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Covariance Operators of Skew Distributions

By

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In this paper we extend the concept of a skew distribution on a real Hilbert space H defined in [4] and [6] to that on a complex Hilbert space K with an antiunitary involution Γ, and show the following result which stems from Theorem .3 in [4]. Whenever a skew distribution m is given on even or infinite dimensional (K, Γ), any two of the following conditions imply the other one:

(i) m is a factor distribution;

(ii) m is $\mathscr{U}$-invariant; and

(iii) any pair of Γ-invariant orthogonal subspaces are independent with respect to m.

In the appendix we give a correspondence of a pair of Fock and anti-Fock representations to a pair of orthogonal transformations $\{A, {}^tA\}$ with $A^2=-1$ on H.

1. Notations and Definitions

In this section we prepare some notations and definitions from papers [2], [4] and [6] with slight modifications.

Let $\mathfrak{H}$ be a separable complex Hilbert space, $\mathfrak{A}$ a von Neumann algebra on $\mathfrak{H}$ and E a faithful normal trace on $\mathfrak{A}$ with $E(1)=1$. By (K, Γ) we mean a complex Hilbert space K with an antiunitary involution Γ, namely, $(\Gamma\xi|\Gamma\eta)=(\eta|\xi)$ for $\xi, \eta\in K$ and $\Gamma^2=1$. We denote by $F=(F, \mathfrak{A})$ a strongly continuous $(\|F(\xi)\|\leqq\lambda\|\xi\|$ for some $\lambda>0)$ and faithful (if ξ

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$\neq 0$, then $F(\xi)\neq 0$) linear mapping of (K, Γ) to $\mathfrak{A}$ with $F(\Gamma\xi)=F(\xi)^*$. Since E is faithful, $F\neq 0$. For any subset K_0 of K the von Neumann algebra generated by $F(\xi)$, $\xi\in K_0$ is denoted by $\mathfrak{A}(K_0)$. $\mathfrak{A}(F)$ is the union of $\mathfrak{A}(K_0)$ where K_0 runs over all finite subsets of K. Introduce the following equivalence relation into the set $\mathbf{F}$ of such linear mappings. $(F_1, \mathfrak{A}_1)$ and $(F_2, \mathfrak{A}_2)\in\mathbf{F}$ are equivalent if for any finite subset $K_0=\{\xi_1,\cdots, \xi_n\}$ of K there exists an isomorphism τ: $F_1(\xi)\rightarrow F_2(\xi)$, $\xi\in K_0$ of $\mathfrak{A}_1(K_0)$ onto $\mathfrak{A}_2(K_0)$ such that

$$E_1(A)=E_2(\tau(A))$$

for $A\in\mathfrak{A}_1(K_0)$, where E_i is a faithful normal trace on $\mathfrak{A}_i$ with $E_i(1)=1$.

An equivalence class m of $\mathbf{F}$ classified by the above relation is called a *distribution* on (K, Γ). Since algebraic structures of all $(F, \mathfrak{A})\in m$ is preserved by an isomorphism as above, the terminologies for m is utilized as similarly as that for each $(F, \mathfrak{A})$. For $(F, \mathfrak{A})\in m$, let $\mathfrak{H}_F$, π_F and Ω_F be the Hilbert space, the representation and the cyclic unit vector respectively such that

$$E(A)=(\pi_F(A)\Omega_F\,|\,\Omega_F)$$

for $A\in\mathfrak{A}(F)$. Denote

$$\pi_F(\mathfrak{A})=\{\pi_F(A):\ A\in\mathfrak{A}(F)\}''.$$

Then it is easily seen that $(F, \mathfrak{A})$ and $(\pi_F\circ F, \pi_F(\mathfrak{A}))$ are equivalent. The latter is called the *standard* representative of m. From this we may assume that for every $(F, \mathfrak{A})\in m$,

$$\mathfrak{A}=\{F(\xi):\ \xi\in K\}''$$

in the following. If $\mathfrak{A}$ is a factor for $(F, \mathfrak{A})\in m$, m is called a *factor* distribution.

Let K_1 be a Γ-invariant subspace of K such that $\Gamma_1=\Gamma|K_1$. For distributions m on (K, Γ) and m_1 on (K_1, Γ_1) choose suitable representatives $(F, \mathfrak{A})\in m$ and $(F_1, \mathfrak{A}_1)\in m_1$. If $\mathfrak{A}_1\subset\mathfrak{A}$ and $F(\xi)=F_1(\xi)$ for $\xi\in K_1$, then m is called an *extension* of m_1 and denoted by $m_1=m|K_1$.

Since for any distribution m on (K, Γ) there is $\lambda>0$ such that

$$E(F(\xi)^*F(\xi))\leqq\lambda\|\xi\|^2$$

for all $\xi\in K$, there exists a positive operator t on K such that

$$E(F(\eta)^*F(\xi))=(t\xi\,|\,\eta),$$

which is called the *covariance* operator of m. It is easily seen that $t=\Gamma t\Gamma$.

A family $\{K_\iota\colon \iota\in I\}$ of subspaces of K is said to be *independent* with respect to m, if

$$E(A_{\iota_1}\cdots A_{\iota_n})=E(A_{\iota_1})\cdots E(A_{\iota_n})$$

for any $A_\iota\in\mathfrak{A}(F_\iota)$ with $F_\iota\in m\,|\,K_\iota$ and for any $\iota_1,\cdots,\ \iota_n\in I$.

A distribution is *skew*, if for any $(F,\ \mathfrak{A})\in m$

$$[E(\xi),\ F(\eta)]_+\in\mathfrak{A}\cap\mathfrak{A}'$$

and if $F\in m$ implies $-F\in m$. It follows from the last condition that

$$E(F(\xi_1)\cdots F(\xi_{2n+1}))=0.$$

A unitary operator on $(K,\ \Gamma)$ which commutes with Γ is called a *Bogoliubov transformation.* $\mathcal{U}$ is a set of Bogoliubov transformations on $(K,\ \Gamma)$ whose commutant is the algebra of scalar operators. A skew distribution m is called to be *$\mathcal{U}$-invariant* if $(F,\ \mathfrak{A})\in m$ implies $(F\circ U,\ \mathfrak{A})\in m$ and if

$$[F(U\xi),\ F(U\eta)]_+=[F(\xi),\ F(\eta)]_+,$$

here $(F\circ U)(\xi)=F(U\xi)$.

A self dual CAR algebra $\mathfrak{A}_{SDC}(K,\ \Gamma)$ over $(K,\ \Gamma)$ is a *-algebra generated by $B(\xi)$, $\xi\in K$, its adjoint $B(\xi)^*$, $\xi\in K$ and the identity which satisfy the following three relation: $B(\xi)$ is linear in ξ, $[B(\xi),\ B(\eta)]_+=(\xi\,|\,\Gamma\eta)1$ and $B(\xi)^*=B(\Gamma\xi)$. If K has a finite dimension, $\mathfrak{A}_{SDC}(K,\ \Gamma)$ has a finite dimension. Irrespective of the dimension of K, $\mathfrak{A}_{SDC}(K,\ \Gamma)$ has a unique C^*-norm and $\overline{\mathfrak{A}}_{SDC}(K,\ \Gamma)$ denotes its C^*-completion.

A state φ on $\overline{\mathfrak{A}}_{SDC}(K,\ \Gamma)$ satisfying the following relation is called a *quasifree* state:

$$\varphi(B(\xi_1)\cdots B(\xi_{2n+1}))=0,$$

$$\varphi(B(\xi_1)\cdots B(\xi_{2n}))=\sum \mathrm{sgn}(s)\prod_{j=1}^{n}\varphi(B(\xi_{s(2j-1)})B(\xi_{s(2j)})),$$

where $n=1, 2,\cdots$, the sum is over all permutations s satisfying

$$s(1)<s(3)<\cdots<s(2n-1) \quad \text{and} \quad s(2j-1)<s(2j)$$

for $j=1,\cdots, n$ and $\mathrm{sgn}(s)$ is the signature of s.

2. Results

Lemma 1. *Let t be a bounded positive operator. If $(t\xi|\eta)=0$ for any pair of ξ and η in K with $(\xi|\eta)=(\xi|\Gamma\eta)=0$, then t is a scalar operator.*

Proof. Choose a complete orthonormal system $\{\xi_\iota\colon \iota\in I\}$ of K with $\xi_\iota=\Gamma\xi_\iota$. It follows from the hypothesis that $(t\xi_\iota|\xi_\kappa)=0$ for ι, $\kappa\in I$ with $\iota\neq\kappa$, and hence $t\xi_\iota=\lambda_\iota\xi_\iota$ for some $\lambda_\iota\geqq 0$, $\iota\in I$. For any ι_0 and ι_1 in I with $\iota_0\neq\iota_1$, put

$$\eta_{\iota_0}=\xi_{\iota_0}-\xi_{\iota_1},\ \eta_{\iota_1}=\xi_{\iota_0}+\xi_{\iota_1},$$

and $\eta_\iota=\xi_\iota$ for $\iota\neq\iota_0$ and $\iota\neq\iota_1$. Then $\Gamma\eta_\iota=\eta_\iota$ for $\iota\in I$ and $\{\eta_\iota\colon \iota\in I\}$ is a complete orthogonal system. It follows from the hypothesis that $t\eta_{\iota_0}=\mu\eta_{\iota_0}$ for some $\mu\geqq 0$. On the other hand

$$t\eta_{\iota_0}=t(\xi_{\iota_0}-\xi_{\iota_1})=\lambda_{\iota_0}\xi_{\iota_0}-\lambda_{\iota_1}\xi_{\iota_1}.$$

Thus $\lambda_{\iota_0}=\mu=\lambda_{\iota_1}$. Repeating the similar argument for each pair of elements in I, we get $\lambda_\iota=\mu$ for $\iota\in I$. Consequently $t=\mu 1$.

Corollary. *Let t be the covariance operator of a skew distribution m on (K, Γ). If $[m(\xi), m(\eta)]_+=0$ for any pair of ξ and η in K with $(\xi|\eta)=(\xi|\Gamma\eta)=0$, then t is a scalar operator.*

Proof. It is clear from

$$2(t\xi|\eta)=E([F(\xi), F(\eta)^*]_+)$$

for $(F, \mathfrak{A})\in m$.

It should be noted that, since the underlying Hilbert space $\mathfrak{H}$ of $\mathfrak{A}$ for $(F, \mathfrak{A})\in m$ is assumed to be separable, $\mathfrak{A}(F)'$ is generated by a countable family of elements. Furthermore, since F is assumed to be faithful, it follows that K is separable.

Lemma 2. *A skew distribution m with the covariance operator t has a representative $(F, \mathfrak{A}) \in m$ such that there exist a locally compact space Z, a positive measure ν whose carrier is Z, a ν-measurable field $\zeta \to \mathfrak{H}(\zeta)$ of Hilbert spaces, a ν-measurable field $\zeta \to \mathfrak{A}(\zeta)$ of von Neumann algebras, a ν-measurable field $\zeta \to E_\zeta$ of finite normal traces and a ν-measurable operator valued function $\zeta \to t(\zeta)$ with $t(\zeta) \in \mathbf{B}(K)$ and that*

$$\mathfrak{H} = \int^{\oplus} \mathfrak{H}(\zeta) d\nu(\zeta), \ \mathfrak{A} = \int^{\oplus} \mathfrak{A}(\zeta) d\nu(\zeta), \ E = \int^{\oplus} E_\zeta d\nu(\zeta),$$

$$[F(\xi), \ F(\eta)^*]_- = 2\int^{\oplus} (t(\zeta)\xi \mid \eta) 1(\zeta) d\nu(\zeta)$$

and

$$t = \int t(\zeta) E_\zeta(1(\zeta)) d\nu(\zeta),$$

where $\mathbf{B}(K)$ is a full operator algebra on K and $1(\zeta)$ is the identity in $\mathfrak{A}(\zeta)$.

Proof. According to the reduction theory and the separability of $\mathfrak{H}$, we can conclude that there exist a locally compact space Z which satisfies the second axiom of countability, a positive measure ν whose carrier is Z, a ν-measurable field $\zeta \to \mathfrak{H}(\zeta)$ of non zero Hilbert spaces on Z and a ν-measurable field $\zeta \to \mathfrak{A}(\zeta)$ of factor von Neumann algebras over $\mathfrak{H}(\zeta)$ on Z such that $\mathfrak{A}$ is spatially isomorphic to

$$\int^{\oplus} \mathfrak{A}(\zeta) d\nu(\zeta) \quad \text{over} \quad \int^{\oplus} \mathfrak{H}(\zeta) d\nu(\zeta).$$

Since K is separable, it contains a Γ-invariant countable dense $\mathbf{Q}$ linear subset K_0 where $\mathbf{Q}$ denotes the field of rational complex numbers. Since F is strongly continuous, a $*$-algebra $\mathfrak{A}_0$ generated by $\{F(\xi): \xi \in K\}$ and the identity has a countable base with respect to the uniform topology. Hence we may associate with $T \in \mathfrak{A}_0$ a ν-measurable field $\zeta \to T(\zeta)$ of operators with $T(\zeta) \in \mathfrak{A}(\zeta)$ such that

$$T = \int^{\oplus} T(\zeta) d\nu(\zeta), \ \|T(\zeta)\| \leqq \|T\|$$

and a mapping $\Phi_\zeta: T \to T(\zeta)$ is a $*$-homomorphism. Put $F_\zeta(\xi) = \Phi_\zeta(F(\xi))$ for $\xi \in K$. Then

$$F(\xi)=\int^{\oplus} F_{\zeta}(\xi)\,d\nu(\zeta)$$

and F_{ζ} belongs to some skew distribution m_{ζ} on (K, Γ). It follows that

$$[F(\xi), F(\eta)^{*}]_{+}=\int^{\oplus} \lambda_{\zeta}(\xi, \eta)1(\zeta)\,d\nu(\zeta)$$

for some complex number $\lambda_{\zeta}(\xi, \eta)\in\mathbf{C}$. Since K is separable, it contains a countable dense linear subset K_0 on $\mathbf{Q}$. Then there is a ν-null set $N\subset Z$ such that

$$[F_{\zeta}(\xi), F_{\zeta}(\eta)^{*}]_{+}=\lambda_{\zeta}(\xi, \eta)1(\zeta)$$

and $\|F_{\zeta}(\xi)\|\leqq\|F(\xi)\|$ for $\zeta\notin N$ and $\xi, \eta\in K_0$. Since

$$\begin{aligned}\int^{\oplus} \lambda_{\zeta}(\lambda\xi_1+\xi_2, \eta)1(\zeta)\,d\nu(\zeta)&=[F(\lambda\xi_1+\xi_2), F(\eta)^{*}]_{+}\\&=\lambda[F(\xi_1), F(\eta)^{*}]_{+}+[F(\xi_2), F(\eta)^{*}]_{+}\\&=\int^{\oplus}\{\lambda\lambda_{\zeta}(\xi_1, \eta)+\lambda_{\zeta}(\xi_2, \eta)\}1(\zeta)\,d\nu(\zeta),\end{aligned}$$

$$\int^{\oplus} \lambda_{\zeta}(\xi, \xi)1(\zeta)\,d\nu(\zeta)=[F(\xi), F(\xi)^{*}]_{+}\geqq 0$$

and

$$|\lambda_{\zeta}(\xi, \xi)|\leqq\|[F(\xi), F(\xi)^{*}]_{+}\|$$

for $\xi, \xi_1, \xi_2, \eta\in K_0$ and $\lambda\in\mathbf{Q}$, it follows that λ_{ζ} is a positive definite bounded bilinear functional on K_0 as well as on K. Hence a bounded positive operator $t(\zeta)$ exists on K for $\zeta\in Z-N$ satisfying

$$[F_{\zeta}(\xi), F_{\zeta}(\eta)^{*}]_{+}=\lambda_{\zeta}(\xi, \eta)1(\zeta)=2(t(\zeta)\xi\,|\,\eta)1(\zeta)$$

for $\xi, \eta\in K$. Define $t(\zeta)=0$ for $\zeta\in N$. Then the mapping $\zeta\rightarrow t(\zeta)$ is a ν-measurable function on Z with values in $\mathbf{B}(K)$ and

$$[F(\xi), F(\eta)^{*}]_{+}=2\int^{\oplus}(t(\zeta)\xi\,|\,\eta)1(\zeta)\,d\nu(\zeta)$$

for any $\xi, \eta\in K$. Since E is a faithful normal trace with $\|E\|=1$ over $\mathfrak{A}$, there exists a ν-measurable field $\zeta\rightarrow E_{\zeta}$ of finite faithful normal traces on Z such that

$$E=\int^{\oplus} E_{\zeta}\,d\nu(\zeta).$$

It follows that

$$2(t\xi\,|\,\eta)=E([F(\xi),\ F(\eta)^*]_+)=2\int(t(\zeta)\xi\,|\,\eta)E_\zeta(1(\zeta))\,d\nu(\zeta)$$

and hence

$$t=\int E_\zeta(1(\zeta))t(\zeta)\,d\nu(\zeta).$$

Definition. A distribution m is a *canonical* skew distribution, if for $(F,\ \mathfrak{A})\in m$

$$[F(\xi),\ F(\eta)^*]_+=(\xi\,|\,\eta)1.$$

Remark 1. If m is a factor skew distribution with the covariance operator t, then

$$[F(\xi),\ F(\eta)^*]_+=2(t\xi\,|\,\eta)1.$$

Since $t=\Gamma t\Gamma$, K is a pre-Hilbert space with respect to the inner product $(\ |\ )_t$ defined by

$$(\xi\,|\,\eta)_t=(2t\xi\,|\,\eta)$$

for $\xi,\ \eta\in K$. Denote by K_t the completion of K with respect to $(\ |\ )_t$. Since Γ can be extended to K_t, we shall denote it by the same letter Γ. Then

$$[F(\xi),\ F(\eta)^*]_+=(\xi\,|\,\eta)_t1,$$

that is, F is a canonical skew distribution on $(K_t,\ \Gamma)$, which generates a self dual CAR algebra $\mathfrak{A}_{SDC}(K_t,\ \Gamma)$. Choosing a standard representative $(F,\ \mathfrak{A})$, we know that $\mathfrak{A}$ is a hyperfinite II_1 factor, if K is of infinite dimension.

If the underlying Hilbert space $\mathfrak{H}$ of a von Neumann algebra $\mathfrak{A}$ is separable, then $\mathfrak{A}$ has a countable generator $\mathfrak{M}$. Let $\mathfrak{B}_0$ and $\mathfrak{B}$ denote $*$-algebras algebraically generated by $\mathfrak{M}$ on $\mathbf{Q}$ and $\mathbf{C}$ respectively. Then $\mathfrak{B}_0$ is countable and the unit ball of $\mathfrak{B}_0$ is uniformly dense in the unit ball of $\mathfrak{B}$. Since $\mathfrak{B}$ is strongly dense in $\mathfrak{A}$, the unit ball of $\mathfrak{B}$ is strongly dense in the unit ball $\mathfrak{A}_1$ of $\mathfrak{A}$ by the Kaplansky's density theorem and hence $\mathfrak{N}_1$ is separable. Since the unit ball of a countably decomposable von Neumann algebra is metrizable by the strong topology, $\mathfrak{N}_1$ satisfies the second axiom of countability. Thus any subset $\mathfrak{N}$ of $\mathfrak{N}_1$ contains a coutable subset $\mathfrak{N}_0$ which is strongly dence in $\mathfrak{N}$.

Utilizing the same notations as in the last lemma, we have the following

Lemma 3. *A necessary and sufficient condition that a skew distribution m be $\mathcal{U}$-invariant is that there be $(F, \mathfrak{N}) \in m$ and $t(\zeta)$ as in Lemma 2 such that*

(i) $E(F(U\xi_1)\cdots F(U\xi_n)) = E(F(\xi_1)\cdots F(\xi_n))$ *for every* $U \in \mathcal{U}$; *and*

(ii) $t(\zeta)$ *is a scalar operator* ν*-almost everywhere.*

Proof. Necessity: By virtue of Lemma 2, there exists a representative $F \in m$ such that

$$2\int^{\oplus}(t(\zeta)U\xi \mid U\eta)1(\zeta)\,d\nu(\zeta) = [F(U\xi), F(U\eta)^*]_+$$

$$= [F(\xi), F(\eta)^*]_+ = 2\int^{\oplus}(t(\zeta)\xi \mid \eta)1(\zeta)\,d\nu(\zeta).$$

Since $\mathfrak{H}$ is separable, $\mathcal{U}$ contains a countable family $\mathcal{U}_0$ which is strongly dense in $\mathcal{U}$ by the preceding discussion. Thus there is a ν-null set $N \subset Z$ such that

$$(t(\zeta)U\xi \mid U\eta) = (t(\zeta)\xi \mid \eta)$$

for any $\zeta \in Z - N$, $U \in \mathcal{U}_0$ and $\xi, \eta \in K_0$. Since $(\mathcal{U}_0)' = \mathcal{U}'$ is the algebra of scalar operators, it follows that $t(\zeta)$ is a scalar operator for $\zeta \in Z - N$.

Sufficiency: If $t(\zeta)$ is a scalar operator ν-almost everywhere, then

$$[F(U\xi), F(U\eta)^*]_+ = 2\int^{\oplus}(t(\zeta)U\xi \mid U\eta)1(\zeta)\,d\nu(\zeta)$$

$$= 2\int^{\oplus}(t(\zeta)\xi \mid \eta)1(\zeta)\,d\nu(\zeta) = [F(\xi), F(\eta)^*]_+.$$

Further, since $F(U\xi)$, $U \in \mathcal{U}$ is strongly continuous, faithful linear mapping in ξ with $F(U\Gamma\xi) = F(U\xi)^*$, U induces an automorphism $\tau(U)$ of $\mathfrak{A}(K_0)$

for any finite K_0 in K such that $\tau(U)F(\xi)=F(U\xi)$, $\xi\in K_0$ and $E(\tau(U)A)=E(A)$, $A\in\mathfrak{A}(K_0)$ by (i).

Lemma 4. *Let B be a canonical skew distribution. If $[\xi_i, \Gamma\xi_i: i=1,\cdots, n]$ and $[\eta_j, \Gamma\eta_j: j=1,\cdots, m]$ are orthogonal, then the quasifree state E with $E(B(\eta)^*B(\xi))=2^{-1}(\xi|\eta)$ on a C^*-algebra of a self dual* CAR *algebra $\mathfrak{A}_{SDC}(K, \Gamma)$ satisfies*

$$E(B(\xi_1)\cdots B(\xi_n)B(\eta_1)\cdots B(\eta_m))=E(B(\xi_1)\cdots B(\xi_n))E(B(\eta_1)\cdots B(\eta_m)),$$

where $[\omega_k: k=1,\cdots, l]$ denotes the subspace spanned by $\omega_1,\cdots, \omega_l$.

Proof. If n or m are odd, then the left side and at least one of the factors in the right side are 0 due to $E(B(\xi_i)B(\eta_j))=E(B(\eta_j)B(\xi_i))=0$. If n and m are even,

$$E(B(\xi_1)\cdots B(\xi_n)B(\eta_1)\cdots B(\eta_m))=\sum_{\sigma}\operatorname{sgn}(\sigma)\prod_{j=1}^{k}E(B(\xi_{\sigma(2j-1)})B(\xi_{\sigma(2j)}))$$

where $2k=n+m$, $\xi_{n+i}=\eta_i$ for $i=1,\cdots, m$, sgn is the signature of the permutation σ satisfying

$$\sigma(1)<\sigma(3)<\cdots<\sigma(2k-1) \text{ and } \sigma(2j-1)<\sigma(2j)$$

for $j=1,\cdots, k$. If there is j with $1\leqq j\leqq k$ such that $1\leqq\sigma(2j-1)\leqq n$ and $n+1\leqq\sigma(2j)\leqq n+m$, then

$$\prod_{j=1}^{k}E(B(\xi_{\sigma(2j-1)})B(\xi_{\sigma(2j)}))=0.$$

Therefore we have only to consider the sum over all permutations σ satisfying

$$1\leqq\sigma(2j-1)<\sigma(2j)\leqq n$$

for $j=1,\cdots, 2^{-1}n$ and

$$n+1\leqq\sigma(2i-1+n)<\sigma(2i+n)\leqq n+m$$

for $i=1,\cdots, 2^{-1}m$. Let s and t denote the permutations

$$s=\begin{pmatrix}1 & 2\cdots & n\\ \sigma(1) & \sigma(2)\cdots & \sigma(n)\end{pmatrix} \text{ and } t=\begin{pmatrix}n+1 & n+2\cdots & n+m\\ \sigma(n+1) & \sigma(n+2)\cdots & \sigma(n+m)\end{pmatrix}.$$

Then $\mathrm{sgn}(\sigma)=\mathrm{sgn}(s)\mathrm{sgn}(t)$ for σ considered. Consequently

$$E(B(\xi_1)\cdots B(\xi_n)B(\eta_1)\cdots B(\eta_m))$$

$$=\sum_{\sigma}\mathrm{sgn}(\sigma)\{\prod_{j=1}^{n/2}E(B(\xi_{\sigma(2j-1)})B(\xi_{\sigma(2j)}))\}\{\prod_{i=1}^{m/2}E(B(\xi_{\sigma(2i-1+n)})B(\xi_{\sigma(2i+n)})\}$$

$$=\{\sum_{s}\mathrm{sgn}(s)\prod_{j=1}^{n/2}E(B(\xi_{s(2j-1)})B(\xi_{s(2j)}))\}$$

$$\{\sum_{t}\mathrm{sgn}(t)\prod_{i=1}E(B(\eta_{t(2i-1)})B(\eta_{t(2i)}))\}$$

$$=E(B(\xi_1)\cdots B(\xi_n))E(B(\eta_1)\cdots B(\eta_m)),$$

where s and t satisfy the condition in the definition of a quasifree state.

Since E is faithful and ν has the carrier Z, it follows that a function $\zeta\rightarrow E_\zeta(1(\zeta))$ is ν-measurable and $0<E_\zeta(1(\zeta))<+\infty$ ν-almost everywhere. Define a probability measure μ on Z by $d\mu(\zeta)=E_\zeta(1(\zeta))d\nu(\zeta)$. Then μ and ν are equivalent and hence the μ-measurability and the ν-measurability coincide.

Theorem. *Let m be a skew distribution on (K, Γ) whose dimension is even or infinite. Any two of the following conditions imply the remaining one:*

(i) *m is a factor distribution;*

(ii) *m is $\mathcal{U}$-invariant; and*

(iii) *any pair of Γ-invariant orthogonal subspaces are independent with respect to m.*

In this case m is the canonical skew distribution up to a scalar constant.

Proof. (i) and (ii) imply (iii): According to (i), (ii) and Lemma 3, the covariance operator t of m is a scalar operator, say $2t=\lambda 1$ for $\lambda>0$. Put $B(\xi)=\lambda^{-1/2}F(\xi)$. Again by (i)

$$[B(\xi), B(\eta)^*]_+=E([B(\xi), B(\eta)^*]_+)1=\lambda^+E([F(\xi), F(\eta)^*]_+)1=(\xi\,|\,\eta)1,$$

hence B is a canonical skew distribution. Then $B(\xi)$, $\xi\in K$ generate a self dual CAR algebra $\mathfrak{A}_{SDC}(K, \Gamma)$, we have

$$
\begin{aligned}
&E(F(\xi_1)\cdots F(\xi_n)F(\eta_1)\cdots F(\eta_m)) \\
&\quad =\lambda^{(n+m)/2}E(B(\xi_1)\cdots B(\xi_n)B(\eta_1)\cdots B(\eta_m)) \\
&\quad =\lambda^{(n+m)/2}E(B(\xi_1)\cdots B(\xi_n))E(B(\eta_1)\cdots B(\eta_m)) \\
&\quad =E(F(\xi_1)\cdots F(\xi_n))E(F(\eta_1)\cdots F(\eta_m))
\end{aligned}
$$

for any orthogonal subspaces $[\xi_i, \Gamma\xi_i: i=1,\dots, n]$ and $[\eta_j, \Gamma\eta_j: j=1,\dots, m]$ by Lemma 4.

(ii) and (iii) imply (i): Employing the same notations as in Lemma 2, we find a μ-measurable field $\zeta\to F_\zeta(\xi)$ of operators on Z and a μ-measurable operator valued function $\zeta\to t(\zeta)$ with the remaining properties in Lemma 2 such that

$$F(\xi)=\int^{\oplus} F_\zeta(\xi)\,d\nu(\zeta)$$

and

$$t=\int t(\zeta)\,d\nu(\zeta).$$

It follows from (ii) and Lemma 3 that $t(\zeta)$ is a scalar operator, say $2t(\zeta)=\lambda(\zeta)1$ and $\lambda(\zeta)>0$ for $\zeta\in Z-N$ and $\mu(N)=0$. Putting $B_\zeta(\xi)=\lambda(\zeta)^{-1/2}F_\zeta(\xi)$ for $\zeta\in Z-N$ and $B_\zeta(\xi)=0$ for $\zeta\in N$, we have a μ-measurable field $\zeta\to B_\zeta(\xi)$ of canonical skew distributions. According to Lemma 4

$$
\begin{aligned}
&E(F(\xi_1)\cdots F(\xi_{2n})F(\eta_1)\cdots F(\eta_{2m})) \\
&\quad =\int\lambda(\zeta)^{n+m}E_\zeta(B_\zeta(\xi_1)\cdots B_\zeta(\xi_{2n})B_\zeta(\eta_1)\cdots B_\zeta(\eta_{2m}))\,d\nu(\zeta) \\
&\quad =\int\lambda(\zeta)^{n+m}E_\zeta(B(\xi_1)\cdots B_\zeta(\xi_{2n}))E_\zeta(B_\zeta(\eta_1)\cdots B_\zeta(\eta_{2m}))\,d\nu(\zeta).
\end{aligned}
$$

On the other hand

$$
\begin{aligned}
&E(F(\xi_1)\cdots F(\xi_{2n}))E(F(\eta_1)\cdots F(\eta_{2m})) \\
&\quad =\int\lambda(\zeta)^{n}E_\zeta(B_\zeta(\xi_1)\cdots B_\zeta(\xi_{2n}))\,d\nu(\zeta)\int\lambda(\zeta)^{m}E_\zeta(B_\zeta(\eta_1)\cdots B_\zeta(\eta_{2m}))\,d\nu(\zeta).
\end{aligned}
$$

Selecting ξ_i and η_j being mutually orthogonal such that $\xi_{n+i}=\Gamma\xi_i$ and $\eta_{m+j}=\Gamma\eta_j$. Then by (iii)

$$\int\{\lambda(\zeta)/2\}^{n+m}\,d\mu(\zeta)=\int\{\lambda(\zeta)/2\}^{n}\,d\mu(\zeta)\int\{\lambda(\zeta)/2\}^{m}\,d\mu(\zeta)$$

for any integers $n\geqq 0$ and $m\geqq 0$. Therefore $\lambda(\zeta)$ is constant μ-almost everywhere and hence $\lambda(\zeta)=\lambda$ for some $\lambda>0$ μ-almost everywhere. Let $B(\xi)$ be an operator in $\mathfrak{A}$ which corresponds to $\zeta\rightarrow B_\zeta(\xi)$. Then

$$F(\xi)=\int^{\oplus} F_\zeta(\xi)d\nu(\zeta)=\int^{\oplus}\lambda^{1/2}B_\zeta(\xi)d\nu(\zeta)=\lambda^{1/2}B(\xi)$$

and hence B is a canonical skew distribution on (K, Γ). Taking a standard representative $(F, \mathfrak{A})\in m$, we can conclude that the von Neumann algebra $\mathfrak{A}$ is a factor.

(iii) and (i) imply (ii): By (iii), if $(\xi|\eta)=(\xi|\Gamma\eta)=0$, then

$$(t\xi|\eta)=E(m(\eta)^*m(\xi))=E(m(\eta)^*)E(m(\xi))=0.$$

It follows from Lemma 1 that $2t=\lambda 1$ for some $\lambda>0$. Put $B(\xi)=\lambda^{-1/2}F(\xi)$. Then B is a canonical skew distribution by (i). Since

$$\begin{aligned}E(F(\xi_1)\cdots F(\xi_m))&=\lambda^n E(B(\xi_1)\cdots B(\xi_{2n}))\\&=\lambda^n\sum_\sigma \operatorname{sgn}(\sigma)\prod_{j=1}^n E(B(\xi_{\sigma(2j-1)})B(\xi_{\sigma(2j)}))\\&=\lambda^n\sum_\sigma \operatorname{sgn}(\sigma)\prod_{j=1}^n E(B(U\xi_{\sigma(2j-1)})B(U\xi_{\sigma(2j)}))\\&=\lambda^n E(B(U\xi_1)\cdots B(U\xi_{2n}))=E(F(U\xi_1)\cdots F(U\xi_{2n})),\end{aligned}$$

it follows from Lemma 3 that m is $\mathcal{U}$-invariant.

Remark 2. In case where the last theorem is valid, the covariance operator of m is a scalar operator. Making use of the results of Segal [5], we have that if $A\in\mathfrak{A}$ and $\tau(U)A=A$ for all $U\in\mathcal{U}$, then A is a scalar operator, where $\tau(U)$ is the *-automorphism induced by

$$\tau(U)F(\xi)=F(U\xi).$$

3. Appendix

Let K be a complex Hilbert space and Γ be an antiunitary involution. A projection e on K is called a *basis* projection if $\Gamma e\Gamma=1-e$. Restricting the coefficient to the real number field and define the inner product $(\ |\ )_r$ by

$$(\xi\,|\,\eta)_r=\Re(\xi\,|\,\eta),$$

we have a real Hilbert space $(K, (\ |\ )_r)$, which we denote by K_r. Here $\Re$ denotes the real part. Then the operator $\mathbf{i}$ of multiplying i on K_r is an orthogonal transformation on K_r with $\mathbf{i}^2=-1$. Since the set $\{\xi\in K\colon \Gamma\xi=\xi\}$ forms a real Hilbert space for the inner product $(\ |\ )_r$, we designate it by H. Then $\mathbf{i}H$ is a set $\{\xi\in K\colon \Gamma\xi=-\xi\}$ and $K_r=H\oplus\mathbf{i}H$. The complexification $H+iH$ of H with the inner product

$$(\xi_1+i\xi_2\,|\,\eta_1+i\eta_2)=(\xi_1\,|\,\eta_1)-(\xi_2\,|\,\eta_2)+i\{(\xi_2\,|\,\eta_1)-(\xi_1\,|\,\eta_2)\}$$

is naturally isomorphic to K. For any orthogonal transformation v on H there corresponds a Bogoliubov transformation u on K such that

$$u(\xi_1+i\xi_2)=v\xi_1+iv\xi_2$$

for ξ_1, $\xi_2\in H$. Conversely, since every Bogoliubov transformation is reduced by H, the restriction v of u onto H is an orthogonal transformation on H. Since $v=1$ iff $u=1$, this correspondence is bijective. Let Λ be an orthogonal transformation on H with $\Lambda^2=-1$ and e be the projection onto the complex subspace of K such that

$$eK=\{\xi-i\Lambda\xi\colon \xi\in H\}.$$

Then e is a basis projection, $\Gamma(\xi+i\Lambda\xi)=\xi-i\Lambda\xi$ and

$$(1-e)K=\{\xi+i\Lambda\xi\colon\ \xi\in H\}.$$

Proposition 1. *There is a bijection between the family of pairs* $\{e, 1-e\}$ *of basis projections on* (K, Γ) *and the family of pairs* $\{\Lambda, {}^t\Lambda\}$ *of orthogonal transformations with* $\Lambda^2=-1$ *on* H.

Proof. With Λ satisfying ${}^t\Lambda\Lambda=\Lambda{}^t\Lambda=1$ and $\Lambda^2=-1$, we associate basis projections e and $1-e$ as above.

Suppose that e is a basis projection. Let H be the real Hilbert space $\{\eta+\Gamma\eta\colon\ \eta\in eK\}$. Choose $\xi\in H$. If $\xi=\eta+\Gamma\eta$ and $\xi=\eta'+\Gamma\eta'$ for η and $\eta'\in eK$. Then $\eta-\eta'=\Gamma(\eta-\eta')$ and $\eta-\eta'\in eK$, which implies that $\eta=\eta'$. Thus we may define a transformation Λ_0 on H by

$$\Lambda_0(\eta+\Gamma\eta)=i(\eta-\Gamma\eta)=i\eta+\Gamma(i\eta)$$

for $\eta \in ek$. Then Λ_0 is an orthogonal transformation with $\Lambda_0^2 = -1$. Since for $\xi = \eta + \Gamma\eta$ and $\eta \in eK$ we have

$$\xi + i\Lambda_0\xi = 2\Gamma\eta \in (1-e)K \quad \text{and} \quad \xi - i\Lambda_0\xi = 2\eta \in eK,$$

it follows that Λ_0 is associated with e and $1-e$. Hence the mapping $\{\Lambda, {}^t\Lambda\} \to \{e, 1-e\}$ defined above is onto. Suppose now that e is a basis projection associated with a given orthogonal transformation Λ with $\Lambda^2 = -1$ such that

$$eK = \{\xi - i\Lambda\xi : \xi \in H\},$$

and that Λ_0 is associated with e as above. Since, then, for any $\xi \in H$ we have $\eta = \xi - i\Lambda\xi \in eK$ and $\Gamma\eta = \xi + i\Lambda\xi \in (1-e)K$, it follows that

$$\eta + \Gamma\eta = 2\xi \text{ and } i(\eta - \Gamma\eta) = 2\Lambda\xi.$$

Therefore $\Lambda_0\xi = \Lambda\xi$ for $\xi \in H$, that is, $\Lambda_0 = \Lambda$.

Remark 3. It is clear that a self dual CAR algebra $\mathfrak{A}_{SDC}(K, \Gamma)$ on (K, Γ) and a Clifford algebra $\mathfrak{A}_{CLI}(H)$ coincide, if K is of even or infinite dimension.

A family $\{K_\iota : \iota \in I\}$ of Γ-invariant subspaces of (K, Γ) is said to *be independent with respect to* a quasifree state φ on $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ if

$$\varphi(A_{\iota_1} \cdots A_{\iota_n}) = \varphi(A_{\iota_1}) \cdots \varphi(A_{\iota_n})$$

for any $A_\iota \in \overline{\mathfrak{A}}_{SDC}(K_\iota, \Gamma)$ and for any $\iota_1, \dots, \iota_n \in I$, where $\overline{\mathfrak{A}}_{SDC}(K_\iota, \Gamma)$ is the C^*-subalgebra of $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ generated by $B(\xi)$, $\xi \in K$ and the identity. It is shown in [2] that there is a one to one correspondence between an operator s on (K, Γ) with $0 \leqq s \leqq 1$ and $\Gamma s \Gamma = 1 - s$, and a quasifree state φ on $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$ as the following

$$(s\xi \mid \eta) = \varphi(B(\eta)^* B(\xi)).$$

Proposition 2. *Let φ be a quasifree state on $\overline{\mathfrak{A}}_{SDC}(K, \Gamma)$. Then φ is central if and only if any pair of Γ-invariant subspaces are independent with respect to φ.*

Proof. The only if part is clear from Lemma 4. It suffices to show the if part. If ξ and η are non zero vectors with $(\xi \mid \eta) = (\xi \mid \Gamma\eta)$, then

$$(s\xi\,|\,\eta)=\varphi(B(\eta)^*B(\xi))=\varphi(B(\eta)^*)\varphi(B(\xi))=0.$$

It follows from Lemma 1 that s is a scalar operator, that is, $s=2^{-1}1$.

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