Publ. RIMS Kyoto Univ.
Vol. 6 (1970/71), 461-476

Kripke Models and Intermediate Logics

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In [10], Kripke gave a definition of the semantics of the intuitionistic logic. Fitting [2] showed that Kripke's models are equivalent to algebraic models (i.e., pseudo-Boolean models) in a certain sense. As a corollary of this result, we can show that any partially ordered set is regarded as a (characteristic) model of a intermediate logic.[1] We shall study the relations between intermediate logics and partially ordered sets as models of them, in this paper.

We call a partially ordered set, a *Kripke model*.[2] At present we don't know whether any intermediate logic has a Kripke model. But Kripke models have some interesting properties and are useful when we study the models of intermediate logics. In §2, we shall study general properties of Kripke models. In §3, we shall define the *height* of a Kripke model and show the close connection between the height and the *slice*, which is introduced in [7]. In §4, we shall give a model of LP_n which is the least element in n-th slice $\mathcal{S}_n$ (see [7]).

§1. Preliminaries

We use the terminologies of [2] on algebraic models, except the use of 1 and 0 instead of $\bigvee$ and $\bigwedge$, respectively. But on Kripke models, we give another definition, following Schütte [13].[3]

Definition 1.1. *If M is a non-empty partially ordered set, then*

Received September 8, 1970.

1) These models are studied in *e.g.*, Segerberg [14] and Gabbay-de Jongh [3]. We deal with only propositional logics in this paper.

2) This terminology is different from that in [2].

3) In this paper, the word *algebraic models* is used to denote pseudo-Boolean algebras.

we say M is a Kripke model.[4] *Let M be a Kripke model which is partially ordered by a relation* $\leq$. *Suppose that W is a mapping from all the pairs of formulas and elements in M to* $\{t, f\}$. *W is called an M-valuation, if W satisfies the following conditions. For any u, v in M,*

1) *if* $W(p, u)=t$ *and* $u \leq v$ *then* $W(p, v)=t$, *where p is any propositional variable,*
2) $W(A \wedge B, u)=t$ *iff* $W(A, u)=t$ *and* $W(B, u)=t$,
3) $W(A \vee B, u)=t$ *iff* $W(A, u)=t$ *or* $W(B, u)=t$,
4) $W(A \supset B, u)=t$ *iff for any r in M such that* $u \leq r$ $W(A, r)=f$ *or* $W(B, r)=t$,
5) $W(\neg A, u)=t$ *iff for any r in M such that* $u \leq r$ $W(A, r)=f$.

Let W be any M-valuation. We say a formula A is *valid in* (M, W), if $W(A, u)=t$ for any u in M. If for any M-valuation W, A is valid in (M, W), we say A is *valid in M*.

Following theorem is due to Fitting [2].

Theorem 1.2. 1) *For any Kripke model M and any M-valuation W, there is a pseudo-Boolean algebra P and an assignment f of P such that for any formula A, A is valid in* (M, W) *iff* $f(A)=1$.[5]

2) *Conversely, suppose that a pseudo-Boolean algebra P and its assignment f are given. Then there is a Kripke model M and an M-valuation W such that for any formula A, A is valid in* (M, W) *iff* $f(A)=1$.

Proof. We sketch Fitting's proof.

1) Suppose that M and W are given. If a subset N of M satisfies the following condition

$$\text{if } u \in N \text{ and } u \leq v \text{ then } v \in N,$$

4) Kripke's original definition says that M is a non-empty set with a *transitive*, *reflexive* relation, but for our purposes we have only to deal with partially ordered sets, since for any set M with a transitive, reflexive relation there is a partially ordered set N such that for any formula A, A is valid in M iff A is valid in N

5) In [2], the word *homomorphism* is used, instead of *assignment*.

we say N is *closed*. Let P be the class of all closed subsets of M. Then we can prove that P is pseudo-Boolean algebra with respect to set intersection and set union. As for zero element we take the empty set. Define an assignment f of P by $f(p)=\{u;\ W(p,u)=t\}$ for any propositional variable p. Then it is clear that our theorem holds for this f and P.

2) Suppose P and f are given. Let M be the class of all prime filters of P. Clearly, M can be partially ordered by set inclusion $\subseteq$. Define an M-valuation W by

$$W(p,u)=t \quad \text{iff} \quad f(p)\in u.$$

Now, it is easy to verify that our theorem holds for this M and W.

As a corollary of Theorem 1.2, we can obtain that

Corollary 1.3. 1) *For any Kripke model M, there is a pseudo-Boolean algebra P such that for any formula A, A is valid in M iff A is valid in P.*

2) *For any pseudo-Boolean algebra P, there is a Kripke model M such that for any formula A, A is valid in P if A is valid in M.*

We don't know whether the converse of Corollary 1.3.2 holds and whether any intermediate logic has a Kripke model. But we shall show in Corollary 1.5 that if P is finite then the converse holds. This implies that any finite intermediate logic has a Kripke model.

We write P_M (or M_P) for the pseudo-Boolean algebra (or Kripke model) constructing from a Kripke model M (or a pseudo-Boolean algebra P) by the method of Fitting. We know that A is valid in M_P iff A is valid in P_{M_P} by Corollary 1.3.1. Now, we define a mapping f from P to P_{M_P} by the condition that

$$f(a)=\{F;\ a\in F \text{ and } F\in\mathcal{T}(P)\},$$

where $\mathcal{T}(P)$ denote the set of all prime filters of P. It is clear that f is an isomorphism from P into P_{M_P}.

Lemma 1.4. *If P is finite, then f is a mapping onto P_{M_P}.*

Proof. Let U be any element in P_{M_P}. We say that an element F

in U is *minimal*, when if G is a subset of F then $G=F$ for any G in U. Since P is finite, U is also finite. Hence for any G in U there is a minimal element F such that F is a subset of G. Let $F_1, \cdots, F_k$ be all the minimal elements in U. Define U_i $(1\leq i\leq k)$ by

$$U_i=\{G;\ F_i \text{ is a subset of } G \text{ and } G\in\mathcal{T}(P)\}.$$

It is clear that $U=\bigcup_{i=1}^{k} U_i$, since U is in P_{M_P}. Let $F_i=\{a_{ij};\ 1\leq j\leq n_i\}$. Then we write $(F_i)_*$ for $\bigcap_{j=1}^{n_i} a_{ij}$. It is easy to see that $G\in U_i$ iff $(F_i)_* \in G$. So, $f((F_i)_*)=U_i$. Hence $f(\bigcup_{i=1}^{k}(F_i)_*)=\bigcup_{i=1}^{k} f((F_i)_*)=\bigcup_{i=1}^{k} U_i=U$, since $f(a\cup b)=f(a)\cup f(b)$ for any $a, b\in P$. Thus we have Lemma 1. 4.

So, we obtain

Corollary 1. 5. *If P is finite, then the converse of Corollary 1. 2. 3 holds.*[6]

In §3, we shall prove that if a pseudo-Boolean algebra P is in $\mathcal{S}_n$ $(n<\omega)$, P_{M_P} is also in $\mathcal{S}_n$.

§2. Properties of Kripke Models

We shall henceforth write a *model* for a Kripke model and a *logic* for an intermediate **logic**. We write $L(M)$ for the logic characterized by a model M, *i.e.*, the set of formulas which are valid in M. We write $\leq_M$ for the relation which orders a model M. Following the notation in [7], we write $L_1\subset L_2$ if a logic L_1 is included by a logic L_2, as a set of formulas.

Definition 2. 1. *Let M be a model. A subset N of M is called a submodel of M if N is closed with respect to $\leq_M$, i.e., for any a, b in M, if $a\in N$ and $a\leq_M b$ then $b\in N$. $\leq_N$ is a restriction of $\leq_M$ to N.*

We can prove easily that

Lemma 2. 2. *Let N be a submodel of M. If two M-valuation W and W' satisfy the following condition*

6) See Dummett-Lemmon [1] Lemma 2.

$W(p, a) = W'(p, a)$ *for any* $a \in N$ *and any propositional variable* p,

then $W(A, a) = W'(A, a)$ *for any* $a \in N$ *and any formula* A.

Corollary 2.3. *If* N *is a submodel of* M, *then* $L(M) \subset L(N)$.

Proof. Let W be any N-valuation. Define a mapping W^* by

$$W^*(p, a) = \begin{cases} W(p, a) & \text{if } a \in N, \\ f & \text{otherwise.} \end{cases}$$

It is easy to verify that W^* is really an M-valuation. Suppose that $A \notin L(N)$. Then there is $a \in N$ such that $W(A, a) = f$ for some N-valuation W. By Lemma 2.2, $W^*(A, a) = f$. Hence $A \notin L(M)$.

Definition 2.4. *Suppose that* M_i *is a submodel for any* $i \in I$. *The set* $\{M_i;\ i \in I\}$ *is called a covering of* M, *if* $M = \bigcup_{i \in I} M_i$.

Theorem 2.5. *If* $\{M_i;\ i \in I\}$ *is a covering of* M, *then* $L(M) \supset \subset \bigcap_{i \in I} L(M_i)$, *where* $\bigcap_{i \in I} L(M_i)$ *denotes the intersection of* $L(M_i)$*'s as logics.*[7]

Proof. By Corollary 2.3, for any $i \in I$ $L(M) \subset L(M_i)$. So, $L(M) \subset \bigcap_{i \in I} L(M_i)$. Suppose that $A \notin L(M)$. Then there is $a \in M$ and an M-valuation W such that $W(A, a) = f$. By the definition of covering, $a \in M_i$ for some $i \in I$. Define an M_i-valuation V by restricting the domain of the second argument of W to M_i. Then it is easy to see that $V(A, a) = W(A, a) = f$. Thus $A \notin L(M_i)$ for some $i \in I$.

Now, we define two operations on models, following the operations defined in [6].

Definition 2.6. *Let* M *and* N *be models such that* $M \cap N$ *is empty. The model* $M \uparrow N$ *is a set* $M \cup N$ *with a relation* $\leq_{M \uparrow N}$ *defined below. For any* $a, b \in M \cup N$,

$a \leq_{M \uparrow N} b$ *iff either* 1) $a \leq_M b$ *and* $a, b \in M$ *or* 2) $a \leq_N b$ *and* $a, b \in N$ *or* 3) $a \in M$ *and* $b \in N$.

7) See Hosoi [8].

If both M and N are isomorphic (as a partially ordered set) to some model L, we write $L \uparrow L$ for $M \uparrow N$.

Definition 2.7. *Let M_i be a model for any $i \in I$, such that $M_i \cap M_j$ is empty if $i \neq j$. The model $(M_i)_{i \in I}$ is a set $\bigcup_{i \in I} M_i$ with a relation $\leq$ defined below.*
For any $a, b \in \bigcup_{i \in I} M_i$,
$a \leq b$ iff there is $i \in I$ such that $a, b \in M_i$ and $a \leq_{M_i} b$.[8)]

We sometimes write $(M)_{M \in \mathfrak{T}}$ for $(M_i)_{i \in I}$, if $\mathfrak{T}$ is the ordered set $\{M_i;\ i \in I\}$. If each M_i is isomorphic to some L and the cardinal of I is σ then we write L^σ for $(M_i)_{i \in I}$. We remark that $P_{M \uparrow N} = P_M \uparrow P_N$ and *the direct product of* P_{M_i} $(i \in I) = P_{(M_i)_{i \in I}}$.

Corollary 2.8. $L((M_i)_{i \in I}) \supset\subset \bigcap_{i \in I} L(M_i)$.

Proof. Because $\{M_i;\ i \in I\}$ is a covering of $(M_i)_{i \in I}$. (See [14].)

Define a model S'_n for $1 \leq n < \omega$, which is totally linear ordered set with n elements. It is easy to see that $P_{S'_n} = S_n$ where S_n is a pseudo-Boolean model defined by Gödel [2]. So, henceforth we write S_n also for the Kripke model S'_n.

Lemma 2.9. *Let M be a model. If $\exists a \in M \forall b \in M\ a \leq_M b$ holds, then M is of the form $S_1 \uparrow N$. (For the sake of brevity, we say M is of the form $S_1 \uparrow N$ even if $M = S_1$).*

Proof. Let a be an element in M such that for any $b \in M$ $a \leq_M b$ holds. Let N be a submodel which is equal to $M - \{a\}$. Then it is clear that M is isomorphic to $S_1 \uparrow N$.

Mckay [11] proved that for any pseudo-Boolean algebra P, there are pseudo-Boolean algebras P_i $(i \in I)$ such that $P \supset\subset \bigcap_{i \in I} S_1 \uparrow P_i$. We give another proof of this result for Kripke models.[*)]

Theorem 2.10. *For any model M there exist models N_i $(i \in I)$ such that $L(M) \supset\subset \bigcap_{i \in I} L(S_1 \uparrow N_i)$.*

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8) This notion is defined also in [9]. Henceforth, we sometimes abbreviate $\leq_M$ as $\leq$, when a fixed model M is considered.

*) Henceforth, a pseudo-Boolean model P denotes the set of formulas valid in P as well as a pseudo-Boolean algebra, whenever no confusions seem to occur.

Proof. For any $a \in M$, we write M_a for the submodel $\{b; a \leq_M b\}$. Clearly, $\{M_a; a \in M\}$ is a covering of M. Hence by Theorem 2.5, $L\{M\} \supset \subset \bigcap_{a \in M} L(M_a)$. Moreover, each M_a is of the form $S_1 \uparrow N_a$ by Lemma 2.9.

It should be remarked that in contrast with the above theorem, the following statement is false. *For any model M, there exists a model N such that $L(M) \supset \subset L(S_1 \uparrow N)$.*

The following theorem is useful, when we compare one logic with another logic. Let f be a surjective mapping from M to N such that 1) for any $a, b \in M$ if $a \leq_M b$ then $f(a) \leq_N f(b)$, and 2) for $a \in M$ and any $c \in N$ if $f(a) \leq_N c$ then there is $b \in M$ such that $f(b) = c$ and $a \leq_M b$. Then we say f is an *embedding of M into N*. If there is an embedding of M into N, we say *M is embeddable in N.*

Theorem 2.11. *If M is embeddable in N then $L(M) \subset L(N)$.*[9]

Proof. Suppose $A \notin L(N)$. Then there is an N-valuation W and an element $b \in N$ such that $W(A, b) = f$. Define an M-valuation V by

$$V(p, a) = W(p, f(a)) \quad \text{for any propositional variable } p \text{ and any } a \in M,$$

where f is an embedding of M into N. We can show that V is really an M-valuation, and that $V(B, a) = W(B, f(a))$ for any formula B. Let c be an element in M such that $f(c) = b$. Now, $V(A, c) = W(A, b) = f$. So, $A \notin L(M)$.

Corollary 2.12. 1) *If M_1 is embeddable in M_2 and M_2 is embeddable in M_3, then M_1 is embeddable in M_3.*

2) *Let g be a surjective mapping from a set J to a set I. Suppose that M_j is embeddable in N_i for any $j \in J$ and any $i \in I$ such that $g(j) = i$. Then $(M_j)_{j \in I}$ is embeddable in $(N_i)_{i \in I}$.*

3) *Suppose that M_1 and N_1 are embeddable in M_2 and N_2, respectively. Then $M_1 \uparrow N_1$ is embeddable in $M_2 \uparrow N_2$.*

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9) We can prove this theorem by using Theorem 4.6 in [9]. In [9], an embedding is called a *strongly isotone* mapping.

§3. Height of Models

In this section, we shall define the *height* $h(M)$ of a given model M, and prove that $L(M)$ is in the n-th slice $\mathcal{S}_n$ iff $h(M)=n$, for $n\leq\omega$. We say *a model M is in $\mathcal{S}_n$* if $L(M)\in\mathcal{S}_n$ (or equivalently, $P_M\in\mathcal{S}_n$).

Lemma 3.1. *Suppose that $M_i\in\mathcal{S}_{n_i}$ for $i\in I$. Then $(M_i)_{i\in I}\in\mathcal{S}_n$, where $n=\sup\{n_i;\ i\in I\}$. (n and n_i may be ω.)*

Intuitively, the *height* of a model M is the maximal m such that $a_1<a_2<\cdots<a_m$ and each a_i is in M, where $a<b$ means $a\leq b$ and $a\neq b$. To make the definition precise, we need some preparations. Suppose that a model M is given. For any $a, b\in M$ such that $a\leq b$, we say a sequence $\alpha=\langle a_1, \cdots, a_m\rangle$ ($m\geq 1$) of elements in M is *a chain from a to b* if 1) $a_1=a$ and $a_m=b$ and 2) $a_i<a_{i+1}$ for $1\leq i<m$. In such a case we define $l(\alpha)=m$. For any $a, b\in M$ such that $a\leq b$, define a mapping d by

$$d(a, b)=\sup\{l(\alpha);\ \alpha \text{ is a chain from } a \text{ to } b\}.$$

We note that if $a<b$ then $d(a, b)\geq 2$. For the sake of brevity, let $d(a, b)=0$ if $a\nleq b$.

Definition 3.2. *The height h is a mapping from the class of all models to $\{1, 2, \cdots, \omega\}$, which is defined by*

$$h(M)=\sup\{d(a, b);\ a, b\in M\}.$$

We remark that $h(M)\geq 1$, since $d(a, a)=1$.

Lemma 3.3. *Let M be a model. If $h(M)=n$, then $M\in\mathcal{S}_n$, where $n<\omega$.*

Proof. We prove our lemma by induction on n.

1) Case $n=1$.

If there exist $a, b\in M$ such that $a<b$, then $h(M)\geq d(a, b)\geq 2$. So, for any $a, b\in M$ if $a\leq b$ then $a=b$. Therefore $M=(S_1)^\sigma$ where σ is the cardinal of M. So $L(M)\supset\subset L(S_1)$. This means $M\in\mathcal{S}_1$.

2) Case $n>1$.

For each $a\in M$, define a submodel M_a of M by $M_a=\{b;\ a\leq b\}$.

By the proof of Theorem 2.10, $\{M_a;\ a\in M\}$ is a covering of M and each M_a is of the form $S_1\uparrow N_a$. We first prove that

(3.1) $h(M_a)\leq h(M)$ for any $a\in M$ and there is $b\in M$ such that $h(M_b)=h(M)>1$.

Since M_a is a subset of M, $d(b,c)\leq h(M)$ for any $b,c\in M_a$. So, $h(M_a)\leq h(M)$. We can find $b',c'\in M$ such that $b'\leq c'$ and $d(b',c')=h(M)$, since $h(M)$ is finite. So $h(M_{b'})\geq d(b',c')=h(M)$. Thus, $h(M_{b'})=h(M)$. Next, we can show that

(3.2) for any $a\in M$, if $h(M_a)\neq 1$ then $h(N_a)=h(M_a)-1$.

Now, by (3.1) and (3.2), if $h(M_a)\neq 1$ then $h(N_a)=n_a\leq n-1$ and there is b such that $h(N_b)=n-1$. By the hypothesis of induction, $N_a\in\mathcal{S}_{n_a}$. Since $P_{M_a}=P_{S_1}\uparrow P_{N_a}=S_1\uparrow P_{N_a}$ and $P_{N_a}\in\mathcal{S}_{n_a}$, $P_{M_a}\in\mathcal{S}_{n_a+1}$ by Theorem 6.2 in [7]. That is,

(3.3) if $h(M_a)>1$ then $M_a\in\mathcal{S}_{n_a+1}$, where $n_a+1\leq n$ and
if $h(M_a)=1$ then $M_a\in\mathcal{S}_1$.

By (3.1), $\max\{n_a+1;\ a\in M\}=n$. Thus by (3.3), Lemma 3.1 and Theorem 2.10, $M\in\mathcal{S}_n$.

Lemma 3.4. *If there is a chain α in M such that $l(\alpha)=n+1$, then P_n is not valid in M $(n\geq 1)$, where P_n is defined inductively by*

$$P_1=((p_1\supset p_0)\supset p_1)\supset p_1\,,$$
$$P_{k+1}=((p_{k+1}\supset P_k)\supset p_{k+1})\supset p_{k+1}\,.$$

Proof. Let α be $\langle a_1,\cdots,a_{n+1}\rangle$. We define an M-valuation W by

$$W(p_0,b)=f \qquad \text{for any } b\in M,$$
$$W(p_i,b)=\begin{cases} t & \text{if } a_{n-i+1}<b \qquad (1\leq i\leq n)\\ f & \text{otherwise.}\end{cases}$$

For the sake of brevity, let $P_0=p_0$. Now, we prove by induction on i that $W(P_i,a_{n-i+1})=f$ for $0\leq i\leq n$.

1) $i=0$. $W(P_0,a_{n+1})=W(p_0,a_{n+1})=f$ by the assumption.

2) $i>0$. By the hypothesis of induction, $W(P_{i-1},a_{n-i+2})=f$.

Since $a_{n-i+1}<a_{n-i+2}$, $W(p_i, a_{n-i+2})=t$. So $W(p_i \supset P_{i-1}, a_{n-i+1})=f$. Since $W(p_i, b)=t$ for $a_{n-i+1}<b$, $W((p_i \supset P_{i-1}) \supset p_i, a_{n-i+1})=t$. But $W(p_i, a_{n-i+1}) =f$. Hence $W(P_i, a_{n-i+1})=f$. If we take n for i, then we have $W(p_n, a_1) =f$. This means that P_n is not valid in M.

Corollary 3.5. *If $h(M)=\omega$, then $M \in \mathcal{S}_\omega$.*

Proof. It can be easily proved that if $h(M)=\omega$ then for any $2 \leq n < \omega$ there is a chain α in M such that $l(\alpha)=n$. Then we have $M \notin \mathcal{S}_m$ for any $m<\omega$ by Lemma 3.4.

Putting these results together, we obtain

Corollary 3.6. *For any $n \leq \omega$, $h(M)=n$ iff $M \in \mathcal{S}_n$.*

Next, we shall prove that if a pseudo-Boolean algebra P is in $\mathcal{S}_n$ $(n<\omega)$, then M_P is also in $\mathcal{S}_n$.

Lemma 3.7. *Let P be a pseudo-Boolean algebra in $\mathcal{S}_n$ $(n<\omega)$. Then there is no set of prime filters $\{F_i;\ 0 \leq i \leq n\}$ of P such that*

(3.4) $$F_n \subsetneqq F_{n-1} \subsetneqq \cdots \subsetneqq F_0 .$$

($F_i \subsetneqq F_j$ means that F_i is a proper subset of F_j).

Proof. Suppose that a set of prime filters $\{F_i;\ 0 \leq i \leq n\}$ satisfies (3.4). We prove that there is an assignment f of P such that

(3.5) 1) $f(P_0) \in P - F_0$ and

2) $f(P_k) \in F_{k-1} - F_k$ for any k such that $1 \leq k \leq n$,

where P_i is the formula defined in Lemma 3.4. We define f by induction. Define $f(p_0)=a_0$, where $a_0=0$. Then it is clear that $f(P_0)= f(p_0)=0 \in P-F_0$. Suppose that we define $f(p_i)$ for $0 \leq i \leq k < n$ such that $f(P_k)=b \in F_{k-1}-F_k$. Since $F_{k+1} \subsetneqq F_k$, we can take an element a_{k+1} out of $F_k - F_{k+1}$. We define $b_{k+1}=a_{k+1} \cup (a_{k+1} \supset b)$ and $f(p_{k+1})=b_{k+1}$. We first show

(3.6) $$b_{k+1} \in F_k - F_{k+1} .$$

Since F_k is a filter, $a_{k+1} \leq b_{k+1}$ and $a_{k+1} \in F_k$, so $b_{k+1} \in F_k$. If $b_{k+1} \in F_{k+1}$, then either $a_{k+1} \in F_{k+1}$ or $a_{k+1} \supset b \in F_{k+1}$, since F_{k+1} is prime. But a_{k+1}

$\in F_{k+1}$ contradicts the hypothesis. So, $a_{k+1} \supset b \in F_{k+1}$. Then $a_{k+1} \in F_k$, $a_{k+1} \supset b \in F_k$, $a_{k+1} \cap (a_{k+1} \supset b) \leq b$, and hence $b \in F_k$. But this contradicts the assumption. Thus $b_{k+1} \notin F_{k+1}$. Next we show that

(3.7) $\qquad b_{k+1} \supset b = b.$

Since $a_{k+1} \cap (b_{k+1} \supset b) \leq b_{k+1} \cap (b_{k+1} \supset b) \leq b$, $(b_{k+1} \supset b) \leq (a_{k+1} \supset b) \leq b_{k+1}$. So $b_{k+1} \supset b = b_{k+1} \cap (b_{k+1} \supset b) \leq b$. Hence $b_{k+1} \supset b = b$, since $b_{k+1} \supset b \geq b$ always holds. By (3.6) and (3.7), $f(P_{k+1}) = b_{k+1} \in F_k - F_{k+1}$. If we take n for k in (3.5), we have $f(P_n) \in F_{n-1} - F_n$. Since $1 \in F_n$, $f(P_n) \neq 1$. So P_n is not valid in P. But this contradicts $P \in \mathcal{S}_n$.

By Lemma 3.7, if $P \in \mathcal{S}_n$ then $h(M_P) \leq n$. But by Corollary 1.3, $L(M_P) \subset P$. So $h(M_P) = n$. This means $M_P \in \mathcal{S}_n$.

§4. Applications of Kripke Models

In this section, we shall study about models of the logic LP_n, which is defined by adding axiom schema P_n (see Lemma 3.4) to the intuitionistic propositional logic.[10] It is proved in [7] that S_n is the greatest and LP_n is the least element in $\mathcal{S}_n$. We now know that a model M is in $\mathcal{S}_n$ iff $h(M) = n$ and that the *Kripke model* S_n is a linearly ordered set with n elements. So, it is natural to ask what models the least element LP_n has.

First we introduce the monotonic descending sequence of models $\{R_{nm};\ m < \omega\}$ and show that this sequence coverges to LP_n. Moreover we show $\{R_{nm};\ n < \omega\}$ converges to the logic D_{m-1} which is discussed in Gabby—de Jongh [3]. We give an axiomatization of R_{nm}. We also give a model of LQ_n, which is introduced in Hosoi [8].

We need some preparations.

Definition 4.1. *Define a mapping w by the condition that for any model M such that $d(a, b)$ is finite for $a, b \in M$,*

$$w(M) = \sup[\textit{the cardinal of } \{b;\ d(a, b) = 2\};\ a \in M].$$

10) Hereafter, we sometimes write $LJ + A_1 + \cdots + A_m$ for the logic which is obtained by adding axiom schemata $A_1, \cdots, A_m$ to the intuitionistic logic.

Definition 4.2. *If a model M satisfies the following conditions, we call M a m-tree model.*

1) *There is a least element in M with respect to $\leq$.*
2) *For any a, b, c in M, if $b\leq a$ and $c\leq a$ then either $b\leq c$ or $c\leq b$.*
3) $w(M)\leq m\leq\omega$.

We write $\mathcal{U}_{nm}$ $(m\leq\omega, n<\omega)$ for the class of all models M such that $h(M)=n$ and M is an m-tree model. Remark that if a submodel M of an m-tree model satisfies the condition 1), then M is also an m-tree model. Any m-tree model is also an n-tree model for $m\leq n$.

An element $a\in M$ is said to be *maximal* if $a\leq b$ implies $a=b$ for any $b\in M$.

Definition 4.3. *Let $M\in\mathcal{U}_{nm}$. We define a model M^* as follows.*

1) *If $n=1$, then $M^*=M$.*

2) *Suppose $n>1$. Let $\{a_i;\ i\leq s\}$ be all maximal elements in M. (Since $M\in\mathcal{U}_{nm}$, s is at most ω). Now M^* is a set $M\cup\{a_{ij};\ i\leq s$ and $1\leq j\leq n-d(a_0, a_i)\}$, where a_0 is the least element and $a_{ij}\notin M$, with a relation $\leq_{M^*}$ such that $a\leq_{M^*}b$ iff either 1) $a, b\in M$ and $a\leq_M b$ or 2) $a\in M$, $a\leq_M a_i$ and $b=a_{ij}$ or 3) $a=a_{ij}$, $b=a_{ik}$ and $j\leq k$.*

Clearly if $M\in\mathcal{U}_{nm}$ then $M^*\in\mathcal{U}_{nm}$.

Lemma 4.4. *If $M\in\mathcal{U}_{nm}$ for some m, n, then $L(M^*)\subset L(M)$.*

Proof. Define a mapping f from M^* to M by

$$f(a)=\begin{cases} a & \text{if } a\in M \\ a_i & \text{if } a=a_{ij} \text{ for some } j. \end{cases}$$

Since f is an embedding of M^* into M, $L(M^*)\subset L(M)$ by Theorem 2.11.

Let $M\in\mathcal{U}_{nm}$. M is said to be *complete* if $d(a_0, a)=n$ for any maximal element a of M. It is trivial that M^* is complete. Now, we define a special complete element in $\mathcal{U}_{nm}$.

Definition 4.5. *Define a model R_{nm} $(n<\omega, m\leq\omega)$ recursively as follows.*

$$R_{1m}=S_1\,,\quad R_{k+1m}=S_1\uparrow(R_{kn})^m.$$

Clearly, R_{nm} is complete and is in $\mathcal{U}_{nm'}$ for any $m'\geq m$.

Lemma 4.6. *R_{nm} is the least element in $\mathcal{U}_{nm}$.*

Proof. By Lemma 4.4 and the above remark, we have only to prove that $L(R_{nm})\subset L(M)$ for any complete element M in $\mathcal{U}_{nm}$. We shall show that R_{nm} is embeddable in M for any complete element M in $\mathcal{U}_{nm}$, by induction on n. For $n=1$, the identity mapping on M is an embedding of R_{1m} into M, since $M\in\mathcal{U}_{1m}$ iff $M=S_1=R_{1m}$. Suppose $n>1$. By Definition 4.2, M is of the form $S_1\uparrow(M_i)_{i\leq k}$ for some $k\leq m$ and each M_i is in $\mathcal{U}_{n-1m}$ since M is complete. By the assumption, R_{n-1m} is embeddable in M_i for any i. So, $(R_{n-1m})^m$ is embeddable in $(M_i)_{i\leq k}$ by Corollary 2.12, 2) and hence R_{nm} is embeddable in M by Corollary 2.12, 3). Thus $L(R_{nm})\subset L(M)$.

Corollary 4.7. *If $m\geq m'$ and $n\geq n'$, then $L(R_{nm})\subset L(R_{n'm'})$. Moreover if $m>m'$, $L(R_{nm})\subsetneqq L(R_{nm'})$ and if $n>n'$, $L(R_{nm})\subsetneqq L(R_{n'm})$.*

Proof. Since $R_{n'm'}$ is a submodel of $R_{nm'}$, by Corollary 2.3 $L(R_{nm'})\subset L(R_{n'm'})$. By Lemma 4.6 $L(R_{nm})\subset L(R_{nm'})$. Let A_k be the formula introduced by [3], i.e.,

$$A_k=\bigwedge_{i=0}^{k+1}((p_i\supset\bigvee_{j\neq i}p_j)\supset\bigvee_{j\neq i}p_j)\supset\bigvee_{i=0}^{k+1}p_i\,.$$

Suppose $m>m'$. Then by [3], $A_{m'-1}\in L(R_{nm'})$ but $A_{m'-1}\notin L(R_{nm})$. Suppose $n>n'$. Then $P_{n'}\in L(R_{n'm})$ but $P_{n'}\notin L(R_{nm})$, since $h(R_{km})=k$ for any $k<\omega$. So our proof is completed.

Using the idea of Kripke [10], we have the following lemma.[11)]

Lemma 4.8. *Let M be a model in S_n, which is of the form $S_1\uparrow N$ and $w(M)\leq m<\omega$. Then there is a model M' in $\mathcal{U}_{nm}$ such that $L(M')\subset L(M)$.*

Proof. A chain α from a to b is called *proper*, where $\alpha=\langle a_1, \cdots, a_k\rangle$, if $d(a_i, a_{i+1})=2$ for any i such that $1\leq i<k$. Let a_0 be the least element of M. We define a model M' by the condition 1) $M'=\{\alpha;$

11) See also [1] and [5].

α *is a proper chain from* $a_0\}$ and 2) for any $\alpha=\langle a_1, \cdots, a_k\rangle$ and $\beta=\langle b_1, \cdots, b_h\rangle$, $\alpha\leq_{M'}\beta$ iff $k\leq h$ and $b_i=a_i$ for any $i\leq k$. Since $w(M)\leq m$, $w(M')\leq m$. It can be easily proved that M' is a m-tree model and $h(M')=n$. Hence $M'\in\mathcal{U}_{nm}$. We now prove that $L(M')\subset L(M)$. Define a mapping f from M' to M, by $f(\alpha)=a$ if α is a chain from a_0 to a. Then f is an embedding of M' into M. So $L(M')\subset L(M)$.

Corollary 4.9. *Let M be a model in S_n, such that $w(M)\leq m\leq\omega$. Then $L(R_{nm})\subset L(M)$.*

Proof. By Theorem 2.10, there are models N_i's such that $L(M)\supset\subset\bigcap_{i\in I}L(S_1\uparrow N_i)$. Furthermore we can take such $S_1\uparrow N_i$'s as submodels of M, so $w(S_1\uparrow N_i)\leq m$. By Lemma 4.6 and Corollary 4.8, $L(R_{nm})\subset L(S_1\uparrow N_i)$ for any $i\in I$. Hence $L(M)\supset\subset\bigcap_{i\in I}L(S_1\uparrow N_i)\supset L(R_{nm})$.

Theorem 4.10. 1) $LP_n\supset\subset\bigcap_{m<\omega}L(R_{nm})$ $(1\leq n<\omega)$.

2) $D_m\supset\subset\bigcap_{n<\omega}L(R_{nm+1})$ $(0\leq m<\omega)$, *where D_k is a logic defined by adding axiom schema A_k to intuitionistic logic.* (See [3]).

Proof. 1) By Mckay [12] Theorem 2.2, LP_n has the *finite model property*. So there are finite *Kripke* models M_i's such that $LP_n\supset\subset\bigcap_{i\in I}L(M_i)$.[12] Clearly $h(M_i)=n_i\leq n$. Let $w(M_i)$ be m_i. Since M_i is finite, $m_i<\omega$. By Corollary 4.9, $L(R_{n_im_i})\subset L(M_i)$. So $LP_n\supset\bigcap_{m<\omega}L(R_{nm})$. Clearly, $LP_n\subset\bigcap_{m<\omega}L(R_{nm})$. 2) can be proved similarly as 1) by using the argument [3], since each D_k has the finite model property.

Corollary 4.11. 1) $LP_n\supset\subset L(R_{n\omega})$. 2) $LJ\supset\subset\bigcap_{n,m<\omega}L(R_{nm})$.

Proof. 1) Clearly $LP_n\supset L(R_{n\omega})$. By Corollary 4.9 and Theorem 4.10, $LP_n\supset\subset\bigcap_{m<\omega}L(R_{nm})\supset L(R_{n\omega})$. 2) Trivial.

In [3], an axiomatization of the logic D_m is given, i.e., $D_m\supset\subset LJ+A_m$. Using this fact, we can obtain an axiomatization of R_{nm}.

Theorem 4.12. $L(R_{nm})\supset\subset LJ+P_n+A_{m-1}$ *for* $1\leq m<\omega$.

Proof. Since $A_{m-1}\in D_{m-1}$, $L(R_{nm})\supset LJ+P_n+A_{m-1}$ by Theorem 4.10.

12) See Corollary 1.5.

Conversely, let P be the Lindenbaum algebra of $LJ+P_n+A_{m-1}$. Since $P\in\mathcal{S}_n$, M_P is also in $\mathcal{S}_n$ by Lemma 3.7. So if $A\notin P$ then there is an M_P-valuation W such that A is not valid in (M_P, W). Using the same method as in [3], we can prove that there is a model M such that $A\notin L(M)$ and $L(M)\supset L(R_{n'm'})$ for some $n'\leq n$ and $m'\leq m$. Hence $A\notin L(R_{nm})$ by Corollary 4.7. Thus we have $L(R_{nm})\subset LJ+P_n+A_{m-1}$.

As a corollary of Theorem 4.10, cwe an give a model of LQ_n $(2\leq n<\omega)$, which is obtained by adding axiom schema Q to LP_n, where $Q=\neg p\vee\neg\neg p$.[13] It is proved in Theorem 4.16 in Hosoi [8] that LQ_n does not have a finite model if $n\geq 3$. First we have

Lemma 4.13. *Let $S_1\uparrow M$ be a finite model, in which Q is valid. Then M is of the form $N\uparrow S_1$.*

Proof. Suppose that both a and b are distinct maximal elements in $S_1\uparrow M$. Define $S_1\uparrow M$-valuation W by

$$W(p,c)=\begin{cases} t & \text{if } c=a \\ f & \text{otherwise.}\end{cases}$$

It is easy to verify that $W(Q,a_0)=f$, where a_0 is the least element of $S_1\uparrow M$. This contradicts that $Q\in L(S_1\uparrow M)$. So, $S_1\uparrow M$ has only one maximal element. Thus, M is of the form $N\uparrow S_1$.

Theorem 4.14. $LQ_{n+1}\supset\subset\bigcap_{m<\omega}L(R_{nm}\uparrow S_1)\supset\subset L(R_{n\omega}\uparrow S_1)$. *In other words, there exists a pseudo-Boolean model P of LP_n such that $LQ_{n+1}\supset\subset P\uparrow S_1$.*

Proof. By Mckay [12], LQ_{n+1} has the finite model property. So we can take finite models M_i's of the form $S_1\uparrow N_i$ such that $LQ_{n+1}\supset\subset\bigcap_{i\in I}L(M_i)$. By Lemma 4.13, M_i is of the form $M_i'\uparrow S_1$. Clearly $M_i'\in\mathcal{S}_{n_i}$ and $w(M_i')=m_i$ for some $n_i\leq n$ and $m_i<\omega$. So, by Lemma 4.4, Corollary 4.6 and Lemma 4.8 R_{nm_i} is embeddable in M_i'. Hence $R_{nm_i}\uparrow S_1$ is embeddable in M_i by Corollary 2.12, 3). So $L(M_i)\supset L(R_{nm_i}\uparrow S_1)$ and hence $LQ_{n+1}\supset\bigcap_{m<\omega}L(R_{nm}\uparrow S_1)$. Since $R_{n\omega}\uparrow S_1$ is embeddable in $R_{nm}\uparrow S_1$,

13) See Definition 4.11 and Lemma 4.12 in [8].

$\bigcap_{m<\omega} L(R_{nm} \uparrow S_1) \supset L(R_{n\omega} \uparrow S_1)$. Clearly $L(R_{n\omega} \uparrow S_1) \supset LQ_{n+1}$. Remark that $LQ \supset \subset LJ + Q \supset \subset \bigcap_{n,m<\omega} L(R_{nm} \uparrow S_1)$.

Note Added in Proof (March 5, 1971):

C. G. Mckay defined a sequence of models J_n' in "A note on the Jaśkowski sequence" Z. Math. Logik Grundlagen Math. 13 (1967) and proved that $\bigcap_{n<\omega} J_n' \supset \subset LJ$. But this is not the case. For, by the results of Gabbay-de Jongh [3], $D_1 \subset \bigcap_{n<\omega} J_n'$ and $LJ \subsetneqq D_1$. Mckay stated in his letter to the author, dated 25th September 1970, that his result is incorrect.

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