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On Quasifree States of the Canonical Commutation Relations (I)

By

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Abstract

A self-dual CCR algebra is defined and arbitrary quasifree state is realized in a Fock type representation of another self-dual CCR algebra of a double size as a preparation for a study of quasi-equivalence of quasifree states.

§ 1. Introduction

A necessary and sufficient condition for the quasi-equivalence of two quasifree representations of the canonical anticommutation relations (CAR) has been derived in [11] for the gauge invariant case and in [3] for the general case. We shall derive an analogous result for the canonical commutation relations (CCR) in this series of papers.

A quasifree state of CCR and Bogoliubov automorphisms have been extensively studied ([5]~[10], [12], [13]). We shall use the formulation developped in [2].

In section 2, we review the formulation in [2]. A self-dual algebra is defined when a linear space K, an antilinear involution Γ of K and a hermitian form γ on K satisfying $\gamma(\Gamma f, \Gamma g)=-\gamma(f, g)^*$ are given. In section 3, we define a quasifree state in terms of a nonnegative hermitian form S on K such that $S(f, g)-S(\Gamma g, \Gamma f)=\gamma(f, g)$. In section 4, the structure of S relative to (K, γ, Γ) is analyzed.

In section 5, basic properties of a Fock representation are stated and a result in [1] is quoted. A Fock type representation is defined as a generalization of a Fock representation to the case of degenerate γ (i.e.

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the case with nontrivial center). In section 6, a quasifree state is realized as the restriction of Fock type state of a CCR algebra for $(\hat{K}_S, \hat{\gamma}_S, \hat{\Gamma}_S)$ where $\hat{K}_S$ is about twice as large as K.

An application to the quasi-equivalence of quasifree states will be made in a subsequent paper [5].

§ 2. Basic Notions

Let K be a complex linear space and $\gamma(f, g)$ be a hermitian form for $f, g \in K$. Let Γ be an antilinear involution ($\Gamma^2=1$) satisfying $\gamma(\Gamma f, \Gamma g)=-\gamma(g, f)$. A *self-dual* CCR *algebra* $\mathfrak{A}(K, \gamma, \Gamma)$ over (K, γ, Γ) is the quotient of the complex free $*$algebra generated by $\mathrm{B}(f)$, $f \in K$, its conjugate $\mathrm{B}(f)^*$, $f \in K$ and an identity 1 over (the two-sided $*$ ideal generated by) the following relations:

(1) $\mathrm{B}(f)$ is complex linear in f,

(2) $\mathrm{B}(f)^*\mathrm{B}(g)-\mathrm{B}(g)\mathrm{B}(f)^*=\gamma(f, g)1$,

(3) $\mathrm{B}(\Gamma f)^*=\mathrm{B}(f)$.

Any one-to-one linear mapping U of K onto K satisfying $\gamma(Uf, Ug)=\gamma(f, g)$ and $\Gamma U=U\Gamma$ preserves the above relations (1)$\sim$(3) and hence there exists a unique $*$ automorphism $\tau(U)$ of $\mathfrak{A}(K, \gamma, \Gamma)$ satisfying $\tau(U)\mathrm{B}(f)=\mathrm{B}(Uf)$. U and $\tau(U)$ shall be called a *Bogoliubov transformation* and a *Bogoliubov $*$ automorphism*.

Any operator P on K satisfying

(1) $P^2=P$,

(2) $\gamma(f, Pf)>0$, if $Pf\neq 0$,

(3) $\gamma(Pf, g)=\gamma(f, Pg)$,

(4) $\Gamma P\Gamma=1-P$,

is called a *basis projection*. Such P is linear.

Let L be a complex pre-Hilbert space. A CCR *algebra* $\mathfrak{A}_{\mathrm{CCR}}(L)$ over L is the quotient of the free $*$algebra generated by $(\mathrm{a}^\dagger, f)$, (f, a), $f\in L$ and an identity by (the two-sided $*$ ideal generated by) the following relations:

(1) $(\mathrm{a}^\dagger, f)$ is complex linear in f,

(2) $(f, \mathrm{a})=(\mathrm{a}^\dagger, f)^*$,

(3) $[(f, a), (a^\dagger, g)]=(f, g)_L$,
$[(a^\dagger, f), (a^\dagger, g)]=[(f, a), (g, a)]=0$.

Let P be a basis projection. Then the mapping $\alpha(P)$ from $\mathfrak{A}(K, \gamma, \Gamma)$ to $\mathfrak{A}_{CCR}(PK)$ defined by

$$\alpha(P)(B(f_1)\cdots B(f_n))=(\alpha(P)B(f_1))\cdots(\alpha(P)B(f_n)), \tag{2.1a}$$

$$\alpha(P)B(f)=(a^\dagger, Pf)+(P\Gamma f, a) \tag{2.1b}$$

is a $*$ isomorphism of $\mathfrak{A}(K, \gamma, \Gamma)$ onto $\mathfrak{A}_{CCR}(PK)$.

Let $\mathfrak{A}$ be a $*$ algebra with an identity. A state φ of $\mathfrak{A}$ is a complex valued linear functional over $\mathfrak{A}$ satisfying $\varphi(1)=1$ and $\varphi(A^*A)\geqq 0$ for all $A\in\mathfrak{A}$. Associated with every state φ, there exists a triplet $\mathfrak{H}_\varphi$, π_φ, Ω_φ of a Hilbert space, a representation of $\mathfrak{A}$ by densely defined closable operators $\pi_\varphi(A)$, $A\in\mathfrak{A}$ and a unit vector Ω_φ, cyclic for $\pi_\varphi(\mathfrak{A})$, such that $\varphi(A)=(\Omega_\varphi, \pi_\varphi(A)\Omega_\varphi)$, $\pi_\varphi(A)^*\supset\pi_\varphi(A^*)$ and the domain of $\pi_\varphi(A)$ is $\pi_\varphi(\mathfrak{A})\Omega_\varphi$.

Let Re K denote the set of $f\in K$ such that $\Gamma f=f$. It is a real linear space. $f\in\text{Re } K$ if and only if $B(f)^*=B(f)$.

Let φ be a state of $\mathfrak{A}(K, \gamma, \Gamma)$ such that $\pi_\varphi(B(f))$ is essentially selfadjoint for all $f\in\text{Re } K$. Let $W_\varphi(f)=\exp i\,\overline{\pi_\varphi(B(f))}$, $f\in\text{Re }K$. We shall call such state φ over $\mathfrak{A}(K, \gamma, \Gamma)$ as a *regular state* if $W_\varphi(f)$ satisfies the Weyl-Segal relations:

$$W_\varphi(f)W_\varphi(g)=W_\varphi(f+g)\exp\frac{1}{2}\gamma(g, f). \tag{2.2}$$

Let φ be a regular state over $\mathfrak{A}(K, \gamma, \Gamma)$. Let N_φ be the set of $f\in K$ with $\pi_\varphi(B(f))=0$, which is a linear subset of K. Let Re $N_\varphi=N_\varphi\cap$ Re K. The collection of distances

$$d_\Psi(f, f')\equiv\sup_{|t|\leqq 1}\|\{W_\varphi(tf)-W_\varphi(tf')\}\Psi\|,\ \Psi\in\mathfrak{H}_\varphi, \tag{2.3}$$

defines a vector topology on Re K/Re N_φ, which we shall denote by τ_φ. It also induces a vector topology on (Re K/Re N_φ)$+i$(Re K/Re N_φ)$=K/N_\varphi$, which will be denoted again by τ_φ. The topology induced by one distance d_Ψ for a cyclic Ψ is mutually equivalent and is equivalent to τ_φ [4]. (The cyclicity here refers to $W_\varphi(f)$, $f\in\text{Re } K$.)

§ 3. Quasifree States

Definition 3.1. *A state φ on $\mathfrak{A}(K, \gamma, \Gamma)$ satisfying the following relations is called a quasifree state*:

$$\varphi(\mathrm{B}(f_1)\cdots\mathrm{B}(f_{2n-1}))=0 \tag{3.1}$$

$$\varphi(\mathrm{B}(f_1)\cdots\mathrm{B}(f_{2n}))=\sum\prod_{j=1}^{n}\varphi(\mathrm{B}(f_{s(j)})\mathrm{B}(f_{s(j+n)})) \tag{3.2}$$

where $n=1, 2\cdots$ and the sum is over all permutations s satisfying $s(1)<s(2)<\cdots<s(n)$, $s(j)<s(j+n)$, $j=1,\cdots, n$.

Lemma 3.2. *For any state over $\mathfrak{A}(K, \gamma, \Gamma)$, the hermitian form defined by*

$$\varphi(\mathrm{B}(f)^*\mathrm{B}(g))=S(f, g), \tag{3.3}$$

is positive semidefinite (*i.e.* $S(f, f)\geqq 0$) *and satisfies*

$$\gamma(g, f)=S(g, f)-S(\Gamma f, \Gamma g). \tag{3.4}$$

Proof. The positivity of φ implies the positive semidefiniteness of S.

$$\begin{aligned} S(\Gamma f, \Gamma g)&=\varphi(\mathrm{B}(f)\mathrm{B}(g)^*)=\varphi(\mathrm{B}(g)^*\mathrm{B}(f))-\gamma(g, f)1\\ &=S(g, f)-\gamma(g, f). \end{aligned}$$

Q. E. D.

Lemma 3.3. *The hermitian form*

$$(g, f)_S\equiv S(g, f)+S(\Gamma f, \Gamma g) \tag{3.5}$$

is positive semi-definite and satisfies

$$(\Gamma g, \Gamma f)_S=(f, g)_S, \tag{3.6}$$

$$|\gamma(g, f)|^2\leqq(f, f)_S(g, g)_S. \tag{3.7}$$

It is positive definite if γ is non-degenerate.

Proof. From Lemma 3.2,

$$S(f, f)\geqq 0, \quad S(\Gamma f, \Gamma f)\geqq 0.$$

Hence $(g, f)_S$ is positive semidefinite. We also have

$$(\Gamma g, \Gamma f)_S = S(\Gamma g, \Gamma f) + S(f, g) = (f, g)_S.$$

By the Schwarz inequality,

$$\begin{aligned} |\gamma(g, f)| &\leqq |S(g, f)| + |S(\Gamma f, \Gamma g)| \\ &\leqq S(g, g)^{\frac{1}{2}} S(f, f)^{\frac{1}{2}} + S(\Gamma f, \Gamma f)^{\frac{1}{2}} S(\Gamma g, \Gamma g)^{\frac{1}{2}} \\ &\leqq (S(g, g) + S(\Gamma g, \Gamma g))^{\frac{1}{2}} (S(f, f) + S(\Gamma f, \Gamma f))^{\frac{1}{2}} \\ &= (g, g)_S^{\frac{1}{2}} (f, f)_S^{\frac{1}{2}}. \end{aligned}$$

If $(f, f)_S = 0$, we have $\gamma(f, g) = 0$ for all g. If γ is non-degenerate, we have $f = 0$. Therefore, $(f, g)_S$ is positive definite. Q. E. D.

Lemma 3.4. *The set N_S of $f \in K$ satisfying $(f, f)_S = 0$ is a Γ-invariant subspace of K such that $S(f, g) = \gamma(f, g) = 0$ for any $f \in N_S$ and any $g \in K$. If S is related to a state φ by* (3.3), *then $\pi_\varphi(\mathrm{B}(f)) = 0$ is equivalent to $f \in N_S$. ($N_S = N_\varphi$ for a regular φ.)*

Proof. From the positive semidefiniteness of $(g, f)_S$, it follows that $(g, f)_S = 0$ for any $g \in K$ whenever $f \in N_S$. Hence N_S is a subspace of K. By (3.6), N_S is Γ-invariant. From (3.7), $\gamma(f, g) = 0$ for any $g \in K$ whenever $f \in N_S$. This implies that $\mathrm{B}(f)$, $f \in N_S$ commutes with all $\mathrm{B}(g)$, $g \in K$. In addition, $0 \leqq S(f, f) \leqq (f, f)_S = 0$ which implies $\|\pi_\varphi(\mathrm{B}(f)) \Omega_\varphi\|^2 = S(f, f) = 0$ for $f \in N_S$. Therefore $f \in N_S$ implies $\pi_\varphi(\mathrm{B}(f)) = 0$. Conversely, $\pi_\varphi(\mathrm{B}(f)) = 0$ implies $S(f, f) = \|\pi_\varphi(\mathrm{B}(f))\Omega_\varphi\|^2 = 0$, $S(\Gamma f, \Gamma f) = \|\pi_\varphi(\mathrm{B}(f))^* \Omega_\varphi\|^2 = 0$, and hence $(f, f)_S = 0$. Q. E. D.

Lemma 3.5. *For any positive semidefinite hermitian $S(g, f)$ on $K \times K$ satisfying* (3.4), *there exists a unique quasifree state φ_S satisfying* (3.3). *Any quasifree state is regular.*

Proof. The existence will be seen from Lemma 5.3 and Corollary 6.2. The uniqueness is immediate from (3.1) and (3.2). The regularity will be seen from Corollary 5.6.

Definition 3.6. *Let $\mathfrak{H}_S$, π_S, Ω_S denote the Hilbert space, the repre-*

sentation and the cyclic unit vector canonically associated with the quasifree state φ_S *through the relation*

(3.8) $$\varphi_S(A)=(\Omega_S,\ \pi_S(A)\Omega_S),\ A\in\mathfrak{A}(K,\ \gamma,\ \Gamma).$$

If S *commutes with a Bogoliubov transformation* U, *then a unitary operator* $\mathrm{T}_S(U)$ *on* $\mathfrak{H}_S$ *is defined by*

(3.9) $$\mathrm{T}_S(U)\pi_S(A)\Omega_S=\pi_S(\tau(U)A)\Omega_S$$

and the continuity. (S *is defined in Lemma* 4.2.)

§ 4. Structure of $(S,\ K,\ \gamma,\ \Gamma)$

Definition 4.1. K_S *denotes the completion of* K/N_S *with respect to the positive hermitian form induced on* K/N_S *by* $(f,\ g)_S$. K/N_S *is identified with a dense subset of* K_S. The Hilbert space topology on K/N_S is denoted by τ_S.

Lemma 4.2. (1) *There exists an antiunitary involution* Γ_S *on* K_S *such that* $\overline{\Gamma f}=\Gamma_S\bar{f}$ *for all* $f\in K$ *where* $\bar{f}\equiv f+N_S\in K/N_S$.

(2) *There exists a bounded operator* γ_S *on* K_S *such that*

(4.1) $$\gamma(f,\ g)=(\bar{f},\ \gamma_S\bar{g})_S$$

for $f,\ g\in K$. *It satisfies*

(4.2) $$\gamma_S^*=\gamma_S,\ \Gamma_S\gamma_S\Gamma_S=-\gamma_S\quad \text{and}\quad \|\gamma_S\|_S\leqq 1.$$

(3) *There exists a bounded operator* S *on* K_S *such that*

(4.3) $$S(f,\ g)=(\bar{f},\ S\bar{g})_S$$

for $f,\ g\in K$. *It satisfies*

(4.4) $$S^*=S,\ \Gamma_S S\Gamma_S=1-S,\quad 0\leqq S\leqq 1,$$

and

(4.5) $$S-\Gamma_S S\Gamma_S=\gamma_S.$$

Proof. Due to the Γ-invariance of N_S and (3.6), $\Gamma_S\bar{f}=\overline{\Gamma f}$ defines an antilinear isometric operator on K/N_S and hence the closure Γ_S of

$\bar{\Gamma}_S$ is defined on all vectors in K_S and $(\Gamma_S g, \Gamma_S f)_S=(f, g)_S$ for all $f, g\in K_S$. Since $\Gamma^2=1$, we have $\Gamma_S^2=1$ and hence Γ_S is an antiunitary involution on K_S.

(3.7) and Lemma 3.4 imply the existence of γ_S satisfying (4.1) and $\|\gamma_S\|_S\leqq 1$. Since $\gamma(f, g)$ is hermitian, we have $\gamma_S^*=\gamma_S$. Since $\gamma(\Gamma f, \Gamma g)=-\gamma(g, f)$, we have $\Gamma_S\gamma_S\Gamma_S=-\gamma_S$.

From the positivity $S(\Gamma f, \Gamma f)\geqq 0$ of S, we have $0\leqq S(f, f)\leqq \|f\|_S^2$ for $f\in K$. This together with Lemma 3.4 imply the existence of S satisfying (4.3), $S^*=S$ and $0\leqq S\leqq 1$. From (3.5), we have $S+\Gamma_S S\Gamma_S=1$ and from (3.4), we have (4.5). Q. E. D.

Definition 4.3. *Let E_+, E_- and E_0 be the spectral projection of γ_S for* $(0, +\infty)$, $(-\infty, 0)$ *and* $\{0\}$, *respectively. Let $K_\pm=E_\pm K_S$ and $K_0=E_0K_S$.*

Lemma 4.4. *$\Gamma_S E_\pm\Gamma_S=E_\mp$, $\Gamma_S E_0\Gamma_S=E_0$, $\Gamma_S K_\pm=K_\mp$ and $\Gamma_S K_0=K_0$*

Proof. This follows from $\Gamma_S\gamma_S\Gamma_S=-\gamma_S$. Q. E. D.

§ 5. Fock Representations

Definition 5.1. *A quasifree state φ_S is called a Fock state if the operator S of Lemma 4.2 is a basis projection on K_S. S in such a case will be written generally as P. The associated representation π_P is called a Fock representation.*

Lemma 5.2. *If P is a basis projection of (K, γ, Γ), then the quasifree state φ_P of $\mathfrak{A}(K, \gamma, \Gamma)$ for $P(f, g)=\gamma(f, Pg)$, if it exists, is a Fock state.*

Remark. In this case γ is automatically non-degenerate and $N_P=0$. P originally given on K is a restriction to K of the operator P on K_P defined by Lemma 4.2 and we have $\gamma(f, Pg)=(f, \gamma_P Pg)_P=(f, Pg)_P$ for $f, g\in K$. Therefore the appearance of two P is probably not confusing.

We shall summarize known properties of a Fock state in the following

3 lemmas.

Lemma 5.3. *Let P be a basis projection for (K, γ, Γ). A state φ of $\mathfrak{A}(K, \gamma, \Gamma)$ satisfying*

$$\varphi(\mathrm{B}(f)\mathrm{B}(\Gamma f))=0, \quad f \in PK, \tag{5.1}$$

exists, is unique and is a quasifree state φ_P.

Proof. By splitting $\mathrm{B}(f)$ as a sum $\mathrm{B}(Pf)+\mathrm{B}((1-P)f)$ and bringing $\mathrm{B}(Pf)$ to the left of any other $\mathrm{B}((1-P)f')$ with a help of the commutation relations, any element A in $\mathfrak{A}(K, \gamma, \Gamma)$ can be written as $A=\sum\mathscr{P}_i \mathrm{B}(f_i)+\sum\mathrm{B}(g_j)\mathscr{P}'_j+\lambda$ where $f_i\in(1-P)K$, $g_j\in PK$. Since (5.1) implies $\varphi(Q\mathrm{B}(f))=\varphi(\mathrm{B}(g)Q)=0$ for $f\in(1-P)K$, $g\in PK$ and $Q\in\mathfrak{A}(K, \gamma, \Gamma)$ by the Schwarz inequality, we have $\varphi(A)=\lambda$. Hence the uniqueness.

The well known Fock state of $\mathfrak{A}_{\mathrm{CCR}}(PK)$ gives the quasifree state φ_P through the identification of $\mathfrak{A}_{\mathrm{CCR}}(PK)$ with $\mathfrak{A}(K, \gamma, \Gamma)$ via $\alpha(P)$. φ_P clearly satisfies (5.1). Q. E. D.

Lemma 5.4. *Let $f\in\mathrm{Re}\,K$ and $D_0\equiv\pi_P[\mathfrak{A}(K, \gamma, \Gamma)]\Omega_P$. D_0 is a dense set of entire analytic vectors of $\mathrm{B}(f)$. The sum*

$$\sum_{n=0}^{\infty} n!^{-1}i^n\pi_P(\mathrm{B}(f))^n \tag{5.2}$$

converges on D_0. Its closure, denoted by $\mathrm{W}_P(f)$, is unitary and satisfies

$$\mathrm{W}_P(f_1)\mathrm{W}_P(f_2)=\mathrm{W}_P(f_1+f_2)\exp(1/2)\gamma(f_2, f_1), \tag{5.3}$$

$$(\Omega_P, \mathrm{W}_P(f)\Omega_P)=\exp-(1/2)\gamma(f, Pf). \tag{5.4}$$

$f\to\mathrm{W}_P(f)$ is continuous with respect to a norm $\gamma(f, Pf)^{1/2}$ on Re *K and the strong operator topology on $\mathfrak{H}_P$.*

Proof. Let $(\mathfrak{H}_P)_n$ be the subspace of $\mathfrak{H}_P$ generated by $\prod_{j=1}^{n}\pi_P(\mathrm{B}(g_j))\Omega_P$, $g_j\in PK$. If $\Psi\in\sum_{n=0}^{N}(\mathfrak{H}_P)_n$, then

$$\|\pi_P(\mathrm{B}(f))\Psi\|\leqq\sqrt{2}\,(N+1)^{1/2}\gamma(f, (2P-1)f)^{1/2}\|\Psi\|. \tag{5.5}$$

[This follows from a well known calculation: Let $\{f_j\}$ be a complete orthonormal basis of PK with $f_0=Pf$. Then $\Phi=\prod_{\nu=1}^{n}\pi_P(\mathrm{B}(f_{j_\nu}))\Omega_P(n\leqq N)$

is a complete orthonormal basis of $\sum_{n=0}^{N} (\mathfrak{H}_P)_n$, for which $\pi_P(\mathrm{B}(Pf))\Phi$ is also mutually orthogonal and $\|\pi_P(\mathrm{B}(Pf))\Phi\| = (k+1)^{1/2}\gamma(f, Pf)^{1/2}\|\Phi\|$ where k is the number of ν with $j_\nu = 0$. Hence $\|\pi_P(\mathrm{B}(Pf))\Psi\| \leqq (N+1)^{1/2} \gamma(f, Pf)^{1/2}\|\Psi\|$. A similar calculation with $f_0 = \Gamma(1-P)f$ leads to $\|\pi_P(\mathrm{B}[(1-P)f])\Psi\| \leqq N^{1/2}\gamma(f, (P-1)f)^{1/2}\|\Psi\|$.]

From (5.5) we have $\lim_{n\to\infty} \|\pi_P(\mathrm{B}(f))^n\Psi\|^{1/n}/n = 0$ for $\Psi \in D_0 = \bigcup_N \{\sum_{n=0}^{N} (\mathfrak{H}_P)_n\}$. Hence all such Ψ is an entire analytic vector for $\pi_P(\mathrm{B}(f))$, $f \in \mathrm{Re}\, K$, (5.2) applied on Ψ converges absolutely, the closure $\bar{\pi}_P(\mathrm{B}(f))$ of $\pi_P(\mathrm{B}(f))$ is selfadjoint, $\mathrm{W}_P(f) = \exp i\bar{\pi}_P(\mathrm{B}(f))$, and $\mathrm{W}_P(f)$ is unitary. [14]. (5.4) follows from $(\Omega_P, \pi_P(\mathrm{B}(f))^{2n}\Omega_P) = (2n)!\,2^{-n}n!^{-1}\gamma(f, Pf)^n$.

By the commutation relations, we have

$$(5.6) \qquad n!^{-1}\mathrm{B}(f_1+f_2)^n = \sum_{k+l+2m=n} k!^{-1}\mathrm{B}(f_1)^k l!^{-1}\mathrm{B}(f_2)^l m!^{-1}2^{-m}\gamma(f_2, f_1)^m.$$

From the previous result and the Schwarz inequality, $\sum k!^{-1}l!^{-1}(\mathrm{B}(f_1)^k\Phi, \mathrm{B}(f_2)^l\Psi)$ is absolutely convergent for Φ, $\Psi \in D_0$ and hence we obtain from (5.6) the equality (5.3) for a matrix element between two vectors Φ and Ψ in a dense set D_0. Hence (5.3) holds.

From (5.3) and (5.4), we have

$$(5.7) \qquad d_P(f_1, f_2)^2 \equiv \|\{\mathrm{W}_P(f_1) - \mathrm{W}_P(f_2)\}\Omega_P\|^2$$
$$= 2\{1 - (\exp -(1/4)\|f_2 - f_1\|_P^2)\cos(i/2)\gamma(f_2, f_1)\}$$

where $\|f\|_P^2 \equiv \gamma(f, [2P-1]f)$, which is $2\gamma(f, Pf)$ for $f \in \mathrm{Re}\, K$, and $\gamma(f_2, f_1) = \gamma(\Gamma f_2, \Gamma f_1) = -\gamma(f_2, f_1)^*$ is pure imaginary for f_1, $f_2 \in \mathrm{Re}\, K$. Since $\gamma(f_1, f_1) = 0$ for $f_1 \in \mathrm{Re}\, K$, we have from (3.7)

$$(5.8) \qquad |\gamma(f_2, f_1)| = |\gamma(f_2 - f_1, f_1)| \leqq \|f_2 - f_1\|_P \|f_1\|_P.$$

Hence $f \to \mathrm{W}_P(f)\Omega_P$ is continuous. By (5.3) and (3.7), this implies the continuity of $f \to \mathrm{W}_P(f)\Psi$ for $\Psi = \mathrm{W}_P(g)\Omega_P$, $g \in \mathrm{Re}\, K$. Since $\pi_P(\mathrm{B}(g)) = \lim_{t\to 0} (it)^{-1}(\mathrm{W}(tg) - 1)$ on D_0 for $f \in \mathrm{Re}\, K$, and since $\mathrm{W}_P(g_1)\cdots\mathrm{W}_P(g_n)\Omega_P$ is a multiple of $\mathrm{W}_P(\sum g_j)\Omega_P$, finite linear combinations of $\mathrm{W}_P(g)\Omega_P$, $g \in \mathrm{Re}\, K$, are dense in $\mathfrak{H}_P$. Therefore $f \to \mathrm{W}_P(f)$ is continuous. Q. E. D.

Lemma 5.5. *Let* Re K_P *be the real Hilbert space obtained by the* *completion of* Re K *with respect to the inner product* $(f_1, f_2)_P = \gamma(f_1, (2P-1)f_2)$, $f_1, f_2 \in \mathrm{Re}\, K$. *If* $f = \lim f_n$, $f_n \in \mathrm{Re}\, K$, *then* $\mathrm{W}_P(f) \equiv \lim \mathrm{W}_P(f_n)$ *exists and does not depend on* $\{f_n\}$ *for a fixed* f.

Let H_1 *be a linear subset of* Re K_P. *Denote by* $H_1^\perp$ *the set of vectors* $f \in \mathrm{Re}\, K_P$ *such that* $(f, \gamma_P g)_P = 0$ *for all* $g \in H_1$. *Let* $\mathrm{R}_P(H_1)$ *be the von Neumann algebra generated by* $\mathrm{W}_P(f)$, $f \in H_1$. *Let* $\bar{H}_1$ *denote the closure of* H_1 *in* Re K_P. *Then*

(0) $\mathrm{R}_P(\mathrm{Re}\, K_P)$ *is irreducible and* $\mathrm{R}_P(0)$ *is trivial*,

(i) $\mathrm{R}_P(H_1) = \mathrm{R}_P(\bar{H}_1)$,

(ii) $\mathrm{R}_P(H_1)' = \mathrm{R}_P(H_1^\perp)$,

(iii) $(\mathrm{R}_P(H_1) \cup \mathrm{R}_P(H_2))'' = \mathrm{R}_P(H_1 + H_2)$,

(iv) $(\mathrm{R}_P(H_1) \cap \mathrm{R}_P(H_2))'' = \mathrm{R}_P(\bar{H}_1 \cap \bar{H}_2)$,

(v) Ω_P *is cyclic for* $\mathrm{R}_P(H_1)$ *if and only if* $\bar{P}(H_1 + iH_1)$ *is dense in* $\bar{P}K_P$. ($\bar{P}$ *is the closure of* P *on* K_P.)

(vi) Ω_P *is separating for* $\mathrm{R}_P(H_1)$ *if and only if* $\bar{P}(H_1^\perp + iH_1^\perp)$ *is dense in* $\bar{P}K_P$,

(vii) $\mathrm{R}_P(H_1)$ *is a factor if and only if* $\bar{H}_1 \cap H_1^\perp$ *is* 0.

Proof. The existence of the unique limit $\mathrm{W}_P(f)$ for $f \in \mathrm{Re}\, K_P$ follows from Lemma 5.4. The von Neumann algebra $\mathrm{R}_P(H_1)$ is $\mathrm{R}(H_1/\mathrm{Re}\, K_P)$ in the notation of [1], where $(f_1, f_2)_S$ and $\gamma(f_1, f_2)$ are respectively (f_1, f_2) and $(f_1, \beta f_2)$. (i)$\sim$(iv) and (vii) follow from Theorem 1 of [1]. (0) and (v) follow from Lemma 5.1 of [1]. (vi) follows from (v) and (ii). Q. E. D.

Corollary 5.6. *A Fock representation is regular and irreducible.*

This is due to Lemmas 5.4 *and* 5.5.

The Fock representation defined above is applicable only for the case of non-degenerate γ. We now consider its generalization to the case of degenerate γ.

Definition 5.7. *A quasifree state* φ_S *in called a Fock type state if* $N_S = 0$ *and the spectrum of the operator* S *in Lemma* 4.2 *is contained in* $\{0, 1/2, 1\}$. *The corresponding representation is called a Fock type* *representation.*

Lemma 5.8. *Let K, γ, Γ be given. Let $\Pi(f_1, f_2)$ be a positive semidefinite hermitian form on K satisfying* (3.4), *where S is to be replaced by Π. Assume that $N_\Pi=0$ and the spectrum of the operator Π defined by Lemma* 4.2 *is contained in* $\{0, 1/2, 1\}$. *Let $E_\pm$, E_0 be defined as in Definition* 4.3. *Let*

$$\tilde{K}_\Pi=K_\Pi\oplus E_0K_\Pi, \tag{5.9}$$

$$\tilde{\Gamma}_\Pi(f\oplus g)=\Gamma_\Pi f\oplus\Gamma_\Pi g, \tag{5.10}$$

$$\tilde{\gamma}_\Pi(f_1\oplus g_1,\ f_2\oplus g_2)=(f_1,\ \gamma_\Pi f_2)_\Pi+i\{(g_1,\ f_2)_\Pi-(f_1,\ g_2)_\Pi\}. \tag{5.11}$$

Let $\tilde{\mathfrak{A}}=\mathfrak{A}(\tilde{K}_\Pi, \tilde{\gamma}_\Pi, \tilde{\Gamma}_\Pi)$ and identify $\mathfrak{A} \equiv \mathfrak{A}(K, \gamma, \Gamma)$ with the subalgebra $\mathfrak{A}(K\oplus 0, \tilde{\gamma}_\Pi, \tilde{\Gamma}_\Pi)$ of $\tilde{\mathfrak{A}}$. Let

$$\tilde{\Pi}(f\oplus g)=\{E_+f+(E_0f-ig)/2\}\oplus\{(iE_0f+g)/2\}, \tag{5.12}$$

$$\tilde{\Pi}(h_1,\ h_2)=\tilde{\gamma}_\Pi(h_1,\ \tilde{\Pi}h_2). \tag{5.13}$$

Then $\varphi_{\tilde{\Pi}}$ is a Fock state of $\tilde{\mathfrak{A}}$ and its restriction to $\mathfrak{A}$ is the Fock type state φ_Π.

Proof. $\tilde{\Gamma}_\Pi$ is an antiunitary involution of $\tilde{K}_\Pi$ and $\tilde{\gamma}_\Pi$ is a hermitian form satisfying $\tilde{\gamma}_\Pi(\tilde{\Gamma}h_1, \tilde{\Gamma}h_2)=-\tilde{\gamma}_\Pi(h_1, h_2)^*$. From (5.12), it follows that $\tilde{\Gamma}\tilde{\Pi}\tilde{\Gamma}+\tilde{\Pi}=1$, $\tilde{\Pi}^2=\tilde{\Pi}$,

$$\begin{aligned}\tilde{\gamma}_\Pi(f_1\oplus g_1,\ &\tilde{\Pi}(f_2\oplus g_2))\\ &=(f_1,\ \Pi f_2)_\Pi+(g_1,\ \Pi g_2)_\Pi+i\{(g_1,\ \Pi f_2)_\Pi-(f_1,\ \Pi g_2)_\Pi\}\\ &=\tilde{\gamma}_\Pi(\tilde{\Pi}(f_1\oplus g_1),\ f_2\oplus g_2),\end{aligned} \tag{5.14}$$

and

$$\tilde{\gamma}_\Pi(f\oplus g,\ \tilde{\Pi}(f\oplus g))\geqq 0. \tag{5.15}$$

Therefore $\tilde{\Pi}$ is a basis projection and $\varphi_{\tilde{\Pi}}$ is a Fock state.

The restriction of $\varphi_{\tilde{\Pi}}$ to $\mathfrak{A}$ is φ_Π as is seen from (5.14). Q. E. D.

Corollary 5.9. *For any Π in Lemma* 5.8, *the Fock type state φ_Π*

exists. The commutant $\pi_{\Pi}(\mathfrak{A})'$ *is abelian and is generated by* $\pi_{\Pi}(\mathrm{B}(f))$, $f\in E_0K_{\Pi}$.

Proof. From Lemmas 5.8 and 5.5 (ii), the following computation suffices: If $f\oplus g\in(K\oplus 0)^{\perp}$, then $(f,\ \gamma_{\Pi}(1-E_0)f_1)_{\Pi}=(g,\ E_0f_1)_{\Pi}=0$ for all $f_1\in K$ and hence $f\in E_0K_{\Pi}$ and $g=0$. Q. E. D.

§ 6. A Realization of a Quasifree State on a Fock Type Representation

Lemma 6.1. (1) *Let*

$$K_S'=K_S\oplus K_S, \tag{6.1}$$

$$\gamma_S'(f_1\oplus g_1,\ f_2\oplus g_2)=(f_1,\ \gamma_S f_2)_S-(g_1,\ \gamma_S g_2)_S, \tag{6.2}$$

$$\Gamma_S'=\Gamma_S\oplus\Gamma_S. \tag{6.3}$$

Then Γ_S' *is an antilinear involution and* γ_S' *is a hermitian form satisfying* $\gamma_S'(\Gamma_S'h_1,\ \Gamma_S'h_2)=-\gamma_S'(h_1,\ h_2)^*$. *If* $N_S=N_{S'}$ *and* $\tau_S=\tau_{S'}$, *then there exists a one-to-one linear map* U *of* K_S' *onto* $K_{S'}'$ *such that* $Uh=h$ *for* $h=(f+N_S)\oplus(g+N_S)$, $f,\ g\in K$. *It satisfies* $U\Gamma_S'=\Gamma_{S'}'U$ *and* $\gamma_S'(h_1,\ h_2)=\gamma_{S'}'(Uh_1,\ Uh_2)$.

(2) *Let*

$$\begin{aligned}(f_1\oplus g_1,\ f_2\oplus g_2)_S'=&(f_1,\ f_2)_S+(g_1,\ g_2)_S\\&+2(f_1,\ S^{1/2}(1-S)^{1/2}g_2)_S\\&+2(g_1,\ S^{1/2}(1-S)^{1/2}f_2)_S.\end{aligned} \tag{6.4}$$

Then it is a Γ_S'*-invariant positive semidefinite form satisfying*

$$|\gamma_S'(h_1,\ h_2)|\leqq\|h_1\|_S'\|h_2\|_S'. \tag{6.5}$$

The kernel N_S' (*i.e. the set of* h *satisfying* $\|h\|_S'=0$) *consists of* $f\oplus -f$, $f\in E_0K_S$. *If* $N_S=N_{S'}$ *and* $\tau_S=\tau_{S'}$, *then* $N_{S'}'=UN_S'$.

(3) (6.4), γ_S' *and* Γ_S' *induce on* K_S'/N_S' *a positive definite inner product* $(\hat{h}_1,\ \hat{h}_2)_S^{\wedge}\equiv(h_1,\ h_2)_S'$, *a hermitian form* $\hat{\gamma}_S(\hat{h}_1,\ \hat{h}_2)\equiv\gamma_S'(h_1,\ h_2)$ *and an antilinear involution* $\hat{\Gamma}_S\hat{h}\equiv(\Gamma_S h)^{\wedge}$ *satisfying* $(\hat{\Gamma}_S\hat{h}_1,\ \hat{\Gamma}_S\hat{h}_2)_S^{\wedge}=(\hat{h}_2,\ \hat{h}_1)_S^{\wedge}$ *and* $\hat{\gamma}_S(\hat{\Gamma}_S\hat{h}_1,\ \hat{\Gamma}_S\hat{h}_2)=-\hat{\gamma}_S(\hat{h}_2,\ \hat{h}_1)$ *where* $\hat{h}=h+N_S'\in K_S'/N_S'$. *The closure*

of $\hat{\gamma}_S$ *and* $\hat{\Gamma}_S$ *on the completion* $\hat{K}_S$ *of* K'_S/N'_S, *denoted by the same letter, satisfy the same properties.* $\hat{\Gamma}_S$ *is antiunitary and there exists an operator* $\hat{\gamma}_S$ *such that*

$$\hat{\gamma}_S(h_1, h_2)=(h_1, \hat{\gamma}_S h_2)_S^{\hat{}}, \tag{6.6}$$

$$\hat{\gamma}_S^*=\hat{\gamma}_S, \quad \hat{\Gamma}_S\hat{\gamma}_S\hat{\Gamma}_S=-\hat{\gamma}_S. \tag{6.7}$$

If $N_S=N_{S'}$ *and* $\tau_S=\tau_{S'}$, *then* U *of* (2) *induccs a one-to-one linear map of* $\hat{K}_S$ *onto* $\hat{K}_{S'}$ *such that* $\hat{U}\hat{\Gamma}_S=\hat{\Gamma}_{S'}\hat{U}$ *and* $\hat{\gamma}_{S'}(\hat{U}\hat{h}_1, \hat{U}\hat{h}_2)=\hat{\gamma}_S(\hat{h}_1, \hat{h}_2)$.

(4) *Let*

$$\Pi_S=(1/2)(1+\hat{\gamma}_S). \tag{6.8}$$

Then $\hat{\Gamma}_S\Pi_S\hat{\Gamma}_S=1-\Pi_S$, $\Pi_S^*=\Pi_S$ *and the spectrum of* Π_S *is contained in* $\{0, 1/2, 1\}$.

(5) *For* $f\in K$, *let* $[f]\equiv(\bar{f}\oplus 0)+N'_S$ *and identify* K'_S/N'_S *with a dense subset of* $\hat{K}_S$. *Then*

$$\hat{\gamma}_S([f], [g])=\gamma(f, g). \tag{6.9}$$

$$([f], \Pi_S[g])_S^{\hat{}}=S(f, g). \tag{6.10}$$

(6) *If* $N_S=N_{S'}$ *and* $\tau_S=\tau_{S'}$, *then* $\tau_{\Pi_S}=\tau_{\Pi_{S'}}$ *and eigenspaces of* Π_S *and* $\Pi_{S'}$ *for an eigenvalue* $1/2$ *are mapped by* $\hat{U}$.

Proof. (1) The properties of Γ'_S and γ'_S are immediate. Since K_S and $K_{S'}$ is the completion of $K/N_S=K/N_{S'}$ with respect to $\tau_S=\tau_{S'}$, there is a natural identification map U which is linear. If $f_j, g_j\in K$ and $h_j=(f_j+N_S)\oplus(g_j+N_S)$, then

$$\gamma'_S(h_1, h_2)=\gamma(f_1, f_2)-\gamma(g_1, g_2)=\gamma'_{S'}(h_1, h_2),$$

$$\Gamma'_S h_1=(\Gamma f_1+N_S)\oplus(\Gamma g_1+N_S)=\Gamma'_{S'}h_1.$$

Since such f_j and g_j are dense in K_S, these equalities imply $U\Gamma_S=\Gamma_{S'}U$ and $\gamma_S(h_1, h_2)=\gamma_{S'}(Uh_1, Uh_2)$.

(2) (6.4) is obviously a Γ'_S-invariant hermitian form. We have

$$(f\oplus g, f\oplus g)'_S=\|S^{1/2}f+(1-S)^{1/2}g\|_S^2+\|(1-S)^{1/2}f+S^{1/2}g\|_S^2\geqq 0. \tag{6.11}$$

We also have

$$\gamma'_S(f_1\oplus g_1, f_2\oplus g_2)=(S^{1/2}f_1+(1-S)^{1/2}g_1, S^{1/2}f_2+(1-S)^{1/2}g_2)_S \tag{6.12}$$

$$-((1-S)^{1/2}f_1+S^{1/2}g_1,\ (1-S)^{1/2}f_2+S^{1/2}g_2)_S$$

due to $\gamma_S=2S-1$, which implies

$$\begin{aligned}|\gamma'_S(f_1\oplus g_1, f_2\oplus g_2)| &\leqq \|S^{1/2}f_1+(1-S)^{1/2}g_1\|_S\|S^{1/2}f_2+(1-S)^{1/2}g_2\|_S \\ &\quad+\|(1-S)^{1/2}f_1+S^{1/2}g_1\|_S\|(1-S)^{1/2}f_2+S^{1/2}g_2\|_S \\ &\leqq \|f_1\oplus g_1\|'_S\|f_2\oplus g_2\|'_S.\end{aligned}$$

By (6.11), $\|f\oplus g\|'_S=0$ is equivalent to $(2S-1)f=0$ and $f+g=0$. Namely N_S consists of $f\oplus -f$, $f\in E_0K_S$. E_0K_S is the set of $f\in K_S$ such that $(f, \gamma_S g)_S=0$ for all $g\in K_S$. If $N_S=N_{S'}$ and $\tau_S=\tau_{S'}$, then there is a natural identification of K_S with $K_{S'}$ which identifies E_0K_S with $E'_0K_{S'}$ due to $(f, \gamma_S g)_S=\gamma(f, g)=(f, \gamma_{S'}g)_{S'}$ for $f, g\in K$. (E_0 and E'_0 are orthogonal eigenprojections of S and S' for an eigenvalue 1/2. Since the orthogonality refers to different inner product, E_0 and E'_0 need not be the same.) This implies $N'_{S'}=UN'_S$.

(3) Immediate from (1) and (2).

(4) Let $\hat{K}_S^+$, $\hat{K}_S^-$ and $\hat{K}_S^0$ be the subspace of $\hat{K}_S$ generated by $\{S^{1/2}f \oplus -(1-S)^{1/2}f\}^\wedge$, $\{(1-S)^{1/2}f\oplus -S^{1/2}f\}^\wedge$ and $\{E_0f\oplus E_0f\}^\wedge$, respectively, where f runs over K_S. It is easily seen that they are mutually orthogonal and altogether generate $\hat{K}_S$. For h_σ, $h'_\sigma\in\hat{K}_S^\sigma$ we have $\hat{\gamma}_S(h_\sigma, h'_{\sigma'})=\sigma\delta_{\sigma\sigma'}(h_\sigma, h'_{\sigma'})^\wedge_S$ where $\sigma=+$, $-$ or 0. Therefore $\hat{\gamma}_S h_\sigma=\sigma h_\sigma$ and the spectrum of Π_S is contained in $\{0, 1/2, 1\}$.

(5) Immediate from definitions.

(6) From the proof of (4) and the last part of the proof of (2), it follows that $\hat{K}_S^0$ for S and S' are mapped by U if $N_S=N_{S'}$ and $\tau_S=\tau_{S'}$.

The topology τ_{Π_S} is the strong topology of $\hat{K}_S$. Let $(f_\alpha\oplus g_\alpha)^\wedge$ be a Cauchy net relative to τ_{Π_S} where f_α, $g_\alpha\in K_S$. $S^{1/2}f_\alpha+(1-S)^{1/2}g_\alpha\equiv F_\alpha$ and $(1-S)^{1/2}f_\alpha+S^{1/2}g_\alpha\equiv G_\alpha$ are Cauchy in K_S. Therefore $f_\alpha+g_\alpha=\{S^{1/2}+(1-S)^{1/2}\}^{-1}(F_\alpha+G_\alpha)$ and $(2S-1)(f_\alpha-g_\alpha)=\{S^{1/2}+(1-S)^{1/2}\}(F_\alpha-G_\alpha)$ are Cauchy. Conversely, if $f_\alpha+g_\alpha$ and $(2S-1)(f_\alpha-g_\alpha)$ are Cauchy in K_S, then F_α and G_α are Cauchy and hence $(f_\alpha\oplus g_\alpha)^\wedge$ is Cauchy in $\hat{K}_S$.

If $N_S=N_{S'}$ and $\tau_S=\tau_{S'}$, then the properties of a net f_α being Cauchy relative to τ_S and $\tau_{S'}$ are the same. Furthermore, $\gamma_S=2S-1$

and $(f, \gamma_S g)_S=(f, \gamma_{S'}g)_{S'}$ imply that $(2S-1)g_\alpha$ is Cauchy relative to τ_S if and only if $(2S'-1)g_\alpha$ is Cauchy relative to $\tau_{S'}$ by the duality.

Combining above two sets of arguments, we see that $(f_\alpha \oplus g_\alpha)^\wedge$ is Cauchy relative to τ_{Π_S} if and only if $(f_\alpha \oplus g_\alpha)^\wedge$ is Cauchy relative to $\tau_{\Pi_{S'}}$. Q. E. D.

Corollary 6.2. *The map $f \in K \to [f] \in K_S$ inducces a $*$ homomorphism α_S of $\mathfrak{A}(K, \gamma, \Gamma)$ into $\mathfrak{A}(\hat{K}_S, \hat{\gamma}_S, \hat{\Gamma}_S)$. The restriction of a Fock typc state φ_{Π_S} of $\mathfrak{A}(\hat{K}_S, \hat{\gamma}_S, \hat{\Gamma}_S)$ to $\alpha_S\mathfrak{A}(K, \gamma, \Gamma)$ gives a quasifree state φ_S of $\mathfrak{A}(K, \gamma, \Gamma)$ through $\varphi_{\Pi_S}(\alpha_S A)=\varphi_S(A)$.*

This is immediate from Lemma 6.1.

Remark 6.3. It is possible to realize φ_S directly in a Fock representation in the following manner: Define $K'_S=K_S\oplus K_S$, $\Gamma'_S=\Gamma_S\oplus\Gamma_S$,

$$\gamma''_S(f_1\oplus g_1, f_2\oplus g_2)=(f_1, \gamma_S f_2)_S-(g_1, \gamma_S g_2)_S$$
$$+i\{(g_1, E_0 f_2)_S-(f_1, E_0 g_2)_S\}$$

and

$$(f_1\oplus g_1, f_2\oplus g_2)''_S=(f_1, f_2)_S+(g_1, g_2)_S$$
$$+2(f_1, (1-E_0)S^{1/2}(1-S)^{1/2}g_2)_S$$
$$+2(g_1, (1-E_0)S^{1/2}(1-S)^{1/2}f_2)_S. \tag{6.13}$$

Then (6.13) is positive definite and

$$|\gamma''_S(h_1, h_2)|\leqq\|h_1\|''_S\|h_2\|''_S.$$

Let K''_S be the completion of K'_S relative to $\|h\|''_S$, $\bar{\gamma}''_S$ and Γ''_S be the closure of γ''_S and Γ'_S, $\bar{\gamma}''_S(h_1, h_2)=(h_1, \gamma''_S h_2)''_S$ and $P_S=(\gamma''_S+1)/2$. Then P_S is a basis projection. Let α''_S be the $*$ homomorphism of $\mathfrak{A}(K, \gamma, \Gamma)$ into $\mathfrak{A}(K''_S, \bar{\gamma}''_S, \Gamma''_S)$ induced by $f\to\bar{f}\oplus 0$. Then the restriction of the Fock state φ_{P_S} of $\mathfrak{A}(K''_S, \bar{\gamma}''_S, \Gamma''_S)$ to $\alpha''_S\mathfrak{A}(K, \gamma, \Gamma)$ induces the quasifree state φ_S of $\mathfrak{A}(K, \gamma, \Gamma)$.

This method has a defect that a canonical identification map U can not be defined between K''_S, $\bar{\gamma}''_S$, Γ''_S and $K''_{S'}$, $\bar{\gamma}''_{S'}$, $\Gamma''_{S'}$ even if $N_S=N_{S'}$ and $\tau_S=\tau_{S'}$, due to the dependence of the operator E_0 on S.

Lemma 6.4. *Let φ_S be a quasifree state of $\mathfrak{A}(K, \gamma, \Gamma)$. The induced topology τ_{φ_S} on K is the same as τ_S of Definition* 4.1.

Proof. Denote $W_{\varphi_S}(f)$ by $W_S(f)$. Since Ω_S is cyclic for $\mathfrak{A}(K, \gamma, \Gamma)$ and $\pi_S(A)\Omega_S$, $A \in \mathfrak{A}(K, \gamma, \Gamma)$, is entire for $\overline{\pi_S(B(f))} = \lim_{t\to 0} \text{-}it^{-1}(W_S(tf) - 1)$, $f \in \text{Re}\, K$, Ω_S is cyclic for R_S.

By [5], it is known that τ_φ is a vector topology and is given by one distance $d_\Psi(f_1, f_2)$ for a cyclic Ψ. Therefore it is enough to show the equivalence of $\|f\|_S^2 \to 0$ and

$$d_{\Omega_S}(f, 0) = 2\{1 - \exp(-\|f\|_S^2/4)\} \to 0,$$

where (5.7) is used. This equivalence is obvious. Q. E. D.

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