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Asymptotic Ratio Set and Property L'_{λ}

By

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Abstract

Powers' property L_{λ} is strengthened by requiring the simultaneous validity over a finite number of states. It is then shown that a von Neumann algebra R on a separable space has the modified property—called the property L'_{λ}—if and only if $\lambda(1-\lambda)^{-1}$ is in the asymptotic set $r_{\infty}(R)$, where $0 \leqq \lambda \leqq 1/2$. It is also noted that any finite continuous von Neumann algebra has the property $L_{1/2}$.

The closedness of $r_{\infty}(R)$ for any von Neumann algebra R on a separable space follows as a corollary.

§1. Introduction

Powers has introduced the following property of a von Neumann algebra to reformulate his earlier classification theory of factors [5, 6].

Definition 1.1. *A von Neumann algebra R has the property L_{λ} $(0 \leqq \lambda \leqq 1/2)$ if, for every $\varepsilon > 0$ and any normal state ω of R, there exists an operator N in R satisfying the following conditions*:

(a) $N^2 = 0$, $N^*N + NN^* = 1$.

(b) *For any $Q \in R$,*

$$|(1-\lambda)\omega(QN) - \lambda\omega(NQ)| \leqq \varepsilon \|Q\|. \tag{1.1}$$

The present author and Woods have introduced the asymptotic ratio set $r_{\infty}(R)$ as an invariant for R under $*$-isomorphisms [1]. It consists of all $x \in [0, \infty)$ such that $R \sim R \otimes R_x$ ($\sim$ denotes a $*$-isomorphism), where R_x, $x \in [0, \infty)$, is a specific one parameter family of factors and $R_x \sim R_{(x^{-1})}$ for $x \neq 0$.

It is immediately seen that R has the property L_{λ} if $x_{\lambda} \equiv \lambda(1-\lambda)^{-1} \in r_{\infty}(R)$. The converse is not true for $\lambda = 1/2$ (Lemma 6.1) but the

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situation for $\lambda\neq 1/2$ is not known. (The converse holds for $\lambda=0$.)

To find the properties similar to Definition 1.1 and equivalent to $\lambda(1-\lambda)^{-1}\in r_\infty(R)$, we strengthen the property L_λ as follows:

Definition 1.2. *R has the property L'_λ if for every $\varepsilon>0$ and a finite number of normal states $\omega_1, \cdots, \omega_n$ of R, there exists an operator N in R satisfying the following conditions*:

(a) $N^2=0$, $N^*N+NN^*=1$.

(b) *For any $Q\in R$ and $j=1, \cdots, n$,*

$$|(1-\lambda)\omega_j(QN)-\lambda\omega_j(NQ)|\leqq\varepsilon\|Q\|. \tag{1.2}$$

Obviously the property L'_λ implies the property L_λ. For this strengthened property, we have

Theorem 1.3. *A von Neumann algebra R on a separable space has the property L'_λ if and only if $\lambda(1-\lambda)^{-1}\in r_\infty(R)$.*

The property $L'_{1/2}$ for a finite von Neumann algebra R on a separable space can be phrased as the existence of a weakly central sequence of type I_2 factors. Theorem 1.3 for this case is slightly stronger than Theorem 1 (also see Theorem 2) in [7].

§2. Property L_λ and Type

Lemma 2.1. *If a von Neumann algebra R has the property L_λ, $\lambda\neq 1/2$, then R does not have a finite part.*

Proof. Assume that ϕ is a normal normalized finite trace on R. Since $N^*N+NN^*=1$, we have $\phi(N^*N)=\phi(NN^*)=1/2$. From the property L_λ with $Q=N^*$, we have

$$|\phi(N^*N)|<|1-2\lambda|^{-1}\varepsilon \tag{2.1}$$

for arbitrary $\varepsilon>0$. This is in contradiction with $\phi(N^*N)=1/2$ if $\lambda\neq 1/2$. Q. E. D.

Lemma 2.2. *If a von Neumann algebra R has the property L_λ, $0<\lambda<1/2$, then R does not have a semifinite part.*

Proof. Assume that ϕ is a normal semifinite trace in R and E

be a projection in R such that $0<\phi(E)<\infty$. Let

(2.2) $$\omega(Q)=\phi(E)^{-1}\phi(EQE), \qquad Q\in R$$

is a state on R and has the following properties.

(2.3) $$\omega(EQ)=\omega(QE)=\omega(Q),$$

(2.4) $$\omega(Q_1EQ_2)=\omega(Q_2EQ_1).$$

From the property L_λ, we have

(2.5) $$|(1-\lambda)\omega(N^*N)-\lambda\omega(NN^*)|<\varepsilon,$$

(2.6) $$|(1-\lambda)\omega(EN^*N)-\lambda\omega(NEN^*)|<\varepsilon,$$

(2.7) $$|(1-\lambda)\omega(N^*EN)-\lambda\omega(NN^*E)|<\varepsilon.$$

By using (2.3) in (2.6) and (2.7), adding $(1-\lambda)$ times (2.6) and λ times (2.7) together, using the triangle inequality and (2.4), we obtain

(2.8) $$|(1-\lambda)^2\omega(N^*N)-\lambda^2\omega(NN^*)|<\varepsilon.$$

We also have

(2.9) $$\omega(N^*N)+\omega(NN^*)=1,$$

from the property (a) for N.

From (2.5) and (2.9), we have $|\lambda-\omega(N^*N)|<\varepsilon$ and $|(1-\lambda)-\omega(NN^*)|<\varepsilon$. Substituting these into (2.8), we have

(2.10) $$|\lambda(1-\lambda)(1-2\lambda)|<[1+\lambda^2+(1-\lambda)^2]\varepsilon.$$

Since $\varepsilon>0$ is arbitrary, $\lambda=0$ or 1 or 1/2, which contradicts with $0<\lambda<1/2$. Q.E.D.

Lemma 2.3. *If a von Neumann algebra R has the property $L_{1/2}$, then R does not have a discrete part.*

Proof. Let E be an abelian projection in R and ϕ be a normal state of $R, \phi(E)\neq 0$. Let $\phi_1(Q)=\phi(EQE)\phi(E^{-1})$. Since $R_E=$center of $R_E=$(center of $R)_E$, there exists a central element $F(N)$ for each $N\in R$ such that $\|F(N)\|\leqq\|N\|$ and $EF(N)=ENE$. Let N be such that $N^2=0$, $NN^*+N^*N=1$ and

$$(2.11) \qquad |\phi_1(QN)-\phi_1(NQ)| \leqq \varepsilon\|Q\|$$

for all $Q\in R$. Since $NN^*N=N$ and $N(NN^*)=0$, we have from (2.11) with $Q=NN^*F(N)^*$

$$(2.12) \qquad |\phi_1(F(N)^*N)| \leqq \varepsilon.$$

From (2.11) with $Q=N^*E$, we also have

$$(2.13) \qquad |\phi_1(N^*EN)-\phi_1(NN^*)| \leqq \varepsilon.$$

From (2.11) with $Q=N^*$, we have

$$(2.14) \qquad |\phi_1(N^*N)-\phi_1(NN^*)| \leqq \varepsilon.$$

Since $NN^*+N^*N=1$ and $\phi_1(F(N)^*N)=\phi_1(N^*EN)$, we have from (2.12), (2.13) and (2.14)

$$|\phi_1(1)| \leqq 5\varepsilon.$$

Since ε is arbitrary positive number, this is a contradiction. Q.E.D.

Corollary 2.4. *If a von Neumann algebra R has the property L'_λ, then the following conclusions hold.*

(1) *If $0<\lambda<1/2$, then R is purely infinite.*

(2) *If $\lambda=0$, then R is properly infinite.*

(3) *If $\lambda=1/2$, then R is continuous.*

This follows trivially from Lemmas 2.1, 2.2, and 2.3 because the property L'_λ implies the property L_λ.

§3. Sufficiency

Lemma 3.1. *If $R\sim R\otimes R_x$, $x=\lambda(1-\lambda)^{-1}$, $0\leqq\lambda\leqq 1/2$, then R has the property L'_λ.*

Proof. Let $H=H_a\otimes H_b$, $R=R_x\otimes R_b$. Let a normal state ω_l of R and $\varepsilon>0$ be given. Since ω is normal, there exist $\Omega_{jl}\in H$ such that for $Q\in R$

$$(3.1) \qquad \omega_l(Q)=\sum_j(\Omega_{jl}, Q\Omega_{jl}).$$

Since $\Sigma\|\Omega_{jl}\|^2=\omega_l(1)=1$, there exists N_l for any given $\varepsilon'>0$ such that

(3.2) $$\sum_{j>N_l} \|\Omega_{jl}\|^2 < \varepsilon'.$$

Let $H_a = \otimes(H_k, \Phi_k)$, $R_x = \otimes(M_k, \Phi_k)$, dim $H_k = 4$, Sp $(\Phi_k, M_k) = (1-\lambda, \lambda)$. By Lemma 3.1 of [4], there exists K for any $\varepsilon'' > 0$ such that

(3.3) $$\|\Omega_{jl} - \Omega'_{jl}\| < \varepsilon'', \quad \Omega'_{jl} = \Omega''_{jl} \otimes \{\bigotimes_{k>K} \Phi_k\},$$

for $j = 1, \cdots, N_l$ where $\Omega''_{jl} \in \{\bigotimes^{K}_{k=1} H_k\} \otimes H_b$. We set

(3.4) $$\omega''_l(Q) = \sum_{j=1}^{N_l} (\Omega'_{jl}, Q\Omega'_{jl}), \quad \omega'_l(Q) = \omega''_l(1)^{-1} \omega''_l(Q)$$

for $Q \in R$.

Let $k > K$, $H_k = H_{k1} \otimes H_{k2}$, $M_k = \mathscr{B}(H_{k1}) \otimes 1$ and

(3.5) $$\Phi_k = \lambda^{1/2} \Phi_{11} \otimes \Phi_{21} + (1-\lambda)^{1/2} \Phi_{12} \otimes \Phi_{22}, \quad \|\Phi_{ij}\| = 1.$$

Let N and N' be operators in M_k and M'_k such that

(3.6) $$N\Phi_{1i} \otimes \Phi_{2j} = \delta_{i1} \Phi_{12} \otimes \Phi_{2j},$$

(3.7) $$N'\Phi_{1i} \otimes \Phi_{2j} = \delta_{j1} \Phi_{1i} \otimes \Phi_{22}.$$

Then we have

(3.8) $$(1-\lambda)^{1/2} N\Phi_k = \lambda^{1/2} (N')^* \Phi_k,$$

(3.9) $$\lambda^{1/2} N^* \Phi_k = (1-\lambda)^{1/2} N' \Phi_k.$$

Hence we have

$$\begin{aligned}(1-\lambda)(\Omega'_{jl}, QN\Omega'_{jl}) &= (1-\lambda)^{1/2} \lambda^{1/2} (\Omega'_{jl}, Q(N')^* \Omega'_{jl}) \\ &= (1-\lambda)^{1/2} \lambda^{1/2} (N'\Omega'_{jl}, Q\Omega'_{jl}) \\ &= \lambda(N^* \Omega'_{jl}, Q\Omega'_{jl}) \\ &= \lambda(\Omega'_{jl}, NQ\Omega'_{jl}).\end{aligned}$$

Therefore we have

(3.10) $$(1-\lambda)\omega'_l(QN) = \lambda \omega'_l(NQ).$$

From (3.1)-(3.4), we have $\|\omega_l - \omega'_l\| < \varepsilon$ for sufficiently small ε' and ε''. Hence N has the properties (a) and (b) of Definition 1.2.

§4. Necessity - Properly Infinite Case

Lemma 4.1. *If R has the property L_λ, then for any $\varepsilon>0$ and a normal state ω of R, there exist a type I_2 factor N_1 in R and a normal state ω_1 of R satisfying the following conditions*:

(1) $R=N_1\otimes\widetilde{R}$ *where* $\widetilde{R}=N_1'\cap R$.

(2) $\omega_1(Q_1Q_2)=\omega_{10}(Q_1)\omega(Q_2)$ *for* $Q_1\in N_1, Q_2\in\widetilde{R}$, *where* ω_{10} *is a state on* N_1 *such that* $\mathrm{Sp}(\omega_{10}, N_1)=(1-\lambda, \lambda)$.

(3) $\|\omega-\omega_1\|\leqq\varepsilon$.

Proof. Let N be the operator in R satisfying (a) and (b) with ε replaced by ε_1 in definition 1.2. Let N_1 be the type I_2 factor generated by N. Since $Q_{ij}\equiv\sum_k u_{ki}Qu_{jk}\in\widetilde{R}$ for any $Q\in R$, where $u_{11}=N^*N$, $u_{22}=NN^*$, $u_{21}=N$, $u_{12}=N^*$, and since $Q=\sum_{i,j}Q_{ij}u_{ij}$, we have (1).

Let $\tilde{\omega}$ be the restriction of ω to $\widetilde{R}$. Let ω_λ be the state on N_1 defined by $\omega_\lambda(N)=\omega_\lambda(N^*)=0$, $\omega_\lambda(N^*N)=\lambda$, $\omega_\lambda(NN^*)=1-\lambda$. Let $\omega_1=\omega_\lambda\otimes\tilde{\omega}$. Then ω_1 satisfies (2) by construction.

By setting $Q=N^*N\widetilde{Q}$ and $NN^*\widetilde{Q}$, $\widetilde{Q}\in\widetilde{R}$, in (1.1) and adding the resulting equation, we obtain

$$|\omega(\widetilde{Q}N)|\leqq 2\varepsilon_1\|\widetilde{Q}\|. \tag{4.1}$$

By setting $Q=N^*\widetilde{Q}$ in (1.1) and using $NN^*+N^*N=1$, we obtain

$$|\omega(\widetilde{Q}N^*N)-\lambda\omega(\widetilde{Q})|\leqq\varepsilon_1\|\widetilde{Q}\|, \tag{4.2}$$

$$|\omega(\widetilde{Q}NN^*)-(1-\lambda)\omega(\widetilde{Q})|\leqq\varepsilon_1\|\widetilde{Q}\|. \tag{4.3}$$

Hence for $Q=\sum_{ij}Q_{ij}u_{ij}$, we have

$$|\omega(Q)-\omega_1(Q)|\leqq\varepsilon_1[2\|Q_{12}\|+2\|Q_{21}\|+\|Q_{11}\|+\|Q_{22}\|]. \tag{4.4}$$

Since $\|Q\|\geqq\|Q_{ij}\|$ (Lemma 2.3 of [2]), we have

$$|\omega(Q)-\omega_1(Q)|\leqq 6\varepsilon_1\|Q\|. \tag{4.5}$$

By taking $\varepsilon_1=\varepsilon/6$, we have (4). Q.E.D.

Lemma 4.2. *If R has the property L'_λ, then for any $\varepsilon>0$, a normal state ω of R and a finite number of σ-weakly continuous* *linear functionals* ϕ_j *on* R, *there exist a type* I_2 *factor* N_1 *in* R, *a normal state* ω_1 *of* R *and* σ*-weakly continuous linear functionals* ϕ_{j1} *on* R *satisfying the following conditions*:

(1) $R=N_1\otimes\widetilde{R}$ *where* $\widetilde{R}=N'_1\cap R$.

(2) $\omega_1(Q_1Q_2)=\omega_{10}(Q_1)\omega(Q_2)$ *for* $Q_1\in N_1$, $Q_2\in\widetilde{R}$, *where* ω_{10} *is a state on* N_1 *such that* Sp $(\omega_{10}, N_1)=(1-\lambda, \lambda)$.

(3) $\|\omega-\omega_1\|\leqq\varepsilon$.

(4) $\phi_{j1}(Q_1Q_2)=\omega_{10}(Q_1)\phi_j(Q_2)$ *for* $Q_1\in N_1$, $Q_2\in\widetilde{R}$.

(5) $\|\phi_j-\phi_{j1}\|\leqq\varepsilon$.

Proof. Any σ-weakly continuous linear functional ϕ_j can be decomposed into a linear combination of 4 states: $\phi_j=\lambda_a\phi_{ja}-\lambda_b\phi_{jb}+i(\lambda_c\phi_{jc}-\lambda_d\phi_{jd})$, $\lambda_a, \lambda_b, \lambda_c, \lambda_d\in[0, \infty)$. Hence, if $\phi_{ja}, \phi_{jb}, \phi_{jc}$ and ϕ_{jd} are approximated by $\phi'_{ja}, \phi'_{jb}, \phi'_{jc}$, and ϕ'_{jd} having the property (4) above (ϕ_j replaced by ϕ_{ja} etc.), then ϕ_j approximated by $\phi_{j1}=\lambda_a\phi'_{ja}-\lambda_b\phi'_{jb}+\lambda_b\phi'_{jb}+i(\lambda_c\phi'_{jc}-\lambda_d\phi'_{jd})$, which have the property (4).

Due to Property L'_λ, we can find N_1, ω_1 and $\phi'_{ja}, \phi'_{jb}, \phi'_{jc}, \phi'_{jd}$ as in the proof of Lemma 4.1. Q.E.D.

Lemma 4.3. *Let* R *be a properly infinite von Neumann algebra having the property* L'_λ. *For any* $\varepsilon_j>0$, $j\in N$, *a countable number of normal states* $\phi_j\in N$, *of* R *and another normal state* ω *of* R, *there exist mutually commuting type* I_2 *factors* N_j, $j\in N$, *in* R, *normal states* ϕ_{jk}, $j, k\in N$, $j\leqq k$, *of* R and *normal states* ω_j, $j\in N$, *of* R *satisfying the following conditions*:

(1) $R=N_1\otimes\cdots\otimes N_n\otimes\widetilde{R}_n$, $\widetilde{R}_n=(\bigcup_{j=1}^{n}N_j)'\cap R$.

(2) $\omega_n(Q_1\cdots Q_{n+1})=\{\prod_{j=1}^{n}\omega_{j0}(Q_j)\}\omega(Q_{n+1})$ *for* $Q_j\in N_j$, $j\leqq n$, *and* $Q_{n+1}\in\widetilde{R}_n$, *where* ω_{j0} *is a state on* N_j *with* $\mathrm{Sp}(\omega_{j0}, N_j)=(1-\lambda, \lambda)$.

(3) $\|\omega_{j-1}-\omega_j\|\leqq\varepsilon_j$, $j\in N$, *where* $\omega_0=\omega$.

(4) $\phi_{jn}(Q_1Q_2)=\omega_{n0}(Q_1)\phi_j(Q_2)$ *for* $Q_1\in N_n$ *and* $Q_2\in N'_n\cap R$, $n\geqq j$.

(5) $\|\phi_j-\phi_{jn}\|<\varepsilon_n$ for $j\leqq n$.

Proof. We find N_k, ω_k and ϕ_{jk} by induction on k. For $k=1$, such N_1, ω_1 and ϕ_{11} exists by Lemma 4.2. Assume now that N_k, ω_k and ϕ_{jk} are chosen for $j\leqq k\leqq n$.

For any state ϕ of R, let ϕ^n_{lm} be a σ-weakly continuous linear functional on $\widetilde{R}_n$ defined by $\phi^n_{lm}(Q)=\phi(Qu^n_{lm})$ where u^n_{lm} is a matrix unit of $N_1\otimes\cdots\otimes N_n$. We then have $\phi(Q)=\sum_{lm}\phi^n_{lm}(Q^n_{lm})$ where $Q^n_{lm}=\sum_k u^n_{kl}\,Qu^n_{mk}$. If $\widehat{\phi}^n_{lm}$ is a σ-weakly continuous linear functional on $\widetilde{R}_n$ satisfying $\|\phi^n_{lm}-\widehat{\phi}^n_{lm}\|<\varepsilon'$ and $\widehat{\phi}^n_{lm}(Q_1Q_2)=\omega_{n0}(Q_1)\phi^n_{lm}(Q_2)$ for $Q_1\in N'_{n+1}$, $Q_2\in N'_{n+1}\cap\widetilde{R}_n$, then $\widehat{\phi}(Q)=\sum_{lm}\widehat{\phi}^n_{lm}(Q^n_{lm})$ satisfies $\|\phi-\widehat{\phi}\|<\varepsilon''$ and $\widehat{\phi}(Q_1Q_2)=\omega_{n0}(Q_1)\phi(Q_2)$ for $Q_1\in N_{n+1}$, $Q_2\in N'_{n+1}\cap R$, where $\varepsilon''\to 0$ as $\varepsilon'\to 0$. $\widehat{\phi}$ is automatically a state of R.

Since R is properly infinite, $\widetilde{R}_n$ is properly infinite. Hence $\widetilde{R}_n\sim(N_1\otimes\cdots\otimes N_n)\otimes\widetilde{R}_n=R$. (1 in $\widetilde{R}_n$ has a partition into a finite number (2^n) of projections which are mutually equivalent.)

We can now use Lemma 4.2 for $\widetilde{R}_n$, $\widetilde{\omega}_n=\omega_n|\widetilde{R}_n=\omega|\widetilde{R}_n$ and $(\phi_j)^n_{lm}$, $j\leqq n+1$ and obtain N_{n+1}, $(\widehat{\phi_j})^n_{lm}$ and $\widetilde{\omega}'_{n+1}$ from which $\phi_{j(n+1)}$ is constructed as above $\widehat{\phi}$ and ω_{n+1} is constructed by $\omega_{n+1}=\widetilde{\omega}'_{n+1}\otimes\omega'_n$ where ω'_n is the restriction of ω_n to $N_1\otimes\cdots\otimes N_n$ which is the same as $\bigotimes_{j=1}^n\omega_{j0}$. We have $\|\omega_{n+1}-\omega_n\|\leqq\|\widetilde{\omega}'_{n+1}-\widetilde{\omega}\|\leqq\varepsilon_n$. Q.E.D.

Lemma 4.4. *Let R be a properly infinite von Neumann algebra on a separable space having the property L'_λ. Let $\varepsilon_j=2^{-j}\varepsilon$, $\varepsilon>0$ and ϕ_j be a countable dense subset of the set of all normal states. Then N_j and ω_j in Lemma 4.3 have the following properties*:

(α) *There exists a normal state $\omega_\infty=\lim_j\omega_j$ such that $\|\omega_\infty-\omega\|<\varepsilon$.*

(β) *$R=(\widetilde{R}_\infty\cup N_\infty)''$ where $N_\infty=(\bigcup_{j=1}^\infty N_j)''$, $\widetilde{R}_\infty=N'_\infty\cap R$.*

(γ) *$\omega_\infty(Q_1Q_2)=\omega_\infty(Q_1)\omega(Q_2)$ for $Q_1\in N_\infty$, $Q_2\in\widetilde{R}_\infty$.*

(δ) *$\omega_\infty(Q_1\cdots Q_n)=\prod_{j=1}^n\omega_{j0}(Q_j)$ for $Q_j\in N_j$.*

Proof. By (3) of Lemma 4.3, ω_j is a Cauchy sequence. Hence $\omega_\infty=\lim\omega_j$ exists and is a normal state of R. We also have $\|\omega_\infty-\omega\|\leqq\sum_{j=1}^\infty\|\omega_j-\omega_{j-1}\|<\varepsilon$. Thus we have ($\alpha$).

(δ) is already satisfied for ω_j, $j\geqq n$. Hence it holds for ω_∞.

If $Q_1\in N_1\otimes\cdots\otimes N_n$, then ($\gamma$) is satisfied by ω_j, $j\geqq n$ and hence by ω_∞. Since $N_\infty=(\bigcup_{j=1}^\infty N_j)''$, we have ($\gamma$).

To prove (β), we first define

(4.6) $$\phi_j^{nk}(Q_1Q_2)=\omega_{n+k}(Q_1)\phi_j(Q_2)$$

for $Q_1\in(\bigcup_{l=0}^{k}N_{n+l})''$ and $Q_2\in(N_1\cup\cdots\cup N_{n-1}\cup\widetilde{R}_{k+n})''$ where $n\geqq j$. Then we have

(4.7) $$\begin{aligned}\|\phi_j-\phi_j^{nk}\|&\leqq\|\phi_j-\phi_{j,n}\|+\sum_{l=1}^{k}\|\phi_j^{nl}-\phi_j^{n(l-1)}\|\\&\leqq\|\phi_j-\phi_{j,n}\|+\sum_{l=1}^{k}\|\phi_j-\phi_{j,n+l}\|\\&\leqq\sum_{l=0}^{k}\varepsilon_{n+l}<2^{1-n}\varepsilon.\end{aligned}$$

Here we used $\|\phi_a\otimes\phi_1-\phi_a\otimes\phi_2\|=\|\phi_1-\phi_2\|$.

Let v_{ij}^n be a matrix unit in N_n such that $\omega_{n0}(v_{11}^n)=\lambda$, $\omega_{n0}(v_{22}^n)=1-\lambda$, $\omega_{n0}(v_{ij}^n)=0$ for $i\neq j$. For $Q\in R$ define

(4.8) $$\tau_{ij}(N_n)Q=\sum_k v_{ki}^nQv_{jk}^n,$$

(4.9) $$\tau_nQ=\lambda\tau_{11}(N_n)Q+(1-\lambda)\tau_{22}(N_n)Q,$$

(4.10) $$\tau_{n,k}=\prod_{l=0}^{k}\tau_{n+l}.$$

By lemma 2.3 of [2], we have $\|\tau_{kl}(N_n)Q\|\leqq\|Q\|$ and hence $\|\tau_nQ\|\leqq\|Q\|$, $\|\tau_{n,k}Q\|\leqq\|Q\|$.

From our choice of V_{ij}^n, it follows that

(4.11) $$\phi_j^{nk}(\tau_{m,l}Q)=\phi_j^{nk}(Q)$$

if $n\leqq m$, $m+l\leqq n+k$. (Use $Q=\sum_{ij}v_{ij}^n\tau_{ij}(N_n)Q$.) We also have

(4.12) $$\tau_{n,k}Q\in(\bigcup_{l=0}^{k}N_{n+l})'\cap R.$$

From (4.7), we have

(4.13) $$|\phi_j(\tau_{m,l}Q)-\phi_j(Q)|<2^{-m}\varepsilon$$

for $j\leqq m$.

Since $\tau_{m,l}Q$ is uniformly bounded, $\bigcap_{k\_0}(\bigcup_{l>k}\tau_{m,l}Q)^-$ is non-empty where the closure is relative to the weak topology which is the same as the σ-weak topology for uniformly bounded set. Let Q_m be an element in this set. $\|Q_m\|\leqq\sup_l\|\tau_{m,l}Q\|\leqq\|Q\|$.

From (4.12), we have

(4.14) $$Q_m\in(\bigcup_{l=0}^{\infty}N_{m+l})'\cap R.$$

Since $(\bigcup_{n=1}^{m-1} N_n)''$ is a finite type I factor, the right hand side of (4.14) is the same as $(N_1 \cup \cdots \cup N_{m-1} \cup \widetilde{R}_\infty)''$. Hence

(4.15) $$Q_m \in (N_\infty \cup \widetilde{R}_\infty)''.$$

From (4.13) and the σ-weak continuity of ϕ_j, we have

(4.16) $$|\phi_j(Q_m) - \phi_j(Q)| \leqq 2^{-m}\varepsilon$$

for $j \leqq m$. Since $\{\phi_j\}$ is dense in the normal states of R and since Q_m is uniformly bounded, (4.16) implies

(4.17) $$Q = \text{w-}\lim_{m\to\infty} Q_m \in (N_\infty \cup \widetilde{R}_\infty)''.$$

This proves (β). Q. E. D.

Lemma 4.5. *Let R be a properly infinite von Neumann algebra on a separable space having the property L'_λ. Then $R \sim R \otimes R_x$ with $x = \lambda(1-\lambda)^{-1}$.*

Proof. Let ω be a normal state of R. By Lemma 4.4, we have a state ω_∞ and von Neumann subalgebras $\widetilde{R}_\infty$ and N_∞ such that $R = (\widetilde{R}_\infty \cup N_\infty)''$, $\widetilde{R}_\infty \subset N'_\infty$, ω_∞ is a product state for $(\widetilde{R}_\infty, N_\infty)$ and $\|\omega - \omega_\infty\| < \varepsilon$. Let F be the central carrier of ω_∞. Then this shows that $FR = F\widetilde{R}_\infty \otimes FN_\infty$ and further the property (δ) shows that $FN_\infty \sim R_x$ with $x = \lambda(1-\lambda)^{-1}$. Hence

(4.18) $$FR \sim F\widetilde{R}_\infty \otimes R_x \sim F\widetilde{R}_\infty \otimes R_x \otimes R_x \sim FR \otimes R_x.$$

For any central projection $F_1 \neq 0$, there exists a normal state ω whose central carrier is contained in F_1 and hence $\omega(F_1) = 1$. We then have $\omega_\infty(F_1) > 1 - \varepsilon$. Hence $F_1 F \neq 0$. Namely, for any central projection $F_1 \neq 0$, there exists a central subprojection $F_2 (= FF_1) \neq 0$ such that $F_2 R \sim F_2 R \otimes R_x$.

If (4.18) holds for F, then it obviously holds for any central subprojection of F. If (4.18) holds for $F = F_a$ and F_b, then it holds for $F_a F_b$ and hence it holds for $F_a \vee F_b = F_a(1-F_b) \oplus F_a F_b \oplus F_b(1-F_a)$. If (4.18) holds for any infinite (or finite) family of mutually disjoint central projections F_α, then it holds for their sum $\oplus_\alpha F_\alpha$. If (4.18) holds for any increasing chain of central projections F_α, then it holds for its limit: $\lim F_\alpha = \bigoplus_\alpha F_\alpha[\prod_{\beta<\alpha}(1-F_\beta)]$. Therefore, there exists a largest central projection F such that $FR\sim FR\otimes R_\tau$ and $(1-F)$ has no central subprojection E such that $ER\sim ER\otimes R_x$. By the above result, we then have $1-F=0$. Hence $R\sim R\otimes R_x$. Q.E.D.

§5. Necessity - Finite Case

By Corollary 2.4, we have only to consider $L'_{1/2}$. Let R be a finite von Neumann algebra in a separable Hilbert space and ϕ be a faithful normal normalized finite trace on R. We first recall some properties which we shall be using.

1° We define $\|A\|_2=\phi(A^*A)^{1/2}$. We have

$$\|A\|_2=\|A^*\|_2, \tag{5.1}$$

$$\begin{aligned} |\phi(Q_1AQ_2)| &= |\phi(Q_2Q_1A)| \\ &\leqq \phi(Q_2Q_1Q_1^*Q_2^*)^{1/2}\phi(A^*A)^{1/2} \\ &\leqq \|Q_1\|\,\|Q_2\|\,\|A\|_2, \end{aligned} \tag{5.2}$$

$$\begin{aligned} \|Q_1AQ_2\|_2 &= \phi([AQ_2]^*Q_1^*Q_1[AQ_2])^{1/2} \\ &\leqq \|Q_1\|\,\|AQ_2\|_2 = \|Q_1\|\,\|Q_2^*A^*\|_2 \\ &\leqq \|Q_1\|\,\|Q_2^*\|\,\|A^*\|_2 \\ &\leqq \|Q_1\|\,\|Q_2\|\,\|A\|_2. \end{aligned} \tag{5.3}$$

2° Let ω be a normal state of R. Then $\omega(Q)=\Sigma(\Omega_j, Q\Omega_j)$ where Ω_j is a vector in the representation space of R associated with ϕ. Let n be such that $\sum_{j>n}\|\Omega_j\|^2<\varepsilon/2$. Let $A_j\in R$ be such that $|\phi(A_j^*QA_j)-(\Omega_j, Q\Omega_j)|\leqq(2n)^{-1}\varepsilon\|Q\|$. Then $|\omega(Q)-\phi(AQ)|\leqq\varepsilon\|Q\|$ with $A=\sum_{j=1}^n A_jA_j^*$ and $|\phi(A)-1|\leqq\varepsilon$. Namely a state of the form $\phi(AQ)$ with $A\geqq 0$ and $\phi(A)=1$ is dense in the set of all normal state of R.

3° Let P_1 and P_2 be two projections. Then there exists an angle operator ([3], Appendix)

$$\Theta=\int_0^{\pi/2}\theta dE(\theta) \tag{5.4}$$

in the center of $\{P_1, P_2\}''$ and partial isometries $u_{ij}(i, j=1, 2)$ such that

(5.5) $u_{ij}u_{kl}=\delta_{jk}u_{il},\ u_{11}+u_{22}=E((0, \pi/2)),$

(5.6) $P_1=u_{11}+P_1\wedge P_2+P_1\wedge(1-P_2),$

(5.7) $P_2=P_1\wedge P_2+(1-P_1)\wedge P_2+\sum_{i,j}v_iv_ju_{ij},$

(5.8) $v_1=\cos\Theta,\ v_2=\sin\Theta,$

where $P_1\wedge P_2$ denotes the projection on $P_1H\cap P_2H$, $E(\{0\})=P_1\wedge P_2+(1-P_1)\wedge(1-P_2)$, $E(\{\pi/2\})=P_1\wedge(1-P_2)+(1-P_1)\wedge P_2$. We shall use the spectral projection $E(\Delta)$ for a Borel subset Δ of $[0, \pi/2]$ and the following operator

(5.9) $U=\cos\Theta+(\cos\Theta)^{-1}[P_1, P_2]$

which has the property

(5.10) $UP_2U^*=P_1UU^*,$

(5.11) $UU^*=U^*U=1-E(\{\pi/2\}).$

By using the measure $\mu(\Delta)\equiv\phi(E(\Delta))$ and $\mu_1(\Delta)=\phi(E(\Delta)P_1)$, we have

(5.12) $\|[P_1, P_2]\|_2^2=\int_0^{\pi/2}\cos^2\theta\sin^2\theta\, d\mu(\theta),$

(5.13) $\|P_2(U^*-1)P_1E(\Delta)\|_2^2=\int_\Delta(1-\cos\theta)^2 d\mu_1(\theta).$

(5.14) $\|P_2P_1E(\Delta)\|_2^2=\int_\Delta\cos^2\theta\, d\mu_1(\theta),$

(5.15) $\|(1-P_2)P_1E(\Delta)\|_2^2=\int_\Delta\sin^2\theta\, d\mu_1(\theta).$

Lemma 5.1. *Let R be a finite von Neumann algebra on a separable space. Let N_0 be a finite type I factor in R and N be an operator in R satisfying $N^2=0$ and $N^*N+NN^*=1$. Assume that $\|[N, Q]\|_2\leqq\varepsilon\|Q\|$ for every $Q\in N_0$. Then there exists N_ε in $(N_0'\cap R)$ such that $N_\varepsilon^2=0$, $N_\varepsilon^*N_\varepsilon+N_\varepsilon N_\varepsilon^*=1$ and $\|N-N_\varepsilon\|_2\leqq\varepsilon'(\varepsilon)$ where* $\varepsilon'(\varepsilon)\to0$ as $\varepsilon\to0$.

Proof. Let e_{ij} be a matrix unit of $N_0(i, j=1\cdots n)$. Let

(5.16) $N'=\sum_j e_{j1}Ne_{1j}.$

Then we have

(5.17) $$\|N'-N\|_2 \leqq \sum_j \|[N, e_{1j}]\|_2 \leqq n\varepsilon.$$

Let us consider two projections $P_1 = e_{11}$ and $N^*N = P_2$. Then use (5.9) to define

(5.18) $$W = NU^* e_{11} E([0, \pi/4]).$$

From (5.12), $\|[N^*N, e_{11}]\|_2 \leqq 2\varepsilon$ and $\mu(\Delta) \geqq \mu_1(\Delta)$ for any Δ, we have

(5.19) $$\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\mu_1(\theta) \leqq 4\varepsilon^2.$$

From (5.13) and (5.14) we obtain, by using $N^*N = P_2^* P_2$,

(5.20) $$\|N(U^*-1)e_{11}E([0, \pi/4])\|_2^2 = \int_{[0, \pi/4]} (1-\cos\theta)^2 d\mu_1(\theta)$$
$$\leqq \int_{[0, \pi/4]} \cos^2\theta \sin^2\theta \, d\mu_1(\theta),$$

(5.21) $$\|Ne_{11}\{1-E([0, \pi/4])\}\|_2^2 = \int_{(\pi/4, \pi/2]} \cos^2\theta \, d\mu_1(\theta)$$
$$\leqq 2\int_{(\pi/4, \pi/2]} \cos^2\theta \sin^2\theta \, d\mu_1(\theta).$$

Hence we have as a sum of orthogonal vectors

(5.22) $$\|W - Ne_{11}\|_2 = 2\sqrt{2}\,\varepsilon.$$

From the definition, W is partially isometric and

(5.23) $$W^* W = e_{11} E([0, \pi/4]).$$

We now consider two projections $P'_1 = E([\pi/4, \pi/2])e_{11}$ and $P'_2 = WW^*$. From (5.22), we have

(5.24) $$\|P'_2 - Ne_{11}N^*\|_2 \leqq 4\sqrt{2}\,\varepsilon.$$

We also have from $NN^* = (1-P_2)$ and (5.15)

(5.25) $$\|N^* e_{11}\{1-E((\pi/4, \pi/2])\}\|_2^2$$
$$= \int_{[0, \pi/4]} \sin^2\theta \, d\mu_1(\theta) \leqq 8\varepsilon^2.$$

Therefore

(5.26) $$\|[P'_1, Ne_{11}N^*]\|_2 \leqq \|[e_{11}, Ne_{11}N^*]\|_2 + 4\sqrt{2}\,\varepsilon$$
$$\leqq (2+4\sqrt{2})\varepsilon,$$

(5.27) $$\|[P'_1, P'_2]\|_2 \leqq (2+12\sqrt{2})\varepsilon.$$

We now define

(5.28) $$W' = E'([0, \pi/4]) P'_1 U' W.$$

As before we obtain $\|W' - P'_1 W\|_2 \leqq (24+2\sqrt{2})\varepsilon$ from (5.27). Since $\|(e_{11} - P'_1) W\|_2 \leqq 2\sqrt{2}\varepsilon$ by (5.25), we have

(5.29) $$\|W' - e_{11} W\|_2 \leqq (24+4\sqrt{2})\varepsilon.$$

From (5.23) and the definition, we have

(5.30) $$(W')^* W' \subset E([0, \pi/4]) e_{11},$$

(5.31) $$W'(W')^* = E'([0, \pi/4]) E((\pi/4, \pi/2]) e_{11}.$$

Since the entire construction is done in $\{N, N^*, e_{11}\}''$, we have $W' \in R$.

We now estimate the size of the projection

(5.32) $$P = e_{11} - W'(W')^* - (W')^* W'.$$

From (5.22) and (5.29), we have

(5.33) $$\|W' - e_{11} N e_{11}\|_2 \leqq (24+6\sqrt{2})\varepsilon.$$

From $N^*N + NN^* = 1$ and $\|[e_{11}, N]\|_2 \leqq \varepsilon$, we obtain

(5.34) $$\|P\|_2 \leqq (98+24\sqrt{2})\varepsilon.$$

Let P_a and P_b be two equivalent projections in R with the sum P. Let W'' be a partial isometry in R such that $P_a = W''(W'')^*$, $P_b = (W'')^* W''$. Let

(5.35) $$N_{11} = W' + W'' \in R.$$

From (5.33) and (5.34), we have

(5.36) $$\|N_{11} - e_{11} N e_{11}\|_2 \leqq (122+30\sqrt{2})\varepsilon.$$

By construction, $N_{11}^2 = 0$, $N_{11} N_{11}^* + N_{11}^* N_{11} = e_{11}$.

Let

(5.37) $$N_\varepsilon = \sum_j e_{j1} N_{11} e_{1j}.$$

Then we have $N_\varepsilon^2 = 0$, $N_\varepsilon N_\varepsilon^* + N_\varepsilon^* N_\varepsilon = 1$, $N_\varepsilon \in N'_0 \cap R$ and

(5.38) $$\|N-N_\varepsilon\|_2 \leqq n\|N_{11}-e_{11}Ne_{11}\|_2+\|N'-N\|_2 \leqq \varepsilon'(\varepsilon)$$

where $\varepsilon'(\varepsilon)\to 0$ as $\varepsilon\to 0$ by (5.17) and (5.36). Q.E.D.

Lemma 5.2. *Let R be a finite von Neumann algebra on a separable space having the property $L'_{1/2}$. Let N_0 be a finite type I factor in R and $\widetilde{R}=N'_0\cap R$. Let $\varepsilon>0$ and normal states $\omega_1, \cdots, \omega_n$ of $\widetilde{R}$ be given. Then there exists an operator $\widetilde{N}$ in $\widetilde{R}$ such that*

(a) $\widetilde{N}^2=0$, $\widetilde{N}\widetilde{N}^*+\widetilde{N}^*\widetilde{N}=1$.

(b) *For $Q\in\widetilde{R}$ and $j=1, \cdots, n$,*

(5.39) $$|\omega_j(Q\widetilde{N})-\omega_j(\widetilde{N}Q)| \leqq \varepsilon\|Q\|.$$

Proof. Let C_α be a finite number of operators in N_0 such that $C_\alpha\geqq 0$, $\phi(C_\alpha)=1$ and N_0 is the linear span of $\{C_\alpha\}$. ($\alpha=1, \cdots, m$.)

Let

(5.40) $$\bar{\omega}_j(Q)=(\omega_j\otimes\phi_0)(Q),\ j=1, \cdots, n$$

where ϕ_0 is the normalized trace on N_0 and $Q\in R=\widetilde{R}\otimes N_0$. Let

(5.41) $$\bar{\omega}_{n+\alpha}(Q)=\phi(C_\alpha Q),\ \alpha=1, \cdots, m,$$

where $Q\in R$.

By the property $L'_{1/2}$, there exists for any given $\varepsilon''>0$, an operator N in R such that $N^2=0$, $NN^*+N^*N=1$ and

(5.42) $$|\bar{\omega}_j(QN)-\bar{\omega}_j(NQ)| \leqq \varepsilon''\|Q\|$$

for $j=1, \cdots, n+m$ and $Q\in R$.

From (5.42) with $j=n+\alpha$, we have $|\phi([N,C_\alpha]Q)|\leqq\varepsilon''\|Q\|$ and hence, by setting $Q=[N, C_\alpha]^*$, we obtain

(5.43) $$\|[N, C_\alpha]\|_2^2=\|[N, C_\alpha]^*\|_2^2 \leqq 2\varepsilon''\|C_\alpha\|.$$

Hence we have

(5.44) $$\|[N, Q]\|_2 \leqq \varepsilon'''\|Q\|$$

for all $Q\in N_0$ where $\varepsilon'''\to 0$ as $\varepsilon''\to 0$.

We can now use Lemma 5.1. There exists an operator N_ε in $\widetilde{R}$ such that $N_\varepsilon^2=0$, $N_\varepsilon N_\varepsilon^*+N_\varepsilon^*N_\varepsilon=1$ and

(5. 45) $$\|N-N_\varepsilon\|_2 \leqq \varepsilon'(\varepsilon''').$$

For each $\bar{\omega}_j$ and $\varepsilon>0$, there exists an operator $A_j \in \widetilde{R}$ such that $A_j \geqq 0$, $\phi(A_j)=1$ and

(5. 46) $$\|\bar{\omega}_j(Q)-\phi(A_jQ)\| \leqq (\varepsilon/12)\|Q\|$$

for all $Q\in R$. We then have

(5. 47) $$\|\phi(A_jQN)-\phi(A_jQN_\varepsilon)\| \leqq \|A_j\|\|Q\|\varepsilon'(\varepsilon'''),$$

(5. 48) $$\|\phi(A_jNQ)-\phi(A_jN_\varepsilon Q)\| \leqq \|A_j\|\|Q\|\varepsilon'(\varepsilon''').$$

We choose ε'' such that $\|A_j\|\varepsilon'(\varepsilon''') \leqq \varepsilon/6$ for $j=1, \cdots, n$ and $\varepsilon'' \leqq \varepsilon/3$. From (5. 46), (5. 47), (5. 48) and (5. 42), we obtain

(5. 49) $$\|\bar{\omega}_j(QN_\varepsilon)-\bar{\omega}_j(N_\varepsilon Q)\| \leqq \varepsilon\|Q\|$$

for $j=1, \cdots, n$ and $Q\in R$.

By restricting (5. 49) to $Q\in\widetilde{R}$, we obtain (5. 39) for $\widetilde{N}=N_\varepsilon$.

Q. E. D.

Lemma 5. 3. *Let R be a finite von Neumann algebra on a separable space having the property $L'_{1/2}$. Then $R\sim R\otimes R_1$.*

Proof. Now the proof is exactly the same as the entire section 4 except the use of $\widetilde{R}_n\sim(N_1\otimes\cdots\otimes N_m)\otimes\widetilde{R}_n=R$ in the proof of Lemma 4. 3 is replaced by the use of Lemma 5. 2. Q. E. D.

Proof of Theorem 1. 3. Let R be a von Neumann algebra on a separable Hilbert space and F be its central projection such that FR is finite and $(1-F)R$ is properly infinite. If R has the property L'_λ, $0\leqq\lambda<1/2$, then $F=0$ from Corollary 2. 4 and we obtain $R\sim R\otimes R_x$, $x=\lambda(1-\lambda)^{-1}$ from Lemma 4. 5. (The proof for $\lambda=0$ is immediate from Corollary 2. 4.) If R has the property $L'_{1/2}$, then FR and $(1-F)R$ separately have property $L'_{1/2}$. (Replace N by FN and $(1-F)N$.) Hence, by Lemma 4. 5 and Lemma 5. 3, we have $FR\sim FR\otimes R_1$, $(1-F)R\sim(1-F)R\otimes R_1$. Therefore $R\sim R\otimes R_1$. Q.E.D.

§6. Supplementary Remarks

Lemma 6.1. *Any finite continuous von Neumann algebra R has the property $L_{1/2}$.*

Proof. Let $\varepsilon>0$ and a normal state ω of R be given. Let F be the central carrier of ω. Then FR has a faithful normal normalized finite trace ϕ. There exists $A\in R$ such that $A\geqq 0$, $\phi(A)=1$ and

$$(6.1)\qquad |\omega(Q)-\phi(AQ)|\leqq(\varepsilon/4)\|Q\|$$

for all $Q\in R$.

A in FR has a spectral decomposition (relative to ϕ). Let E_j, $j=1, 2, \cdots$ be spectral projections of A for the intervals

$$(6.2)\qquad [(j-1)\varepsilon/2,\ j\varepsilon/2).$$

E_j vanishes for $j>(2\|A\|/\varepsilon)+1$. Let

$$(6.3)\qquad A_\varepsilon=-\varepsilon/4+\sum_{j=1}^{\infty}(j\varepsilon/2)E_j.$$

We have

$$(6.4)\qquad \|A-A_\varepsilon\|\leqq\varepsilon/4.$$

If $E_j\neq 0$, let E_{j1} and E_{j2} be equivalent projections in R with the sum E_j and N_j be a partial isometry such that $N_jN_j^*=E_{j1}$, $N_j^*N_j=E_{j2}$. Let $N=\sum_j N_j$. Then we have $N^2=0$, $N^*N+NN^*=1$ and

$$(6.5)\qquad [N,\ A_\varepsilon]=0.$$

From (6.1), (6.4) and (6.5), we have

$$(6.6)\qquad |\omega(QN)-\omega(NQ)|\leqq\varepsilon\|Q\|$$

for all $Q\in R$. Q.E.D.

Lemma 6.2. $\mathrm{r}_\infty(R)$ *is closed for any von Neumann algebra R on a separable space.*

Proof. Let $\lambda_n\to\lambda$ and R has the property L'_{λ_n}. Then it has obviously the property L_λ. From Theorem 1.3 if $\lambda_n(1-\lambda_n)^{-1}\in\mathrm{r}_\infty(R)$, then R has the property L'_{λ_n} and hence R has the property L'_λ, which implies

$\lambda(1-\lambda)^{-1}\in r_\infty(R)$, $0\leqq\lambda\leqq1/2$. Q.E.D.

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