Publ. RIMS, Kyoto Univ.
7 (1971/72), 57-67

On a Mixed Problem for Hyperbolic Equations with Discontinuous Boundary Conditions

By

Kazunari HAYASHIDA*

1. Mixed problems for hyperbolic equations have been studied by many authors. For variable coefficients the powerful tool is the theorem of Hille-Yosida ([3], [4], [5], [7]). In this case the mixed problems have been treated completely by Mizohata [5] and Ikawa [3].

On the other hand Čehlov [2] has shown the existence of a weak solution for mixed problems under discontinuous boundary conditions. He has imposed the assumption that the space domain is a half space and the equation is the wave equation. His method is the Fourier-Laplace transformation.

In this note we consider a mixed problem under discontinuous boundary conditions of Dirichlet or Neumann type. We proceed mainly along the lines of [3] and [5].

2. Let Ω be a bounded domain in the n-dimensional Euclidean space R^n with boundary $\partial\Omega$ of class C^∞. We assume that $\partial\Omega$ consists of two measurable sets $\partial_1\Omega$ and $\partial_2\Omega$ having no common points. Further let us assume that

$$\partial_2\Omega \cap \overline{\partial_1\Omega} = \phi. \tag{2.1}$$

We set

$$(u, v)_k=\int_\Omega \sum_{|\alpha|\leq k} D^\alpha u\ \overline{D^\alpha v}\ dx,$$
$$||u||_k^2=(u, u)_k$$

and

Received November 12, 1970.
Communicated by S. Mizohata.
* Department of Mathematics, Nagoya University, Chikusa-ku, Nagoya, Japan.

$$(u, v)=(u, v)_0, \quad \|u\|=\|u\|_0.$$

Let us denote by $H^k(\Omega)$ the Sobolev space with norm $\| \ \|_k$ and by $K(\Omega)$ the completion of all u each of which belongs to $C^\infty(\bar{\Omega})$ and vanishes in a neighborhood of $\partial_1\Omega$ with $H^1(\Omega)$ norm.

Consider the elliptic operator L of second order on $\bar{\Omega}\times[0, T]$:

$$L=-\sum_{i,j}\frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right)+\sum_i b_i(x, t)\frac{\partial}{\partial x_i}+c(x, t), \tag{2.2}$$

where the coefficients are all in $C^\infty(\bar{\Omega}\times[0, T])$. We assume that $a_{ij}(x)$ $(=a_{ji}(x))$ are real valued and positive definite on $\bar{\Omega}$ and we consider the following equation for real valued $h_i(x)$, $h(x)\in C^\infty(\bar{\Omega})$:

$$\frac{\partial^2}{\partial t^2}u+Lu+\left(2\sum_i h_i(x)\frac{\partial}{\partial x_i}+h(x)\right)\frac{\partial}{\partial t}u=f. \tag{2.3}$$

Let us impose the following boundary condition

$$B_1u(x, t)=u(x, t)=0 \quad \text{on} \quad \partial_1\Omega\times[0, T], \tag{2.4}$$

$$B_2u(x, t)=\left\{\frac{d}{dn}-\langle h, \gamma\rangle\frac{\partial}{\partial t}+\sigma(x, t)\right\}u(x, t)=0 \quad \text{on } \partial_2\Omega\times[0, T], \tag{2.5}$$

where

$$\frac{d}{dn}=\sum_{i,j}a_{ij}(x)\cos(\nu, x_j)\frac{\partial}{\partial x_i} \qquad (\nu \text{ is the exterior mormal vector}),$$

$$\langle h, \gamma\rangle=\sum_i h_i(x)\cos(\nu, x_i)$$

and $\sigma(x)$ is C^∞ on $\partial_2\Omega$. The equation (2.3) has been considered in [3] and [5] under the boundary condition $B_1u=0$ or $B_2u=0$ on $\partial\Omega\times[0, T]$.

Now we define the boundary condition (2.5) in the weak sense as follows:

Definition 2.1. *Let $u(., t)$ be in $H^1(\Omega)$ and $(Lu)(., t)$ be in $L^2(\Omega)$ for $0\leq t\leq T$. Further we assume that u is in $\mathscr{E}_t^1(H^1(\Omega))[0, T]$.*[1] *Then u is said to satisfy the boundary condition* (2.5) *weakly on $\partial_2\Omega\times[0, T]$,*

1) For the Banach space E the letter $\mathscr{E}_t^k(E)[0, T]$ means the set of E-valued functions which are k-times continuously differentiable in $0\leqq t\leqq T$.

if the following equality holds on $[0, T]$;

$$\left(\left\{-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)\right\}u, \varphi\right)=\sum_{i,j}\left(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_j}\right) + \int_{\partial_2\Omega}\left(\sigma u - \langle h, \gamma\rangle \frac{\partial u}{\partial t}\right)\bar{\varphi}\, dS \tag{2.6}$$

for any $\varphi \in K(\Omega)$.

In addition we define the boundary condition for vector functions as follows.

Definition 2.2. *Let* $U=\{u, v\}$ *be in* $H^1(\Omega)\times H^1(\Omega)$ *and* Lu *be in* $L^2(\Omega)$. *Then* U *is said to satisfy the boundary condition* (B_2) *on* $\partial_2\Omega$, *if the following equality holds for any* $\varphi\in K(\Omega)$;

$$\left(\left\{-\frac{\partial}{\partial x_i}\left(a_{ij}\frac{\partial}{\partial x_j}\right)u, \varphi\right)=\sum_{i,j}\left(a_{ij}\frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_j}\right) + \int_{\partial_2\Omega}(\sigma u - \langle h, \gamma\rangle v)\bar{\varphi}\, dS. \tag{2.7}$$

In this note we shall prove the following theorems, where we assume that $f(x, t)\in \mathcal{E}_t^0(K(\Omega))[0, T]$ and $f(x, 0)$ has compact support in Ω.

Theorem 1. *Suppose that* $u_0(x)$, $v_0(x)\in K(\Omega)$, $Lu_0\in L^2(\Omega)$ *and* $\{u_0, v_0\}$ *satisfies the boundary condition* (B_2) *on* $\partial_2\Omega$. *Then there is a unique solution* $u(x, t)\in \mathcal{E}_t^1(K(\Omega))[0, T]\cap \mathcal{E}_t^2(L^2(\Omega))[0, T]$ *of the equation* (2.3) *satisfying*

$$u=u_0,\ u_t=v_0 \quad \text{on } t=0 \ (\textit{initial condition}) \tag{2.8}$$

and

$$B_2 u=0 \quad \textit{weakly on } \partial_2\Omega\times[0, T]. \tag{2.9}$$

Theorem 2. *In addition to the assumption of Theorem* 1, *assume that* $u_0\in H^2_{\mathrm{loc}}(\bar{\Omega}-S)$,[2] *where* S *is the boundary of* $\partial_1\Omega(\partial_2\Omega)$. *Then* $u(x, t)$ *also belongs to* $H^2_{\mathrm{loc}}(\bar{\Omega}-S)$. *Thus the solution satisfies* $B_2u=0$

2) The space $H^2_{\mathrm{loc}}(\bar{\Omega}-S)$ is the set of functions belonging to locally H^2 in $\bar{\Omega}-S$.

in the interior of $\partial_2\Omega$.

Remark. Here we have assumed (2.1). Hence if $\partial_1\Omega$ is an $(n-2)$-dimensional compact manifold, our theorem holds. With difference method, Babaeva and Namazov [1] has shown the existence of the solution for our problem also when $\partial_2\Omega$ is an $(n-2)$-dimensional compact manifold.

3. Let us consider the space $H=K(\Omega)\times L^2(\Omega)$ with the inner product

$$(U_1, U_2)_H=\sum_{i,j}\left(a_{ij}(x)\frac{\partial u_1}{\partial x_i}, \frac{\partial u_2}{\partial x_j}\right)+(v_1, v_2)$$
$$+\int_{\partial_2\Omega}\sigma(x)u_1\bar{u}_2dS+c_1(u_1, u_2),$$

where $U_i=\{u_i, v_i\}$ $(i=1, 2)$ and c_1 is a sufficiently large constant depending only on a_{ij} and σ. We denote by $\|U\|_H$ the H-norm of U. Obviously the space H is complete and by the well-known interpolation relation (see e.g. [6]) the norm $\|U\|_H$ is equivalent to $\|u\|_1+\|v\|_0$ $(U=\{u, v\})$.

The formulation in this section is radically due to the book of H.G. Garnir.[3)]

Set the operator $A(t)$ in such a way that

$$A(t)=\begin{pmatrix} 0 & 1 \\ -L & -M \end{pmatrix}, \tag{3.1}$$

where $M=2\sum_i h_i(x)\dfrac{\partial}{\partial x_i}+h(x)$ (see (2.3)). Then $A(t)$ is a closed operator from H to itself having the following definition domain

(3.2) $D(A(t))=\{U=\{u, v\}\,|\, u, v\in K(\Omega), Lu\in L^2(\Omega)$

and U satisfies the boundary condition (B_2) on $\partial_2\Omega$ in the sense of Definition 2.2$\}$.

---

3) *Les Problèmes aux Limites de la Physique Mathématique*, Birkhäuser, 1958.

Since $D(A(t))$ is independent of t, we write simply by $D(A)$.

Remark. Mizohata [5] and Ikawa [3] have set

$$H=H_0^1(\Omega)\times L^2(\Omega)$$

and

$$D(A)=(H^2(\Omega)\cap H_0^1(\Omega))\times H_0^1(\Omega)$$

for the case of the Dirichlet type boundary condition. They have set also for the case of the Neumann type boundary condition as follows:

$$H=H^1(\Omega)\times L^2(\Omega)$$

and

$$D(A)=\Big\{U=\{u,\ v\}\ \Big|\ u\in H^2(\Omega),\ v\in H^1(\Omega) \text{ and } \frac{d}{dn}u-\langle h,\gamma\rangle v+\sigma u=0 \quad \text{on } \partial\Omega\Big\}.$$

Lemma 1. *There is a positive constant c_2 depending only on $A(t)$ and $\sigma(x)$ such that it holds that for any $U\in D(A)$,*

$$|(U,\ A(t)U)_H+(A(t)U,\ U)_H|\leqq c_2\|U\|_H^2.$$

Proof. We easily see

$$\begin{aligned}(3.3)\qquad &(U,\ A(t)U)_H+(A(t)U,\ U)_H\\ &=\sum_{i,j}\left(a_{ij}\frac{\partial u}{\partial x_i},\ \frac{\partial v}{\partial x_j}\right)+(v,\ -Lu-Mv)\\ &\quad+\int_{\partial_2\Omega}\sigma u\bar{v}\,dS+c_1(u,\ v)\\ &\quad+\sum_{i,j}\left(a_{ij}\frac{\partial v}{\partial x_i},\ \frac{\partial u}{\partial x_j}\right)+(-Lu-Mv,\ v)\\ &\quad+\int_{\partial_2\Omega}\sigma v\bar{u}\,dS+c_1(v,\ u).\end{aligned}$$

Since U satisfies the boundary condition (B_2) on $\partial_2\Omega$ (see Definition 2.2), we have

$$(3.4)\qquad \sum_{i,j}\left(a_{ij}\frac{\partial u}{\partial x_i},\frac{\partial v}{\partial x_j}\right)+\int_{\partial_2\Omega}\sigma\ u\bar{v}\ dS$$

$$=(Lu,\ v)-\left(\sum_i b_i\frac{\partial u}{\partial x_i}+cu,\ v\right)$$

$$+\int_{\partial_2\Omega}<h,\ \gamma>v\bar{v}\,dS.$$

Further it is easily seen that

$$(Mv,\ v)+(v,\ Mv)=2\int_{\partial_2\Omega}<h,\ \gamma>v\bar{v}\,dS \tag{3.5}$$

$$-2\sum_i\left(\frac{\partial h_i}{\partial x_i}v,\ v\right)+(hv,\ v)+(v,\ hv).$$

Combining (3.4), (3.5) and (3.3), we have proved the lemma.

Lemma 2. *If λ is real and $|\lambda|\geqq c_2$, we have for any $U\in D(A)$*

$$\|(\lambda I-A(t))U\|_H\geqq(|\lambda|-c_2)\|U\|_H.$$

Proof. We easily see

$$\|(\lambda I-A(t))U\|_H^2\geqq\lambda^2\|U\|_H^2-\lambda\{(U,\ A(t)U)_H+(A(t)U,\ U)_H\}.$$

By Lemma 1 we get

$$\|(\lambda I-A(t))U\|_H^2\geqq(\lambda^2-|\lambda|c_2)\|U\|_H^2$$

$$\geqq\{(|\lambda|-c_2)^2+c_2(|\lambda|-c_2)\}\|U\|_H^2.$$

Now set for any $\varphi,\ \psi\in K(\Omega)$

$$B_t[\varphi,\ \psi]=\sum_{i,j}\left(a_{ij}\frac{\partial\varphi}{\partial x_i},\ \frac{\partial\psi}{\partial x_j}\right) \tag{3.6}$$

$$+\left(\sum_i b_i\frac{\partial\varphi}{\partial x_i}+c\varphi,\ \psi\right)+\int_{\partial_2\Omega}\sigma\varphi\bar{\psi}\,dS$$

$$-\lambda\int_{\partial_2\Omega}<h,\ \gamma>\varphi\bar{\psi}\,dS$$

$$+\lambda(M\varphi,\ \psi)+\lambda^2(\varphi,\ \psi).$$

Then using the interpolation relation for the trace of functions (see e.g. [6]), we see that there is a positive constant c_3 such that if λ is real and $|\lambda|\geqq c_3$, it holds for any $\varphi\in K(\Omega)$,

$$|B_t[\varphi,\ \varphi]|\geqq c_3^{-1}\|\varphi\|_1^2.$$

It is easily seen that

$$|B_t[\varphi, \psi]| \leqq c_4 \|\varphi\|_1 \|\psi\|_1 \quad \text{for any } \varphi, \psi \in K(\Omega).$$

Hence by the theorem of Lax-Milgram we have the following

Lemma 3. *For any given anti-linear functional l on $K(\Omega)$ there is a unique solution $u \in K(\Omega)$ of the equation*

$$B_t[u, \varphi] = l(\varphi) \quad \textit{for any } \varphi \in K(\Omega).$$

From Lemma 3 we immediately see that

Lemma 4. *If λ is real and $|\lambda| \geqq c_3$, then for any $F \in H$ there is a unique solution $U \in D(A)$ of the equation*

$$(\lambda I - A(t))U = F. \tag{3.7}$$

Proof. Put $U = \{u, v\}$ and $F = \{f, g\}$. Then the equation (3.7) is equivalent to

$$v = \lambda u - f \tag{3.8}$$

and

$$Lu + \lambda(\lambda + M)u = g + (\lambda + M)f. \tag{3.9}$$

Let us put in Lemma 3

$$l(\varphi) = ((\lambda + M)f + g, \varphi) - \int_{\partial_2 \Omega} \langle h, \gamma \rangle f \bar{\varphi} \, dS.$$

Then l satisfies the assumption of Lemma 3 by the well-known inequality. Thus there is a $u \in K(\Omega)$ such that $B_t[u, \varphi] = l(\varphi)$ for any $\varphi \in K(\Omega)$. In particular, taking φ as in $C_0^\infty(\Omega)$, we see that (3.9) holds and $Lu \in L^2(\Omega)$. Hence we get from (3.6), (3.8) and (3.9)

$$\sum_{i,j} \left(a_{ij} \frac{\partial u}{\partial x_i}, \frac{\partial \varphi}{\partial x_j} \right) + \int_{\partial_2 \Omega} u \bar{\varphi} \, dS$$

$$- \int_{\partial_2 \Omega} \langle h, \gamma \rangle v \bar{\varphi} \, dS$$

$$= \left(- \left\{ \sum_{i,j} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right) \right\} u, \varphi \right) \quad \text{for any } \varphi \in K(\Omega).$$

By Definition 2.2, this equality implies that U satisfies the boundary condition (B_2) on $\partial_2\Omega$. Thus $U\in D(A)$. Therefore we have completed the proof.

Let us rewrite by new c_2 the maximum of c_2 and c_3. Then combining Lemmas 2 and 4, we obtain

Lemma 5. *If* $|\lambda|\geqq c_2$, *then it holds that*

$$\|(\lambda I-A(t))^{-1}\|_H\leqq\frac{1}{|\lambda|-c_2}.$$

4. In this section we shall prove that $D(A)$ is dense in H. Let us denote by $C^\infty_{(0)}(R^n_-)$ the set of C^∞ functions on $x_n\leqq 0$ having compact support there. Then we have

Lemma 6. *For* u *in* $C^\infty_{(0)}(R^n_-)$ *there is a sequence* $\{\varphi_i\}$ *in* $C^\infty_{(0)}(R^n_-)$ *such that*

(i) $\varphi_i\to u$ *in* $H^1(R^n_-)$

(ii) $\dfrac{\partial}{\partial x_n}\varphi_i=0$ *on* $x_n=0$

and

(iii) *if* $u(x', x_n)$[4] $=0$ *for the fixed* x' *and any* x_n, *then each* $\varphi_i(x', x_n)$ *vanishes also for the* x' *and any* x_n.

The proof of Lemma 6 is familiar, so it is sufficient to show construction of φ_i. The functions φ_i are given as follows;

$$\varphi_i(x', x_n)=\int_{-\infty}^{x_n}\alpha_i(s)\frac{\partial u}{\partial x_n}(x', s)ds,$$

where

$$\alpha_i(s)=\begin{cases}0 & \text{if } s>-\dfrac{1}{i}\\[2ex] 1 & \text{if } s<-\dfrac{2}{i}.\end{cases}$$

For the bounded function $\sigma(x')$ on $x_n=0$, let us take a new sequence

4) Put $x'=(x_1, \cdots, x_{n-1})$.

$\{\varphi_i \exp(x_n\sigma(x'))\}$. Then we have also the following

Lemma 7. *For u in $C^\infty_{(0)}(R^n_-)$ there is a sequence $\{\varphi_i\}$ in $C^\infty_{(0)}(R^n_-)$ such that the properties* (i), (iii) *in Lemma* 6 *hold and in the place of* (ii) *the following property holds*:

(ii) $$\left(\frac{\partial}{\partial x_n}+\sigma(x')\right)\varphi_i=0 \quad on \ x_n=0.$$

Generalizing Lemma 7, we prove the following

Lemma 8. *For any $u\in K(\Omega)$, there is a sequence $\{\varphi_i\}\subset C^\infty(\bar{\Omega})$ satisfying $\left(\frac{d}{dn}+\sigma(x)\right)\varphi_i=0$ on $\partial_2\Omega$ such that each φ_i vanishes in a neighborhood of $\overline{\partial_1\Omega}$ and $\varphi_i\to u$ in $H^1(\Omega)$.*

Proof. We may assume that u is in $C^\infty(\bar{\Omega})$ and $u=0$ in a neighborhood of $\overline{\partial_1\Omega}$. For each point P in $\bar{\Omega}$ let us take an open neighborhood $U(P)$ in such a way that

$$\overline{U(P)}\subset\Omega \qquad \text{for } P\in\Omega,$$
$$u=0 \text{ in } U(P) \quad \text{for } P\in\overline{\partial_1\Omega}$$

and

$$\overline{U(P)}\cap\overline{\partial_1\Omega}=\phi \qquad \text{for } P\in\partial_2\Omega.$$

Since (2.1) holds from our assumption, such a selection of $U(P)$ is possible.

Now there is a finite point set $\{P_1,\cdots,P_N\}$ and the union of $U(P_k)$ $(1\leqq k\leqq N)$ covers $\bar{\Omega}$. Let the function α_k be in $C_0^\infty(U(P_k))$ such that $\sum_{k=1}^N\alpha_k\equiv 1$ in Ω. We assume that the points $P_1,\cdots,P_{N'}(N'\leqq N)$ are in $\partial_2\Omega$. Each subdomain $\overline{U(P_k)\cap\Omega}(1\leqq k\leqq N')$ can be mapped in a one to one C^∞ way into $y_n\leqq 0$[5] such that theo perator $\frac{d}{dn}$ on $U(P_k)$ is transformed into $\frac{\partial}{\partial y_n}$. Applying Lemma 8 for $\alpha_k u$ on $y_n\leqq 0$, we can find a sequence $\{\varphi_i^{(k)}\}$ $\subset C^\infty(\bar{\Omega})$ $(1\leqq k\leqq N')$ having the following property that

5) We denote by $(y_1, \cdots, y_n)$ the new coordinate.

$$\varphi_i^{(k)}=0 \quad \text{in a neighborhood of } \overline{\partial_1\Omega},$$

$$\left(\frac{d}{dn}+\sigma(x)\right)\varphi_i^{(k)}=0 \quad \text{on } \partial_2\Omega \tag{4.1}$$

and

$$\varphi_i^{(k)}\to\alpha_k u \qquad \text{in } H^1(\Omega).$$

Setting

$$\varphi_i=\sum_{k=1}^{N'}\varphi_i^{(k)}+\sum_{k=N'+1}^{N}\alpha_k u, \tag{4.2}$$

we easily see

$$\varphi_i\to u \text{ in } H^1(\Omega).$$

The other properties of $\{\varphi_i\}$ is obvious from (4.1) and (4.2). Hence we have finished the proof.

Finally we have

Lemma 9. *The definition domain* $D(A)$ (*see* (3.2)) *is dense in* H.

Proof. Let the vector function $\{u, v\}$ be in $H(=K(\Omega)\times L^2(\Omega))$. First we take a sequence $\{v_i\}\subset C_0^\infty(\Omega)$ converging to v in $L^2(\Omega)$. Secondly we set $u_i=\varphi_i$ for the sequence $\{\varphi_i\}$ in Lemma 8. Obviously, $\{u_i, v_i\}\to\{u, v\}$ in H. Since $\left(\frac{d}{dn}+\sigma(x)\right)u_i=0$ on $\partial_2\Omega$, we see that (2.7) holds by Green's formula. Thus each $\{u_i, v_i\}$ satisfies the boundary condition (B_2) on $\partial_2\Omega$. Hence $D(A)$ is dense in H.

5. In virtue of Lemma 5 and 9, we can apply the theory of evolution equations quite similarly as in [3] and [5] as follows. Suppose that $F(t)=\{0, f(t)\}$ is in $D(A)$ and $F(t)$, $AF(t)\in\mathscr{E}_t^0(H)[0, T]$. Then for any given $U_0=\{u_0, v_0\}\in D(A)$, there is a unique solution $U(t)=\{u(t), v(t)\}\in D(A)\cap\mathscr{E}_t^1(H)[0, T]$ of the equation

$$\frac{d}{dt}U(t)=AU(t)+F(t) \qquad \text{in } 0<t\leqq T \tag{5.1}$$

with the initial condition $U(0)=U_0$. The equation (5.1) is equivalent to

(2.3). Since $v=u_t$ and (2.7) holds, we see that u satisfies the boundary condition (2.5) weakly on $\partial_2\Omega\times[0, T]$ (see Definition 2.1). Hence Theorem 1 in Section 2 has been shown.

The statement of Theorem 2 is proved quite similarly as in Theorem 1, if we add to the definition domain $D(A)$ the condition that $u\in H^2_{\mathrm{loc}}(\bar{\Omega}-S)$.

Finally, we show the energy inequality for $U(t)\in D(A)\cap\mathscr{E}^1_t(H)[0, T]$. It is easily seen that from Lemma 1

$$\begin{aligned}\frac{d}{dt}\|U(t)\|_H^2&=(U'(t), U(t))_H+(U(t),U'(t))_H\\&=(AU(t)+F(t), U(t))_H+(U(t), AU(t)+F(t))_H\\&\leqq 2c_2\|U(t)\|_H^2+2\|U(t)\|_H\|F(t)\|_H.\end{aligned}$$

From this it follows

$$\|U(t)\|_H\leqq e^{c_2t}\left(\|U(0)\|_H+\int_0^t\|F(s)\|_H ds\right).$$

Hence

$$\|u(t)\|_1+\|u'(t)\|_0\leqq Ce^{c_2t}\left(\|u(0)\|_1+\|v(0)\|_0+\int_0^t\|f(s)\|_0 ds\right).$$

References

[1] Babeava, A.A. and G.K. Namazov, Hyperbolic equations with discontinuous coefficients, degenerate on the initial plane, *Izv. Akad. Nauk Azerbaidzan SSSR Ser. Fiz. -Tehn. Mat. Nauk*, no. 5 (1967), 11-16 (Russian).

[2] Čehlov, V.I., A mixed problem with discontinuous boundary conditions for the wave equation, *Soviet Math. Dokl.* **9** (1968), 1472-1475.

[3] Ikawa, M., Mixed problems for hyperbolic equations of second order, *J. Math. Soc. Japan*, **20** (1968), 580-608.

[4] Lions, J.L., Une remarque sur les applications du théorèmes de Hille-Yosida, *J. Math. Soc. Japan*, **9** (1957), 62-70.

[5] Mizohata, S., Quelque problèmes au bord, du type mixte, pour des équations hyperboliques, *Séminaire sur les équations aux derivées partielles*, Collèdge de France, 1966-1967, 23-60.

[6] ———, *Partial Differential Equations*, Iwanami, 1965 (Japanese).

[7] Yosida, K., On operator theoretical integration of wave equations, *J. Math. Soc. Japan*, **8** (1956), 79-92.