Publ. RIMS Kyoto Univ.
Vol. 6 (1970/71), 483-502

# On the Inverse of Monoidal Transformation

By

Shigeo NAKANO

## Introduction

When we have a complex analytic manifold $X$ and a complex analytic submanifold $M$ of codimension $r \geqq 2$ of $X$, we can form the monoidal transform $\widetilde{X}$ of $X$ with centre $M$. (By a manifold, we shall understand a paracompact connected one through this paper.) $\widetilde{X}$ is a complex analytic manifold with the same dimension $n$ as $X$, there exists a holomorphic mapping $\pi$ from $\widetilde{X}$ onto $X$, and $\pi$ is an analytic homeomorphism between $X-S$ and $X-M$, where $S=\pi^{-1}(M)$. (More properly, we should say $(\widetilde{X}, \pi)$ is the monoidal transform of $X$.) $S$ is an analytic submanifold of $\widetilde{X}$ of codimension 1, and is in a peculiar position in $\widetilde{X}$: The restriction of $\pi$ to $S$ makes $\pi: S \rightarrow M$ an analytic fibre bundle with projective $(r-1)$-space as the standard fibre. (More specifically, $S$ is the normal bundle of $M$ in $X$, with the zero cross section deleted and "divided" by the group $\boldsymbol{C}^*$ operating as multiplication by constants on each fibre.) If we denote the fibre $\pi^{-1}(a)$ by $L_a (a \in M)$, then we have $[S]_{L_a} = [e]^{-1}$, where $[S]$ and $[e]$ denote the complex line bundles defined by the divisor $S$ of $\widetilde{X}$ and the hyperplane $e$ of $\boldsymbol{P}^{r-1} = L_a$ respectively, and $[S]_{L_a}$ denotes the restriction of $[S]$ to $L_a$.

Now the inverse problem of the monoidal transformation is the following: Suppose we have a complex analytic manifold $\widetilde{X}^n$ and a submanifold $S$ of $\widetilde{X}$ of codimension 1. Let $S$ have a structure of a holomorphic fibre bundle over an analytic manifold $M^m$ with projective $(r-1)$-space as a standard fibre $(m+r=n)$. Then under what condi-

Received October 27, 1970.

tion can we find a complex analytic manifold $X^n$ and a holomorphic map $\pi$: $\widetilde{X}\rightarrow X$ so that $X$ contains $M$ as a submanifold and $\widetilde{X}$ is the monoidal transform of $X$ with centre $M$? It is clear that the above condition $[S]_{L_a}=[e]^{-1}$ is necessary. It was first shown by K. Kodaira that this condition is sufficient in case $M$ is a single point and $\widetilde{X}$ is a projective algebraic manifold (Kodaira [4]). In case $M=$ a point, we also have results by H. Grauert [2] and K. Kodaira [5] etc. For $M$ of higher dimension, P. A. Griffiths [3] gave a sufficient condition that the inverse problem can be solved affirmatively. B. G. Moĭšezon [7] gave a necessary and sufficient condition in case $X$ is compact and $M$ has as many independent meromorphic functions as complex dimension of M. A. Lascu [6] treats this problem in abstract algebraic geometry. In this paper, we shall give a necessary and sufficient condition for the existence of $X$, namely, we shall prove the

**Main Theorem.** *Let $\widetilde{X}$ be a complex analytic manifold of complex dimension $n\geqq 3$ and $S$ an analytic submanifold of $\widetilde{X}$ of codimension* 1. *Suppose that $S$ has a structure of an analytic fibre bundle over an analytic manifold $M$ with a projective $(r-1)$-space as the standard fibre and that $r>1$. Denote by $L_a$ the fibre over $a\in M$ in the bundle $S\rightarrow M$, and by $[e]$ the complex line bundle over $L_a\cong\mathbb{P}^{r-1}$ determined by the hyperplane. Then, in order that there exists an $n$-dimensional analytic manifold $X$ containing $M$ and a holomorphic map $\pi$: $\widetilde{X}\rightarrow X$ in such a way that $(\widetilde{X},\pi)$ is the monoidal transform of $X$ with centre $M$ and $S=\pi^{-1}(M)$, it is necessary and sufficient that the following conditions are satisfied*:

($\alpha$) *for any $a\in M$, $[S]_{L_a}=[e]^{-1}$,*

($\beta$) *each $L_a$ has a neighbourhood $V$ (in $\widetilde{X}$) such that $[\mathcal{K}]_V=(S)_V^k$, where $\mathcal{K}$ is the canonical bundle of $V$ and $k$ is a non-negative integer.*

To prove the theorem, we make use of a certain type of cohomology vanishing theorem and this theorem will be proved making use of a variant of a method due to A. Andreotti and E. Vesentini [1].

## §1. A Cohomology Vanishing Theorem

**1.1.** We shall consider the following situation: $V$ is an $n$-dimensional complex analytic manifold and there exist a real valued $C^\infty$ function $\Psi$ and a complex line bundle $\mathcal{B}$ on $V$, with the following properties:

(a) $\Psi$ is plurisubharmonic, i.e. for any point $x$ of $V$, we have

$$\left(\frac{\partial^2\Psi}{\partial x^j\partial\overline{x}^k}\right)\geqq 0 \qquad \text{(positive semi-definite)},$$

where $(x^j)$ is a system of analytic local coordinates around $x$.

(b) For any real value $A$, the set $\{x\,|\,\Psi(x)\leqq A\}$ is empty or compact.

(c) $\mathcal{B}$ is negative, this means the following: $\mathcal{B}$ is defined by a system of transition functions $\{e_{\alpha\beta}\}$ with respect to an open covering $\{U_\alpha\}$ of $V$, and we have a system of positive valued $C^\infty$ functions $\{a_\alpha\}$, $a_\alpha$ being defined in $U_\alpha$ and having the properties

$$a_\alpha/a_\beta=|e_{\alpha\beta}|^2 \qquad \text{in} \quad U_\alpha\cap U_\beta,$$

$$\left(\frac{\partial^2\log a_\alpha}{\partial x^j\partial\overline{x}^{-k}}\right)<0 \qquad \text{at each point of } U_\alpha.$$

(d) The canonical bundle of $V$ is equal to $\mathcal{B}^k$ $(k\geqq 0)$.

The theorem we want to establish in this section is the following

**Theorem 1.** *In the above situation, we have* $H^q(V,\mathcal{O}(\mathcal{B}^{-\varepsilon}))=0$ *for* $\varepsilon=1,2,\cdots,q=1,\cdots,n-1$.

The proof by the method of Carleman estimate will be given in the following.

**1.2.** Let us first establish

**Proposition 1.** *The system of functions* $\{a_\alpha\}$ *can be so chosen that the Kähler metric on* $V$ *given by*

$$ds^2=\sum\frac{\partial^2\log a_\alpha^{-1}}{\partial x^j\partial\overline{x}^k}dx^jd\overline{x}^k \tag{1.1}$$

*is complete.*

*Proof*. Take a real valued $C^\infty$ function $\mu$ of a real variable $\tau$ such that $\mu(\tau)\geqq 0$, $\mu'(\tau)\geqq 0$, $\mu''(\tau)\geqq 0$ and $\int_0^\infty\sqrt{\mu''(\tau)}\,d\tau=+\infty$, and replace given $a_\alpha$ by $a'_\alpha=e^{(-\mu(\Psi))}a_\alpha$. The new system $\{a'_\alpha\}$ satisfies the condition (c) too. Define a Kähler metric on $V$ by (1.1) with $a'_\alpha$ instead of $a_\alpha$ on the right hand side.

If $V$ itself is compact, there is nothing to prove. Assume $V$ is not compact, and take a differentiable arc $\gamma$: $x=x(t)$ $(0\leqq t<1)$ in $V$ which is not contained in any compact subset of $V$. Set $f(t)=\Psi(x(t))$, then $f(t_\nu)\to\infty$ for some sequence $\{t_\nu\}$ of values of $t$. We have:

$$\left(\frac{ds}{dt}\right)^2=\sum_{j,k}\left\{\mu'(f(t))\frac{\partial^2\Psi}{\partial x^j\partial\bar{x}^k}+\mu''(f(t))\frac{\partial\Psi}{\partial x^j}\frac{\partial\Psi}{\partial\bar{x}^k}\right\}\frac{dx^j}{dt}\frac{d\bar{x}^k}{dt}$$
$$+\sum\frac{\partial^2\log a_\alpha^{-1}}{\partial x^j\partial\bar{x}^k}\frac{dx^j}{dt}\frac{d\bar{x}^k}{dt}\geqq\mu''(f(t))\left|\sum_j\frac{\partial\Psi}{\partial x^j}\frac{dx^j}{dt}\right|^2$$
$$\geqq\frac{1}{4}\mu''(f(t))\left|\frac{df}{dt}\right|^2. \qquad (x^j=x^j(x(t)))$$

Then

$$\int_{t=0}^1 ds\geqq\frac{1}{2}\int_0^1\sqrt{\left|\mu''(f((t))\frac{df}{dt}\right|^2}=+\infty.$$

Therefore the proposition holds if we take $a'_\alpha$ instead of the original $a_\alpha$.

From now, we shall assume that $\{a_\alpha\}$ is chosen so that (1.1) is a complete Kähler metric, and we shall fix this metric through this section.

**1.3.** We consider a Hermitian metric $\{b_\alpha\}$ on the fibres of $\mathcal{B}$, that is to say, $b_\alpha$ is a positive valued $C^\infty$ function on $U_\alpha$ and we have

$$b_\alpha/b_\beta=|e_{\alpha\beta}|^2 \qquad \text{in} \quad U_\alpha\cap U_\beta.$$

The system $\{a_\alpha\}$ in **1.1**, **1.2** is an example of such metric.

We denote by $\mathcal{D}^{p,q}(\mathcal{B})$ the $\boldsymbol{C}$-vector space of all $\mathcal{B}$-valued differential forms of type $(p, q)$ with compact supports. (a $\mathcal{B}$-valued differential form $\varphi$ is a system $\{\varphi_\alpha\}$ of differential forms, $\varphi_\alpha$ being defined on $U_\alpha$ and satisfying $\varphi_\alpha=e_{\alpha\beta}\varphi_\beta$ in $U_\alpha\cap U_\beta$.) The metric $\{b_\alpha\}$ on the fibres defines a Hermitian inner product in $\mathcal{D}^{p,q}(\mathcal{B})$

$$(1.2)\qquad (\varphi, \psi)=\int_V \frac{1}{b_\alpha}\varphi_\alpha{}_\wedge *\overline{\psi}_\alpha \qquad\qquad \varphi, \psi\in\mathcal{D}^{p,q}(\mathcal{B}),$$

and makes it a pre-Hilbert space.

With respect to this inner product, the adjoint of the operator $\bar\partial$: $\mathcal{D}^{p,q}(\mathcal{B})\rightarrow\mathcal{D}^{p,q+1}(\mathcal{B})$, is given by the operator $\vartheta$:

$$(1.3)\qquad (\vartheta\varphi)_\alpha=-*D'*\varphi_\alpha,$$

where

$$(1.4)\qquad D'\varphi_\alpha=\partial\varphi_\alpha+\rho_\alpha{}_\wedge\varphi_\alpha,$$

$$(1.5)\qquad \rho_\alpha=-\partial\log b_\alpha.$$

$D'$: $\mathcal{D}^{p,q}(\mathcal{B})\rightarrow\mathcal{D}^{p+1,q}(\mathcal{B})$ is the (1,0)-part of the exterior differentiation defined in terms of the connection $\{\rho_\alpha\}$ on $\mathcal{B}$. As usual we define the Laplace-Beltrami operator $\square$ by

$$(1.6)\qquad \square=\bar\partial\vartheta+\vartheta\bar\partial.$$

We define another norm $\mathrm{n}(\varphi)$ in $\mathcal{D}^{p,q}(\mathcal{B})$:

$$(1.8)\qquad \mathrm{n}(\varphi)^2=(\varphi,\varphi)+(\bar\partial\varphi,\bar\partial\varphi)+(\vartheta\varphi,\vartheta\varphi).$$

The completions of $\mathcal{D}^{p,q}(\mathcal{B})$ with respect to the norms $(\varphi,\varphi)^{1/2}$ and $\mathrm{n}(\varphi)$ are denoted by $\mathcal{L}^{p,q}(\mathcal{B})$ and $W^{p,q}(\mathcal{B})$ respectively.

We shall recall the following propositions:

**Proposition 2.** *$W^{p,q}(\mathcal{B})$ can be considered as the set of elements $\varphi$ of $\mathcal{L}^{p,q}(\mathcal{B})$ for which $\bar\partial\varphi\in\mathcal{L}^{p,q+1}(\mathcal{B})$ and $\vartheta\varphi\in\mathcal{L}^{p,q-1}(\mathcal{B})$. Here $\bar\partial$ and $\vartheta$ are taken in the sense of distributions.*

For the proof we refer the reader to Andreotti-Vesentini [1]. (Proposition 5 of p. 93 in the paper. One can also find detailed exposition in Vesentini [8].) We shall recall that we make use of the completeness of the metric in the proof. We shall further recall that

$$(1.9)\qquad (\bar\partial\varphi,\ \bar\partial\varphi)+(\vartheta\varphi,\vartheta\varphi)\leqq\sigma(\square\varphi,\square\varphi)+\frac{1}{\sigma}(\varphi,\varphi)$$

for any $\sigma>0$ and for any $\mathcal{B}$-valued $C^\infty$ form $\varphi$ such that each term in the formula is defined. This holds because the metric is complete. (Formula (19) of Andreotti-Vesentini [1], p. 93.) The following proposition is an easy consequence of (1. 9).

**Proposition 3.** *If* $\varphi\in\mathcal{L}^{p,q}(\mathcal{B})$ *is of class* $C^{\infty}$ *and* $\square\varphi\in\mathcal{L}^{p,q}(\mathcal{B})$, *then* $\varphi$ *belongs to* $W^{p,q}(\mathcal{B})$. *In particular, if* $\varphi\in\mathcal{L}^{p,q}(\mathcal{B})$ *and* $\square\varphi=0$, *then* $\bar{\partial}\varphi=0$ *and* $\vartheta\varphi=0$.

This is the content of Cor. 6 and Prop. 7 of Andreotti-Vesentini [1] (p. 93), the contents of Theorem 1 (p. 89) and Theorem 21 (p. 94) of the paper are given in the following Propositions:

**Proposition 4.** *If there exists a positive constant* $C$ *such that*

$$(\bar{\partial}\varphi, \bar{\partial}\varphi)+(\vartheta\varphi, \vartheta\varphi)\geqq C(\varphi, \varphi) \tag{1.10}$$

*for all* $\varphi\in\mathcal{D}^{p,q}(\mathcal{B})$, *then for any* $\alpha\in\mathcal{L}^{p,q}(\mathcal{B})$, *there exists one and only one element* $x$ *of* $W^{p,q}(\mathcal{B})$ *satisfying* $\square x=\alpha$.

Here $\square x=\alpha$ is to be understood in the sense of weak solution, i. e., for any $\varphi\in\mathcal{D}^{p,q}(\mathcal{B})$ we have $(\bar{\partial}x, \bar{\partial}\varphi)+(\vartheta x, \vartheta\varphi)=(\alpha, \varphi)$. But since $\square$ is elliptic, if $\alpha$ is $C^{\infty}$ then $x$ is also $C^{\infty}$ and the equality $\square x=\alpha$ holds in strong sense.

**Proposition 5.** *With the above notations, if* $\alpha$ *is* $C^{\infty}$ *with* $\bar{\partial}\alpha=0$, *then* $\beta=\vartheta x$ *is the unique* $C^{\infty}$ *form in* $\mathcal{L}^{p,q-1}(\mathcal{B})$ *which satisfies* $\bar{\partial}\beta=\alpha$ *and* $\vartheta\beta=0$.

We define Green's operator $G$: $\mathcal{L}^{p,q}\rightarrow W^{p,q}$ by $x=G\alpha$. It is also seen that

$$(G\alpha, G\alpha)\leqq C^{-2}(\alpha, \alpha).$$

**1. 4.** We have introduced the connection $\{\rho_{\alpha}\}$ combined with the metric $\{b_{\alpha}\}$. This enables one to define the covariant differentiation $D$: $\mathcal{D}^{p,q}(\mathcal{B})\rightarrow\mathcal{D}^{p+1,q}(\mathcal{B})\oplus\mathcal{D}^{p,q+1}(\mathcal{B})$ by $D=D'+\bar{\partial}$, $D'$ being given in (1. 4). It is straightforward to verify the relation

$$(D^{2}\varphi)_{\alpha}=\Theta\wedge\varphi_{\alpha}, \tag{1.11}$$

where

$$\Theta=\bar{\partial}\rho_\alpha \tag{1.12}$$

is a scalar form on $V$ of type $(1.1)$. We set

$$\chi=\sqrt{-1}\,\Theta=\sqrt{-1}\,\partial\bar{\partial}\log b_\alpha \tag{1.13}$$

and call this the curvature form of the connection $\{\rho_\alpha\}$.

Since the metric of the base manifold $V$ is Kähler, we have the relations

$$L\delta'-\delta'L=\sqrt{-1}\,\bar{\partial},\qquad \Lambda\bar{\partial}-\bar{\partial}\Lambda=-\sqrt{-1}\,\delta'. \tag{1.14}$$

(See, for example, A. Weil [9], p. 44. These formulas are proven for scalar forms there, but they can be applied to $\mathcal{B}$-valued forms too, because the operations are defined for those forms and formulas are of local character.) If we take the adjoint of the first formula of (1.14), we have

$$D'\Lambda-\Lambda D'=-\sqrt{-1}\,\vartheta. \tag{1.15}$$

As (1.11) and (1.12) show, we have $D^2=D'\bar{\partial}+\bar{\partial}D'$. Hence, denoting by $\mathrm{e}(\chi)$ the exterior multiplication by $\chi$, we have

$$\begin{aligned}\mathrm{e}(\chi)\Lambda-\Lambda\mathrm{e}(\chi)&=\sqrt{-1}\,\{(D'\bar{\partial}+\bar{\partial}D')\Lambda-\Lambda(D'\bar{\partial}+\bar{\partial}D')\}.\\&=\sqrt{-1}\,\{D'(\bar{\partial}\Lambda-\Lambda\bar{\partial})+D'\Lambda\bar{\partial}+\bar{\partial}(D'\Lambda-\Lambda D')+\bar{\partial}\Lambda D'\\&\quad-(\Lambda D'-D'\Lambda)\bar{\partial}-D'\Lambda\bar{\partial}-(\Lambda\bar{\partial}-\bar{\partial}\Lambda)D'-\bar{\partial}\Lambda D'\}\\&=(\bar{\partial}\vartheta+\vartheta\bar{\partial})-(D'\delta'+\delta'D'),\end{aligned}$$

that is to say,

$$\mathrm{e}(\chi)\Lambda-\Lambda\mathrm{e}(\chi)=\square-*^{-1}\square*, \tag{1.16}$$

a formula first due to Calabi and Vesentini.

**1.5.** Now we turn to the proof of Theorem 1. It is enough to show $H_K^{n-q}(V,\Omega^n(\mathcal{B}^{\varepsilon}))=0$ $(1\leqq q\leqq n-1)$, because of Serre duality theorem. Here $H_K$ means the cohomology group with compact supports. By condition (d), this is equivalent to saying $H_K^{n-q}(V,\mathcal{O}(\mathcal{B}^{k+\varepsilon}))=0$.

Set $l=k+\varepsilon(>0)$ and apply the considerations in **1.3** and **1.4** to

$\mathscr{B}^l$ instead of original $\mathscr{B}$. We take $b_\alpha$ to be equal to $a_\alpha^l$, where $\{a_\alpha\}$ is the system introduced in **1.2**. (The meaning of $\{b_\alpha\}$ in (1.2) has now changed according to the change $\mathscr{B}\to\mathscr{B}^l$, but $\{a_\alpha\}$ keeps the original meaning for $\mathscr{B}$.)

$$\rho_\alpha=\partial\log b_\alpha=l(\partial\log a_\alpha)=l\rho_\alpha^{(1)}$$
$$\chi=\sqrt{-1}\,\bar\partial\rho_\alpha=l(\sqrt{-1}\,\bar\partial\rho_\alpha^{(1)})=l\chi^{(1)}.$$

If we write $\chi^{(1)}$ as $\chi^{(1)}=\sqrt{-1}\sum\chi_{j\bar k}dx^j\wedge d\bar x^k$, we see $\chi_{j\bar k}=\dfrac{\partial^2\log a_\alpha}{\partial x^j\partial\bar x^k}=-g_{j\bar k}$ according to the choice of the Kähler metric on $V$. Hence $e(\chi^{(1)})=-L$, $e(\chi)=(-l)L$ and (1.16) takes the form

$$(1.18)\qquad \square-*^{-1}\square*=l(\Lambda L-L\Lambda).$$

If $\varphi\in\mathscr{D}^{0,n-q}(\mathscr{B}^l)$, then $(\Lambda L-L\Lambda)\varphi=q\varphi$, and

$$(\bar\partial\varphi,\bar\partial\varphi)+(\vartheta\varphi,\vartheta\varphi)=(\square\varphi,\varphi)=lq(\varphi,\varphi)+(*^{-1}\square*\varphi,\varphi)\geqq lq(\varphi,\varphi)$$

because $(*^{-1}\square*\varphi,\varphi)=(\square*\varphi,*\varphi)\geqq0$. This shows that formula (1.10) holds if we choose $C=lq$.

**Intermediate Proposition.** *For $\varphi\in\mathscr{D}^{0,n-q}(\mathscr{B}^l)$ with $\bar\partial\varphi=0$, we can find $\psi\in\mathscr{L}^{0,n-q-1}(\mathscr{B}^l)$ which is $C^\infty$ and satisfies $\bar\partial\psi=0$, $\vartheta\psi=0$ and $(\psi,\psi)\leqq\frac{4}{C}(\varphi,\varphi)$, where $C=lq$. We have the same result for any choice of Hermitian metric on the fibre, provided* (1.10) *holds with the same value of $C$.*

*Proof.* By Propositions 4 and 5, we see that $\psi=\vartheta G\varphi$ satisfies the requirement except $(\psi,\psi)\leqq\frac{4}{C}(\varphi,\varphi)$. Now since $x=G\varphi$ is in $W^{0,q}(\mathscr{B}^l)$, (1.10) holds for $x$ too. Hence

$$(\bar\partial x,\bar\partial x)+(\vartheta x,\vartheta x)\geqq C(x,x).$$

Apply (1.9) to $x$ with $\sigma=\dfrac{C}{2}$, then we have

$$(\bar\partial x,\bar\partial x)+(\vartheta x,\vartheta x)\leqq\frac{2}{C}(\varphi,\varphi)+\frac{C}{2}(x,x)$$
$$\leqq\frac{2}{C}(\varphi,\varphi)+\frac{1}{2}\{(\bar\partial x,\bar\partial x)+(\vartheta x,\vartheta x)\},$$

whence

$$(\psi, \psi) \leqq (\bar{\partial}x, \bar{\partial}x) + (\vartheta x, \vartheta x) \leqq \frac{4}{C}(\varphi, \varphi).$$

It remains to show that we can take $\psi$ to lie in $\mathcal{D}^{0,q-1}(\mathcal{B}^l)$, because this result means that the $\mathcal{B}^{\varepsilon}$-valued Dolbeault cohomology groups with compact support vanish for type $(0, q)$, $q=1, 2, \cdots, n-1$.

**1.6.** In order to achieve the last point, we proceed as follows: We take a real valued $C^\infty$ function $\lambda$ of a real variable $t$, satisfying the conditions

$$\begin{cases} \lambda(t), \quad \lambda'(t), \quad \lambda''(t) \geqq 0 & \\ \lambda(t) = \lambda'(t) = \lambda''(t) = 0 & \text{for} \quad t>0 \\ \lambda'(t) > 0 & \text{for} \quad t>0 \\ \lambda''(t) > 0 & \text{for} \quad 0<t<1 \\ \lambda''(t) = 0 & \text{for} \quad 1 \leqq t, \end{cases}$$

and introduce new metrics on the fibres of $\mathcal{B}^l$;

$$b_{\alpha,\nu} = e^{-\nu\lambda(\Psi - A)} a_\alpha^l \qquad (\nu = 1, 2, 3, \cdots).$$

Here $A$ is a constant to be determined later. Then the meaning of the inner product changes according as $\nu$ changes. We shall use $(\ ,\ )_\nu$, $\mathcal{L}_\nu^{p,q}$, $W_\nu^{p,q}$, $\vartheta_\nu$, $\Box_\nu \cdots$, for symbols corresponding to the metric $\{b_{\alpha,\nu}\}$, while for original metric $a_\alpha^l = b_{\alpha,0}$, we shall omit the suffix 0.

The change of the curvature form $\chi$ is to be noted. We readily see $\chi_\nu = \chi - \nu\omega$ where $\omega = \sqrt{-1}\,\partial\bar{\partial}\lambda(\Psi - A)$. Hence (1.16), (1.18) take the form

$$\Box_\nu - *^{-1}\Box_\nu * = l(\Lambda L - L\Lambda) + \nu(\Lambda e(\omega) - e(\omega)\Lambda).$$

If we show

(1.19) $$((\Lambda e(\omega) - e(\omega)\Lambda)\varphi, \varphi) \geqq 0 \quad \text{for} \quad \varphi \in \mathcal{D}^{0,n-q}(\mathcal{B}^l),$$

we shall have

(1.20) $$(\Box_\nu \varphi, \varphi)_\nu \geqq lq(\varphi, \varphi) \qquad \varphi \in \mathcal{D}^{0,n-q}(\mathcal{B}^l).$$

As for (1.19), we have, in a small neighbourhood $V$ of a given point,

$$\partial\bar{\partial}\lambda(\Psi-A)=\partial\sum_k\lambda'(\Psi-A)\frac{\partial\Psi}{\partial\bar{x}^k}d\bar{x}^k$$
$$=\sum_{j,k}\left\{\lambda''(\Psi-A)\frac{\partial\Psi}{\partial x^j}\frac{\partial\Psi}{\partial\bar{x}^k}+\lambda'(\Psi-A)\frac{\partial^2\Psi}{\partial x^j\partial\bar{x}^k}dx^j\wedge d\bar{x}^k\right\}.$$

The quantities in { } make up a positive semi-definite matrix. Hence if we take an orthonormal basis $\{\theta_j\}$ of linear differential forms on $U$, we shall have the expressions

$$L=\sqrt{-1}\sum \mathrm{e}(\theta_j)\mathrm{e}(\bar{\theta}_j),$$
$$e(\omega)=\sqrt{-1}\sum w_{j\bar{k}}\mathrm{e}(\theta_j)\mathrm{e}(\theta_k),$$

where $(w_{j\bar{k}})\geqq 0$. We can find a matrix $(t_{lj})$ such that $w_{j\bar{k}}=\sum\bar{t}_{lj}t_{lk}$. The adjoint of $\mathrm{e}(\theta_j)$ is denoted by $\mathrm{i}(\theta_j)$ and is given by $-*\mathrm{e}(\bar{\theta}_j)*$. Because $\{\theta_j\}$ is an orthonormal basis, it can be verified that if $A=(a_1, \cdots, a_r)$, $B=(b_1, \cdots, b_s)$ are set of indices, and if $\theta_{A\wedge}\theta_B$ denotes $\theta_{a_1}\wedge\cdots\wedge\theta_{a_r}\wedge\bar{\theta}_{b_1}\wedge\cdots\wedge\bar{\theta}_{b_s}$, we have

$$(1.21)\qquad \begin{cases}\mathrm{i}(\theta_j)(\theta_{j\wedge}\theta_{A\wedge}\bar{\theta}_B)=\theta_{A\wedge}\bar{\theta}_B \\ \mathrm{i}(\theta_j)(\theta_{A\wedge}\bar{\theta}_B)=0\end{cases}\qquad \text{for } j\notin A,$$

and similarly for $\mathrm{i}(\bar{\theta}_k)$. From this we conclude the relations

$$(1.22)\qquad \begin{cases}\mathrm{e}(\theta_j)\mathrm{i}(\theta_m)+\mathrm{i}(\theta_m)\mathrm{e}(\theta_j)=\delta_{jm}\\ \mathrm{e}(\bar{\theta}_k)\mathrm{i}(\bar{\theta}_m)+\mathrm{i}(\bar{\theta}_m)\mathrm{e}(\bar{\theta}_k)=\delta_{km}\\ \mathrm{e}(\theta_j)\mathrm{i}(\bar{\theta}_k)+\mathrm{i}(\bar{\theta}_k)\mathrm{e}(\theta_j)=0\\ \mathrm{e}(\bar{\theta}_j)\mathrm{i}(\bar{\theta}_k)+\mathrm{i}(\bar{\theta}_k)\mathrm{e}(\theta_j)=0.\end{cases}$$

Now $\Lambda\mathrm{e}(\omega)-\mathrm{e}(\omega)\Lambda=\sum_{j,k,m}w_{jk}\{\mathrm{i}(\bar{\theta}_m)\mathrm{i}(\theta_m)\mathrm{e}(\theta_j)\mathrm{e}(\bar{\theta}_k)-\mathrm{e}(\theta_j)\mathrm{e}(\bar{\theta}_k)\mathrm{i}(\bar{\theta}_m)\mathrm{i}(\theta_m)\}$ can be calculated according to (1.22) and gives

$$\Lambda\mathrm{e}(\omega)-\mathrm{e}(\omega)\Lambda=\sum w_{jk}\{\mathrm{i}(\bar{\theta}_j)\mathrm{e}(\bar{\theta}_k)-\mathrm{e}(\theta_j)\mathrm{i}(\theta_k)\}.$$

Hence if $\varphi$ is a $(0, n-q)$-form with support contained in $U$, we have

$$((\Lambda\mathrm{e}(\omega)-\mathrm{e}(\omega)\Lambda)\varphi, \varphi)=(\sum_{l,j,k}\bar{t}_{lj}\mathrm{i}(\bar{\theta}_j)t_{lk}\mathrm{e}(\bar{\theta}_k)\varphi, \varphi)$$
$$=\sum_l(\sum_k t_{lk}\mathrm{e}(\bar{\theta}_k)\varphi, \sum_j t_{lj}\mathrm{e}(\bar{\theta}_j)\varphi\geqq 0.$$

This is enough to establish (1.19).

So we have shown (1.20) with the constant $lq$ which is independent of $\varphi$ and $\nu$.

**1.7.** Now consider $\varphi\in\mathcal{D}^{0,n-q}(\mathcal{B}^l)$ with $\bar\partial\varphi=0$. Set $K=\operatorname{supp}\varphi$, $A=\max_{x\in K}\Psi(x)$. We take $b_{\alpha,\nu}$ with this value of $A$, and apply the Intermediate Proposition to this case. Then there exists $\psi_\nu\in\mathcal{L}_\nu^{0,n-q-1}(\mathcal{B}^l)$ such that $\vartheta_\nu\psi_\nu=0$, $\bar\partial\psi_\nu=\varphi$ and $(\psi_\nu,\psi_\nu)_\nu\leqq\frac{4}{lq}(\varphi,\varphi)_\nu$. Because $b_{\alpha,\nu}\leqq a_\alpha^l$ and equality holds on the support of $\varphi$,

$$\frac{4}{lq}(\varphi,\varphi)=\frac{4}{lq}(\varphi,\varphi)_\nu\geqq(\psi_\nu,\psi_\nu)_\nu\geqq(\psi_\nu,\psi_\nu), \tag{1.23}$$

and $\{\psi_\nu\}_{\nu=1,2,\cdots}$ form a bounded set in $\mathcal{L}^{0,n-q-1}(\mathcal{B}^l)$. Hence, if we choose a suitable subsequence if necessary, $\psi_\nu$ converges to an element $\psi\in\mathcal{L}^{0,n-q-1}(\mathcal{B}^l)$ weakly. We have $(\psi,\psi)\leqq\frac{4}{lq}(\varphi,\varphi)$ and $\bar\partial\psi=\varphi$ in the sense of distribution.

Set $M=\{x\in V\,|\,\Psi(x)>A+1\}$, then from (1.23) it follows that

$$\int_M a_\alpha^{-l}e^{\nu\lambda(\Psi-A)}(\psi_\nu)_\alpha\wedge*\overline{(\psi_\nu)_\alpha}\leqq(\psi_\nu,\psi_\nu)_\nu\leqq\frac{4}{lq}(\varphi,\varphi). \tag{1.24}$$

On the other hand, there exists a positive constant $c$ such that $\lambda(\Psi-A)\geqq c$ on $M$. Hence the left hand side of (1.24) is not less than $e^{\nu c}\int_M a_\alpha^{-l}(\psi_\nu)_\alpha\wedge*\overline{(\psi_\nu)_\alpha}$. This shows that

$$\int_M a_\alpha^{-l}(\psi_\nu)_\alpha\wedge*\overline{(\psi_\nu)_\alpha}\to0 \qquad \text{for } \nu\to\infty.$$

If we take an element $u\in\mathcal{D}^{0,n-q-1}(\mathcal{B}^l)$ with support contained in $M$, then it follows from the above that $(\psi,u)=\lim_{\nu\to\infty}(\psi_\nu,u)=0$. Hence the support of $\psi$ as a distribution is contained in the compact set $V-M$.

Finally, we can apply to $\psi$ the regularization process as given in Andreotti-Vesentini [1], Lemma 12 p. 97, and obtain a form $\psi_0\in\mathcal{D}^{0,n-q-1}(\mathcal{B}^l)$ with $\bar\partial\psi_0=\varphi$. This completes the proof of Theorem 1.

## §2. Construction of a Plurisubharmonic Function

**2.1.** We consider an $n$-dimensional complex analytic manifold $\widetilde{X}$ and a submanifold $S$ of codimension 1, and assume that the condition $(\alpha)$

in the Main Theorem is satisfied. $S$ is an analytic bundle of projective spaces $\boldsymbol{P}^{r-1}$ over an analytic manifold $M^m$ with $m+r=n$. Take a point $a$ of $M$ and a small coordinate neighbourhood $D$ of $a$. $D$ can be considered as the domain $\{(\zeta^1, \cdots, \zeta^m) \in \boldsymbol{C}^m \mid \sum|\zeta^j|^2<1\}$ in $\boldsymbol{C}^m$, the point $a$ corresponding to the origin and the part $\pi^{-1}(D)$ of $S$ which lies over $D$ has the form $\pi^{-1}(D) \cong D \times \boldsymbol{P}^{r-1}$. We take a set of homogeneous coordinates $(\eta^1, \cdots, \eta^r)$ on $\boldsymbol{P}^{r-1}$. Then $\boldsymbol{P}$ is covered by coordinate neighbourhoods $\{U_\alpha\}$ ; $U_\alpha=\{\eta^\alpha \neq 0\}$, and the set of inhomogeneous coordinates $\{\xi_\alpha^\gamma=\eta^\gamma/\eta^\alpha \mid \gamma=1, \cdots, \hat{\alpha}, \cdots, r\}$ is a set of local coordinates in $U_\alpha$. (When we make use of the notation $\xi_\alpha^\alpha$, it is the constant 1.) Because $[S]_{L_b}=[e]^{-1}$ for any $b \in D$, $[S]_{\pi^{-1}(D)}=q^*[e]^{-1}$, where $q$ is the canonical projection $D \times \boldsymbol{P}^{r-1} \rightarrow \boldsymbol{P}^{r-1}$ and $[e]$ is the complex line bundle on $\boldsymbol{P}^{r-1}$ determined by a hyperplane.

$[e]$ on $\boldsymbol{P}^{r-1}$ can be defined with respect to the open covering $\{U_\alpha\}$ by transition functions

$$\varepsilon_{\alpha\beta}=\eta^\beta/\eta^\alpha=\xi_\alpha^\beta \quad \text{in} \quad U_\alpha \cap U_\beta. \tag{2.1}$$

As the metric on the fibre, we take

$$a_\alpha^{(0)}=\sum_\gamma|\eta^\gamma|^2/|\eta^\alpha|^2=\sum_\gamma|\xi_\alpha^\gamma|^2. \tag{2.2}$$

The curvature form of this metric is the Kähler form associated to the standard Hodge metric of $\boldsymbol{P}^{r-1}$.

On $D \times \boldsymbol{P}$, the bundle $q^*[e]$, which we shall now denote by $[e]$, is defined by (2.1) with respect to $\{D \times U_\alpha\}$. We set

$$\phi(\zeta)=\sum|\zeta^j|^2 \tag{2.3}$$

and take

$$a_\alpha=e^{\phi(\zeta)}a_\alpha^{(0)} \tag{2.4}$$

as the metric on the fibres. The curvature form of this metric is again positive definite.

**2.2.** Local coordinates on $\pi^{-1}(D)$ can be considered as restrictions of local coordinates on $\widetilde{X}$. More precisely, we can choose a finite open covering $\{U_\lambda'\}_{\lambda \in \Lambda}$ of $\boldsymbol{P}$ which refines $\{U_\alpha\}$, open sets $V_\lambda'$ on $\widetilde{X}$ such that $V'_\lambda \cap S = D \times U'_\lambda$, and systems of local coordinates $(z^1_\lambda, \cdots, z^m_\lambda, x^1_\lambda, \cdots, \hat{x}^{\tau(\lambda)}_\lambda, \cdots, x^r_\lambda, y_\lambda)$ on $V'_\lambda$ in the following manner:

(2.5) $\quad \tau$ is a map $\Lambda \to \{1, \cdots, r\}$ such that $U'_\lambda \subset U_{\tau(\lambda)}$.

(2.6) $\quad z^j_\lambda | S = \zeta^j, \qquad x^\gamma_\lambda | S = \xi^\gamma_{\tau(\lambda)}$

(2.7) $\quad y_\lambda = 0$ is the local equation for $S$ in $V'_\lambda$.

(We set $x^{\tau(\lambda)}_\lambda \equiv 1$. Hence (2.6) holds for $\gamma = \tau(\lambda)$ too.)

Furthermore we may assume that the transition function $e_{\lambda\mu} = y_\lambda / y_\mu$ for $[S]$ satisfies

(2.8) $$e_{\lambda\mu}\Big| S = \varepsilon^{-1}_{\lambda\mu},$$

where we write $\varepsilon_{\lambda\mu}$ instead of $\varepsilon_{\tau(\lambda), \tau(\mu)}$. Also we set $a_\lambda = a_{\tau(\lambda)}$.

For the moment, we fix the index $j$ and write $z_\lambda$ instead of $z^j_\lambda$. Then $z_\lambda - z_\mu$ is a holomorphic function on $V'_\lambda \cap V'_\mu$ and is zero on $S$. We set

(2.9) $$f_{\lambda\mu} = y^{-1}_\lambda (z_\lambda - z_\mu),$$

then we have

$$f_{\lambda\nu} = f_{\lambda\mu} + e^{-1}_{\lambda\mu} f_{\mu\nu} \qquad \text{in} \quad V'_\lambda \cap V'_\mu \cap V'_\nu.$$

In other words $\{f_{\lambda\mu}\}$ is a 1-cocycle with values in $\mathcal{O}([S]^{-1})$, with respect to the open covering $\mathfrak{V} = \{V'_\lambda\}$ of $V' = \bigcup_\lambda V'_\lambda$. The restrictions $\varphi_{\lambda\mu} = f_{\lambda\mu}\Big| S$ define an element of $Z^1(\mathfrak{U}, \mathcal{O}([e]))$, where $\mathfrak{U} = \{D \times U'_\lambda\}$. Since $H^1(\boldsymbol{P}^{r-1}, \mathcal{O}([e])) = 0$, we can find holomorphic functions $\varphi_\lambda$ on $D \times U'_\lambda$ such that

$$\varphi_\lambda - \varepsilon_{\lambda\mu}\varphi_\mu = \varphi_{\lambda\mu} \qquad \text{on} \quad D \times (U'_\lambda \cap U'_\mu).$$

We extend $\varphi_\lambda$ to $U'_\lambda$ and denote it by $f_\lambda$, then $z'_\lambda = z_\lambda - y_\lambda f_\lambda$ has the property

(2.10) $\quad z'_\lambda - z'_\mu \equiv 0 \pmod{y^2_\lambda} \qquad$ in $V'_\lambda \cap V'_\mu$.

We set $f'_{\lambda\mu} = y^{-2}_\lambda (z'_\lambda - z'_\mu)$ and proceed as before. This time we make use of $H^1(\boldsymbol{P}^{r-1}, \mathcal{O}([e]^2)) = 0$, and arrive at $z''_\lambda = z'_\lambda - y^2_\lambda f'_\lambda$ with $z''_\lambda - z''_\mu \equiv 0$ (mod. $y^3_\lambda$). Since $H^1(\boldsymbol{P}^{r-1}, \mathcal{O}([e]^k)) = 0$ for any $k > 0$, we can thus proceed as far as we want. Similar procedure can be applied to $x^\alpha_\lambda$ for any fixed $\alpha$ to obtain an approximate holomorphic section of $[S]^{-1}$. Hence,

**Proposition 6.** *Given a positive integer $l$, we may take local coordinates $(z_\lambda, x_\lambda, y_\lambda)$ in such a way that the following hold in addition to* (2.6) *and* (2.7).

$$(2.11)\qquad \begin{cases} z_\lambda^j - z_\mu^j = (y_\lambda)^l f_{\lambda\mu}^j \\ x_\lambda^\alpha - e_{\lambda\mu}^{-1} x_\mu^\alpha = (y_\lambda)^l g_{\lambda\mu}^\alpha \end{cases} \quad \text{in} \quad V'_\lambda \cap V'_\mu.$$

It may be that we have to replace $V'_\lambda$ by a smaller open set, in order to secure that $(z_\lambda^j, x_\lambda^\alpha, y_\lambda)$ form a local coordinate system. But any way $V' = \bigcup_\lambda V'_\lambda$ is an open neighbourhood of $L_a$ in $\widetilde{X}$, and it is enough for our purpose.

**2.3.** Since $\{f_{\lambda\mu}^j\}$, $\{g_{\lambda\mu}^\alpha\}$ are 1-cocycles in $Z^1(\mathfrak{B}, \mathcal{O}([S]^{-l}))$, $Z^1(\mathfrak{B}, \mathcal{O}([S]^{-l-1}))$ respectively, it is clear that we can choose $C^\infty$ functions $f_\lambda^j$, $g_\lambda^\alpha$ in $V'_\lambda$ which satisfy

$$f_\lambda^j - e_{\lambda\mu}^{-l} f_\mu^j = f_{\lambda\mu}^j,$$
$$g_\lambda^\alpha - e_{\lambda\mu}^{-l-1} g_\mu^\alpha = g_{\lambda\mu}^\alpha.$$

We set

$$(2.12)\qquad \begin{cases} Z^j = z_\lambda^j - (y_\lambda)^l f_\lambda^j = z_\mu^j - (y_\mu)^l f_\mu^j, \\ X_\lambda^\alpha = x_\lambda^\alpha - (y_\lambda)^l g_\lambda^\alpha, \end{cases}$$

then we have

$$(2.13)\qquad X_\lambda^\alpha = e_{\lambda\mu}^{-1} X_\mu^\alpha \quad \text{in} \quad V'_\lambda \cap V'_\mu.$$

Define functions $A_\lambda$, $F$ and $\psi$ by

$$(2.14)\qquad \begin{cases} A_\lambda = e^{\sum_j |z^j|^2} \left(\sum_{\alpha=1}^r |X_\lambda^\alpha|^2\right) \quad \text{in} \quad V'_\lambda, \\ F = A_\lambda \cdot |y_\lambda|^2 = A_\mu \cdot |y_\mu|^2, \end{cases}$$

$$(2.15)\qquad \psi = \sum_j |Z^j|^2 + F.$$

Since $A_\lambda = |e_{\lambda\mu}|^{-2} A_\mu$ by (2.13), $F$ is a well-defined function on $V'$. It is clear that $\sum |Z^j|^2$ and $A_\lambda$ reduce, on $S$, to $\phi = \sum |\zeta^j|^2$ and $a_\lambda$ respectively.

**Proposition 7.** *Take $l \geqq 3$, then $\psi$ is plurisubharmonic on the* *part of $V'$ where $F$ is small enough. More specifically, the Levi form of $\psi$ is not less than a positive definite Hermitian form in $(dz^1\cdots dz^m, dy)$ on this part.*

*Proof*. We consider the coordinate neighbourhood $V'_\lambda$ and verify the assertion in this domain. So we shall omit the suffix $\lambda$ for a moment. For definiteness, let us assume $\tau(\lambda)=r$, hence $(z^1, \cdots, z^m, x^1, \cdots, x^{r-1}, y)$ is the coordinate system, while in the expression $\sum|X^\alpha|^2$, the summation extends for $\alpha=1, 2, \cdots, r$.

Direct computation shows that

$$\frac{\partial^2\psi}{\partial z^j\partial\bar{z}^k}=\delta_{jk}+O(|y|),$$

$$\frac{\partial^2\psi}{\partial y\partial\bar{y}}=e^{\Sigma|z^j|^2}(1+\textstyle\sum_{\alpha=1}^{r-1}|x^\alpha|^2)+O(|y|),$$

$$\frac{\partial^2\psi}{\partial x^\alpha\partial\bar{x}^\beta}=e^{\Sigma|z^j|^2}|y|^2(\delta_{\alpha\beta}+O(|y|)),$$

$$\frac{\partial^2\psi}{\partial y\partial\bar{x}^\beta}=e^{\Sigma|z^j|^2}\bar{y}(x^\beta+O(|y|)),$$

$$\frac{\partial^2\psi}{\partial y\partial\bar{z}k}=e^{\Sigma|z^j|^2}(1+\sum_{\alpha=1}^{r-1}|x^\alpha|^2)\{z^k\bar{y}+O(|y|^2)\},$$

$$\frac{\partial^2\psi}{\partial z^j\partial\bar{x}^\beta}=O(|y|^2),\qquad \text{and their conjugates.}$$

By taking $V'$ smaller if necessary, we may assume that $\sum_{\alpha=1}^{r-1}|x_\alpha|^2$ is bounded in $V'$, say $\sum|x_\alpha|^2\leqq G$. Choose $\eta>0$ such that $\eta G<\frac{1}{3}$, then we have

$$\begin{aligned}
&\frac{\partial^2\psi}{\partial y\partial\bar{y}}(dy, d\bar{y})+\sum_\beta\frac{\partial^2\psi}{\partial y\partial\bar{x}^\beta}(dy, d\bar{x}^\beta)+\sum_\alpha\frac{\partial^2\psi}{\partial x^\alpha\partial\bar{y}}(dx^\alpha, d\bar{y})\\
&\quad+\sum_{\alpha,\beta}\frac{\partial^2\psi}{\partial x^\alpha\partial\bar{x}^\beta}(dx^\alpha, d\bar{x}^\beta)\\
&=e^{\Sigma|z^j|^2}\{\sum_{\alpha=1}^{r-1}|(1+\eta)^{-1/2}y\,dx^\alpha+(1+\eta)^{1/2}x^\alpha dy|^2\\
&\quad+(1-\eta\textstyle\sum|x_\alpha|^2+O(|y|))(dy, d\bar{y})+\sum\bar{y}a_\beta(dy, d\bar{x}^\beta)\\
&\quad+\textstyle\sum y\bar{a}_\alpha(dx^\alpha, d\bar{y})+|y|^2\sum\frac{\eta}{1+\eta}\delta_{\alpha\beta}+b_{\alpha\beta}\Big)(dx^\alpha, d\bar{x}^\beta)\}
\end{aligned}$$

$$\geqq e^{\Sigma|z_j|^2}\left\{\frac{2}{3}(dy, d\bar{y})+\sum\bar{y}a_\beta(dy, d\bar{x}^\beta)+\sum y\bar{a}_\beta(dx^\beta, d\bar{y})\right.$$

$$\left.+|y|^2\sum\left(\frac{\eta}{2}\delta_{\alpha\beta}+b_{\alpha\beta}\right)(dx^\alpha d\bar{x}^\beta)\right\},$$

where $a_\alpha$ and $b_{\alpha\beta}$ are quantities of $O(|y|)$. The last expression is $\geqq 0$ provided $|y|$ is small enough. We can deal with terms containing $\frac{\partial^2\psi}{\partial y\partial\bar{z}^k}$ etc. similarly, and we see that the Levi form of $\psi$ is equal to the sum of, say $\frac{1}{2}(\sum_j|dz^j|^2+|dy|^2)$ and a non-negative Hermitian form, if $|y|$ is small enough.

**Proposition 8.** $\left(-\frac{\partial^2\log A_\lambda}{\partial x_\lambda^\alpha\bar{x}_\lambda^\beta}\right)$ *is positive definite for* $|y_\lambda|$ *small enough.*

*Proof.* This is clear since $A_\lambda$ reduces to $a_\lambda$ for $y_\lambda=0$.

**Theorem 2.** *If* $\widetilde{X}$, $S$, $M$ *satisfy condition* ($\alpha$) *of the Main Theorem, then, for any* $a\in M$, *we can find a neighbourhood* $V$ *of* $L_a$ *in* $\widetilde{X}$, *such that the conditions* ($a$), ($b$), ($c$) *in Theorem* 1 *hold for* $V$ *and* $\mathscr{B}=[S]\big|_V$. *If we have condition* ($\beta$) *in addition, then condition* ($d$) *holds too.*

*Proof.* We use the notations in the above. We choose a small positive number $\delta$ such that Propositions 7 and 8 hold for $V=\{x\in\widetilde{X}\,|\,\psi(x)<\delta\}$ and such that $V$ is relatively compact in $V'$. If we set $\Psi=(1-\psi/\delta)^{-1}$, then conditions (a) and (b) are satisfied. If we note $\{e^{-m\psi}A^{-1}\}$ gives a Hermitian metric on the fibres of $[S]$ for any constant $m$, and if we take $m$ big enough, we see that condition (c) is satisfied in view of Propositions 7 and 8. The last assertion of the theorem is trivial.

*Remark.* It is important to note that each fibre of $S$ is either contained in $V$ or does not meet $V$.

## §3. The Proof of the Main Theorem

**3.1.** First we prove the

**Proposition 9.** *Let $V$ be an $n$-dimensional complex analytic manifold and $S$ a submanifold of $V$ of codimension 1. Suppose that $S$ is analytically homeomorphic to $D\times\boldsymbol{P}^{r-1}$, where $D$ is a domain in $\boldsymbol{C}^m$ and $m+r=n$, and that $[S]_S=[e]^{-1}$, $[e]$ being the complex line bundle on $S$ defined by $D\times$(hyperplane). If we have $H^1(V, \mathcal{O}([S]^{-\varepsilon}))=0$ for $\varepsilon=1, 2$, then for any point $a\in D$, there exist a neighbourhood $W$ of $L_a$(=the submanifold of $S$ corresponding to $a\times\boldsymbol{P}$) in $V$, and a holomorphic map $\pi$ from $W$ to $\varDelta=\{(z, y)\in\boldsymbol{C}^m\times\boldsymbol{C}^r \mid |z^j|<\varepsilon\}, |y|^\alpha<\varepsilon\}$ such that $(W, \pi)$ is the monoidal transform of $\varDelta$ with centre $\Gamma$=the linear variety defined by $y^1=\cdots=y^r=0$. We can identify $\Gamma$ with a neighbourhood of $a$ in $D$, and the restriction of $\pi$ to $S$ corresponds to the canonical projection $D\times\boldsymbol{P}\to D$ by these identifications.*

*Proof.* Let $(\zeta^1, \cdots, \zeta^m)$ be coordinates $\boldsymbol{C}^m$ which contains $D$, and let $(\eta^1, \cdots, \eta^r)$ be a set of homogeneous coordinates on $\boldsymbol{P}$. $H^1(V, \mathcal{O}([S]^{-1}))=0$ implies that the restriction $\Gamma(V, \mathcal{O})\to\Gamma(S, \mathcal{O}_S)$ is surjective. Hence there exist holomorphic functions $z^1, \cdots, z^m$ on $V$, whose restrictions on $S$ are $\zeta^1, \cdots, \zeta^m$. $H^1(V, \mathcal{O}([S]^{-2}))=0$ implies that $\Gamma(V, \mathcal{O}([S]^{-1}))\xrightarrow{\rho}\Gamma(S, \mathcal{O}([e]))$ is surjective. The latter contains cross sections corresponding to $\eta^1, \cdots, \eta^r$. (If we define $[e]$ by transition functions $\varepsilon_{\alpha\beta}=\eta^\beta/\eta^\alpha$ as in §2, the cross section corresponding to $\eta^\gamma$ is represented by a system of holomorphic functions $\{\eta^\gamma/\eta^\alpha\}_{\alpha=1,\cdots,r}$ satisfying $\eta^\gamma/\eta^\alpha=\varepsilon_{\alpha\beta}(\eta^\gamma/\eta^\beta)$.) Hence we can find cross sections $f^1, \cdots, f^r$ which are mapped to $\eta^1, \cdots, \eta^r$ by $\rho$.

We may assume we have coordinate neighbourhoods $V'_\lambda$ in $V$, such that $\bigcup_\lambda V'_\lambda\supset L_a$, $V'_\lambda\cap S$ has the form $D_1\times U'_\lambda$, and $S$ is defined in $V'_\lambda$ by a local equation $y_\lambda=0$, where $y_\lambda$ is a member of a local coordinate system in $V'_\lambda$. We also assume that (2.5) and (2.8) hold for these data. The cross section $f^\alpha$ can be expressed as a system $\{f^\alpha_\lambda\}$, where $f^\alpha_\lambda$ is holomorphic in $V'_\lambda$ and

$$f^{\alpha}_{\lambda}=(y_{\lambda}/y_{\mu})^{-1}f^{\alpha}_{\mu} \quad \text{in} \quad V'_{\lambda}\wedge V'_{\mu}.$$

Hence we can associate the holomorphic function on $V'=\bigcup V'_{\lambda}$ which is given by $y_{\lambda}f^{\alpha}_{\lambda}$ in $V'_{\lambda}$. We denote this function by $f^{\alpha}$. $f^{\alpha}$ is in the defining ideal for $S\cap V'$. The fact that the map $\rho$ associates $f^{\alpha}$ to $\eta^{\alpha}$ is now expressed by

$$(3.1) \qquad (f^{\alpha}/y^{\lambda})|_{S}=\eta^{\alpha}/\eta^{\tau(\lambda)}.$$

Now take $C^{n}$ with linear coordinates$(Z^{1},\cdots,Z^{m},Y^{1},\cdots,Y^{r})$ and blow it up with centre $C^{m}$ defined by $Y^{1}=\cdots=Y^{r}=0$. We denote the monoidal transform by $\widetilde{C}$, and define an analytic map $\Phi$ from $V'$ into $\widetilde{C}$ by

$$\Phi(x)=(z^{1}(x),\cdots,z^{m}(x),f^{1}(x),\cdots,f^{r}(x))\times(f^{1}(x):\cdots:f^{r}(x)).$$

In view of (3.1), it is seen that $\Phi$ maps $L_{a}$ onto $(0)\times P^{r-1}$ biholomorphically and the Jacobian of $\Phi$ does not vanish at every point of $L_{a}$. Hence $\Phi$ maps a neighbourhood $W$ of $L_{a}$ onto an open set of $\widetilde{C}$ isomorphically. We can assume that this open set is the transform of a domain $\Delta$ in $C^{n}$ containing the origin, with centre $\Gamma=\Delta\cap C^{m}$. The rest of the Proposition is easy to see.

3.2. The proof of the main theorem is now easy. If we have $\widetilde{X}$, $S$ and $M$ satisfying conditions $(\alpha)$ and $(\beta)$, then by Theorem 2, there exists a neighbourhood $V$ of $L_{a}$ for each $a\in M$, satisfying the conditions of Theorem 1 for $\mathcal{B}=[S]_{V}$. Hence we have the conditions of Proposition 9, and therefore, there exist, for each $a$, a neighbourhood $W_{a}$ of $L_{a}$ in $\widetilde{X}$, a holomorphic map $\pi_{a}$ from $W_{a}$ onto a domain $\Delta_{a}$ in $C^{n}$ and a linear variety $\Gamma_{a}$ of dimension $m$ in $\Delta_{a}$, such that $\pi_{a}\colon W_{a}\to\Delta_{a}$ is the monoidal transform $\Delta_{a}$ with centre $\Gamma_{a}$.

If $W_{a}\cap W_{b}\neq\phi$, we consider the diagram

$$\begin{array}{ccccccc} W_{a} & \supset & W_{a}\cap W_{b} & \xrightarrow{id} & W_{a}\cap W_{b} & \subset & W_{b} \\ \downarrow{\scriptstyle\pi_{a}} & & \downarrow{\scriptstyle\pi_{a}} & & \downarrow{\scriptstyle\pi_{b}} & & \downarrow{\scriptstyle\pi_{b}} \\ \Delta_{a} & \supset & \pi_{a}(W_{a}\cap W_{b}) & \xrightarrow{\phi_{ab}} & \pi_{b}(W_{a}\cap W_{b}) & \subset & \Delta_{b}. \end{array}$$

Since $\pi_{a}$ and $\pi_{b}$ are analytic homeomorphism from $W_{a}-S$ to $\Delta_{a}-\Gamma_{a}$ and

$W_b-S$ to $\Delta_b-\Gamma_b$ respectively, $\phi_{ab}=\pi_b\circ id\circ\pi_a^{-1}$ is defined on $\pi_c(W_a\cap W_b)-\Gamma_a$ and maps it isomorphically onto $\pi_b(W_a\cap W_b)-\Gamma_b$. Because $\Gamma_a$ is of codimension $r\geqq 2$ in $\Delta_a$, $\phi_{ab}$ can be extended to the whole $\pi_c(W_a\cap W_b)$ as a holomorphic mapping. Since we can interchange $a$ and $b$, $\phi_{ab}$ is a biholomorphic homeomorphism from $\pi_a(W_a\cap W_b)$ onto $\pi_b(W_a\cap W_b)$.

Thus we can patch together $\Delta'_a$s and obtain a possibly non-connected analytic manifold $X_1$. In this procedure $\Gamma'_a$s are patched together and $\phi_{ab}$ is nothing but the coordinate transformation on $M$, when we identify $\Gamma_a$ and $\Gamma_b$ with coordinate neighbourhoods around $a$ and $b$ respectively. By a standard connectedness argument, we see that $X_1$ is connected and $\Gamma'_a$s form a submanifold biholomorphically homeomorphic to $M$. We identify this submanifold with $M$.

The open subsets $\bigcup_{a\in M} W_a-S$ of $\widetilde{X}$ and $X_1-M$ of $X_1$ are biholomorphically homeomorphic with each other, the mapping being given by $\pi_a$ on $W_a-S$. Hence we can patch together $\widetilde{X}-S$ and $X_1$ on these parts, and obtain a manifold $X$. The map $\pi$ from $\widetilde{X}$ to $X$, defined by $\pi=\pi_a$ on $W_a$ and by $\pi=id.$ on $\widetilde{X}-S$ makes $X$ the monoidal transform of $X$ with centre $M$, and we have $S=\pi^{-1}(M)$. This completes the proof of the sufficiency part of the Main Theorem.

**3.3.** The necessity part is trivial. As for condition ($\beta$), we remark that, if $(\widetilde{X},\pi)$ is the monoidal transform of $X$ with centre $M$ of codimension $r$, then we have $\mathcal{K}_{\widetilde{X}}=\pi^*\mathcal{K}_X\otimes[S]^{r-1}$, where $\mathcal{K}_X$, $\mathcal{K}_{\widetilde{X}}$ denote the canonical bundles of $X$ and $\widetilde{X}$ respectively.

**3.4.** In our Theorem, conditions ($\alpha$) and ($\beta$) are certainly necessary and sufficient, but, at the moment, the author doesn't know if these are independent. In fact in the works quoted in the introduction, condition ($\beta$) is not explicitly mentioned, and ($\beta$) was used in a technical way in our proof. Hence it may be conjectured that the condition ($\beta$) will follow from ($\alpha$) (and with the value $k=r-1$).

*Note Added in Proof (March 15, 1971)*: After this paper was written, Mr. A. Fujiki and the author noticed that condition ($\alpha$) alone is sufficient to derive the Main Theorem. This supplement will appear in the coming issue of these *Publications.*

### References

[1] Andreotti, A. and E. Vesentini, Carleman estimates for the Laplace-Beltrami equations on complex manifolds, Publ. Math. I. H. E. S. **25** (1965), 81-155.
[2] Grauert, H., Über Modifikationen und exzeptionelle analytische Mengen, Math. Ann. **146** (1962), 331-368.
[3] Griffiths, P. A., The extension problem in complex analysis II. Amer. J. Math. **88** (1966), 365-446.
[4] Kodaira, K., On Kähler varieties of restricted type. Ann. of Math. **60** (1954), 28-48.
[5] ————, On compact analytic surfaces II, Ann. of Math. **78** (1963), 563-626.
[6] Lascu, A., On the contractibility criterion of Castelnuovo-Enriques, Atti Acad. Naz. Lincei, **40** (1966), 1014-1019.
[7] Moĭšezon, B. G., Three papers in Izv. Akad. Nauk SSSR, Ser. Mat. **30** (1966), On $n$-dimensional compact varieties with $n$ algebraically independent meromorphic functions I, II, III, Amer. Math. Transl. Ser. 2. **63** (1967), 51-178.
[8] Vesentini, E., On Levi convexity of complex manifolds and cohomology vanishing theorem, Tata Institute Lecture Notes (1967).
[9] Weil, A., Introduction à l'étude des variétés kähleriennes, Act. Sci. Ind. 1267, Hermann, 1958.