dim E_1$ or $\dim E_{-1}$ is infinite.

We now consider a case where the dimension of the eigenprojection E_1 of U for an eigenvalue 1 is finite and odd. Let $Q_{E_1}(-1)$

$=\prod_j\{\sqrt{2}\mathrm{B}(f_j)\}$ where $\{f_j\}$ is any complete orthonormal set of Γ invariant vectors in E_1K. Then $Q_{E_1}(-1)$ is unitary and implement the Bogoliubov automorphism $\tau(U_1)$ for U_1 which is 1 on E_1K and -1 on $(1-E_1)K$. Since UU_1 has no eigenvalue -1 and an infinite multiplicity for an eigenvalue 1, $\tau(UU_1)$ is inner only if $UU_1\in\mathcal{T}_+$. Since $-U_1\in\mathcal{T}_-$, this implies $-U\in\mathcal{T}_-$.

Finally we consider a case where the dimension of the eigenprojection E_{-1} of U for an eigenvalue -1 is finite and odd. As before $\tau(U_1)$ is inner for U_1 which is 1 on $E_{-1}K$ and -1 on $(1-E_{-1})K$. Since UU_1 has no eigenvalue 1 and an infinite multiplicity for an eigenvalue -1, it is not inner. Q.E.D.

§9. Unitary Implementable Bogoliubov Transformations

Lemma 9.1. *Let P and P' be basis projections. Let* $\sin\theta=|P-P'|$, $0\leqq\theta\leqq\pi/2$. *Let* $E_{\pi/2}=P\wedge(1-P')+(1-P)\wedge P'$, $E_0=P\wedge P'+(1-P)\wedge(1-P')$. *Let*

$$F_\pm=\frac{1}{2}(1-E_{\pi/2}-E_0\pm i(\sin\theta\cos\theta)^{-1}[P,P']),\tag{9.1}$$

$$\mathrm{H}(P'/P)=\theta\{F_+-F_-\}.\tag{9.2}$$

Let $e_1\cdots e_n$ be an orthonormal basis of $\{P\wedge(1-P')\}K$ ($n\leqq\infty$) and U be a unitary operator, determined by the requirement that $Ue_j=\Gamma e_j$, $U\Gamma e_j=e_j$, $Uf=f$ for $f\in(1-E_{\pi/2})K$. Assume that $|P-P'|$ is in the trace class, Let

$$\widehat{\mathrm{R}}(P'/P)=e^{i\mathrm{H}(P'/P)}U.\tag{9.3}$$

$$Q=\left\{\exp\frac{i}{2}(\mathrm{B},\mathrm{H}(P'/P)\mathrm{B})\right\}\prod_{j=1}^n\{\mathrm{B}(e_j)-\mathrm{B}(\Gamma e_j)\}.\tag{9.4}$$

Then $\widehat{\mathrm{R}}(P'/P)\in\mathcal{T}_\sigma$, $\sigma=(-1)^n$, Q is unitary and

$$\widehat{\mathrm{R}}(P'/P)P\widehat{\mathrm{R}}(P'/P)^*=P',\tag{9.5}$$

$$Q\mathrm{B}(f)Q^*=\mathrm{B}((-1)^n\widehat{\mathrm{R}}(P'/P)f),\tag{9.6}$$

$$\varphi_P(QAQ^*)=\varphi_{P'}(A)\tag{9.7}$$

$$\varphi_P(Q)=(\det\cos\theta)^{1/4},\tag{9.8}$$

where the positive quartic root is taken and $A\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$.

Proof. Since $(P-P')^2$ commutes with P and P', θ commutes with P and P'. θ also commutes with Γ. E_0 and $E_{\pi/2}$ are spectral projections of θ for the eigenvalues 0 and $\pi/2$. From $[P, P']^2 = -\sin^2\theta\cos^2\theta$, it follows that $F_+F_-=F_-F_+=0$. Because P and P' are basis projections, $\Gamma F_\pm\Gamma=F_\mp$. Namely $F_\pm$ are partial basis projections.

If $|P-P'|$ is in the trace class, then θ is also in the trace class and hence $\mathrm{H}(P'/P)\in\Sigma$, $e^{i\mathrm{H}(P'/P)}\in\mathcal{T}_+$.

(9.5) follows from a direct calculation. (Also see Appendix.)

Q is unitary. (9.6) is immediate for $f\in(1-E_{\pi/2})K$, $f=e_j$ and $f=\Gamma e_j$ and hence for all f. From (9.5) and (9.6), we have

$$\varphi_P(Q^*AQ)=\varphi_P(\tau((-1)^n\widehat{\mathrm{R}}(P'/P)^*)A) \tag{9.9}$$
$$=\varphi_P(\tau(\widehat{\mathrm{R}}(P'/P)^*)A)=\varphi_{P'}(A).$$

By Definition 3.1, (3.3) and $(g, Pe_k)=(g, P\Gamma e_k)=(g, Pf)=0$ for $g=e_j$ and Γe_j, $k\neq j$ and $f\in(1-E_0)K$, we have for $n\neq0$,

$$\varphi_P(Q)=0. \tag{9.10}$$

Since $Q\in\overline{\mathfrak{A}}_{\mathrm{SDC}}((1-E_0)K,\Gamma)$, we can compute (9.8) by using the Fock state $\varphi_{P(1-E_0)}$ on $\overline{\mathfrak{A}}_{\mathrm{SDC}}((1-E_0)K,\Gamma)$. Hence we may assume $E_0=0$ without loss of generality. If $n=0$ and $E_0=0$, there exists a basis projection E of (K,Γ) commuting with P and P' and a unitary operator u such that $[u, P]=[u, P']=0$ and $uEu^*=1-E$, due to Lemma A. Then it follows that $\mathrm{tr}\,E\mathrm{H}(P'/P)=(1/2)\,\mathrm{tr}\,\mathrm{H}(P'/P)=0$. We can identify $(\mathrm{B}, \mathrm{H}(P'/P)\mathrm{B})$ in $\overline{\mathfrak{A}}_{\mathrm{SDC}}((1-E_0)K,\Gamma)$ with $2(\mathrm{a}^\dagger, E\mathrm{H}(P'/P)\mathrm{a})$ in $\overline{\mathfrak{A}}_{\mathrm{CAR}}(EK)$ and use the formula for $\langle e^K\rangle$ in the Appendix C of [4] where $K=i\mathrm{H}(P'/P)E$ ($(\mathrm{a}^\dagger K\mathrm{a})$ is written as $[K]$ in [4]) and $\rho=(1-P)E$. We have

$$\begin{aligned}\varphi_P(Q)&=\exp\{\mathrm{tr}\log(1+(e^K-1)\rho)\}\\&=\exp\mathrm{tr}_E\log(1+(1-P)(e^K-1)(1-P))\\&=\exp\mathrm{tr}_{E(1-P)}\log(\cos\theta)\\&=\exp\frac{1}{4}\,\mathrm{tr}\log(\cos\theta)=\det(\cos\theta)^{1/4}.\end{aligned} \tag{9.11}$$

where the positive root is to be taken. Q.E.D.

Lemma 9.2. *Let $g\in\mathcal{T}_+^*$ and P be a basis projection. Then*

$$\varphi_P(Q(g))=\det{}_P(P\pi(g)^{-1}P)^{1/2}, \tag{9.12}$$

where $\det_P$ is the determinant taken on the space PK, the branch of the square root is to be determined by an analytic continuation from the value 1 for $g=1$ and the continuity.

Proof. First we consider the case where $\pi(g)=e^{iH}$, $H^*=H$, $\Gamma H^*\Gamma=-H$, H is in the trace class and the continuous inverse image $\pi^{-1}(e^{itH})$, $0\leqq t\leqq 1$ connects 1 and g. Then $\pi(g)$ is a Bogoliubov transformation. Let

$$P'=\pi(g)P\pi(g)^* \tag{9.13}$$

which is again a basis projection. Since H is in the trace class

$$\|P'-P\|_{\mathrm{tr}}\leqq\sum_{n=1}^{\infty}(n!)^{-1}\|\underbrace{[H\cdots[H}_{n},P]\cdots]\|_{\mathrm{tr}}$$
$$\leqq\sum_{n=1}^{\infty}(n!)^{-1}2^n\|H\|_{\mathrm{tr}}\|H\|^{n-1}\|P\|<\infty.$$

Let

$$\widehat{\mathrm{R}}(P/P')^*\pi(g)=V. \tag{9.14}$$

V commutes with the basis projection P by (9.13) and (9.5). Since $V\in\mathcal{T}_+\cup\mathcal{T}_-$, this implies $\det V=+1$ and hence $V\in\mathcal{T}_+$. Let $g'\in\mathcal{T}_+^*$ be such that $\pi(g')=V$. Let $V=e^{iH'}$ where $H'^*=H'$, $\Gamma H'\Gamma=-H'$, $[P,H']=0$. We then have $Q(g')=\pm\exp\frac{i}{2}(\mathrm{B},H'\mathrm{B})$. Under the identification of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ with $\overline{\mathfrak{A}}_{\mathrm{CAR}}(PK)$, $(\mathrm{B},H'\mathrm{B})=2(a^\dagger,H'Pa)-\mathrm{tr}(H'P)$. Therefore

$$\varphi_P(AQ(g'))=\pm\varphi_P(A)\exp-\frac{i}{2}\,\mathrm{tr}(H'P). \tag{9.15}$$

By substituting $A=\widehat{\mathrm{R}}(P/P')$, we obtain

$$\begin{aligned}\varphi_P(Q(g))&=\pm\det(\cos\theta)^{1/4}\exp-\frac{1}{2}\,\mathrm{tr}(\log V)P\\&=\pm\det{}_P[(\cos\theta)V^{-1}P]^{1/2}.\end{aligned} \tag{9.16}$$

Substituting $\cos\theta V^{-1}P=P\cos\theta V^{-1}P$ and $P\cos\theta=P\widehat{\mathrm{R}}(P/P')^*P$, we obtain

$$(9.17) \qquad \varphi_P(Q(g)) = \pm \det{}_P(P\pi(g)^{-1}P)^{1/2}.$$

By absorbing $\pm$ to the ambiguity in the branch of square root, we obtain (9.12).

By substituting $g_n(z)$ such that $\pi(g_n) = \exp i(H_1^{(n)} + zH_2^{(n)})$, $H_j^{(n)*} = H_j^{(n)}$, $\Gamma H_j^{(n)}\Gamma = -H_j^{(n)*}$, making analytic continuation in z from $z=0$ to i and taking limit of $n \to \infty$ in the trace class norm, one obtains (9.12) for most general g. Q.E.D.

Remark. The formula (9.12) can be also obtained by the following method. Consider the case where $\pi(g)-1$ and $S-(1/2)$ are of finite rank and S does not have eigenvalues 0 and 1. Then from (6.24), and (8.18), we have

$$(9.18) \quad \varphi_S(Q(g)) = \varphi_{1/2}[Q(g)\exp -\frac{1}{2}(\mathrm{B}, \log\{S(1-S)^{-1}\}\mathrm{B})]\det(2S)^{1/2}$$
$$= \pm \det(2S)^{1/2} \exp\frac{1}{2} \operatorname{tr}\log[(1+\pi(g)(1-S)S^{-1})/2]$$
$$= \pm \exp\frac{1}{2}\operatorname{tr}\log[S+\pi(g)(1-S)].$$

We can now allow S to take eigenvalues 0 and 1 and to be not of finite rank. (9.18) holds by continuity. If S is the projection P, we have

$$(9.19) \qquad \varphi_P(Q(g)) = \det[P + (1-P)\pi(g)(1-P)]^{1/2}.$$

Since $\det \Gamma A^* \Gamma = \det A$ and $\Gamma\pi(g)^*\Gamma = \pi(g)^{-1}$, we obtain (9.12).

Conversely (9.18) can be obtained from (9.12) by

$$\varphi_S(Q(g)) = \det{}_{P_S}(P_S\pi(g)^{-1}P_S)^{1/2} = \det(1-P_S+P_S\pi(g)^{-1}P_S)^{1/2}$$

where $\pi(g)$ is understood as $\pi(g)\oplus 1$ on $\widehat{K}$. It can be checked easily that this coincides with the above expression.

Note that the formula (12.3) of [4] is a special case of (9.18), where $S = 1-\rho$.

Lemma 9.3. *Let P be a basis projection, g_n be in $\mathcal{T}_+^*$, V be a Bogoliubov transformation $P_n \equiv \pi(g_n)P\pi(g_n)^*$, $P' \equiv VPV^*$. Assume that $\pi(g_n)$ is unitary,* (9.20) $$P\wedge(1-P')=0,$$

(9.21) $$\lim_{n\to\infty}\pi(g_n)=V,$$

(9.22) $$\lim_{n\to\infty}\|P_n-P'\|_{\mathrm{H.S.}}=0,$$

where $\|A\|_{\mathrm{H.S.}}=\|A^*A\|_{\mathrm{tr}}$. *Let* χ_n *be such that*

(9.23) $$\varphi_P(Q(g_n))=\chi_n|\varphi_P(Q(g_n))|.$$

Then

(9.24) $$\mathrm{Q}_P(V)=\lim_{n\to\infty}\pi_P(Q(g_n))\chi_n^{-1}$$

exists and does not depend on the sequence g_n *for a given* V. *It satisfies*

(9.25) $$\mathrm{Q}_P(V^*)=\mathrm{Q}_P(V)^*$$

(9.26) $$\mathrm{Q}_P(V)\pi_P(\mathrm{B}(f))=\pi_P(\mathrm{B}(Vf))\mathrm{Q}_P(V)$$

(9.27) $$(\Omega_P,\mathrm{Q}_P(V)\Omega_P)=\det{}_P(PP'P)^{1/4}>0.$$

If $H\in\Sigma$ *and* $H^*=-H$, *then* $\mathrm{Q}_P(e^H)=\widehat{\mathrm{Q}}(H)\det(Pe^HP)^{-1/2}|\det(Pe^HP)|^{1/2}$.

Proof. Since $\pi(g_n)-1$ is in the trace class, (P_n-P) is in the trace class and hence is in the H.S. class. From (9.22) $(P'-P)$ is also in the H.S. class. Hence $P(1-P')P=P(P'-P)^2P$ is in the trace class, and $(1-P')P=(P-P')P$ is in the H.S. class. From (9.22), it follows

(9.28) $$\|(1-P_n)P-(1-P')P\|_{\mathrm{H.S.}}=\|(P_n-P')P\|_{\mathrm{H.S.}}\to 0.$$

Hence $\|(1-P_n)P\|_{\mathrm{H.S.}}$ is uniformly bounded.

We now have

(9.29) $$\begin{aligned}\|PP_nP-PP'P\|_{\mathrm{tr}}&=\|\{P(1-P_n)-P(1-P')\}(1-P_n)P\\&\quad+P(1-P')\{(1-P_n)P-(1-P')P\}\|_{\mathrm{tr}}\\&\leqq\|(1-P_n)P-(1-P')P\|_{\mathrm{H.S.}}\{\|(1-P_n)P\|_{\mathrm{H.S.}}\\&\quad+\|(1-P')P\|_{\mathrm{H.S.}}\}\to 0.\end{aligned}$$

We also note that (9.22) implies $\|P_n-P_m\|_{\mathrm{H.S.}}\to 0$ as $n, m\to\infty$. Hence $P_{nm}\equiv\pi(g_m)^*P_n\pi(g_m)$ satisfies $\|P_{nm}-P\|_{\mathrm{H.S.}}=\|P_n-P_m\|_{\mathrm{H.S.}}\to 0$ as $n, m\to\infty$. Therefore

(9.30) $$\|PP_{nm}P-P\|_{\mathrm{tr}}=\|P(P_{nm}-P)^2P\|_{\mathrm{tr}}\to 0$$

as $n, m\to\infty$.

Let

(9.31) $$Q_n=\chi_n^{-1}\pi_P(Q(g_n)), \qquad \Psi_n=Q_n\Omega_P.$$

We obtain, from (9.29),

(9.32) $$\begin{aligned}(\Omega_P, \Psi_n) &= |\det{}_P(P\pi(g_n)^{-1}P)|^{1/2}\\ &= \{\det{}_P(P\pi(g_n)P)\det{}_P(P\pi(g_n)^*P)\}^{1/4}\\ &= [\det{}_P(PP_nP)]^{1/4}\to[\det{}_P(PP'P)]^{1/4}\end{aligned}$$

and, from (9.30),

(9.33) $$|(\Psi_n, \Psi_m)|=\det{}_P(PP_{nm}P)^{1/4}\to 1.$$

Due to (9.20), $c\equiv[\det_P(PP'P)]^{1/4}\neq 0$.

Let $\exp i\theta_{nm}=(\Psi_n, \Psi_m)/|(\Psi_n, \Psi_m)|$. If $|1-|(\Psi_n, \Psi_m)||<\varepsilon^2/2$, then $\|(\exp i\theta_{nm})\Psi_n-\Psi_m\|<\varepsilon$ and hence $|e^{i\theta_{nm}}(\Omega_P, \Psi_n)-(\Omega_P, \Psi_m)|<\varepsilon$. If $|(\Omega_P, \Psi_n)-c|<\varepsilon$ and $|(\Omega_P, \Psi_m)-c|<\varepsilon$ in addition, then $|e^{i\theta_{nm}}-1|c<3\varepsilon$ and hence $\|\Psi_n-\Psi_m\|<(1+3/c)\varepsilon$.

Therefore Ψ_n is a Cauchy sequence and has a strong limit $\Psi(V)$.

Let $\Psi=\pi_P(A)\Omega_P$, $A\in\mathfrak{A}_{\mathrm{SDC}}(K, \Gamma)$. Then

$$Q_n\pi_P(A)\Omega_P=\pi_P(\tau[\pi(g_n)]A)Q_n\Omega_P$$

(9.34) $$\begin{aligned}&\|Q_n\pi_P(A)\Omega-\pi_P(\tau[V]A)\Psi(V)\|\\ &\leqq\|A\|\|\Psi_n-\Psi(V)\|+\|\tau(\pi(g_n))A-\tau(V)A\|\to 0,\end{aligned}$$

where (9.21) is used for the second term. Hence Q_n has a strong limit $\mathrm{Q}_P(V)$, which satisfies (9.27) due to (9.32). It also satisfies

(9.35) $$\mathrm{Q}_P(V)\pi_P(A)\Omega_P=\pi_P(\tau(V)A)\mathrm{Q}_P(V)\Omega_P.$$

(9.27) implies $\mathrm{Q}_P(V)\Omega_P\neq 0$. Since π_P is irreducible, (9.35) implies that the range of $\mathrm{Q}_P(V)$ is the whole space. As a strong limit of unitary Q_n, $\mathrm{Q}_P(V)$ is isometric and hence is unitary. (9.35) implies (9.26), which uniquely determines the unitary operator $\mathrm{Q}_P(V)$ up to a multiplicative constant for a given V. The constant is unique due to (9.27). Hence $\mathrm{Q}_P(V)$ does not depend on the sequence.

(9.26) and (9.27) are satisfied when $\mathrm{Q}_P(V^*)^*$ is substituted into

$Q_P(V)$. Hence, by the uniqueness, we immediately have (9.25).

Lemma 9.4. *Let P be a basis projection, V be a Bogoliubov transformation and $P' = V^*PV$. If $(P'-P)$ is in the Hilbert Schmidt class, V is unitarily implementable in the Fock representation π_P.*

Proof. $E_{\pi/2} = P \wedge (1-P') + (1-P) \wedge P'$ is the spectral projection of $(P-P')^2$ for an eigenvalue 1 and hence has a finite dimension. Let $e_1 \cdots e_n$ be an orthonormal basis of $(P \wedge (1-P'))K$. Let U be a unitary operator determined by the requirements $Ue_j = \Gamma e_j$, $U\Gamma e_j = e_j$, $Uf = f$ for $f \in (1-E_{\pi/2})K$. Then U is a Bogoliubov transformation such that $U-1$ is of finite rank. We have $\det U = (-1)^n$. Hence $\tau((-1)^n U)$ is inner and hence is unitarily implementable.

We now consider $V_1 = VU$, $P'' = V_1^* P V_1$. Then $v \equiv |P''-P|$ is in the H.S. class and $P \wedge (1-P'') = (1-P) \wedge P'' = 0$. There exists a monotonically increasing sequence of a finite dimensional spectral projection E_n of v such that $\lim E_n = 1 - E_0$ where E_0 is the eigenprojection of v for an eigenvalue 0. Consider $\mathrm{R}(P''/P) = (1-v^2)^{1/2} - (1-v^2)^{-1/2} \cdot [P, P'']$. Then consider $U_n = (1-E_n) + \mathrm{R}(P''/P)E_n$. We have

$$\| U_n - \mathrm{R}(P''/P) \|_{\mathrm{H.S.}} \leqq \| (1-E_n)((1-v^2)^{1/2}-1) \|_{\mathrm{H.S.}} + \| (1-E_n)(1-v^2)^{-1/2}[P, P''] \|_{\mathrm{H.S.}} \to 0$$

where E_n commutes with P and P'' and $|[P, P'']|^2 = (1-v^2)v^2$ is in the trace class. Hence there exists $Q_P(\mathrm{R}(P''/P))$ on H_P which implements $\tau(\mathrm{R}(P''/P))$.

We now consider $V_2 = VU\mathrm{R}(P''/P)$. It commutes with P and hence φ_P is invariant under $\tau(V_2)$. Hence it is unitarily implementable in π_P.

Q.E.D.

Theorem 6. *Two Fock states* φ_P and $\varphi_{P'}$ *are unitarily equivalent if and only if $(P-P')$ is in the Hilbert Schmidt class.*

Proof. First assume that $P-P'$ is in the Hilbert Schmidt class. Then there exists a Bogoliubov transformation V bringing P' to P, which is unitarily implementable by Lemma 9.4.

Now assume that $P-P'$ is not in the Hilbert Schmidt class. Then

$(P-P')^2$ is not in the trace class. Since P commutes with $(P-P')^2$ and $\Gamma P(P-P')^2\Gamma=(1-P)(P-P')^2$, $(1-P)(P-P')^2=(1-P)P'(1-P)$ is not in the trace class.

By Lemma A, there exists a partial basis projection E and a partial isometry u such that $[E, P]=[E, P']=0$, $(P-P')^2(E+\Gamma E\Gamma)=(P-P')^2$, $[u, P]=[u, P']=0$, $u^*u=E$ and $u^*u=\Gamma E\Gamma$. Then $uEu^*=\Gamma E\Gamma$, $uE(1-P)P'(1-P)u^*=\Gamma E\Gamma(1-P)P'(1-P)$ and $\{E+\Gamma E\Gamma\}\cdot(1-P)P'(1-P)=(1-P)P'(1-P)$. Hence, if $(P-P')^2$ is not in the trace class, then $E(1-P)P'(1-P)$ is not in the trace class.

As a consequence, there exists an infinite number of unit vectors $e_j\in E(1-P)K$, $j=1, 2, \cdots$ such that $(e_j, e_k)=(e_j, P'e_k)=0$ for $j\neq k$ and $\sum_j(e_j, P'e_j)=\infty$. This is proved as follows:

If $E(1-P)P'(1-P)$ has a continuous spectrum Ξ_c, then take a number $\delta>0$ such that $\Xi_c\cap(\delta, 1)\neq\phi$ and take an infinite number of mutually disjoint interval Δ_j in $[\delta, 1]$ with $\Delta_j\cap\Xi_c\neq\phi$. Take any unit vector e_j from $E(\Delta_j)K$ where $E(\Delta_j)$ is the spectral projector of $(1-P)P'(1-P)$ for Δ_j. $(e_j, e_k)=(e_j, P'e_k)=0$ for $j\neq k$ is automatic. Since $(e_j, P'e_j)\geqq\delta$, $\sum_j(e_j, P'e_j)=\infty$.

If $E(1-P)P'(1-P)$ has a purely discrete spectrum, then take e_j to be a complete orthonormal set of eigenvectors of $E(1-P)P'(1-P)$ in $E(1-P)K$. Then $e_j\in(1-P)K$, $(e_j, P'e_k)=(e_j, E(1-P)P'(1-P)e_k)=0$ for $j\neq k$ and $\sum(e_j, P'e_j)=\mathrm{tr}(1-P)P'(1-P)E=\infty$.

Since $e_j\in EK$, $EK\perp\Gamma EK$ and $[E, P']=0$, we have $(e_j, \Gamma e_k)=(e_j, P'\Gamma e_k)=0$ for any j and k.

Let P_j be the projection on the space spanned by Γe_j, $\mathrm{U}_n(\lambda)\equiv\exp i\lambda(P_n-\Gamma P_n\Gamma)$, $\mathrm{U}^{(n)}(\lambda)=\prod_{k=1}^n\mathrm{U}_k(\lambda)$ and $\mathrm{U}(\lambda)=\prod_k\mathrm{U}_k(\lambda)$. We have

$$\det P\mathrm{U}^{(n)}(\lambda)P=\exp i\lambda^n, \tag{9.36}$$

$$\begin{aligned}\det P'\mathrm{U}^{(n)}(\lambda)P'&=\prod_{j=1}^n\det\{P'\exp i\lambda(P_j-\Gamma P_j\Gamma)P'\}\\&=(\exp i\lambda n)\prod_{j=1}^n[1+(e^{-i\lambda}-1)(e_jP'e_j)]^2.\end{aligned}\tag{9.37}$$

From (9.36), it follows that $Q_P(\mathrm{U}^{(n)}(\lambda))e^{-i\lambda n}$ has a strong limit $Q_P(\mathrm{U}(\lambda))$. It also follows from the proof of Lemma 9.3 that approach to the limit is uniform locally in λ and hence $Q_P(\mathrm{U}(\lambda))$ is continuous in λ. Hence $(\Phi, Q_P(U(\lambda))\Phi)\neq 0$ for sufficiently small λ for a given Φ. Thus for $\Phi\in\mathfrak{H}_P$, there exists λ such that

(9.38) $$\lim_{n\to\infty}(\Phi, Q_P(U^{(n)}(\lambda))e^{-i\lambda n}\Phi)\neq 0$$

and $\cos\lambda\neq 1$.

Let $g_n\in\mathcal{I}_+^*$ be such that $\pi(g_n)=U^{(n)}(\lambda)$. Due to (9.37), it is necessary for the existence of a nonzero limit

$$\lim_{n\to\infty}\varphi_{P'}(Q(g_n)e^{-i\lambda n})\neq 0$$

that

(9.39) $$\sum(e^{-i\lambda}-1)(e_j, P'e_i)<\infty.$$

This implies that φ_P and $\varphi_{P'}$ can not be unitarily equivalent. Q.E.D.

Theorem 7. *A Bogoliubov automorphism $\tau(V)$ is unitarily implementable in the Fock representation π_P if and only if $(1-P)VP$ is in the Hilbert Schmidt class.*

Proof. We note that

(9.40) $$\varphi_P(\tau(V)A)=\varphi_{V^*PV}(A).$$

Hence, if V is unitarily implementable, $|V^*PV-P|$ is in the Hilbert Schmidt class. Hence $P|V^*PV-P|^2=(PV^*(1-P))((1-P)VP)$ is in the trace class, which implies $(1-P)VP$ is in the Hilbert Schmidt class.

Conversely, if $(1-P)VP$ is in the trace class, then $P|V^*PV-P|^2$ and $\Gamma\{P|V^*PV-P|^2\}\Gamma=(1-P)|V^*PV-P|^2$ are both in the trace class. Hence V^*PV-P is in the Hilbert Schmidt class and V is unitarily implementable for π_P. Q.E.D.

Theorem 8. *A Bogoliubov automorphism $\tau(U)$ is unitarily implementable for all Fock representations if and only if $U-1$ or $U+1$ is in the Hilbert Schmidt class, where* $\dim K\neq$odd.

Proof. "If" part is immediate from Theorem 7. We may assume that $\dim K$ is infinite. [The case where $\dim K=$odd is not considered because there is no Fock representation].

For "only if" part, we have to show that if $PU(1-P)$ is in the Hilbert Schmidt class for all basis projection P, then $U-1$ or $U+1$ is in the Hilbert Schmidt class where U satisfies $U^*=U^{-1}$, $\Gamma U\Gamma=U$.

Let Δ be any measurable subset of $\{e^{i\theta}; 0<\theta<\pi\}$ and E_1 be any subprojection of the spectral projection of U for the set Δ such that $[U, E_1]=0$. Assume that E_1 has an infinite dimension and $E_0=1-E_1-\Gamma E_1\Gamma$ has an infinite or an even dimension. Let $E=E_1+\Gamma E_1\Gamma$.

There exists an antiunitary involution T (a complex conjugation) on EK, commuting with the spectral projections of UE_1 and with Γ. Let P_1 be the subprojection of E for the subspace spanned by $f+i\Gamma Tf$, $f\in E_1K$. Then $(E-P_1)K$ is spanned by $f-i\Gamma Tf$, $f\in E_1K$ and $\Gamma P_1\Gamma=E-P_1$. Hence there exists a basis projection $P>P_1$.

Since $U(f+i\Gamma Tf)=Uf+i\Gamma TU^*f$, we have

$$(1-P)U(f+i\Gamma Tf)=\frac{1}{2}[(U-U^*)f-i\Gamma T(U-U^*)f].$$

Therefore

$$\|(1-P)U(f+i\Gamma Tf)\|^2/\|f+i\Gamma Tf\|^2=\|(U-U^*)f/2\|^2/\|f\|^2.$$

Since $(1-P)UPP_1=(1-P)UP_1$ must be in the Hilbert Schmidt class, $(U-U^*)E_1$ must be in the Hilbert Schmidt class. [Note that $2^{-1/2}(f_j+i\Gamma Tf_j)$ is an orthonormal basis of P_1K if f_j is an orthonormal basis of E_1K.]

In order that $(U-U^*)E_1$ is in the H.S. class for any E_1, it is necessary that U has a purely discrete spectrum and its accumulation points are at most 1 and -1.

Next assume that $Uf_j=e^{i\alpha_j}f_j$, $Ug_j=e^{i\beta_j}g_j$, $j=1,2,\cdots$, $0\leqq\alpha_j\leqq\pi$, $0\leqq\beta_j\leqq\pi$, $|\alpha_j-\beta_k|\geqq\alpha(>0)$, $(f_j,f_k)=(g_j,g_k)=\delta_{jk}$ and $(\Gamma g_j,g_k)=(\Gamma g_j,f_k)=(f_j,\Gamma f_k)=0$. Further assume that the orthogonal complement of the set of all f_j, g_j, Γf_j, Γg_j, $j=1,2,\cdots$ has an infinite or even dimension.

Let P_1 be the subspace spanned by (f_j+g_j) and $(\Gamma f_j-\Gamma g_j)$, $j=1,2,\cdots$. Then there exists a basis projection $P\geqq P_1$. We have

$$(1-P)U(f_j+g_j)=(e^{i\alpha_j}-e^{i\beta_j})(f_j-g_j)/2.$$

Therefore,

$$\|(1-P)UP\|_{\mathrm{H.S.}}^2 \geqq \sum_j \|(1-P)U(f_j+g_j)\|^2/2 \geqq \sum_j \sin^2[(\alpha_j-\beta_j)/2] \geqq \sum_j \sin^2(\alpha/2)=\infty.$$

Thus, the spectrum of U can not have more than one accumulation points nor points with an infinite multiplicity.

From the above two conclusion, we see that $U-1$ or $U+1$ must be compact.

If $U+1$ is compact, then $(-U)-1$ is compact and $\tau(-1)$ is unitarily implementable in all Fock representation.

If $U-1$ is compact and an eigenvalue 1 has a finite multiplicity, there exists an infinite number of eigenvectors f_j of U belonging to an eigenvalue $e^{i\alpha_j}$ such that $0<\alpha_j<\pi$ and $\sum \alpha_j<\infty$. Let E be the projection for the subspace spanned by all f_j and Γf_j and $W=UE+(1-E)$. Then $\tau(W)$ is an inner Bogoliubov automorphism by Lemma 8.3 and an eigenvalue 1 of UW^* has an infinite multiplicity.

Thus we may restrict our attention to the case where $U-1$ is compact and an eigenvalue 1 of U has an infinite multiplicity. In this case Δ can be taken the whole set $\{e^{i\theta}; 0<\theta<\pi\}$ and hence $U-U^*$ is in the H.S. class. This implies that $U-1$ is in the H.S. class.

Q.E.D.

§10. Pseudo Fock States

Lemma 10.1. *Let P be a partial basis projection with the Γ codimension 1. Let e_0 be a fixed Γ invariant unit vector in $(1-P-\Gamma P\Gamma)K$. Let π_P on $\mathfrak{H}_P$ be the Fock representation of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(PK+\Gamma PK, \Gamma)$. Then there exists an irreducible representation $\pi_{(P,e_0)}$ of $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ on $\mathfrak{H}_P$ uniquely determined by the following requirements:*

$$\pi_{(P,e_0)}(\mathrm{B}(f))=2^{-1/2}(e_0,f)\mathrm{T}_P(-1)+\pi_P[\mathrm{B}(Pf+\Gamma P\Gamma f)]. \tag{10.1}$$

Proof. Since $\pi_{(P,e_0)}(\mathrm{B}(f))$ given by (10.1) satisfies the defining properties (1), (2), (3) of a selfdual CAR algebra, it automatically has a unique extension to the whole $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$. Q.E.D.

Definition 10.2. *A pseudo Fock state* $\varphi_{(P,e_0)}$ *of* $\overline{\mathfrak{A}}_{\mathrm{SDC}}(K,\Gamma)$ *is defined by*

$$\varphi_{(P,e_0)}(A)=(\Omega_P,\pi_{(P,e_0)}(A)\Omega_P), \tag{10.2}$$

where P, e_0 *and* $\pi_{(P,e_0)}$ *are given in Lemma* 10.1 *and* Ω_P *is the cyclic vector corresponding to the Fock state* φ_P *of* $\overline{\mathfrak{A}}_{\mathrm{SDC}}(PK+\Gamma PK,\Gamma)$.

Lemma 10.3. *Let* P *be a partial basis projection with a* Γ *codimension* 1 *and*

$$S=(1/2)(1+P-\Gamma P\Gamma). \tag{10.3}$$

Then

$$\varphi_S=(1/2)\{\varphi_{(P,e_0)}+\varphi_{(P,-e_0)}\}. \tag{10.4}$$

Pure states $\varphi_{(P,e_0)}$ *and* $\varphi_{(P,-e_0)}$ *are not unitarily equivalent.* R_S *is not a factor and its center is generated by* $\pi_S(\mathrm{B}(e_0))\mathrm{T}_P(-1)$, *where* $\mathrm{T}_P(-1)$ *is a unitary operator in* $\pi_S(\mathfrak{A}(PK+\Gamma PK,\Gamma))''$ *satisfying* $\mathrm{T}_P(-1)\pi_S(\mathrm{B}(f))\mathrm{T}_P(-1)=\pi_S(\mathrm{B}(-f))$ *for* $f\in PK+\Gamma PK$.

Proof. Any element A in $\mathfrak{A}_{\mathrm{SDC}}(K,\Gamma)$ can be written as $A_1+A_2\mathrm{B}(e_0)=A$ where A_1 and A_2 are in $\mathfrak{A}_{\mathrm{SDC}}(PK+\Gamma PK,\Gamma)$. Both sides of (10.4) give $\varphi_P(A_1)$ and hence (10.4) holds.

If $A\in\overline{\mathfrak{A}}_{\mathrm{SDC}}(PK+\Gamma PK,\Gamma)$, then $\pi_{(P,e_0)}(A)=\pi_{(P,-e_0)}(A)$. The set of all such $\pi_{(P,e_0)}(A)$ is irreducible by Lemma 4.3. Therefore any unitary operator satisfying $W\pi_{(P,e_0)}(A)W^*=\pi_{(P,-e_0)}(A)$ must be a multiple of the identity. However, $\pi_{(P,-e_0)}(\mathrm{B}(e_0))=-2^{-1/2}\mathrm{T}_P(-1)\neq\pi_{(P,e_0)}(\mathrm{B}(e_0))$. Therefore $\pi_{(P,e_0)}$ and $\pi_{(P,-e_0)}$ are not unitarily equivalent.

From this, it follows that

$$\mathfrak{H}_S=\mathfrak{H}_{\varphi(P,e_0)}\oplus\mathfrak{H}_{\varphi(P,-e_0)} \tag{10.5}$$

$$\Omega_S=2^{-1/2}(\Omega_{\varphi(P,e_0)}\oplus\Omega_{\varphi(P,-e_0)}) \tag{10.6}$$

$$\pi_S(\mathrm{B}(f))=\pi_{\varphi(P,e_0)}(\mathrm{B}(f))\oplus\pi_{\varphi(P,-e_0)}(\mathrm{B}(f)) \tag{10.7}$$

and $\pi_S(\mathrm{B}(e_0))(\mathrm{T}_P(-1)\oplus\mathrm{T}_P(-1))$, which is $2^{-1/2}$ on $\mathfrak{H}_{\varphi(P,e_0)}$ and $-2^{-2/1}$ on $H_{\varphi(P,-e_0)}$ generates the center of R_S. The operator $\mathrm{T}_P(-1)\oplus\mathrm{T}_P(-1)$, which belongs to $\pi_S(\overline{\mathfrak{A}}_{SDC}(PK+\Gamma PK,\Gamma))''$, can be characterized up to a multiplicative constant by its anticommutation property with $\mathrm{B}(f)$, $f\in PK+\Gamma PK$.

Theorem 9. *Let E be a partial basis projection with a finite odd Γ codimension and T be a Hilbert Schmidt operator such that $TE=ET=T$. Let*

$$S=T+1-\Gamma T\Gamma+(1/2)(1-E-\Gamma E\Gamma). \tag{10.8}$$

Then R_S is not a factor. Conversely, if R_S is not a factor, then S is of the form given by (10.8).

Proof. Let $e_1\cdots e_{2n+1}$ be a complete orthonormal system of Γ invariant vectors in $(1-E-\Gamma E\Gamma)K$ and E_0 be the projection on the subspace spanned by $e_{2j}+ie_{2j+1}$, $j=1,\cdots,n$. By setting $E_1=E+E_0$, $T_1=T+(1/2)E_0$, we obtain a case where the partial basis projection E_1 has a Γ-codimension 1.

Let

$$S'=\Gamma E_1\Gamma+(1/2)(1-E_1-\Gamma E_1\Gamma).$$

Then $S^{1/2}-(S')^{1/2}$ is in the Hilbert Schmidt class and hence R_S and $R_{S'}$ are $*$ isomorphic. By Lemma 10.3, where we set $P=\Gamma E_1\Gamma$, $R_{S'}$ is not a factor and hence R_S is not a factor.

If S is of the form given by (10.8) where the Γ codimension of E is finite and even and T is as before. Then the same argument as above shows that R_S is $*$ isomorphic to $R_{S'}$ where $S'=\Gamma E_1\Gamma$ is a basis projection. Hence R_S is a factor.

If S is not of the form given by (10.8) where the Γ-codimension of E is finite, then $S^{1/2}(1-S)^{1/2}$ is not in the Hilbert Schmidt class. Let $P_S'\equiv2(S\oplus(1-S))-P_S$. Then $P_S'-P_S$ is not in the Hilbert Schmidt class. In the proof of Lemma 4.11, Ψ_-, if nonvanishing, is a vector giving a vector states $\varphi_{P_S'}$ in the representation space associated with φ_{P_S}. By Theorem 6, we have $\Psi_-=0$ and hence from the proof of Lemma 4.11, R_S must be a factor. Q.E.D.

Appendix: Angle between Two Projections

We state a result concerning an angle operator between two projections which is essentially taken from [1]. If one of two projections has either dimension 1 or codimension 1, then the nonzero eigenvalue of the angle operator coincides with the geometrical angle between corresponding subspaces.

Theorem 10. *Let P_1 and P_2 be projection operators on a complex Hilbert space K. Let $\theta(P_1, P_2)$ be defined by*

$$0 \leqq \theta(P_1, P_2) \leqq \pi/2, \tag{A.1}$$

$$\sin\theta(P_1, P_2) = |P_1 - P_2|. \tag{A.2}$$

Let $\mathrm{E}(0)$ *and* $\mathrm{E}(\pi/2)$ *denote eigenprojections of* $\theta(P_1, P_2)$ *for eigenvalues* 0 *and* $\pi/2$, $E = \mathrm{E}(0) + \mathrm{E}(\pi/2)$, *and*

$$v_1 = \cos\theta(P_1, P_2), \quad v_2 = \sin\theta(P_1, P_2). \tag{A.3}$$

Let

$$\mathrm{R}(P_1/P_2) = v_1 + v_1^{-1}[P_1, P_2], \tag{A.4}$$

$$\mathrm{I}(P_1, P_2) = v_1^{-1}(P_1 + P_2 - 1). \tag{A.5}$$

Let

$$\mathrm{u}_{11}(P_1/P_2) = P_1(1-E), \tag{A.6}$$

$$\mathrm{u}_{22}(P_1/P_2) = (1-P_1)(1-E), \tag{A.7}$$

$$\mathrm{u}_{12}(P_1/P_2) = (v_1 v_2)^{-1} P_1 P_2 (1-P_1), \tag{A.8}$$

$$\mathrm{u}_{21}(P_1/P_2) = (v_1 v_2)^{-1}(1-P_1) P_2 P_1. \tag{A.9}$$

Let $P \wedge P'$ denote the projection on $PK \cap P'K$ if P and P' are projections. Let $\mathfrak{R}$ be the von Neumann algebra $\{P_1, P_2\}''$ generated by P_1 and P_2 and $\mathfrak{Z}$ be its center $\mathfrak{R} \cap \mathfrak{R}'$.

Then $\mathfrak{Z}$ is generated by $\theta(P_1, P_2) = \theta(P_2, P_1)$, EP_1 and EP_2. $\mathfrak{R}$ is generated by its center $\mathfrak{Z}$ und $\mathrm{u}_{ij}(P_1/P_2)$, $i, j = 1, 2$ satisfying

$$\mathrm{u}_{ij}(P_1/P_2)^* = \mathrm{u}_{ji}(P_1/P_2), \tag{A.10}$$

$$\mathrm{u}_{ij}(P_1/P_2)\mathrm{u}_{kl}(P_1/P_2) = \delta_{jk}\mathrm{u}_{il}(P_1/P_2). \tag{A.11}$$

$\mathfrak{R}E$ is commutative and is generated by four minimal projections $P_1 \wedge P_2$, $P_1 \wedge (1-P_2)$, $(1-P_1) \wedge P_2$ and $(1-P_1) \wedge (1-P_2)$ where

$$\mathrm{E}(0) = P_1 \wedge P_2 + (1-P_1) \wedge (1-P_2),$$
$$\mathrm{E}(\pi/2) = P_1 \wedge (1-P_2) + (1-P_1) \wedge P_2 .$$

$\mathfrak{R}(1-E)$ is a tensor product of the center $\mathfrak{Z}(1-E)$ and the type I_2 factor generated by the matrix unit $\mathrm{u}_{ij}(P_1/P_2)$. Relative to this matrix unit, we have

$$P_1(1-E) = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{A.12}$$

$$P_2(1-E) = \begin{pmatrix} v_1^2, & v_1 v_2 \\ v_1 v_2, & v_2^2 \end{pmatrix}, \tag{A.13}$$

$$\mathrm{R}(P_1/P_2)(1-E) = \begin{pmatrix} v_1, & v_2 \\ -v_2, & v_1 \end{pmatrix}, \tag{A.14}$$

$$\mathrm{I}(P_1, P_2)(1-E) = \begin{pmatrix} v_1, & v_2 \\ v_2, & -v_1 \end{pmatrix}. \tag{A.15}$$

The operator $\mathrm{R}(P_1/P_2)$ *satisfies*

$$\mathrm{R}(P_1/P_2)^* = \mathrm{R}(P_2/P_1), \tag{A.16}$$

$$\mathrm{R}(P_1/P_2)\mathrm{R}(P_1/P_2)^* = \mathrm{R}(P_1/P_2)^*\mathrm{R}(P_1/P_2) = 1 - E(\pi/2), \tag{A.17}$$

$$\mathrm{R}(P_1/P_2)P_2\mathrm{R}(P_1/P_2)^* = P_1 - P_1 \wedge (1-P_2), \tag{A.18}$$

$$\mathrm{R}(P_1/P_2)^*P_1\mathrm{R}(P_1/P_2) = P_2 - P_2 \wedge (1-P_1). \tag{A.19}$$

The operator $\mathrm{I}(U_1, U_2)$ *satisfies*

$$\mathrm{I}(P_1, P_2)^* = \mathrm{I}(P_1, P_2) = \mathrm{I}(P_2, P_1), \tag{A.20}$$

$$\mathrm{I}(P_1, P_2)^2 = 1 - E(\pi/2), \tag{A.21}$$

$$\mathrm{I}(P_1, P_2)\mathrm{u}_{ij}(P_1/P_2)\mathrm{I}(P_1, P_2) = \mathrm{u}_{ij}(P_2/P_1). \tag{A.22}$$

Proof. Since $-1 \leqq P_1 - P_2 \leqq 1$, we have $0 \leqq |P_1 - P_2| \leqq 1$ and hence $\theta(P_1, P_2)$ is uniquely well defined by (A. 2) and (A. 1). By a direct calculation,

$$[(P_1-P_2)^2, P_1] = [(P_1-P_2)^2, P_2] = 0 \tag{A.23}$$

and hence $\theta(P_1, P_2) \in \mathfrak{Z}$.

If $(P_1 - P_2)f = f$, then $(f, P_1 f) \leqq \|f\|$ and $(f, P_2 f) \geqq 0$ imply

$\|P_2 f\|=0$ and hence $P_2 f=0$, $P_1 f=f$. If $(P_1-P_2)f=-f$, we obtain $P_2 f=f$, $P_1 f=0$. Converses are obviously true. Hence we have

$$\text{(A. 24)} \qquad \mathrm{E}(\pi/2)=P_1\wedge(1-P_2)+(1-P_1)\wedge P_2 .$$

Next assume $P_1 f=P_2 f$. Let $g_1=P_1 f$, $g_2=(1-P_1)f$. Then $P_1 g_1 =g_1=P_2 g_1$ and $(1-P_1)g_2=g_2=(1-P_2)g_2$. Hence $g_1\in(P_1\wedge P_2)K$, $g_2\in\{(1-P_1)\wedge(1-P_2)\}K$ and $f=g_1+g_2$. Conversely, such f satisfies $(P_1-P_2)f=0$. Hence

$$\text{(A. 25)} \qquad \mathrm{E}(0)=P_1\wedge P_2+(1-P_1)\wedge(1-P_2).$$

Obviously $P_1\wedge P_2$, $(1-P_1)\wedge P_2$, $P_1\wedge(1-P_2)$ and $(1-P_1)\wedge(1-P_2)$ belong to $\mathfrak{Z}$.

From (A. 23), (A. 24) and definitions, we have

$$\text{(A. 26)} \qquad \mathrm{u}_{ij}(P_1/P_2)E=E\,\mathrm{u}_{ij}(P_1/P_2)=0.$$

By using identities

$$\text{(A. 27)} \qquad P_1(P_1-P_2)^2=P_1(1-P_2)P_1=P_1-P_1P_2P_1$$

$$\text{(A. 28)} \qquad P_2(P_1-P_2)^2=P_2(1-P_1)P_2=P_2-P_2P_1P_2$$

$$\text{(A. 29)} \qquad (1-P_1)(P_1-P_2)^2=(1-P_1)P_2(1-P_1),$$

$$\text{(A. 30)} \qquad (1-P_2)(P_1-P_2)^2=(1-P_2)P_1(1-P_2),$$

we obtain (A. 10) and (A. 11). This also shows that $\mathrm{u}_{ij}(P_1/P_2)$ are everywhere defined bounded operators. [The range of $P_1P_2(1-P_1)$ and $(1-P_1)P_2P_1$ is in $(1-E)K$, where $(v_1v_2)^{-1}$ is uniquely defined].

By using (A. 27) and (A. 29), we have (A. 13). (A. 12), (A. 14) and (A. 15) are immediate from the definition. (A. 16) ~ (A. 22) are obtained from (A. 12) ~ (A. 15).

$\mathfrak{R}$ is generated by P_1 and P_2 and hence by $\theta(P_1, P_2)$, EP_1, EP_2 and $\mathrm{u}_{ij}(P_1/P_2)$. Since $E\mathrm{u}_{ij}(P_1/P_2)=0$, $\mathfrak{R}E$ is generated by $E\theta(P_1, P_2)$, EP_1, EP_2 and hence as is stated in the Lemma.

On $(1-E)K$, $\mathrm{u}_{ij}(P_1/P_2)$ generates a type I_2 factor and hence $\mathfrak{R}(1-E)$ is as is stated in the Lemma and $\mathfrak{Z}$ is generated by $\theta(P_1, P_2)$, EP_1 and EP_2. Q.E.D.

As an immediate application of Theorem 10, we have

Theorem 11. *Let P_1 and P_2 be basis projections on K relative to Γ. Then*

$$\Gamma\theta(P_1, P_2)\Gamma=\theta(P_1, P_2), \tag{A. 31}$$

$$\Gamma \mathrm{R}(P_1/P_2)\Gamma=\mathrm{R}(P_1/P_2), \tag{A. 32}$$

$$\Gamma \mathrm{I}(P_1, P_2)\Gamma=-\mathrm{I}(P_1, P_2), \tag{A. 33}$$

$$\Gamma \mathrm{u}_{ij}(P_1/P_2)\Gamma=-\mathrm{u}_{ji}(P_1/P_2) \qquad (i\neq j), \tag{A. 34}$$

$$\Gamma \mathrm{u}_{ii}(P_1/P_2)\Gamma=\mathrm{u}_{jj}(P_1/P_2) \qquad (i\neq j), \tag{A. 35}$$

$$\Gamma(P_1\wedge P_2)\Gamma=(1-P_1)\wedge(1-P_2), \tag{A. 36}$$

$$\Gamma(P_1\wedge(1-P_2))\Gamma=(1-P_1)\wedge P_2. \tag{A. 37}$$

There exists an antiunitary involution T which commutes with $\theta(P_1, P_2)$, $\mathrm{u}_{ij}(P_1, P_2)$ and Γ.

The linear operator $\widehat{\mathrm{R}}(P_1, P_2)$ defined by

$$\widehat{\mathrm{R}}(P_1/P_2)\mathrm{E}(\pi/2)=T\Gamma\mathrm{E}(\pi/2), \tag{A. 38a}$$

$$\widehat{\mathrm{R}}(P_1/P_2)(1-\mathrm{E}(\pi/2))=\mathrm{R}(P_1/P_2), \tag{A. 38b}$$

is unitary, commutes with $\theta(P_1, P_2)$ and Γ and satisfies

$$\widehat{\mathrm{R}}(P_1, P_2)P_2\widehat{\mathrm{R}}(P_1, P_2)^*=P_1. \tag{A. 39}$$

Γ on $(1-E)K$ is given by

$$\Gamma(1-E)=T\varepsilon(\mathrm{u}_{12}(P_1/P_2)-\mathrm{u}_{21}(P_1/P_2)) \tag{A. 40}$$

where ε is a linear operator, commutes with $\theta(P_1, P_2)$, $\mathrm{u}_{ij}(P_1, P_2)$, T ana Γ and satisfies $\varepsilon^*=-\varepsilon$, $\varepsilon^2=E-1$. The multiplicity of $\theta(P_1, P_2)$ at any point in $(0, \pi/2)$ is a multiple of 4.

Proof. From $\Gamma P_i\Gamma=1-P_i$ and definitions, we obtain (A. 31) ~ (A. 37). We shall prove the existence of the operator T and its property.

Let e_1 be any Γ invariant vector. Let $\mathrm{K}(e_1)$ be a closed real linear space generated by $\{\sum\mathcal{P}_{ij}u_{ij}(P_1/P_2)+\mathcal{P}E\}e_1$ where $\mathcal{P}_{ij}$ and $\mathcal{P}$ are any bounded selfadjoint operator in $\mathfrak{Z}$. Then $\mathrm{K}(e_1)+i\mathrm{K}(e_1)$ is a closed subspace of K, containing e_1 and invariant under Γ and $\mathfrak{R}$. Furthermore, for any Ψ_1 and Ψ_2 in $\mathrm{K}(e_1)$, (Ψ_1, Ψ_2) is real. Note that

$(e_1, \mathcal{P}_{ij}\mathrm{u}_{ij}(P_1/P_2)e_1)=0$ if $i\neq j$ due to (A. 34).

If mutually orthogonal subspaces $\mathrm{K}(e_\nu)+i\mathrm{K}(e_\nu)$ having such properties are given for $\nu<\nu_0$, then by choosing any Γ invariant vector e_{ν_0} in $(\bigcup_{\nu<\nu_0}\{\mathrm{K}(e_\nu)+i\mathrm{K}(e_\nu)\})^{\perp}$, we can obtain $\mathrm{K}(e_{\nu_0})+i\mathrm{K}(e_{\nu_0})$, which is orthogonal to $\mathrm{K}(e_\nu)+i\mathrm{K}(e_\nu)$, $\nu<\nu_0$ and has such properties. By induction, the total Hilbert space is a direct sum of such $\mathrm{K}(e_\nu)+i\mathrm{K}(e_\nu)$. Let $T\sum(f_\nu+ig_\nu)=\sum(f_\nu-ig_\nu)$ for $f_\nu, g_\nu\in\mathrm{K}(e_\nu)$. Then T is an antiunitary involution commuting with $\theta(P_1, P_2)$, $\mathrm{u}_{ij}(P_1, P_2)$ and Γ.

The statements concerning $\widehat{\mathrm{R}}(P_1/P_2)$ and ε are immediate where ε is defined by $\Gamma T(\mathrm{u}_{12}(P_1/P_2)-\mathrm{u}_{21}(P_1/P_2))$. Since $T\varepsilon$ restricted to $(1-E)K$ is an antiunitary operator, commuting with $\theta(P_1, P_2)$ and P_1 and satisfying $(T\varepsilon)^2=-(1-E)$, $\theta(P_1, P_2)$ restricted to $P_1(1-E)$ has an even multiplicity. Since $\theta(P_1, P_2)$ restricted to $1-P_1$ has the same multiplicity as $\theta(P_1, P_2)$ restricted to P_1 due to $\Gamma P_1\Gamma=1-P_1$, and $[\theta(P_1, P_2), \Gamma]=0$, the multiplicity of $\theta(P_1, P_2)$ at any point in $(0, \pi/2)$ must be a multiple of 4. Q.E.D.

Lemma A. *Let P and P' be basis projections. Then there exists a partial basis projection F and a partial isometry u, both commuting with P and P', such that $F+\Gamma F\Gamma=1-\mathrm{E}(0)-\mathrm{E}(\pi/2)$, $u^*u=F$ and $uu^*=\Gamma F\Gamma$.*

Proof. Use the notation in the proof of Theorem 11. The operator ε has at most three eigenvalues 0, i and $-i$. The eigenprojection for 0 is $1-\mathrm{E}(0)-\mathrm{E}(\pi/2)$. Let F be an eigenprojection for i. Since $[\Gamma, \varepsilon]=0$, $\Gamma F\Gamma$ must be an eigenprojection for $-i$ and hence F is a partial basis projection commuting with $\mathfrak{R}$.

Next we modify the construction of $\mathrm{K}(e_\nu)$ as follows. We restrict our attention to $(1-\mathrm{E}(0)-\mathrm{E}(\pi/2))K$. Let $\mathrm{K}(e_\nu)$, $\nu<\nu_0$ be given. Then choose a unit vector e'_{ν_0} in $F(\bigcup_{\nu<\nu_0}\{\mathrm{K}(e_\nu)+i\mathrm{K}(e_\nu)\})^{\perp}$. Let $\sqrt{2}\,e_{\nu_0}=e'_{\nu_0}+\Gamma e'_{\nu_0}$ and $\sqrt{2}\,e_{\nu_0+1}=i(e'_{\nu_0}-\Gamma e'_{\nu_0})$. Since $\varepsilon e'_{\nu_0}=ie'_{\nu_0}$, $\varepsilon\Gamma e'_{\nu_0}=-i\Gamma e'_{\nu_0}$ and $[\varepsilon, \mathfrak{R}]=0$, $(\mathfrak{R}e'_{\nu}, \mathfrak{R}\Gamma e'_{\nu_0})=0$ and hence $\mathrm{K}(e_{\nu_0})\perp\mathrm{K}(e_{\nu_0+1})$. Note that $(\bigcup_{\nu<\nu_0}\{\mathrm{K}(e_\nu)+i\mathrm{K}(e_\nu)\})^{\perp}$ is invariant under F and $F\neq 0$ on this subspace unless $F+\Gamma F\Gamma=1-E$ is 0 on this subspace, which occurs only if this subspace is EK.

We define u' to be 1 on $K(e_{\nu_0})$, -1 on $K(e_{\nu_0+1})$ and 0 on EK. Then $u = u'F$ commutes with $\mathfrak{K}$ and $u^*u=F$, $uu^*=\Gamma F\Gamma$. Q.E.D.

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