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A Remark on Bures Distance Function for Normal States

By

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Abstract

An inequality between the norm of the difference of two vector states of a W^*-algebra and the infimum distance between vectors representing the states in a fixed representation is derived.

For a normal state ω of a W^*-algebra R and a normal representation π of R on a Hilbert space H_π, $\mathrm{S}(\pi, \omega)$ denotes the set of all vectors x in H_π such that $(x, \pi(Q)x) = \omega(Q)$ for all $Q \in R$. (We do not assume $\omega(1) = 1$ in accordance with Bures.) Let

$$\mathrm{d}_\pi(\omega, \omega') = \inf\{\|x-y\|;\ x \in \mathrm{S}(\pi, \omega),\ y \in \mathrm{S}(\pi, \omega')\}$$

whenever $\mathrm{S}(\pi, \omega)$ and $\mathrm{S}(\pi, \omega')$ are non-empty and $\mathrm{d}_\pi(\omega, \omega') = \sqrt{2}$ otherwise. Let

$$\mathrm{d}(\omega, \omega') = \inf_\pi \mathrm{d}_\pi(\omega, \omega').$$

A trivial calculation shows

$$\mathrm{d}(\omega, \omega') \geqq [\omega(1) + \omega'(1)]^{-1} \|\omega - \omega'\|.$$

Bures has shown [1] that

$$\mathrm{d}(\omega, \omega')^2 \leqq \|\omega - \omega'\|.$$

We shall derive a similar inequality for a fixed representation π, which implies the equivalence of topologies induced by the norm and $\mathrm{d}_\pi(\omega, \omega')$ on the set of all vector states in the representation π.

If R is a von Neumann algebra on a Hilbert space H and $x \in H$, we write $\omega_x(Q) = (x, Qx)$ for $Q \in R$. If π is a representation of R on H_π and $x \in H_\pi$, we also write $\omega_x(Q) = (x, \pi(Q)x)$ for $Q \in R$.

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Our aim is the proof of the following:

Theorem 1. *For all* $x, y \in H_\pi$,

$$d_\pi(\omega_x, \omega_y)^2 \leqq 2\|\omega_x - \omega_y\|. \tag{1}$$

We start with technical lemmas.

Lemma 2. $\omega_x = \omega_y$ *if and only if there exists a partial isometry* $W \in R'$ *such that* $Wx = y$, $W^*y = x$, $WH = \overline{Ry}$, $W^*H = \overline{Rx}$.

This is essentially Lemma 3, Chap. I, §4.1 in [2] where $\mathfrak{R} = \overline{Rx}$, $\mathfrak{R}_1 = \overline{Ry}$, $\mathscr{B} = R \big| \mathfrak{R}$, $\mathscr{B}_1 = R \big| \mathfrak{R}_1$, Φ and Φ_1 are restriction maps of R to $\mathfrak{R}$ and $\mathfrak{R}_1$.

Lemma 3. *Let* $\{F_\alpha\}$ *be a partition of* 1 *by central projections of* $\pi(R)$. *For* $x, y \in H_\pi$,

$$\|\omega_x - \omega_y\| = \sum_\alpha \|\omega_{F_\alpha x} - \omega_{F_\alpha y}\|, \tag{2}$$

$$d_\pi(\omega_x, \omega_y)^2 = \sum_\alpha d_\pi(\omega_{F_\alpha x}, \omega_{F_\alpha y})^2. \tag{3}$$

Proof. For the norm, the following computation proves (2)

$$\begin{aligned}
\|\omega_x - \omega_y\| &= \sup\{\mathrm{Re}[\omega_x(Q) - \omega_y(Q)]\,;\ \|Q\| \leqq 1, Q \in \pi(R)\} \\
&= \sup\{\mathrm{Re}\sum_\alpha [\omega_{F_\alpha x}(F_\alpha Q) - \omega_{F_\alpha y}(F_\alpha Q)]\,;\ \|Q\| \leqq 1, Q \in \pi(R)\} \\
&= \sum_\alpha \sup\{\mathrm{Re}[\omega_{F_\alpha x}(Q_\alpha) - \omega_{F_\alpha y}(Q_\alpha)]\,;\ \|Q_\alpha\| \leqq 1, Q_\alpha \in \pi(R)F_\alpha\} \\
&= \sum_\alpha \sup\{\mathrm{Re}[\omega_{F_\alpha x}(Q) - \omega_{F_\alpha y}(Q)]\,;\ \|Q\| \leqq 1, Q \in \pi(R)\} \\
&= \sum_\alpha \|\omega_{F_\alpha x} - \omega_{F_\alpha y}\|.
\end{aligned}$$

For d_π, we denote the representation $F_\alpha \pi(Q)$ of $Q \in R$ on $F_\alpha H_\pi$ by $F_\alpha \pi$. Then,

$$\begin{aligned}
d_\pi(\omega_x, \omega_y)^2 &= \inf\{\|x' - y'\|^2;\ x' \in S(\pi, \omega_x),\ y' \in S(\pi, \omega_y)\} \\
&= \inf\{\sum_\alpha \|F_\alpha x' - F_\alpha y'\|^2;\ x' \in S(\pi, \omega_x),\ y' \in S(\pi, \omega_y)\} \\
&= \sum_\alpha \inf\{\|x'_\alpha - y'_\alpha\|^2;\ x'_\alpha \in S(F_\alpha \pi, \omega_{F_\alpha x}),\ y'_\alpha \in S(F_\alpha \pi, \omega_{F_\alpha y})\} \\
&= \sum_\alpha \inf\{\|x'_\alpha - y'_\alpha\|^2;\ x'_\alpha \in S(\pi, \omega_{F_\alpha x}),\ y'_\alpha \in S(\pi, \omega_{F_\alpha y})\} \\
&= \sum_\alpha d_\pi(\omega_{F_\alpha x}, \omega_{F_\alpha y})^2.
\end{aligned}$$

Q.E.D.

Corollary 4. *Let* $x' \in S(\pi, \omega_x)$, $y' \in S(\pi, \omega_y)$, $\varepsilon > 0$ *and*

(4) $$\|x'-y'\|^2<\mathrm{d}_\pi(\omega_x, \omega_y)^2+\varepsilon.$$

Let F be a central projection of $\pi(R)$. Then

(5) $$\|Fx'-Fy'\|^2<\mathrm{d}_\pi(\omega_{Fx}, \omega_{Fy})^2+\varepsilon.$$

Proof. From (4) and Lemma 3, we have

$$\|Fx'-Fy'\|^2+\|(1-F)x'-(1-F)y'\|^2$$
$$<\mathrm{d}_\pi(\omega_{Fx}, \omega_{Fy})^2+\mathrm{d}_\pi(\omega_{(1-F)x}, \omega_{(1-F)y})^2+\varepsilon.$$

On the other hand, we have $Fx'\in\mathrm{S}(\pi, \omega_{Fx})$, $Fy'\in\mathrm{S}(\pi, \omega_{Fy})$, and similar equations for $1-F$. Therefore,

$$\|Fx'-Fy'\|^2\geqq\mathrm{d}_\pi(\omega_{Fx}, \omega_{Fy})^2,$$
$$\|(1-F)x'-(1-F)y'\|^2\geqq\mathrm{d}_\pi(\omega_{(1-F)x}, \omega_{(1-F)y})^2.$$

Hence we have (5). Q.E.D.

Lemma 5. *For $x, y, z\in H_\pi$,*

(6) $$\mathrm{d}_\pi(\omega_x, \omega_y)+\mathrm{d}_\pi(\omega_y, \omega_z)\geqq\mathrm{d}_\pi(\omega_x, \omega_z).$$

Proof. Let $\varepsilon>0$. Let $\alpha_\beta\in H_\pi$, $\alpha, \beta=x, y, z$ be such that $\alpha_\beta\in\mathrm{S}(\pi, \omega_\alpha)$ and

(7) $$\|\alpha_\beta-\beta_\alpha\|^2<\mathrm{d}_\pi(\omega_\alpha, \omega_\beta)^2+\varepsilon.$$

By Lemma 2, there exists a partial isometry W in R' such that $Wy_x=y_z$, $W^*y_z=y_x$. There exists then a central projection F of R' and partial isometries W_1 and W_2 in R' such that

$$F(1-WW^*)=W_1^*W_1, \quad F(1-W^*W)\geqq W_1W_1^*,$$
$$(1-F)(1-W^*W)=W_2^*W_2, \quad (1-F)(1-WW^*)\geqq W_2W_2^*.$$

Let

$$W_1'=(W_1+W^*)F, \quad W_2'=(W_2+W)(1-F).$$

Then W_1' and W_2' are isometric on FH and $(1-F)H$ and satisfies

(8) $$W_1'y_z=Fy_x, \quad W_2'y_x=(1-F)y_z.$$

Let

$$x'=Fx_y+W_2'x_y, \ y'=Fy_x+(1-F)y_z, \ z'=W_1'z_y+(1-F)z_y.$$

Then we obtain $x'\in\mathrm{S}(\pi, \omega_x)$, $y'\in\mathrm{S}(\pi, \omega_y)$, $z'\in\mathrm{S}(\pi, \omega_z)$ and

(9) $\|x'-y'\|^2=\|F(x_y-y_x)\|^2+\|(1-F)(x_y-y_x)\|^2=\|x_y-y_x\|^2,$

(10) $\|y'-z'\|^2=\|F(y_z-z_y)\|^2+\|(1-F)(y_z-z_y)\|^2=\|y_z-z_y\|^2,$

where (8) and isometry of W_1' and W_2' on FH and $(1-F)H$ are used.

From (9), (10) and (7), we obtain

$$\begin{aligned} d_\pi(\omega_x, \omega_z) &\leqq \|x'-z'\| \leqq \|x'-y'\|+\|y'-z'\| \\ &\leqq d_\pi(\omega_x, \omega_y)+d_\pi(\omega_y, \omega_z)+\varepsilon'(\varepsilon) \end{aligned}$$

where $\varepsilon'(\varepsilon)\to 0$ as $\varepsilon\to 0$. Hence we have (6). Q.E.D.

Lemma 6. *For $x\in H_\pi$ and $Q\in\pi(R)^+$,*

(11) $d_\pi(\omega_x, \omega_{Qx})^2\leqq\|\omega_x-\omega_{Qx}\|.$

If $\omega_x\leqq n\omega_y$ for some $n>0$, then (11) *holds.*

Proof. Due to the proof of Proposition 1.12 of [1].

Lemma 7. *For $x\in H_\pi$ and $Q\in\pi(R')$,* (11) *holds.*

Proof. For $Q_1\in\pi(R)$, we have

$$\omega_{Qx}(Q_1^*Q_1)=\|QQ_1x\|^2\leqq\|Q\|^2\omega_x(Q_1^*Q_1).$$

Hence Lemma 6 implies (11). Q.E.D.

Proof of Theorem 1. Let e_1 and e_2 be an orthonormal basis of M. Let π' be the representation $\pi'(Q)=\pi(Q)\otimes 1$ on $H'=H_\pi\otimes M$. Let $0<\varepsilon<1$ and

$$z(\varepsilon)=(1-\varepsilon)^{1/2}x\otimes e_1+\varepsilon^{1/2}y\otimes e_2.$$

Since $\varepsilon^{-1}\omega_{z(\varepsilon)}\geqq\omega_{y\otimes e_2}$, there exists $A\in\pi(R)^+$ such that $\widehat{A}=A\otimes 1$ satisfies $\omega_{\widehat{A}z(\varepsilon)}=\omega_y$, by Sakai's Radon Nikodym theorem [3]. By Lemma 2, there exists a partial isometry W in $\pi'(R)'$ such that $W(y\otimes e_1)=\widehat{A}z(\varepsilon)$. By the proof of Proposition 1.12 in [1], we have

(12) $\|\widehat{A}z(\varepsilon)-z(\varepsilon)\|^2\leqq\|\omega_y-\omega_{z(\varepsilon)}\|.$

Let U_{ij} be defined by $(\Phi, U_{ij}\Psi)=(\Phi\otimes e_i, W(\Psi\otimes e_j))$ for all $\Phi, \Psi\in H_\pi$. Then $U_{ij}\in\pi(R)'$ due to $W\in\pi'(R)'=(\pi(R)\otimes 1)'$. Further, $(W^*W)\cdot(y\otimes e_1)=y\otimes e_1$ implies $(U_{11}^*U_{11}+U_{21}^*U_{21})y=y$. Let $y'=U_{11}y$, $y''=U_{21}y$. We then have

(13) $$\omega_y=\omega_{y'}+\omega_{y''}.$$

The equation (12) now reads as

(14) $$\|\omega_y-\omega_{z(\varepsilon)}\|\geqq\{\|W(y\otimes e_1)-x\otimes e_1\|-\|z(\varepsilon)-x\otimes e_1\|\}^2$$
$$\geqq\|y'-x\|^2+\|y''\|^2-\varepsilon'(\varepsilon)$$

where $\varepsilon'(\varepsilon)\to 0$ as $\varepsilon\to 0$. We also have

(15) $$\|\omega_y-\omega_{z(\varepsilon)}\|\leqq\|\omega_y-\omega_x\|+\|\omega_x-\omega_{z(\varepsilon)}\|\leqq\|\omega_y-\omega_x\|+\varepsilon''(\varepsilon)$$

where $\varepsilon''(\varepsilon)\to 0$ as $\varepsilon\to 0$.

By Lemma 6, we have from (13),

(16) $$\mathrm{d}_\pi(\omega_y,\omega_{y'})^2\leqq\|\omega_y-\omega_{y'}\|=\|\omega_{y''}\|=\|y''\|^2.$$

By definition,

(17) $$\mathrm{d}_\pi(\omega_{y'},\omega_x)^2\leqq\|y'-x\|^2.$$

By Lemma 5, we have

(18) $$\mathrm{d}_\pi(\omega_x,\omega_y)^2\leqq\{\mathrm{d}_\pi(\omega_x,\omega_{y'})+\mathrm{d}_\pi(\omega_{y'},\omega_y)\}^2$$
$$\leqq 2\mathrm{d}_\pi(\omega_x,\omega_{y'})^2+2\mathrm{d}_\pi(\omega_{y'},\omega_y)^2.$$

Collecting (14) ~ (18) together, and taking the limit of $\varepsilon\to 0$, we have (1). Q.E.D.

Remark. If $\pi(R)'$ is properly infinite, then $\omega_{z(\varepsilon)}$ in the above proof can be realized as a vector state in H_π. Hence we immediately obtain an improved version of the inequality:

(19) $$\mathrm{d}_\pi(\omega_x,\omega_y)^2\leqq\|\omega_x-\omega_y\|.$$

If $\pi(R)$ is finite and has a trace vector φ in H_π, then (19) can be proved as follows:

By the proof of Theorem 1 and Lemma 5, $\mathrm{d}_\pi(\omega_x,\omega_y)$ is continuous in ω_x and ω_y relative to the norm topology on states. Hence it is enough to prove (19) for a dense set of states ω_x and ω_y. We shall consider vector states $\omega_{A\varphi}$ and $\omega_{B\varphi}$, $A, B\in\pi(R)^+$, which are dense in the set of all normal states of $\pi(R)$ and hence in the set of π-vector states of R.

Let E_+ and E_- be spectral projection of $A-B$ for $(0,\infty)$ and

$(-\infty, 0)$. For $Q=E_+-E_-$, we have $\|Q\|=1$, $Q(A-B)=(A-B)Q=|A-B|$. Since ω_φ is a trace, we have

$$\begin{aligned}\omega_{A\varphi}(Q)-\omega_{B\varphi}(Q)&=\omega_\varphi(Q(A^2-B^2))\\&=\frac{1}{2}\{\omega_\varphi(Q(A-B)(A+B))+\omega_\varphi((A-B)Q(A+B))\}\\&=\omega_\varphi(|A-B|(A+B))\\&=\omega_{|A-B|^{1/2}\varphi}(E_+(A+B)E_++E_-(A+B)E_-).\end{aligned}$$

Since $E_+(A+B)E_+\geqq E_+(A-B)E_+=E_+|A-B|$ and $E_-(A+B)E_-\geqq E_-(B-A)E_-=|A-B|E_-$, we obtain

$$\begin{aligned}\|\omega_{A\varphi}-\omega_{B\varphi}\|&\geqq\omega_\varphi(|A-B|^2)=\|A\varphi-B\varphi\|^2\\&\geqq \mathrm{d}_\pi(\omega_{A\varphi}-\omega_{B\varphi}).\end{aligned}$$

Q.E.D.

Combining above conclusions, we see that if the finite part $F\pi(R)$ of $\pi(R)$ is not smaller than its commutant $F\pi(R)'$, then (19) holds.

On the other hand, (19) is not true in general as can be seen from an example of $\pi(R)=\mathcal{B}(H_\pi)$, $\dim H_\pi=2$.

References

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