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On Quasifree States of the Canonical Commutation Relations (II)

By

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Abstract

A necessary and sufficient condition for the quasi-equivalence of two quasifree primary representations of the canonical commutation relations is derived.

§ 1. Introduction

A quasifree state of the self-dual CCR algebra $\mathfrak{A}(K, \gamma, \Gamma)$, which is a slight generalization of conventional canonical commutation relations, has been discussed in the preceding work [1]. In the present paper, we derive a necessary and sufficient condition for the quasi-equivalence of representations associated with quasifree states, when the representation is primary (i.e. the associated von Neumann algebra is a factor).

We believe that the following features of the present analysis is worth mentioning.

(1) Despite of many marked differences on mathematical structure between the present case of CCR and the case of CAR [2], such as unbounded B(f) for CCR and bounded B(f) for CAR, the indefinite metric γ for the test function space K of CCR and the definite metric for K of CAR, and the difference in the details of the final statement, the two cases can be treated by essentially the same technique, yielding quite a similar results.

(2) For CAR, there is a unique C^* norm for the $*$ algebra generated by B(f). In the present case, there is no intrinsic topology in the $*$ algebra generated by B(f). As a result, the topology induced by the representation plays an important role and serves as an invariant in the

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quasi-equivalence classification of quasifree states.

(3) On a Hilbert space of a definite metric, natural vector topologies are induced by the metric. On a Hilbert space of an indefinite metric, the natural topology is generally too weak. Indeed, it is the weak topology by its algebraic dual in the present case. It seems that a Hilbert space with an indefinite metric equipped with a certain class of vector topology is more canonical object to study. Problems concerning the structure (K, γ, Γ) and Hilbert Schmidt class operators on K, which we have treated with a help of ordinary tools on a Hilbert space with a definite metric, might serve as a testing ground for any general theory of a Hilbert space with an indefinite metric.

The study of quasifree states is probably of no direct physical interest. However, we know an example of the mathematical structure of free Bose gas analysed in [3], which turned out to be common to a large class of systems [4～7]. Our hope is that a complete analysis of CCR and CAR in the present paper and in [2] presents similar useful examples.

In section 2, we obtain simpler properties of the von Neumann algebra associated with quasi-free states. In section 3, a quasifree state is viewed as a KMS state relative to a Bogoliubov automorphism. In section 4, a bilinear Hamiltonian is introduced which is used in section 5 to discuss the unitary implementability of a Bogoliubov transformation on a Fock type representation. In section 6, a necessary and sufficient condition for the quasi-equivalence of two quasifree primary states of CCR is obtained as the main Theorem.

We shall freely use the notation in [1].

§ 2. Simple Properties of von Neumann Algebras Associated with Quasifree States

Lemma 2.1. *Let ω' be a mapping of K'_S onto itself given by*

$$\omega'(f \oplus g) = \Gamma_S g \oplus \Gamma_S f. \tag{2.1}$$

ω' leaves N'_S invariant and induces a mapping ω of K'_S/N'_S onto itself. Its closure, denoted again by ω, as a mapping of $\hat{K}_S$ onto itself is an

antilinear involution satisfying.

$$[\omega, \hat{\Gamma}_S]=0, \tag{2.2}$$

$$(\omega h_1, \omega h_2)_S^{\wedge}=(h_2, h_1)_S^{\wedge}, \tag{2.3}$$

and

$$\hat{\gamma}_S(\omega h_1, \omega h_2)=\hat{\gamma}_S(h_2, h_1). \tag{2.4}$$

Let $\tau(\omega)$ *be a mapping of* $\mathfrak{A}(\hat{K}_S, \hat{\gamma}_S, \hat{\Gamma}_S)$ *onto itself given by*

$$\tau(\omega)\sum_i c_i \mathrm{B}(h_1^i)\cdots\mathrm{B}(h_{n_i}^i)=\sum_i c_i^* \mathrm{B}(\omega h_1^i)\cdots\mathrm{B}(\omega h_{n_i}^i). \tag{2.5}$$

Then it is a conjugate $*$ *automorphism. There exists an antiunitary involution* $\mathrm{T}_{\Pi_S}(\omega)$ *on* $\mathfrak{H}_{\Pi_S}$ *uniquely determined by*

$$\mathrm{T}_{\Pi_S}(\omega)\pi_{\Pi_S}(A)\Omega_{\Pi_S}=\pi_{\Pi_S}(\tau(\omega)A)\Omega_{\Pi_S}. \tag{2.6}$$

Proof. From the definition (2.1), if follows that ω' is an antilinear involution leaving N_S' invariant and hence the same is true for ω. (2.2) follows immediately from $[\omega', \Gamma_S']=0$. (2.3) follows from the antiunitarity of Γ_S relative to $(f, g)_S$ and (6.4) of [1]. (2.4) follows from (6.2) of [1] and $\Gamma_S\gamma_S\Gamma_S=-\gamma_S$.

By the antilinearity of ω, (2.2) and (2.4), $\tau(\omega)$ defined by (2.5) preserves the three relations in the definition of a selfdual CCR algebra (Section 2 in [1]) and hence $\tau(\omega)$ is a conjugate $*$ automorphism.

Due to (2.3) and (2.4), we have $\varphi_{\Pi_S}(\tau(\omega)A)=\varphi_{\Pi_S}(A)^*$, from which the existence of $\mathrm{T}_{\Pi_S}(\omega)$ follows. Q. E. D.

Lemma 2.2. *The set of* $h\in\hat{K}_S$ *satisfying* $\hat{\gamma}_S([f], h)=0$ *for all* $f\in\mathrm{Re}\,K$ *is* $\{E_0K_S\oplus K_S\}^\wedge$.

Proof. If $f\in\mathrm{Re}\,K$, $f_1\in E_0K_S$ and $g_1\in K_S$, then $\gamma_S'(f\oplus 0, f_1\oplus g_1)=0$. Hence $\hat{\gamma}_S([f], h)=0$ for $h\in\{E_0K_S\oplus K_S\}^\wedge$.

Let f, g be elements of $(1-E_0)K_S$ in the domain of $(2S-1)^{-1}$. Let

$$\begin{aligned}\mathrm{h}(f\oplus g)\equiv&(2S-1)^{-1}(f+2S^{1/2}(1-S)^{1/2}g)\\&\oplus-(2S-1)^{-1}(g+2S^{1/2}(1-S)^{1/2}f).\end{aligned} \tag{2.7}$$

Then we have

(2.8) $$\gamma'_S(f_1\oplus g_1,\ h(f_2\oplus g_2))=(f_1\oplus g_1,\ f_2\oplus g_2)'_S.$$

Since $\hat{\gamma}_S|[E_0K\oplus E_0K]=0$, since the restriction of $\hat{\gamma}_S^2$ to the orthogonal complement of $[E_0K\oplus E_0K]$ is 1 and since $[(1-E_0)K\oplus(1-E_0)K]$ is orthogonal to $[E_0K\oplus E_0K]$, we have from (2.8)

(2.9) $$\{\mathrm{h}(f_2\oplus g_2)\}^\wedge=\hat{\gamma}_S\{(f_2\oplus g_2)^\wedge\}.$$

Therefore $\hat{\gamma}_S\overline{[K]}$ contains vectors $\{\mathrm{h}(f\oplus 0)\}^\wedge$, where f is any element of $(1-E_0)K_S$ in the domain of $(2S-1)^{-1}$. Therefore $\hat{\gamma}_S\overline{[K]}+\{E_0K_S\oplus K_S\}^\wedge$ is dense in $\{K_S\oplus K_S\}^\wedge$ and hence in $\hat{K}_S$.

Since $(\|f\oplus g\|'_S)^2=\|f+g\|_S^2$ for $f\in E_0K_S$ and $g\in K_S$, $\{E_0K_S\oplus K_S\}^\wedge$ is closed. Therefore it is the orthogonal complement of $\hat{\gamma}_S\overline{[K]}$. Q.E.D.

Lemma 2.3. *Let R_S be the von Neumann algebra generated by spectral projections of all $\pi_{\Pi_S}(\alpha_S\mathrm{B}(f))$, $f\in\mathrm{Re}\,K$ on the representation space of $\mathfrak{A}(\hat{K}_S,\ \hat{\gamma},\ \hat{\Gamma}_S)$ associated with φ_{Π_S}. Then the following conditions are equivalent:*

(1) *Ω_{Π_S} is cyclic for R_S.*

(2) *Ω_{Π_S} is separating for R_S.*

(3) *S does not have an eigenvalue 0.*

(4) *S does not have an eigenvalue 1.*

Proof. Using the notation of Lemma 5.5 of [1], we have

$$R_S=R_{\tilde{\Pi}_S}(H_1)$$

with $H=\mathrm{Re}(\hat{K}_S\oplus\hat{E}_0\hat{K}_S)$, $H_1=[\mathrm{Re}\,K]\oplus 0$, $\bar{H}_1=\{\mathrm{Re}\,K_S\oplus 0\}^\wedge\oplus 0$ where $\hat{E}_0$ is the eigenprojection of Π_S for an eigenvalue 1/2. (Since $\|f\oplus 0\|'_S=\|f\|_S$, $\{K_S\oplus 0\}^\wedge$ is closed.) Since $\hat{E}_0\hat{K}_S\subset\{K_S\oplus 0\}^\wedge$, $\bar{H}_1^\perp\subset\mathrm{Re}\,\hat{K}_S$ and hence we obtain from Lemma 2.2 $H_1^\perp=\mathrm{Re}\{(E_0K_S\oplus K_S)^\wedge\oplus 0\}$

(1)→(3): Assume that (3) does not hold and $Sf=0$ for $f\in K_S$, $f\neq 0$. Then $\mathrm{h}(0\oplus f)=0\oplus f$ and hence $\Pi_S\{(0\oplus f)^\wedge\}=(0\oplus f)^\wedge$. Further $(0\oplus f,\ g\oplus 0)'_S=0$ for any $g\in K_S$. This implies $((0\oplus f)^\wedge\oplus 0,\ [g]\oplus 0)_{\tilde{\Pi}_S}=0$ due to $\hat{E}_0\ \{(0\oplus f)^\wedge\}=0$. Therefore $\tilde{\Pi}_S(H_1+iH_1)$ is not dense in $\tilde{\Pi}_S\hat{K}_S$ and Ω_{Π_S}, which can be identified with $\Omega_{\tilde{\Pi}_S}$, is not cyclic due to Lemma 5.5 (v) of [1].

(3)→(1): Let

(2.10) $$\mathrm{k}_+(f)=S^{1/2}f\oplus\{-(1-S)^{1/2}f\}$$

(2.11) $$\mathrm{k}_-(f)=\{-(1-S)^{1/2}f\}\oplus S^{1/2}f.$$

If $f\in(1-E_0)K_S$ is in the domain of $(2S-1)^{-1}$, we have

(2.12) $$\mathrm{h}(\mathrm{k}_\pm(f))=\pm\mathrm{k}_\pm(f).$$

Hence $\{\mathrm{k}_+(f)\}^\wedge\in\hat{E}_+\hat{K}_S$, $\{\mathrm{k}_-(f)\}^\wedge\in\hat{E}_-\hat{K}_S$, where $\hat{E}_+$ and $\hat{E}_-$ are eigen-projections of Π_S for eigenvalues 1 and 0. By the continuity, this holds for any $f\in K_S$. Further,

(2.13) $$\mathrm{k}_+(S^{1/2}f)+\mathrm{k}_-((1-S)^{1/2}f)=(2S-1)f\oplus 0,$$

(2.14) $$\mathrm{k}_-(S^{1/2}f)+\mathrm{k}_+((1-S)^{1/2}f)=0\oplus(2S-1)f.$$

Therefore the set of $\mathrm{k}_+(f)^\wedge+\mathrm{k}_-(g)^\wedge$ is dense in $(1-\hat{E}_0)\ \hat{K}_S$ and hence $\mathrm{k}_+(f)^\wedge$ and $\mathrm{k}_-(f)^\wedge$ are dense in $\hat{E}_+\hat{K}_S$ and $\hat{E}_-\hat{K}_S$, respectively.

Since

(2.15) $$\Pi_S\{(f\oplus 0)^\wedge\}=(1/2)(1+\hat{\gamma}_S)\{(f\oplus 0)^\wedge\}=\mathrm{k}_+((2S-1)^{-1}S^{1/2}f)^\wedge$$

for $f\in(1-E_0)K_S$ and since $f\in K_S\to\mathrm{k}_+(f)^\wedge\in\hat{K}_S$ is continuous, $\Pi_S\ \{(K_S\oplus 0)^\wedge\oplus 0\}=\tilde{\Pi}_S\{((1-E_0)K_S\oplus 0)^\wedge\oplus 0\}+\tilde{\Pi}_S(\hat{E}_0\hat{K}_S\oplus 0)$ is dense in $\tilde{\Pi}_S\tilde{K}_{\Pi_S}$ if (3) holds.

(3) $\rightleftarrows$ (4). This follows from $\Gamma_S S\Gamma_S=1-S$.

(4) $\rightleftarrows$ (2). By Lemma 5.5 (ii) of [1], Lemma 2.1 and Lemma 2.2

$$R_S'=T_{\Pi_S}(\omega)R_S\,T_{\Pi_S}(\omega).$$

Furthermore, we have $\varphi_{\Pi_S}(\tau(\omega)A^*)=\varphi_{1-S}(A)$ for $A\in\alpha_S\mathfrak{A}(K,\ \gamma,\ \Gamma)$. Therefore by (3) $\rightleftarrows$ (1), Ω_{Π_S} is cyclic for R_S' if and only if (4) holds. Since Ω_{Π_S} is separating for R_S if and only if it is cyclic for R_S', we have (4) $\rightleftarrows$ (2). Q. E. D.

Lemma 2.4. *The center of* R_S *is generated by* exp $i\pi_{\Pi_S}(\mathrm{B}(h))$, $h\in\mathrm{Re}\,(E_0K_S\oplus 0)^\wedge$. *In particular,* R_S *is a factor if and only if* $K_0=0$.

Proof. From the beginning part of the proof of Lemma 2.3, we have $\bar{H}_1\cap H_1^\perp=(E_0K_S\oplus 0)^\wedge\oplus 0$ and hence this Lemma follows from Lemma 5.5 of [1].

Lemma 2.5. *Ω_{Π_S} is separating for the center of R_S.*

Proof. Since $\{(\mathrm{Re}\, E_0 K_S \oplus 0)^\wedge \oplus 0\}^\perp = \mathrm{Re}\, \hat{K}_S \oplus 0$ and since $\tilde{\Pi}_S(\hat{K}_S \oplus 0)$ is dense in $\tilde{\Pi}_S \tilde{K}_{\Pi_S}$, we obtain the Lemma by Lemma 5.5 (vi) of [1].

Q. E. D.

§ 3. KMS Conditions

Lemma 3.1. *Suppose that S does not have an eigenvalue* 0. *Let*

$$H_S = \log\{S(1-S)^{-1}\}. \tag{3.1}$$

Then $\exp iH_S$ *is a Bogoliubov transformation on K_S and φ_S is $\tau(\exp iH_S)$ invariant.*

Proof. Since $0 < S < 1$, we have $H_S^* = H_S$. Since S commute with γ_S, we have $\gamma(e^{iH_S}f, e^{iH_S}g) = \gamma(f, g)$. Since $\Gamma_S S \Gamma_S = 1 - S$, we have $\Gamma_S H_S \Gamma_S = -H_S$. Therefore $\exp iH_S$ is a Bogoliubov transformation.

Since $S(f, g)$ is invariant under this transformation due to $[S, H_S] = 0$, φ_S is $\tau(\exp iH_S)$ invariant.

Q. E. D.

Definition 3.2. *Suppose that S does not have an eigenvalue* 0. *Let Θ_S be an infinitesimal generator defined by*

$$\exp it\Theta_S = T_S(\exp itH_S), \tag{3.2}$$

where $T_S(\cdot)$ is defined on $\mathfrak{H}_S$ by Definition 3.6 of [1].

Lemma 3.3. *Suppose that S does not have an eigenvalue* 0 *and identify $\mathfrak{H}_S$, $\pi_S(A)$ and Ω_S with $\mathfrak{H}_{\Pi_S}$, $\pi_{\Pi_S}(\alpha_S A)$ and Ω_{Π_S}. If $A \in R_S$ then*

$$\mathrm{T}_{\Pi_S}(\omega) A \Omega_{\Pi_S} = e^{-\Theta_S/2} A^* \Omega_{\Pi_S}. \tag{3.3}$$

Proof. Let $f \in D(S^{-1/2})$. Then

$$\begin{aligned} \mathrm{T}_{\Pi_S}(\omega)\pi_{\Pi_S}[\mathrm{B}((f\oplus 0)^\wedge)]\Omega_{\Pi_S} &= \pi_{\Pi_S}[\mathrm{B}((0\oplus \Gamma_S f)^\wedge)]\Omega_{\Pi_S} \\ &= \pi_{\Pi_S}[\mathrm{B}(((1-S)^{1/2}S^{-1/2}\Gamma_S f\oplus 0)^\wedge)]\Omega_{\Pi_S} \\ &= e^{-\Theta_S/2}\pi_{\Pi_S}[\mathrm{B}((f\oplus 0)^\wedge)]^*\Omega_{\Pi_S}, \end{aligned}$$

where we have used

$$\pi_{\Pi_S}[\mathrm{B}(k_-(S^{-1/2}\Gamma_S f)^\wedge)]\Omega_{\Pi_S}=0$$

not only for $f\in(1-E_0)K_S\cap D(S^{-1/2})$ but also for $f\in E_0K_S$. By using this result repeatedly, and by using the commutativity of $\mathrm{T}_{\Pi_S}(\omega)\pi_{\Pi_S}[\mathrm{B}((f\oplus 0)^\wedge)]\mathrm{T}_{\Pi_S}(\omega)$ with $e^{-\Theta_S/2}\pi_{\Pi_S}[\mathrm{B}((g\oplus 0)^\wedge)]e^{\Theta_S/2}$, we obtain

$$\begin{aligned}(3.4)\qquad &\mathrm{T}_{\Pi_S}(\omega)\pi_{\Pi_S}[\mathrm{B}((f_1\oplus 0)^\wedge)]\cdots\pi_{\Pi_S}[\mathrm{B}((f_n\oplus 0)^\wedge)]\Omega_{\Pi_S}\\ &=e^{-\Theta_S/2}\pi_{\Pi_S}[\mathrm{B}((f_n\oplus 0)^\wedge)]^*\cdots\pi_{\Pi_S}[\mathrm{B}((f_1\oplus 0)^\wedge)]^*\Omega_{\Pi_S}.\end{aligned}$$

If the support of the Fourier transform of $e^{iH_St}f$ is in $[-l, l]$, then $e^{-\Theta_S/2}$ is bounded by $e^{-nl/2}$ on $\pi_{\Pi_S}[\mathrm{B}((f\oplus 0)^\wedge)]^n\Omega_{\Pi_S}$. By the estimate (5.5) of [1], we have the convergence of

$$(3.5)\qquad e^{-\Theta_S/2}\mathrm{W}_S(f)\,\Omega_{\Pi_S}=\sum_{n=0}^{\infty}n!^{-1}i^n e^{-\Theta_S/2}\pi_{\Pi_S}[\mathrm{B}((f\oplus 0)^\wedge)]^n\Omega_{\Pi_S}$$

where f is assumed to be in $\operatorname{Re}K_S$ and $\mathrm{W}_S(f)$ denotes $\mathrm{W}_{\varphi_S}(f)=\mathrm{W}_{\tilde{\Pi}_S}(\{(f\oplus 0)^\wedge\oplus 0\})$. Therefore

$$(3.6)\qquad \mathrm{T}_{\Pi_S}(\omega)\mathrm{W}_S(f)\Omega_{\Pi_S}=e^{-\Theta_S/2}\mathrm{W}_S(-f)\Omega_{\Pi_S}.$$

A linear combination of $W_S(f)$ such that $e^{iH_St}f$ has a Fourier transform with a compact support, is dense in R_S. Therefore

$$(e^{-\Theta_S/2}\Psi,\ A^*\Omega_{\Pi_S})=(\Psi,\ T_{\Pi_S}(\omega)A\Omega_{\Pi_S})$$

holds for any $A\in R_S$ and Ψ in the domain of $e^{-\Theta_S/2}$. This implies that $A^*\Omega_{\Pi_S}\in \mathrm{D}(e^{-\Theta_S/2})$ and (3.3) holds. Q. E. D.

Corollary 3.4. *φ_S is a KMS state of $(K_S, \gamma_S, \Gamma_S)$ for the automorphism $\tau(\exp itH_S)$.*

Proof. This follows from the antiunitarity of $\mathrm{T}_{\Pi_S}(\omega)$.∎

Corollary 3.5. *Let $j(A)=T_{\Pi_S}(\omega)A\mathrm{T}_{\Pi_S}(\omega)$ for $A\in R_S$. Then $(\Omega_{\Pi_S}, Aj(A)\Omega_{\Pi_S})\geqq 0$.*

§ 4. Bilinear Hamiltonian

Lemma 4.1. *Assume that γ is non-degenerate. Let K_1 be finite dimensional subspace of K. Then there exists a Γ-invariant finite dimensional* *subspace* K_1' *of* K *such that* $K_1' \supset K_1$, *and the restriction of the hermitian form* γ *to* K_1' *is nondegenerate. Further, there exists a basis projection* P *for* (K_1', γ, Γ).

Proof. Let $f_1 \cdots f_n$ be a linearly independent basis of K_1. Let $K_2 = K_1 + \Gamma K_1$ and f_j' be a maximal linearly independent subset of $f_k + \Gamma f_k$, $i(f_k - \Gamma f_k)$, $k=1, \ldots, n$. f_j' is a complete set of linearly independent Γ invariant vectors of K_2.

Since f_j' is invariant, $\gamma(f_{j_1}', f_{j_2}')^* = \gamma(f_{j_2}', f_{j_1}') = \gamma(\Gamma f_{j_2}', \Gamma f_{j_1}') = -\gamma(f_{j_1}', f_{j_2}')$ and hence $\gamma(f_{j_1}', f_{j_2}')$ is purely imaginary. In particular, $\gamma(f_j', f_j') = 0$.

If γ restricted to K_2 is not identically 0, let f_{j_1}' and f_{j_2}' be a pair such that $\gamma(f_{j_1}', f_{j_2}') = 0$ and set $e_1 = f_{j_1}'$, $e_2 = i[\gamma(f_{j_1}', f_{j_2}')]^{-1} f_{j_2}'$, $f_j'' = f_j' - i\gamma(e_2, f_j')e_1 + i\gamma(e_1, f_j')e_2$. Then e_1 and e_2 are Γ invariant, $\gamma(e_1, e_2) = i$ and $\gamma(e_1, f_j'') = \gamma(e_2, f_j'') = 0$. Apply the same procedure to f_j''. Repeat this process until we obtain $e_{2k-1} e_{2k}$, $k=1 \cdots l$ and $f_j^{(l+1)}$, $j > 2l$, such that $\gamma(f_j^{(l+1)}, f_{j'}^{(l+1)}) = \gamma(e_j, f_{j'}^{(l+1)}) = 0$, $\gamma(e_j, e_{j'}) = 0$ unless $(j, j') = (2k-1, 2k)$ or $(2k, 2k-1)$ and $\gamma(e_{2k-1}, e_{2k}) = i$, $k=1, \ldots, l$.

Next, let g_k, $k=1 \cdots s$ be a maximal linearly independent subset of $f_j^{(l+1)}$. Let $h_1 \in K$ such that $\gamma(g_1, h_1) \neq 0$. Let h_1' be either one of $\Gamma h_1 + h_1$ and $i(h_1 - \Gamma h_1)$ such that $\gamma(g_1, h_1') \neq 0$. Let $e_{2l+1} = h_1' - i \sum_{k=1}^{l} \{\gamma(e_{2k}, h_1') e_{2k-1} - \gamma(e_{2k-1}, h_1') e_{2k}\}$. We have $\gamma(e_{2l+1}, g_1) = \gamma(h_1', g_1) \neq 0$. Let $e_{2l+2} = i\gamma(e_{2l+1}, g_1)^{-1} g_1$ and $g_k' = g_k - i\gamma(e_{2l+2}, g_k) e_{2l-1} + i\gamma(e_{2l+1}, g_k) e_{2l+2}$. Next apply this procedure to g_k', $k=2, \ldots, s$. After repeating this process s times, we obtain e_j, $j=1, \ldots, 2l+2s$ such that $\Gamma e_j = e_j$, $\gamma(e_j, e_{j'}) = 0$ unless $(j, j') = (2k-1)$, or $(2k, 2k-1)$ and $\gamma(e_{2k-1}, e_{2k}) = i$, $k=1, \ldots, l+s$. Further, the subspace K_1' of K generated by $e_1 \cdots e_{2l+2s}$ and the projection P defined by $Pf = i \sum_k \{\gamma(e_{2k}, f) e_{2k-1} - \gamma(e_{2k-1}, f) e_{2k}\}$ have the desired properties. Q. E. D.

Remark. We see from the above proof that if K has a finite dimension, then dim K is even and there exists a basis projection P for (K, γ, Γ).

Lemma 4.2. *If* γ *is non-degenerate,* $\mathfrak{A}(K, \gamma, \Gamma)$ *is simple (as a*

$*$ algebra) and has a trivial center.

Proof. Let $\mathfrak{J}$ be a non-zero two sided $*$ ideal of $\mathfrak{A}(K, \gamma, \Gamma)$ and A be a non-zero element of $\mathfrak{J}$. A is a polynomial of $\mathrm{B}(f_1), \cdots, \mathrm{B}(f_n)$ for a finite number of $f_1 \cdots f_n \in K$. Let K_1 be the subspace of K generated by $f_1 \cdots f_n$ and apply the preceding Lemma. A can be written as

$$A = \sum_{n=0}^{N} \mathrm{B}(e_1)^n \mathscr{P}_n, \tag{4.1}$$

where $\mathscr{P}_n$ is a polynomial of $\mathrm{B}(e_k)$, $k=2, 3, \cdots$. We may assume that $\mathscr{P}_N \neq 0$ if $A \neq 0$. Let $A_1 = \mathscr{P}_0$ if $N=0$ and $A_1 = [\mathrm{B}(e_2), [\cdots[\mathrm{B}(e_2), A]\cdots]]$ (the N fold commutator with $\mathrm{B}(e_2)$) if $N>0$. Then $A_1 \in \mathfrak{J}$, $A_1 = (-i)^N N! \mathscr{P}_N \neq 0$, and A_1 no longer contains $\mathrm{B}(e_1)$. Repeating this process, we see that $1 \in \mathfrak{J}$ and hence $\mathfrak{J} = \mathfrak{A}(K, \gamma, \Gamma)$. Namely $\mathfrak{A}(K, \gamma, \Gamma)$ is simple as a $*$ algebra.

Next, let A be a central element of $\mathfrak{A}(K, \gamma, \Gamma)$ and (4.1) holds. Assume that $\mathscr{P}_N \neq 0$. If $N \neq 0$, then we have $P_N = (N!)^{-1} i^N [\mathrm{B}(e_2), [\cdots [\mathrm{B}(e_2), A]\cdots] = 0$, which is a contradiction. Hence $N=0$. Repeating this reasoning, we see that A must be a multiple of the identity operator.

Q. E. D.

Definition 4.3. *Assume that γ is non-degenerate. Let H be a finite rank operator on K satisfying*

$$Hf = \sum_{j=1}^{N} \gamma(g_j, f) f_j \tag{4.2}$$

for any $f \in K$. Then $(\mathrm{B}, H\mathrm{B}) \in \mathfrak{A}(K, \gamma, \Gamma)$ is defined by

$$(\mathrm{B}, H\mathrm{B}) = \sum_{j=1}^{N} \mathrm{B}(f_j) \mathrm{B}(\Gamma g_j). \tag{4.3}$$

Remark. An operator H on K is called a finite rank operator if its domain is K and its range has a finite dimension. Any finite rank operator H can be written as $Hf = \sum_{j=1}^{n} e_j(f) f_j$ for all $f \in K$, where $f_j \in K$ and e_j is in the algebraic dual of K. The trace of H is then defined by

$$\mathrm{tr}\, H = \sum_{j=1}^{n} e_j(f_j) \tag{4.4}$$

and is independent of the choice of f_j and e_j for a given H. If H is of finite rank, then AH is of finite rank for any linear operator A defined on the whole K, and if H is given by (4.2), then

$$\text{tr } AH = \sum_{j=1}^{N} \gamma(g_j, Af_j). \tag{4.5}$$

Lemma 4.4. *Assume that γ is non-degenerate.* (B, HB) *is independent of the choice of f_j and g_j for a given H, is linear in H and satisfies*

$$[(\mathrm{B}, H\mathrm{B}), \mathrm{B}(f)] = \mathrm{B}(Hf) - \mathrm{B}(\Gamma H^{\dagger}\Gamma f), \tag{4.6}$$

$$[(\mathrm{B}, H'\mathrm{B}), (\mathrm{B}, H\mathrm{B})] = 2(\mathrm{B}, [\alpha(H'), \ \alpha(H)]\mathrm{B}), \tag{4.7}$$

$$\varphi_S((\mathrm{B}, H\mathrm{B})) = \sum_j \mathrm{S}(\Gamma f_j, \Gamma g_j), \tag{4.8}$$

$$(\mathrm{B}, H\mathrm{B})^* = (\mathrm{B}, H^{\dagger}\mathrm{B}), \tag{4.9}$$

where $H^{\dagger}$ is defined by

$$\gamma(f, H^{\dagger} g) = \gamma(Hf, g), \tag{4.10}$$

which is equivalent to

$$H^{\dagger} f = \sum_{j=1}^{N} \gamma(f_j, f) g_j \tag{4.11}$$

if H satisfies (4.2), *and*

$$\alpha(H) = (1/2)(H - \Gamma H^{\dagger}\Gamma), \tag{4.12}$$

which satisfies

$$\Gamma\alpha(H)^{\dagger}\Gamma = -\alpha(H), \tag{4.13}$$

$$(\mathrm{B}, H\mathrm{B}) = (\mathrm{B}, \alpha(H)\mathrm{B}) - (1/2)\text{tr } H. \tag{4.14}$$

Conversely, if H satisfies

$$\Gamma H^{\dagger}\Gamma = -H, \tag{4.15}$$

then

$$\alpha(H) = H. \tag{4.16}$$

Proof. (4.11) obviously satisfies (4.10) and (4.10) uniquely specifies $H^{\dagger}$ by the non-degeneracy of γ. (4.9) follows immediately from (4.11). We have

$$[(\mathrm{B}, H\mathrm{B}), \mathrm{B}(f)] = \sum_{j=1}^{N} \mathrm{B}(f_j)\gamma(g_j, f) + \gamma(\Gamma f_j, f)\mathrm{B}(\Gamma g_j) = \mathrm{B}(Hf) - \mathrm{B}(\Gamma H^{\dagger}\Gamma f) \tag{4.17}$$

where we have used the equality

$$\Gamma H^{\dagger}\Gamma f = \Gamma \sum_{j=1}^{n} g_j\, \gamma(f_j, \Gamma f) = \sum_{j=1}^{n} \Gamma g_j \gamma(f_j, \Gamma f)^* = -\sum_{j=1}^{n} \Gamma g_j \gamma(\Gamma f_j, f). \tag{4.18}$$

Further, we have (4.8) by (3.3) of [1] and (4.3). Suppose that the same H can be written as in (4.2) in two different ways in terms of f_j, g_j and f'_j, g'_j. Let K_1 be the subspace spanned by f_j, g_j, f'_j, g'_j and K'_1 be as given by Lemma 4.1 Then (B, HB) given by (4.3) can be considered as an element of $\mathfrak{A}(K_1, \gamma, \Gamma)$. Since K_1 has a finite dimension, $(K_1)_S = K_1$, γ_S^{-1} is bounded and the right hand side of (4.8) can be written as

$$\varphi_S((\mathrm{B}, H\mathrm{B})) = \mathrm{tr}\{\gamma_S^{-1}(1-S)H\}, \tag{4.19}$$

which is independent of the choice of f_j and g_j. Since the center of $\mathfrak{A}(K, \gamma, \Gamma)$ is trivial, (4.6) implies that (B, HB) is independent of the choice of f_j, g_j for a given H up to a possible addition of a multiple of an identity and (4.19) then proves that (B, HB) is independent of the choice of f_j, g_j for a given H.

Since

$$\gamma(g, (\Gamma H^{\dagger}\Gamma)^{\dagger} f) = \gamma(\Gamma H^{\dagger}\Gamma g, f) = -\gamma(\Gamma f, H^{\dagger}\Gamma g) = -\gamma(H\Gamma f, \Gamma g) = \gamma(g, \Gamma H \Gamma f),$$

we have $(\Gamma H^{\dagger}\Gamma)^{\dagger} = \Gamma H \Gamma$ and hence (4.13) follows from (4.12). By (4.18), we have

$$(\mathrm{B}, \Gamma H^{\dagger}\Gamma \mathrm{B}) = -\sum \mathrm{B}(\Gamma g_j)\mathrm{B}(f_j) = -(\mathrm{B}, H\mathrm{B}) - \mathrm{tr}\, H. \tag{4.20}$$

Therefore we have (4.14).

By (4.11) we have

$$(\mathrm{B}, H\mathrm{B})^* = \sum_j \mathrm{B}(g_j)\mathrm{B}(\Gamma f_j) = (\mathrm{B}, H^{\dagger}\mathrm{B}).$$

By (4.6), we have

$$[(\mathrm{B}, H'\mathrm{B}), (\mathrm{B}, H\mathrm{B})] = 2\sum \{\mathrm{B}(\alpha H')f_j)\mathrm{B}(\Gamma g_j) + \mathrm{B}(f_j)\mathrm{B}(\alpha(H')\Gamma g_j)\}.$$

By (4.13), we have $\alpha(H')\Gamma g_j = -\Gamma\alpha(H')^\dagger g_j$ and hence

$$[(\mathrm{B}, H'\mathrm{B}), (\mathrm{B}, H\mathrm{B})] = 2(\mathrm{B}, \{\alpha(H')H - H\alpha(H')\}\mathrm{B}).$$

By (4.14), this is the same as $2(\mathrm{B}, [\alpha(H'), \alpha(H)]\mathrm{B})$. Q. E. D.

Remark 4.5. (B, HB) defined by (4.3) and $H^\dagger$ defined by (4.11) are not uniquely determined by H for a general γ.

Lemma 4.6. *For any choice of f_j and g_j satisfying* (4.2), *the formulae* (4.6), (4.7) *and* (4.8) *hold*, $H^\dagger$ *defined by* (4.11) *satisfies* (4.8) *and* (4.10), *and $\alpha(H)$ defined by* (4.12) *satisfies* (4.14). *If $\gamma(f, Hg) = \gamma(Hf, g)$ for all f and g in K, then there exists a choice of f_j and g_j such that $H^\dagger$ defined by* (4.11) *coincides with H.*

Proof. First half follows from the computation in the proof of Lemma 4.4. For the second half, assume that $\gamma(f, Hg) = \gamma(Hf, g)$ for all f and g in K and H is expressed as $Hf = \sum_{j=1}^{N'} \gamma(g'_j, f)f'_j$. Let $H_1 = H - H^\dagger$, where $H^\dagger$ is defined by (4.11) using f'_j and g'_j. From (4.10) and the assumption, $\gamma(H_1 f, g) = 0$ for all f and g. Namely, the range of H_1 is in the Null space of γ (the set of f such that $\gamma(f, g) = 0$ for all $g \in K_1$). Then H_1 has a representation $H_1 f = \sum_{j=1}^{N''} \gamma(g''_j, f)f''_j$ where f''_j is in the Null space of γ. Then $H_1^\dagger$ defined by (4.11) using f''_j and g''_j is 0 as operator due to $\gamma(f''_j, f) = 0$ for all f. Hence we have

$$Hf = \sum_{j=1}^{N'} \gamma(g'_j, f)f'_j + \sum_{j=1}^{N''} \gamma(g''_j, f)f''_j.$$

$H^\dagger$ using this representation is H. Q. E. D.

Lemma 4.7. *Let φ_Π be a Fock type state. Let E_0 be the eigenprojection of Π for an eigenvalue* 1/2. *Let H be a finite rank operator on K_Π such that $E_0 H = HE_0 = 0$. Then H can be represented by* (4.2) *with f_j and g_j in $(1-E_0)K_\pi$ and* (B, HB) *defined by* (4.3) *and $H^\dagger$ defined by* (4.11) *do not depend on the choice of such f_j and g_j. If $\gamma_\Pi(f, Hg) = \gamma_\Pi(Hf, g)$ for all f and g, then $H^\dagger = H$.*

Proof. Since $H = (1-E_0)H(1-E_0)$, we may restrict our attention to $(1-E_0)K$, where γ is non-degenerate. Hence the present Lemma follows from Lemma 4.4. Q. E. D.

§ 5. Unitarily Implementable Bogoliubov Automorphisms

Lemma 5.1. *Let H be a finite rank operator such that $\gamma(f, Hg)=\gamma(Hf, g)$ and $\Gamma H\Gamma=-H$. Fix f_j and g_j in* (4.2) *such that $H^\dagger=H$ and define the corresponding* (B, HB). *Let φ_Π be a Fock type state of (K, γ, Γ). Then $D_0=\pi_\Pi[\mathfrak{A}(K, \gamma, \Gamma)]\Omega_\Pi$ is a dense set of analytic vectors for $\pi_\Pi[(\mathrm{B}, H\mathrm{B})]$. The unitary operator*

$$Q_\Pi(H)=\exp(i/2)\bar{\pi}_\Pi[(\mathrm{B}, H\mathrm{B})] \tag{5.1}$$

satisfies

$$Q_\Pi(H)W_\Pi(f)Q_\Pi(H)^*=W_\Pi(e^{iH}f) \tag{5.2}$$

for $f\in\mathrm{Re}\,K$ and

$$Q_\Pi(H)\bar{\pi}_\Pi(A)Q_\Pi(H)^*=\bar{\pi}_\Pi(\tau(e^{iH})A) \tag{5.3}$$

for $A\in\mathfrak{A}(K, \gamma, \Gamma)$, where $\bar{\pi}_\Pi(A)$ denotes the closure of $\pi_\Pi(A)$.

Proof. We shall use Lemma 5.8 of [1] and identify $\pi_{\tilde{\Pi}}(f\oplus 0)$ and $\Omega_{\tilde{\Pi}}$ with $\pi_\Pi(f)$ and Ω_Π. By Lemma 5.5 (v) of [1], it is easily seen that $\Omega_{\tilde{\Pi}}$ is cyclic for $\pi_\Pi(\mathfrak{A}(K, \gamma, \Gamma))$ and hence we may also identify the whole space $\mathfrak{H}_{\tilde{\Pi}}$ with $\mathfrak{H}_\Pi$. It follows that D_0 is dense.

From (5.5) of [1] we obtain

$$\|\pi_\Pi[(\mathrm{B}, H\mathrm{B})]\Psi\|\leqq[(N+2)(N+1)]^{1/2}G\|\Psi\| \tag{5.4}$$

for a constant G independent of N, where $\Psi\in\sum_{n=0}^{N}(\mathfrak{H}_{\tilde{\Pi}})_n$. Since $\pi_\Pi[(\mathrm{B}, H\mathrm{B})]$ increases N at most by 2, we have

$$\|\pi_\Pi[(\mathrm{B}, H\mathrm{B})]^n\Psi\|\leqq[(N+2n)!N!^{-1}]^{1/2}G^n\|\Psi\|. \tag{5.5}$$

Therefore, $\sum n!^{-1}t^n\|\pi_\Pi[(\mathrm{B}, H\mathrm{B})]^n\Psi\|<\infty$ for small $t>0$ and such Ψ is analytic for $\pi_\Pi[(\mathrm{B}, H\mathrm{B})]$. Since $(\mathrm{B}, H\mathrm{B})^*=(\mathrm{B}, H\mathrm{B})$ by (4.9), the closure $\bar{\pi}_\Pi[(\mathrm{B}, H\mathrm{B})]$ is selfadjoint and

$$Q_\Pi(tH)\Psi=\sum_{n=0}^{\infty}n!^{-1}\{(it/2)\pi_\Pi[(\mathrm{B}, H\mathrm{B})]\}^n\Psi \tag{5.6}$$

for sufficiently small t.

From (5.5) of [1] and (5.4), we also have the convergence of

$$\sum_{n=0}^{\infty}n!^{-1}\pi_\Pi(\mathrm{B}(f))\{(it/2)\pi_\Pi[(\mathrm{B}, H\mathrm{B})]\}^n\Psi \tag{5.7}$$

for small t. Therefore $Q_\Pi(tH)\Psi$, $\Psi \in D_0$ is in the domain of $\bar{\pi}_\Pi(\mathrm{B}(f))$ for small t and (5.7) gives $\bar{\pi}_\Pi(\mathrm{B}(f))Q_\Pi(tH)$.

From the absolute convergence of (5.6) and (5.7) for small t, we have absolute convergence of

$$(5.8) \qquad (Q_\Pi(tH)\Psi, \bar{\pi}_\Pi(\mathrm{B}(f))Q_\Pi(tH)\Phi)$$
$$= \sum_{n,m} (n!m!)^{-1} i^{n-m}(t/2)^{n+m}(\Psi, \pi_\Pi[(\mathrm{B}, H\mathrm{B})^m \mathrm{B}(f)(\mathrm{B}, H\mathrm{B})^n]\Phi)$$

for $\Phi, \Psi \in D_0$ and small t. By re-ordering the summation, we obtain

$$(5.9) \qquad \sum_n n!^{-1}(-it/2)^n(\Psi, \pi_\Pi\{\underbrace{[(\mathrm{B}, H\mathrm{B}), [\cdots[(\mathrm{B}, H\mathrm{B})}_{n}, \mathrm{B}(f)]\cdots]]\}\Phi)$$
$$= \sum_n n!^{-1}(-it)^n(\Psi, \pi_\Pi[\mathrm{B}(H^n f)]\Phi).$$

Since $(\Psi, \pi_\Pi(g)\Phi)$ is continuous in g when g is restricted to a finite dimensional subspace, (5.9) becomes

$$(\Psi, \pi_\Pi[\mathrm{B}(\sum_n n!^{-1}(-itH)^n f)]\Phi) = (\Psi, \pi_\Pi[\mathrm{B}(e^{-itH}f)]\Phi).$$

Therefore we have

$$(5.10) \qquad Q_\Pi(tH)^* \bar{\pi}_\Pi[\mathrm{B}(f)]Q_\Pi(tH)\Phi = \pi_\Pi[\Psi(e^{-itH}f)]\Phi$$

for small t and all $\Phi \in D_0$. Since a unitary transform of a selfadjoint operator is selfadjoint and $\pi_\Pi[\mathrm{B}(e^{-itH}f)]$ is essentially selfadjoint on D_0, we have

$$(5.11) \qquad Q_\Pi(tH)^* \bar{\pi}_\Pi[\mathrm{B}(f)]Q_\Pi(tH) = \pi_\Pi[\mathrm{B}(e^{-itH}f)],$$

for sufficiently small t. By using (5.11) repeatedly, we obtain (5.3) for a general t and for a general A.

From (5.2) of [1] and (5.3), we obtain (5.2). Q. E. D.

Lemma 5.2. *Assume that γ is non-degenerate. Let P be a basis projection and $K = K_P$. Then*

$$(5.12) \qquad (\Omega_P, Q_P(tH)\Omega_P) = \det{}_P(Pe^{-itH}P)^{-1/2},$$

where $\det_P$ *is the determinant of PK and the branch of the square root is determined by the continuity in t.*

Proof. Let

$$f(t)=(\Omega_P,\ Q_P(tH)\Omega_P). \tag{5.13}$$

Then

$$(2/i)f'(t)=(\Omega_P,\ Q_P(tH)\pi_P((\mathrm{B},\ H\mathrm{B}))\Omega_P). \tag{5.14}$$

We decompose f_j as

$$f_j=f'_j+f''_j, \tag{5.15}$$

$$f'_j=e^{-itH}(Pe^{-itH}P)^{-1}Pf_j, \tag{5.16}$$

$$f''_j=f_j-f'_j\in(1-P)K. \tag{5.17}$$

The operator $C_t\equiv Pe^{-itH}P$ is a bounded operator on PK_P, because a finite rank operator H is bounded on PK_P. Since $\gamma(f,\ g)=(f,\ g)_P$ for $f,\ g\in PK$, we have

$$C_t^*C_t=C_t^\dagger C_t=Pe^{itH}Pe^{-itH}P=PP'P \tag{5.18}$$

where

$$P'=e^{itH}Pe^{-itH} \tag{5.19}$$

is another basis projection. Since

$$(f,\ P(1-P')Pf)_P=\gamma(Pf,\ (1-P')Pf)\leqq 0, \tag{5.20}$$

we have

$$PP'P=P-P(1-P')P\geqq P. \tag{5.21}$$

Therefore $(Pe^{-itH}P)^{-1}$ is also bounded on PK_P and

$$\infty>|\det_P(Pe^{-itH}P)^{-1}|=\det_P(PP'P)^{-1/2}\neq 0. \tag{5.22}$$

Therefore f'_j and f''_j are well defined and we have

$$\pi_P(\mathrm{B}(f'_j))^*Q_P(tH)^*\Omega_P=Q_P(tH)^*\pi_P\,(e^{itH}f'_j)^*\Omega_P=0, \tag{5.23}$$

$$\pi_P(\mathrm{B}(f''_j))\Omega_P=0. \tag{5.24}$$

Therefore

$$f'(t)=(i/2)k(t)f(t), \tag{5.25}$$

$$k(t)1=\sum_{j=1}^{N}[\mathrm{B}(f''_j),\ \mathrm{B}(\Gamma g_j)] \tag{5.26}$$

$$= -\sum_{j=1}^{N} \gamma(g_j, f_j'')$$
$$= -\text{tr } H(1-P)\{1-e^{-itH}(Pe^{-itH}P)^{-1}P\}.$$

From $\Gamma H^{\dagger}\Gamma = -H$, we have

$$\text{(5.27)} \qquad \text{tr } H(1-P) = \text{tr } H\Gamma P\Gamma = \sum \gamma(g_j, \Gamma P \Gamma f_j)$$
$$= -\sum \gamma(\Gamma f_j, P\Gamma g_j) = \text{tr } ((\Gamma H^{\dagger}\Gamma)P) = -\text{tr } HP.$$

Hence

$$\text{(5.28)} \qquad k(t) = \text{tr } HP + \text{tr } H(1-P)e^{-itH}(Pe^{-itH}P)^{-1}P$$
$$= \text{tr } He^{-itH}(Pe^{-itH}P)^{-1}P$$
$$= i\frac{d}{dt}\text{tr }_P\log(Pe^{-itH}P)$$

where the trace is taken on PK. Substituting (5.28) into (5.25), we obtain

$$\text{(5.29)} \qquad f(t) = \exp -(1/2)\text{tr }_P\log(Pe^{-itH}P) = \det_P(Pe^{-itH}P)^{-1/2}. \qquad \text{Q. E. D.}$$

Corollary 5.3. *Let φ_Π be a Fock type state such that $K = K_\Pi$. Let H be a finite rank operator satisfying $\gamma(Hf, g) = \gamma(f, Hg)$, $\Gamma H \Gamma = -H$ and $E_0 H = HE_0 = 0$. Then*

$$\text{(5.30)} \qquad (\Omega_\Pi, Q_\Pi(tH)\Omega_\Pi) = \det_{E_+}(E_+ e^{-itH}E_+)^{-1/2}$$

where E_+ is the eigenprojection of Π for an eigenvalue 1.

Proof. Immediate from Lemmas 4.7 and 5.2.

Lemma 5.4. *Let P_1 and P_2 be basis projections for (K, γ, Γ). γ is necessarily non-degenerate. Assume that K is complete with respect to $\|f\|_{P_j} = \gamma(f, (2P_j - 1)f)^{1/2}$, $j = 1, 2$, and that the topologies by these two norms are the same.*

(1) *Let $\theta(P_1, P_2)$ be a non-negative selfadjoint operator on K satisfying*

$$\text{(5.31)} \qquad [\sinh \theta(P_1, P_2)]^2 f = -(P_1 - P_2)^2 f.$$

Such $\theta(P_1, P_2)$ exists, is bounded and commutes with P_1, P_2, $\gamma_{P_1} = 2P_1 - 1$, $\gamma_{P_2} = 2P_2 - 1$ and Γ. $\theta(P_1, P_2)$ and $\theta(P_2, P_1)$ are the same operator.

(2) *Let* $v_2=\sinh\theta(P_1, P_2)$, $v_1=\cosh\theta(P_1, P_2)$ *and* $u_{ij}(P_1/P_2)$ *be bounded operators on* K *satisfying*

$$u_{11}(P_1/P_2)=P_1F_{10},\quad u_{22}(P_1/P_2)=(1-P_1)F_{10} \tag{5.32}$$

$$u_{12}(P_1/P_2)=(v_1v_2)^{-1}P_1P_2(1-P_1), \tag{5.33}$$

$$u_{21}(P_1/P_2)=-(v_1v_2)^{-1}(1-P_1)P_2P_1. \tag{5.34}$$

where $1-F_{10}$ *is the eigenprojection of* $\theta(P_1, P_2)$ *for an eigenvalue* 0.

Such $u_{ij}(P_1/P_2)$ *is unique and satisfies*

$$u_{ij}(P_1/P_2)u_{kl}(P_1/P_2)=\delta_{jk}u_{il}(P_1/P_2), \tag{5.35}$$

$$u_{ij}(P_1/P_2)^*=u_{ji}(P_1/P_2), \tag{5.36}$$

$$\Gamma u_{12}(P_1/P_2)\Gamma=u_{21}(P_1/P_2), \tag{5.37}$$

$$P_2=\sum_{k,l=1}^{2} v_kv_l\,(-1)^{k-1}u_{kl}(P_1/P_2)+(1-F_{10})P_1, \tag{5.38}$$

$$1-P_2=\sum_{k,l=1}^{2} v_{(3-k)}v_{(3-l)}(-1)^k u_{kl}(P_1/P_2)+(1-F_{10})(1-P_1), \tag{5.39}$$

$$\gamma_{P_1}u_{ij}(P_1/P_2)=(-1)^{i-j}u_{ij}(P_1/P_2)\gamma_{P_1}. \tag{5.40}$$

(3) *Let*

$$H(P_1/P_2)=-i\theta(P_1, P_2)(u_{12}(P_1/P_2)+u_{21}(P_1/P_2)). \tag{5.41}$$

Then $H(P_1/P_2)^\dagger=H(P_1/P_2)$, $H(P_1/P_2)^*=-H(P_1/P_2)$ (* *is relative to* $(f, g)_{P_1}$), $\Gamma H(P_1/P_2)H=-H(P_1/P_2)$,

$$U(P_1/P_2)\exp iH(P_1/P_2)=v_1+v_1^{-1}[P_1, P_2] \tag{5.42}$$

is a Bogoliubov transformation for (K, γ, Γ), *and*

$$U(P_1/P_2)^\dagger P_1U(P_1/P_2)=P_2. \tag{5.43}$$

Proof. (1) Since P_2 is bounded with respect to the norm $\|\ \|_{P_2}$, it is also bounded with respect to the norm $\|\ \|_{P_1}$. Therefore (P_1-P_2) is bounded. The operator on the right hand side of (5.31) can be rewritten as

$$-(P_1-P_2)^2=-P_1(1-P_2)P_1-(1-P_1)P_2(1-P_1). \tag{5.44}$$

Since $(f, P_1(1-P_2)P_1f)_{P_1}=\gamma(P_1f, (1-P_2)P_1f)\leqq 0$ for $f\in K$ and $(1-P_1)P_2(1-P_1)=\Gamma P_1(1-P_2)P_1\Gamma$, we have $-(P_1-P_2)^2\geqq 0$. Hence there exists a nonnegative selfadjoint operator $\theta(P_1, P_2)$ satisfying (5.31). $(P_1-P_2)^2$ obviously commutes with Γ. It commutes with P_1 due to (5.44). By symmetry, it commutes with P_2.

Let $K=\max\{\|(P_1-P_2)^2\|_{P_1}, \|(P_1-P_2)^2\|_{P_2}\}$. Let $f_n(x)$ be a sequence of real polynomials of x such that

$$\lim_{n\to\infty} \sup_{x\in[0,K]} |f_n(x)-\sinh^{-1}\sqrt{x}\,|=0.$$

Such f_n exists due to the Weierstrass approximation theorem. Then $\theta(P_1, P_2)=\lim_{n\to\infty} f_n(-(P_1-P_2)^2)=\theta(P_2, P_1)$.

(2) We see that (5.33) is partially isometric by the following computation:

$$\{P_1P_2(1-P_1)\}^*P_1P_2(1-P_1)= (1-P_1)\gamma_{P_1}P_2\gamma_{P_1}P_1P_2(1-P_1)$$

$$=-(1-P_1)P_2P_1P_2(1-P_1)=\{(P_1-P_2)^4-(P_1-P_2)^2\}(1-P_1)$$

$$=v_1^2v_2^2(1-P_1).$$

We also see that $\mathrm{u}_{21}(P_1/P_2)=\Gamma\mathrm{u}_{12}(P_1/P_2)\Gamma$ is also partially isometric and (5.35) holds. From $P_1^*=P_1$, $P_2^*=\gamma_{P_1}P_2\gamma_{P_1}$, $P_1\gamma_{P_1}=P_1$ and $(1-P_1)\gamma_{P_1}=-(1-P_1)$, we have (5.36).

Since $P_1P_2P_1+(1-P_1)P_2(1-P_1)+P_1P_2(1-P_1)+(1-P_1)P_2P_1=P_2$ and $P_1(1-F_{10})=P_2(1-F_{10})$, we have (5.38) and (5.39). (5.40) follows from $\gamma_{P_1}P_1=P_1$ and $(1-P_1)\gamma_{P_1}=-(1-P_1)$.

(3) From (5.36) and $\theta(P_1, P_2)^*=\theta(P_1, P_2)$, we have $\mathrm{H}(P_1/P_2)^*=-\mathrm{H}(P_1/P_2)$. From (5.40) and $[\gamma_{P_1}, \theta(P_1, P_2)]=0$, we have $\mathrm{H}(P_1/P_2)^\dagger=\gamma_{P_1}\mathrm{H}(P_1/P_2)^*\gamma_{P_1}=\mathrm{H}(P_1/P_2)$. From (5.37) and $[\theta(P_1/P_2), \Gamma]=0$, we have $\Gamma H(P_1/P_2)\Gamma=-\mathrm{H}(P_1/P_2)$. Since $(\mathrm{u}_{12}(P_1/P_2)+\mathrm{u}_{21}(P_1/P_2))^2=F_{10}$, $v_1(1-F_{10})=(1-F_{10})$, $[P_1, P_2](1-F_{10})=0$, and $\mathrm{u}_{12}(P_1/P_2)+\mathrm{u}_{21}(P_1/P_2)=(v_1v_2)^{-1}[P_1, P_2]$, we obtain (5.42).

Finally we have

$$\mathrm{U}(P_1/P_2)^\dagger=\gamma_{P_1}\mathrm{U}(P_1/P_2)^*\gamma_{P_1}=v_1-v_1^{-1}[P_1, P_2],$$

$$v_1^2+P_1[P_1, P_2]-[P_1, P_2]P_1-v_1^{-2}[P_1, P_2]P_1[P_1, P_2]=P_2.$$

Hence we have (5.43). Q. E. D.

Lemma 5.5. *Let P_1 and P_2 be basis projections for (K, γ, Γ). Assume that $\| \ \|_j$, $j=1, 2$, give the same topology, with respect to which K is complete and that $\theta(P_1, P_2)$ is in the Hilbert Schmidt class. Then there exists a unique unitary operator $\mathrm{T}(P_1, P_2)$ on $\mathfrak{H}_{P_1}$ such that*

$$\mathrm{T}(P_1, P_2)^* \pi_{P_1}(A) \mathrm{T}(P_1, P_2) = \pi_{P_1}(\tau[\mathrm{U}(P_1/P_2)]A), \tag{5.45}$$

$$(\Omega_{P_1}, \mathrm{T}(P_1, P_2)\Omega_{P_1}) = \det_{P_1}[\mathrm{sech}\ \theta(P_1, P_2)]^{1/2}. \tag{5.46}$$

Proof. Let F_{1r} be the spectral projection of $\theta(P_1, P_2)$ for the infinite interval (r, ∞) relative to the inner product $(f, g)_{P_1}$. Then $\mathrm{H}(P_1/P_2)F_{1r}$ is of finite rank for $r>0$. Therefore it is of the form (4.2). Since $\theta(P_1, P_2)$ commutes with Γ, P_1 and γ_{P_1}, F_{1r} commutes with them and $(\mathrm{H}(P_1/P_2)F_{1r})^\dagger = \mathrm{H}(P_1/P_2)F_{1r}$, $\Gamma_{P_1}\mathrm{H}(P_1/P_2)F_{1r}\Gamma_{P_1} = -\mathrm{H}(P_1/P_2)F_{1r}$. Let

$$U_r = \exp iH(P_1/P_2)F_{1r}, \tag{5.47}$$

$$P_{(r)} = U_r^\dagger P_1 U_r \tag{5.48}$$

$$T_r' = \mathrm{Q}(H(P_1/P_2)F_{1r}). \tag{5.49}$$

$$\alpha_r = (\Omega_{P_1}, T_r'\Omega_{P_1})|(\Omega_{P_1}, T_r'\Omega_{P_1})|^{-1} \tag{5.50}$$

$$T_r = \alpha_r^* T_r'. \tag{5.51}$$

From (5.12) and (5.22), we have

$$\begin{aligned}(\Omega_{P_1}, T_r^*\Omega_{P_1}) &= |(\Omega_{P_1}, (T_r')^*\Omega_{P_1})| \\ &= \det_{P_1}(P_1\{P_2F_{1r} + (1-F_{1r})\}P_1)^{-1/4}. \\ &= \det_{P_1}[\mathrm{sech}\{\theta(P_1, P_2)F_{1r}\}]^{1/2}.\end{aligned} \tag{5.52}$$

In particular, it does not vanish and hence α_r is well defined.

If $\theta(P_1, P_2)$ is in the Hilbert Schmidt class, then $1-\mathrm{sech}\ \theta(P_1, P_2)$ is in the trace class and

$$\lim_{r\to 0}(\Omega_{P_1}, T_r^*\Omega_{P_1}) = \det_{P_1}[\mathrm{sech}\ \theta(P_1, P_2)]^{1/2} > 0. \tag{5.53}$$

Further

$$\begin{aligned}|(T_{r'}^*\Omega_{P_1}, T_r^*\Omega_{P_1})| &= |(\Omega_{P_1}, \mathrm{Q}(\mathrm{H}(P_1/P_2)[F_{1r'} - F_{1r}])\Omega_{P_1})| \\ &= \det_P[\mathrm{sech}\{\theta(P_1, P_2)(F_{1r'} - F_{1r})\}]^{1/2} \to 1\end{aligned} \tag{5.54}$$

as r, $r'\to 0$.

Setting $\alpha_{r,r'}=(T_{r'}^{*}\Omega_{P_1}, T_r^{*}\Omega_{P_1})|(T_{r'}^{*}\Omega_{P_1}, T_r^{*}\Omega_{P_1})|^{-1}$, we see from (5.54) that $T_{r'}^{*}\Omega_{P_1}-\alpha_{r,r'}^{*}T_r^{*}\Omega_{P_1}\to 0$ as r, $r'\to 0$. Substituting this to (5.53), we see that $\alpha_{r,r'}^{*}\to 1$ as r, $r'\to 0$ and hence $T_r^{*}\Omega_{P_1}$ is a Cauchy sequence as $r\to 0$. Let

$$\Omega'_{P_2}=\lim_{r\to+0} T_r^{*}\Omega_{P_1} \tag{5.55}$$

We now have

$$\begin{aligned}(\Omega'_{P_2}, \mathrm{W}_{P_1}(f)\Omega'_{P_2})&=\lim_{r\to+0}(\Omega_{P_1}, \mathrm{W}_{P_1}(\tau(U_r)f)\Omega_{P_1})\\ &=(\Omega_{P_2}, \mathrm{W}_{P_2}(f)\Omega_{P_2}).\end{aligned} \tag{5.56}$$

Since $\{\mathrm{W}_{P_1}(f)\}$ is irreducible, Ω'_{P_2} is cyclic and the vector state of $\pi_{P_1}(\mathfrak{A}(K, \gamma, \Gamma))$ by Ω'_{P_2} is φ_{P_2}.

Let $\mathrm{T}(P_1, P_2)$ be a closed linear operator satisfying

$$\mathrm{T}(P_1, P_2)\mathrm{W}_{P_1}(f)\Omega_{P_1}=\mathrm{W}_{P_1}(\tau(U_r^{\dagger})f)\Omega'_{P_2}. \tag{5.57}$$

Since the mapping defined by (5.57) is isometric and the range is total, such $\mathrm{T}(P_1, P_2)$ exists as a unitary operator.

We have (5.45) from (5.57) and (5.46) from (5.53). Q. E. D.

Corollary 5.6. *Let U be a Bogoliubov transformation on K and P be a basis projection. Then $\tau(U)$ is unitarily implementable on $\mathfrak{H}_P$ if $PU(1-P)U^{\dagger}P$ is in the trace class on K_P.*

Proof. Let $UPU^{\dagger}=P'$, $U_1=U(P/P')$, $U_2=U(P/P')U$. Then $U=U_1^{\dagger}U_2$. Since U_2 commutes with P, φ_P is invariant under U_2 and $\tau(U_2)$ is implementable by a unitary operator $T_P(U_2)$ on $\mathfrak{H}_P$:

$$\mathrm{T}_P(U_2)\pi_p(A)\mathrm{T}_P(U_2)^{*}=\pi_P(\tau(U_2)A). \tag{5.58}$$

Because $PU(1-P)U^{\dagger}P=-(\sinh\theta(P, P'))^2P$ is in the trace class, $\Gamma PU(1-P)U^{\dagger}P\Gamma=-(\sinh\theta(P, P'))^2(1-P)$ is also in the trace class and hence $\theta(P, P')$ is in the H. S. class. Then $\mathrm{T}(P, P')$ implements $\tau(U^{\dagger})$ and $\mathrm{T}(P, P')\mathrm{T}_P(U_2)$ implements $\tau(U)$. Q. E. D.

Lemma 5.7. *Let $(f, g)_1$ and $(f, g)_2$ be positive definite inner products* *on K such that K is complete with respect to both* $\|f\|_1=(f, f)_1^{1/2}$ *and* $\|f\|_2=(f, f)_2^{1/2}$ *and the two norms give the same topology on K.*

(1) *There exists a bounded linear operator* α *on K with a bounded inverse* α^{-1} *such that*

$$(5.59) \qquad (f, g)_2=(\alpha f, \alpha g)_1.$$

(2) *A bounded linear operator Q on K is in the Hilbert Schmidt (resp. trace) class relative to* $(f, g)_2$ *if and only if Q is in the H.S. (resp. trace) class relative to* $(f, g)_1$.

(3) *If Q is in the trace class, thc trace of Q relative to* $(f, g)_1$ *and* $(f, g)_2$ *are the same.*

Proof. (1) If there is no positive a such that $\|f\|_2\leqq a\|f\|_1$ for all $f\in K$, then there exists a sequence f_n such that $\|f_n\|_1=1$ and $\|f_n\|_2\geqq n$. Then $\|f_n/n\|_1\to 8$ while $\|f_n/n\|_2=1$ does not tend to 0. Hence, under our assumption that the two norms induce the same topology on K, there exists $a>0$ such that $\|f\|_2\leqq a\|f\|_1$ for all $f\in K$. Similarly there exists $a'>0$ such that $\|f\|_1\leqq a'\|f\|_2$ for all $f\in K$.

Since $|(f, g)_2|\leqq\|f\|_2\|g\|_2\leqq a^2\|f\|_1\|g\|_1$, there exists by the Riesz theorem a bounded linear operator α_0 such that

$$(f, g)_2=(f, \alpha_0 g)_1.$$

Since $(f, g)_2$ is hermitian and positive, α_0 is hermitian and positive relative to $(f, g)_1$. Setting $\alpha=(\alpha_0)^{1/2}$, where the positive square root is taken relative to $(f, g)_1$ we obtain (5.59) and $\alpha=\alpha^*>0$ relative to $(f, g)_1$.

From $(f, \alpha^2 f)_1=\|f\|_2^2\geqq(a')^{-2}\|f\|_1^2$, we obtain $\alpha^2\geqq(a')^{-2}$ and $\alpha\geqq(a')^{-1}$ relative to $(f, g)_1$ and hence $\|\alpha^{-1}\|_1\leqq a'$. Namely α has a bounded inverse.

(2) Assume that $\{e_\beta\}$ is an orthonormal basis of K relative to $(f, g)_1$. Let $e'_\beta=\alpha^{-1}e_\beta$. Since α^{-1} has a bounded inverse and $\{e_\beta\}$ is total, $\{e'_\beta\}$ is also total in K. Hence $\{e'_\beta\}$ is an orthonormal basis of K relative to $(f, g)_2$.

Let Q^* denote the adjoint of Q relative to $(f, g)_1$ and $|Q|_j$ denote the absolute value of Q relative to $(f, g)_j$. We have $|Q|_1^2=Q^*Q$. From $(f, Qg)_2=(f, \alpha^2 Qg)_1=(Q^*\alpha^2 f, g)_1=(\alpha^{-2}Q^*\alpha^2 f, g)_2$, we obtain $|Q|_2^2=\alpha^{-2}Q^*\alpha^2 Q$.

Assume that Q is in the H.S. class relative to $(f, g)_1$. Then $\mathrm{tr}_1|Q|_1^2 \equiv \sum_\beta \|Qe_\beta\|_1^2<\infty$. It is then known that $\alpha Q\alpha^{-1}$ is also in the H.S. class relative to $(f, g)_1$. Hence

$$\begin{aligned}(5.60)\qquad \mathrm{tr}_2|Q|_2^2 &\equiv \sum_\beta \|Qe_{\beta'}\|_2^2=\sum_\beta (e'_\beta, \alpha^{-2}Q^*\alpha^2 Qe'_\beta)_2\\ &=\sum_\beta (e_\beta, \alpha^{-1}Q^*\alpha^2 Q\alpha^{-1}e_\beta)_1\\ &=\mathrm{tr}_1(\alpha Q\alpha^{-1})^*(\alpha Q\alpha^{-1})<\infty.\end{aligned}$$

and Q is in the H.S. class relative to $(f, g)_2$.

Next assume that Q is in the trace class relative to $(f, g)_1$. Let $Q=W_j|Q|_j$ be the polar decomposition of Q relative to $(f, g)_j$. Let $\tilde{W}_j$ be the adjoint of W_j relative to $(f, g)_j$. Then $\tilde{W}_jQ=|Q|_j$. W_j and $\tilde{W}_j$ are bounded. Hence $|Q|_2=\tilde{W}_2W_1|Q|_1$ is in the trace class relative to $(f, g)_1$ and so is $\alpha|Q|_2\alpha^{-1}$. We now have

$$\begin{aligned}(5.61)\qquad \mathrm{tr}_2|Q|_2 &=\sum_\beta (e'_\beta, |Q|_2 e'_\beta)_2\\ &=\sum_\beta (e_\beta, \alpha|Q|_2\alpha^{-1}e_\beta)_1\\ &=\mathrm{tr}_1\alpha|Q|_2\alpha^{-1}<\infty.\end{aligned}$$

Therefore Q is in the trace class relative to $(f, g)_2$.

(3) The computation of (5.61) shows $\mathrm{tr}_2 Q=\mathrm{tr}_1 Q$ for any Q.

Lemma 5.8. *In Lemma* 5.5, $\theta(P_1, P_2)$ *is in the H. S. class if and only if* P_1-P_2 *is in the H. S. class.*

Proof. If P_1-P_2 is in the H. S. class, $-(P_1-P_2)^2=\sinh^2\theta\,(P_1, P_2)$ is in the trace class and hence $\theta(P_1, P_2)$ is in the H. S. class. If $\theta(P_1, P_2)$ is in the H. S. class, then P_1-P_2 is in the H. S. class, as is obvious from (5.38). Q. E. D.

§ 6. Quasi-equivalence for Non-degenerate Case

In this section, we shall be concerned with S and S' which do not have an eigenvalue 1/2. Since the corresponding Π_S and $\Pi_{S'}$ give Fock states, we shall denote P_S and $P_{S'}$ instead of Π_S and $\Pi_{S'}$.

Definition 6.1. *Two quasifree representations π_S and $\pi_{S'}$ are quasi-equivalent if there exists a (homeomorphic) * isomorphism p from $R_S \equiv \{W_S(f); f \in \mathrm{Re}\, K\}''$ onto $R_{S'} \equiv \{W_{S'}(f); f \in \mathrm{Re}\, K\}''$ such that $pW_S(f) = W_{S'}(f)$ for all $f \in \mathrm{Re}\, K$.*

Lemma 6.2. *Let φ_S and $\varphi_{S'}$ be quasifree states of $\mathfrak{A}(K, \gamma, \Gamma)$ such that S and S' do not have an eigenvalue 1/2. Assume that the following 3 conditions hold:*

(1) $N_S = N_{S'}$.

(2) *The topologies induced by $\|f\|_S$ and $\|f\|_{S'}$ on K are equivalent.*

Identify K_S, γ_S, Γ_S with $K_{S'}$, $\gamma_{S'}$, $\Gamma_{S'}$ by the closure of the identification map $\bar{f} \in K_S \to \bar{f} \in K_{S'}$ for $f \in K$. Identify K'_S, γ'_S, Γ'_S with $K'_{S'}$, $\gamma'_{S'}$ $\Gamma'_{S'}$ through definitions (6.1)~(6.3) of [1]. *Identify $\hat{K}_S$, $\hat{\gamma}_S$, $\hat{\Gamma}_S$ with $\hat{K}_{S'}$, $\hat{\gamma}_{S'}$, $\hat{\Gamma}_{S'}$ due to Lemma* 6.1 (6) *of* [1].

(3) *$P_S - P_{S'}$ is in the Hilbert-Schmidt class on $\hat{K}_S = \hat{K}_{S'}$.*

Then π_S and $\pi_{S'}$ are quasi-equivalent.

If S and S' do not have an eigenvalue 0 *in addition, φ_S and $\varphi_{S'}$ are unitarily equivalent.*

Proof. (1)~(3) imply the unitary equivalence of $(\mathfrak{H}_{P_S}, \pi_{P_S}, \Omega_{P_S})$ and $(\mathfrak{H}_{P_{S'}}, \pi_{P_{S'}}, \Omega_{P_{S'}})$ due to Lemma 6.1 (6) of [1], Lemmas 5.5 and 5.8. The vector Ω_{P_S} and $\Omega_{P_{S'}}$ are separating for the center of R_S and $R_{S'}$ by Lemma 2.5. Therefore π_S and $\pi_{S'}$, which are restrictions of π_{P_S} and $\pi_{P_{S'}}$ to a subalgebra, are quasi-equivalent.

If S and S' do not have an eigenvalue 0, then Ω_{P_S} and $\Omega_{P_{S'}}$ are cyclic for R_S and $R_{S'}$. Hence π_S and $\pi_{S'}$ are unitarily equivalent.

Q. E. D.

Lemma 6.3. *Let $\mathfrak{S}_0$ be the set of positive semidefinite hermitian forms S on K satisfying* (3.4) *of* [1] *and such that the associated operator S does not have an eigenvalue* 1/2. *Define the quasi-equivalence $S \sim S'$ by* *the requirements* (1)~(3) *of Lemma* 6.2. *Then this is an equivalence relation.*

This is obvious from the form of requirements (1)~(3).

If $N_S=N_{S'}$, S and S' do not have an eigenvalue 1/2 and $\dim(K/N_S)<\infty$, then we have a representation of canonical commutation relations for a finite degree of freedom and hence all representations are quasi-equivalent, a well known result of von Neumann. Hence we have a common W^* algebra $R_S=R_{S'}$ generated by $\mathrm{W}_S(f)$, $f\in\mathrm{Re}\,K$, and φ_S can be viewed as the unique state of $R_S=R_{S'}$ satisfying $\varphi_S(\mathrm{W}_S(f))=(\Omega_S, \mathrm{W}_S(f)\Omega_S)$. This makes it meaningful to speak of the norm $\|\varphi_S-\varphi_{S'}\|$ for such S and S'.

Lemma 6.4. *Let* φ_S *and* $\varphi_{S'}$ *be quasifree states of* $\mathfrak{A}(K, \gamma, \Gamma)$ *such that* $N_S=N_{S'}$ *and* S *and* S' *do not have an eigenvalue* 1/2 *nor* 1. *Assume that* $\dim(K/N_S)<\infty$. *Then*

$$\|\varphi_S-\varphi_{S'}\|\geq 2\{1-\det_{P_S}(P_SP_{S'}P_S)^{-1/4}\}. \tag{6.1}$$

Proof. Let ω be as in Lemma 2.1. (2.3) and (2.4) imply $[\omega, \gamma_S]=0$ and hence $[\omega, P_S]=0$. Since ω is the same for S and S' in the present case, we also have $[\omega, P_{S'}]=0$. This implies $\omega\mathrm{H}(P_S/P_{S'})=-\mathrm{H}(P_S/P_{S'})\omega$ and $\tau(\omega)(\mathrm{B}, \mathrm{H}(P_S/P_{S'})\mathrm{B})=-(\mathrm{B}, \mathrm{H}(P_S/P_{S'})\mathrm{B})$. Hence we obtain

$$[\mathrm{T}_{P_S}(\omega), \mathrm{Q}_{P_S}\{\mathrm{H}(P_S/P_{S'})\}]=0. \tag{6.2}$$

Therefore, Ω_{PS} and $\Omega'\equiv Q_{P_S}\{\mathrm{H}(P_S/P_{S'})\}^*\Omega_{P_S}$ are invariant under $\mathrm{T}_{P_S}(\omega)$ and vector states of the representation $\pi(\mathrm{B}(f))\equiv\pi_{P_S}[\mathrm{B}(\bar{f}\oplus 0)]$, $f\in K$ of $\mathfrak{A}(K, \gamma, \Gamma)$ by Ω_{P_S} and Ω' are φ_S and $\varphi_{S'}$.

By Lemma 6.5 of [2], Lemma 3.3 and Corollary 3.5, we obtain

$$\|\varphi_S-\varphi_{S'}\|\geqq 2(1-|(\Omega_{P_S}, \Omega')|). \tag{6.3}$$

By (5.46), we obtain $|(\Omega_{P_S}, \Omega')|=\det_{P_S}(P_SP_{S'}P_S)^{-1/4}$ and hence (6.1).

Q. E. D.

Lemma 6.5. *Let* S *and* S' *be hermitian forms belonging to* $\mathfrak{S}_0$ *such that* $N_S=N_{S'}$ *and* $\tau_S=\tau_{S'}$. *Let*

(6.4) $\chi(S)\equiv\mathrm{Tanh}^{-1}2S^{1/2}(1-S)^{1/2}$, $\sigma(S)=|2S-1|^{-1}(2S-1)$,

(6.5) $\chi(S')\equiv\mathrm{Tanh}^{-1}2(S')^{1/2}(1-S')^{1/2}$, $\sigma(S')=|2S'-1|^{-1}(2S'-1)$.

Then $e^{-\chi(S)}e^{\chi(S')}$ *and* $e^{-\chi(S')}e^{\chi(S)}$ *are bounded and the following conditions are equivalent*:

(1) $P_S-P_{S'}$ *is in the Hilbert Schmidt class.*

(2) $1-\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S')$ *is in the Hilbert Schmidt class.*

(3) $1-\sigma(S')e^{-\chi(S')}e^{\chi(S)}\sigma(S)$ *is in the Hilbert Schmidt class.*

Proof. For each $f\in K_S$, let

$$\|f\|_S^\sharp\equiv\sup_{\|g\|_S=1}|(f,\gamma_S g)_S|=\|(2S-1)f\|_S \tag{6.6}$$

and $K_S^\sharp$ be the completion of K_S relative to (6.6). Let $(f, g)_S^\sharp\equiv(f, (2S-1)^2g)_S$. Consider $\hat{K}'_S\equiv K_S\oplus K_S^\sharp$ equipped with an inner product $(f_1\oplus g_1, f_2\oplus g_2)_S^{\wedge\prime}\equiv(f_1, f_2)_S+(g_1, g_2)_S^\sharp$. If $\tau_S=\tau_{S'}$, K_S is identified with $K_{S'}$. Since $(f,\gamma_S g)_S=(f,\gamma_{S'}g)_{S'}$, $K_S^\sharp$ can be identified with $K_{S'}^\sharp$ and hence $\hat{K}'_S$ with $\hat{K}'_{S'}$.

The mapping from $f\in K_S$ to $(2S-1)f\in K_S$ is isometric as a mapping from $K_S^\sharp$ into K_S. Since $|2S-1|=\mathrm{sech}\,\chi(S)$, $e^{\chi(S)}|2S-1|=2[1+e^{-2\chi(S)}]^{-1}$ is bounded above and below by 2 and 1 both on K_S and on $K_S^\sharp$. Hence the closure of $e^{-\chi(S)}$ is a bounded mapping from $K_S^\sharp$ onto K_S and its inverse is a bounded mapping from K_S onto $K_S^\sharp$. We shall denote the closure and its inverse again by $e^{-\chi(S)}$ and $e^{\chi(S)}$. A similar statement holds for S'. Hence $e^{-\chi(S)}e^{\chi(S')}$ and $e^{-\chi(S')}e^{\chi(S)}$ are bounded on K_S.

Due to

$$2(\|f\oplus g\|_S^\wedge)^2=\|[S^{1/2}+(1-S)^{1/2}](f+g)\|_S^2 + \{\|[S^{1/2}+(1-S)^{1/2}]^{-1}(f-g)\|_S^\sharp\}^2 \tag{6.7}$$

and $\sqrt{2}\geqq S^{1/2}+(1-S)^{1/2}\geqq 1$, the mapping v_S from $f\oplus g\in\hat{K}_S$ to $(f+g)\oplus(f-g)\in\hat{K}'_S$ and its inverse are bounded, where f and g are in K_S. Let $\bar{v}_S$ be the closure of v_S, which is a bounded mapping from $\hat{K}_S$ onto $\hat{K}'_S$ with a bounded inverse. Obviously $\bar{v}_S=\bar{v}_{S'}$.

By a direct computation, we have

$$2\bar{v}_S P_S\bar{v}_S^{-1}=\begin{bmatrix}1 & e^{-\chi(S)}\sigma(S)\\ e^{\chi(S)}\sigma(S) & 1\end{bmatrix}. \tag{6.8}$$

Hence the condition (1) is equivalent to the following two conditions: $(\alpha)\sigma(S)e^{-\chi(S)}-e^{-\chi(S')}\sigma(S')$ is in the H. S. class as a mapping from $K_S^\sharp=K_{S'}^\sharp$ into $K_S=K_{S'}$. $(\beta)\sigma(S)e^{\chi(S)}-e^{\chi(S')}\sigma(S')$ is in the H.S. class as a mapping from K_S into $K_S^\sharp$. Here a mapping Q from a Hilbert space H_1 into another Hilbert space is in the H.S. class if $\sum_\alpha ||Qe_\alpha^1||_2^2<\infty$ where e_α^1 is a complete orthonormal set on H_1 and the norm is in H_2.

Since $||Qf||_S^\sharp/||e^{-\chi(S)}Qf||_S$ is bounded below and above uniformly, (β) is equivalent to the condition (2). The condition (2) implies that $e^{-\chi(S')}\sigma(S')-e^{-\chi(S)}\sigma(S)=(1-\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S'))e^{-\chi(S')}\sigma(S')$ is in the H.S. class as a mapping from $K_S^\sharp$ into K_S (cf. Lemma 5.7), and hence (α). Namely (1) and (2) are equivalent.

By symmetry, (1) and (3) are equivalent. Q. E. D.

Remark 6.6. From the above calculation, we have

$$4||P_S-P_{S'}||_{H.S.}^2=||\beta[1-\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S')]\beta^{-1}||_{H.S.}^2 + ||\beta[1-\sigma(S')e^{-\chi(S')}e^{\chi(S)}\sigma(S)]\beta^{-1}||_{H.S.}^2 \tag{6.9}$$

where the H.S. norm is relative to $(f, g)_S^\wedge$ on the left hand side, relative to $(f, g)_S$ on the right hand side and

$$\beta\equiv\sqrt{2}[1+\exp-2\chi(S)]^{-1/2}(=S^{1/2}+(1-S)^{1/2})$$

satisfies $1\leqq\beta\leqq\sqrt{2}$. As a consequence

$$||P_S-P_{S'}||_{H.S.}^2\geqq(1/8)||1-\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S')||_{H.S.}^2 \tag{6.10}$$

Lemma 6.7. *Assume that K is separable. Let S and S' be in $\mathfrak{S}_0$. Assume that S and S' do not have an eigenvalue* 1, $N_S=N_{S'}$, $\tau_S=\tau_{S'}$ *and $P_S-P_{S'}$ is not in the H.S. class. Then there exists a sequence of Γ invariant finite dimensional subspaces K_n of K_S such that hermitian forms* $\mathrm{S}_n(f, g)=(f, Sg)_S$ *and* $\mathrm{S}_n'(f, g)=(f, S'g)_{S'}$, $f, g\in K_n$, *satisfy* $\lim||\varphi_{S_n}-\varphi_{S_n'}||=2$.

Proof. Let E_+ and E_n be the spectral projection of S for intervals $[1/2, 1]$ and $[1/2+1/n, 1]$, $n=3, 4, \cdots$, respectively. Let f_α be a complete orthonormal basis of E_+K_S relative to $(f, g)_S$ such that $E_nf_\alpha=f_\alpha$

or 0 for all n and α. For any subset A of α, let $\mathrm{K}(A)$ be the subspace of $K_S=K_{S'}$ spanned by f_α and Γf_α, $\alpha\in A$ and $\mathrm{E}(A)$ be the orthogonal projection on $\mathrm{K}(A)$ relative to $(f, g)_S$. Let I_n be the set of α such that $E_n f_\alpha=f_\alpha$ and I denote any finite set of α.

Let $\bar{\gamma}(f, g)=(f, \gamma_S g)_S(=(f, \gamma_{S'}g)_{S'})$. Let $\mathrm{S}(A)$ and $\mathrm{S}'(A)$ be the hermitian forms on $\mathrm{K}(A)$, defined by $\mathrm{S}(A)(f, g)=(f, Sg)_S$, $\mathrm{S}'(A)(f, g)=(f, S'g)_{S'}$. The restriction of $\bar{\gamma}$ and $\Gamma_S=\Gamma_{S'}$ to $\mathrm{K}(A)$ are denoted by the same letters.

By construction of K_S, $N_{\mathrm{S}(A)}=N_{\mathrm{S}'(A)}=0$. Since $\bar{\gamma}(f_\alpha, \Gamma_S f_\alpha)=0$, $\bar{\gamma}(\Sigma c_\alpha f_\alpha, \Sigma c_\alpha f_\alpha)>0$ for $\Sigma c_\alpha f_\alpha\neq 0$ and $\bar{\gamma}(\Sigma c_\alpha \Gamma f_\alpha, \Sigma c_\alpha \Gamma f_\alpha)<0$ for $\Sigma c_\alpha \Gamma f_\alpha\neq 0$, we see that $\mathrm{S}(A)$ and $\mathrm{S}(A')$ belong to $\mathfrak{S}_0$ for $(\mathrm{K}(A), \bar{\gamma}, \Gamma_S)$. We have $\mathrm{S}(A)\mathrm{E}(A)=\mathrm{E}(A)\mathrm{S}\mathrm{E}(A)$ and $\mathrm{S}'(A)\mathrm{E}(A)=\mathrm{E}(A)^{\#}\mathrm{S}'(A)\mathrm{E}(A)$, where $\mathrm{E}(A)^{\#}$ is the adjoint of $\mathrm{E}(A)$ relative to $(f, g)_{S'}$. Since $\tau_S=\tau_{S'}$, there exists an operator α with a bounded inverse such that $(f, g)_{S'}=(f, \alpha g)_S$ and α is hermitian and positive relative to $(f,g)_S$. Then $\mathrm{E}(A)^{\#}=\alpha^{-1}\mathrm{E}(A)\alpha$.

We have $\lim_{I\uparrow I_n}\mathrm{E}(I)=\mathrm{E}(I_n)$ and $\lim_n \mathrm{E}(I_n)=1$. Hence $\lim_n\lim_{I\uparrow I_n}\mathrm{S}(I)\mathrm{E}(I)=S$ and $\lim_n\lim_{I\uparrow I_n}\mathrm{S}'(I)\mathrm{E}(I)=S'$ (as strong limits of operators). Since $[\mathrm{E}(I_n), S]=0$ and $\chi(S)\mathrm{E}(I_n)$ is bounded, we have $\lim_{I\uparrow I_n}\{\sigma(\mathrm{S}(I))\exp\chi(S(I))\}\mathrm{E}(I)=(\sigma(S)\exp\chi(S))\mathrm{E}(I_n)$. (If $f(x)$ is piecewise continuous and its jump points are not eigenvalues of Q, then $\lim Q_I=Q$ implies $\lim_{I\uparrow} f(Q_I)=f(Q)$. [8]) We also have $\lim_{I\uparrow I_n}\{\sigma(\mathrm{S}'(I))\exp-\chi(\mathrm{S}'(I))\}\mathrm{E}(I)=\{\sigma(\mathrm{S}'(I_n))\exp-\chi(\mathrm{S}'(I_n))\}\mathrm{E}(I_n))$ and $\lim_n\{\sigma(\mathrm{S}'(I_n))\exp-\chi(\mathrm{S}'(I_n))\}\mathrm{E}(I_n)=\sigma(S')\exp-\chi(S')$. Since $\|\alpha^{-1}\|^{-1}\mathrm{E}(A)\leqq\mathrm{E}(A)\alpha\mathrm{E}(A)\leqq\|\alpha\|\mathrm{E}(A)$, sech $\chi(\mathrm{S}'(A))$ cosh $\chi(\mathrm{S}(A))=[\mathrm{E}(A)\alpha\mathrm{E}(A)]^{-1}$ (on $\mathrm{K}(A)$) and its inverse are uniformly bounded. Hence $e^{-\chi(\mathrm{S}'(I))}e^{\chi(\mathrm{S}(I))}$ and its inverse $e^{-\chi(\mathrm{S}(I))}e^{\chi(\mathrm{S}'(I))}$ are uniformly bounded due to $1/2\leqq e^{-x}\cosh x\leqq 1$ and hence

$$\lim_n\lim_{I\uparrow I_n}\sigma(\mathrm{S}'(I))e^{-\chi(\mathrm{S}'(I))}e^{\chi(\mathrm{S}(I))}\sigma(\mathrm{S}(I))=\sigma(S')e^{-\chi(S')}e^{\chi(S)}\sigma(S). \tag{6.11}$$

Since x^{-1} is continuous for $x\in[\|\alpha\|^{-1}, \|\alpha^{-1}\|]$, we also obtain as an inverse of (6.11)

$$\lim_n\lim_{I\uparrow I_n}\sigma(\mathrm{S}(I))e^{-\chi(\mathrm{S}(I))}e^{\chi(\mathrm{S}'(I))}\sigma(\mathrm{S}'(I))=\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S'). \tag{6.12}$$

We now have

$$\lim_{n}\lim_{I\uparrow I_n}\|(1-\sigma(\mathrm{S}(I))e^{-\chi(\mathrm{S}(I))}e^{\chi(\mathrm{S}'(I))}\sigma(\mathrm{S}'(I))\|_{H.S.} \geqq \|1-\sigma(S)e^{-\chi(S)}e^{\chi(S')}\sigma(S')\|_{H.S.}=\infty, \tag{6.13}$$

where the H.S. norm is relative to $(f, g)_{S(I)}(=(f, g)_S$ for $f, g\in K(I))$. By Remark 6.6, we have $\lim_n \lim_{I\uparrow I_n} \|P_{\mathrm{S}(I)}-P_{\mathrm{S}'(I)}\|_{H.S.}=\infty$ where the H.S. norm is relative to $(f, g)_{S(I)}$. Since $\det Q\geqq 1+\mathrm{tr}(Q-1)$ for $Q\geqq 1$, we obtain

$$\lim_{m}\lim_{I\uparrow I_n} \det_{P_{S(I)}}(P_{\mathrm{S}(I)}P_{\mathrm{S}'(I)}P_{\mathrm{S}(I)})^{-1/4}=0. \tag{6.14}$$

Hence we can find a finite set $I(n)$ such that $\|\varphi_{\mathrm{S}(I(n))}-\varphi_{\mathrm{S}'(I(n))}\|\geqq 2[1-(1/n)]$ due to Lemma 6.4. Q. E. D.

Lemma 6.8. *Assume that K is separable. Let S and S' be in $\mathfrak{S}_0$. Assume that $N_S=N_{S'}$, $\tau_S=\tau_{S'}$ and $\mathrm{tr}(P_S-P_{S'})^2=-\infty$. Then π_S and $\pi_{S'}$ are not quasi-equivalent.*

Proof. First assume that S and S' do not have an eigenvalue 1. Then Ω_{P_S} and $\Omega_{P_{S'}}$ are cyclic and separating by Lemma 2.3. Hence, if π_S and $\pi_{S'}$ are quasi-equivalent, they are unitarily equivalent and there exists a unitary mapping W_0 from $\mathfrak{H}_{P_S}$ to $\mathfrak{H}_{P_{S'}}$ such that $W_0\mathrm{W}_S(f)W_0^{-1}=\mathrm{W}_{S'}(f)$ for all $f\in\mathrm{Re}\,K$. By continuity, we have

$$W_0\mathrm{W}_{P_S}(f\oplus 0)W_0^{-1}=\mathrm{W}_{P_{S'}}(f\oplus 0)$$

for all $f\in K_S=K_{S'}$.

Since the set of $W'\Omega_{P_S}$ with isometric W' in $(R_S)'$ is total, there exists such W' satisfying $b\equiv(W'\Omega_{P_S}, W_0^{-1}\Omega_{P_{S'}})\neq 0$. The vector states of R_S by Ω_{P_S} and $\Omega_{P_{S'}}$ are denoted by φ_S and $\varphi_{S'}$ (cf. Lemma 6.4). Then

$$\|\varphi_S-\varphi_{S'}\|\leqq 2(1-|b|^2)^{1/2}\equiv 2-\delta. \tag{6.15}$$

We now have a contradiction because there exists a Γ invariant subspace K_0 of K_S such that the restriction of $\varphi_S-\varphi_{S'}$ to the subalgebra generated by $W_{P_S}(f\oplus 0), f\in K_0$ has a norm larger than $2-\delta$, due to Lemma 6.7.

For the general case, let E_1 be the eigenprojection of S for an eigenvalue 1 (relative to $(f, g)_S$) and χ be a positive Hilbert Schmidt class operator on E_1K_S. Let

$$S_{10}=S+(-1+\cosh^2 \chi)E_1+\Gamma_S(\sinh^2\chi E_1)\Gamma_S, \tag{6.16}$$

$$S_1(f, g)=(\bar{f}, S_{10}\bar{g})_S, \quad f, g\in K. \tag{6.17}$$

Then S_1 is in $\mathfrak{S}_0$ for (K, γ, Γ). We have $N_{S_1}=N_S$,

$$(f, g)_{S_1}=(f, \{1+(\cosh 2\chi-1)(E_1+\Gamma_S E_1\Gamma_S)\} g)_S, \tag{6.18}$$

for $f, g\in K_S$, $\tau_{S_1}=\tau_S$ and

$$S_1=S+(-1+\cosh^2\chi \operatorname{sech}2\chi)E_1+\Gamma_S(\sinh^2\chi \operatorname{sech}2\chi)E_1\Gamma_S, \tag{6.19}$$

$$\chi(S_1)=\chi(S)+2\chi(E_1+\Gamma_S E_1\Gamma_S). \tag{6.20}$$

Note that $\chi(S)=0$ on $E_1K_S\oplus\Gamma_S E_1K_S$. Hence we have

$$||1-\sigma(S)e^{-\chi(S)}e^{\chi(S_1)}\sigma(S_1)||^2_{H.S.}=2||(1-e^{2\chi})E_1||^2_{H.S.}<\infty. \tag{6.21}$$

By Lemma 6.2, π_S and π_{S_1} are quasi-equivalent. By (6.19), S_1 does not have an eigenvalue 1. By Lemma 6.5, $S\tilde{_q}S_1$. Similarly, there exists $S_1'\in\mathfrak{S}_0$ such that S_1' does not have an eigenvalue 1, $\pi_{S_1'}$ and $\pi_{S'}$ are quasi-equivalent and $S'\tilde{_q}S_1'$. Since $P_{S_1}-P_{S_1'}=P_S-P_{S'}-(P_S-P_{S_1})+(P_{S'}-P_{S_1'})$ is not in the H.S. class, π_{S_1} and $\pi_{S_1'}$ are not quasi-equivalent by preceding conclusion. Hence π_S and $\pi_{S'}$ are not quasi-equivalent. Q.E.D.

Lemma 6.9. *If S and S' are in $\mathfrak{S}_0$ and quasifree representations π_S and $\pi_{S'}$ are quasi-equivalent, there exists an operator E on $K_S=K_{S'}$ such that E commutes with S, S' and Γ, is an orthogonal projection rela tive both $(f, g)_S$ and $(f, g)_{S'}$, $Sf=S'f$ and $(g, f)_S=(g, f)_{S'}$ for $f\in$ to $(1-E)$ K and EK_S is separable.*

Proof. Since $\tau_S=\tau_{S'}$ due to the quasi-equivalence of π_S and $\pi_{S'}$, there exists an operator α with a bounded inverse such that $(f, g)_{S'}=(\alpha f, \alpha g)_S$ for all $f, g\in K_S=K_{S'}$ and α is hermitian and positive relative to $(f, g)_S$.

We can construct inductively a separable subspace K_μ of K_S for each ordinal $\mu<\mu_S$ in such a way that K_μ is mutually orthogonal relative to both $(f, g)_S$ and $(f, g)_{S'}$, invariant under S, S', Γ and α and $K_S = \oplus K_\mu$. The construction of such K_μ proceeds as follows: Assume that $K_{\mu'}$, $\mu' < \mu$, be given. If the orthogonal complement, relative to $(f, g)_S$, of the union of $K_{\mu'}$, $\mu' < \mu$, is 0, then set $\mu = \mu_S$. Otherwise take a vector f_μ from there and let K_μ be the subspace of K_S spanned by Qf_μ where Q runs over arbitrary polynomials of α, S, S', Γ and 1. The required properties are satisfied inductively.

Let E_μ be an orthogonal projection onto K_μ relative to $(f, g)_S$. $(1-E_\mu)K_S$ is spanned by the union of $K_{\mu'}$, $\mu' \neq \mu$, and hence E_μ is hermitian relative to $(f, g)_{S'}$ due to the α invariance of each $K_{\mu'}$.

Corresponding to the decomposition $K_S = \oplus K_\mu$, we have the decomposition $\hat{K}_S = \oplus \hat{K}_\mu$. Let $\hat{E}_\mu$ be the orthogonal projection onto $\hat{K}_\mu$ relative to $(f, g)_{\hat{S}}$. By similar reason as before, $\hat{E}_\mu$ is hermitian also relative to $(f, g)_{\hat{S}'}$ and commutes with P_S, $P_{S'}$ and $\hat{\Gamma}_S$. Let P_μ, γ_μ, $\hat{\Gamma}_\mu$ be the restrictions of P_S, γ_S, $\hat{\Gamma}_S$ to $\hat{K}_S$. Let $\mathfrak{H}_\mu$, π_μ, Ω_μ be the Fock representation of $\mathfrak{A}(\hat{K}_\mu, \gamma_\mu, \hat{\Gamma}_\mu)$ corresponding to the basis projection P_μ. Then $\mathfrak{H}_{P_S}$, Ω_{P_S} is (unitarily equivalent to) the incomplete infinite tensor product $\otimes(\mathfrak{H}_\mu, \Omega_\mu)$ and $\mathrm{W}_{P_S}(f) = \otimes \mathrm{W}_{P_\mu}(f_\mu)$ for $f = \otimes f_\mu$.

Any normal state φ of $R_S = \{\mathrm{W}_S(f); f \in K_S\}$ is a countable sum of vector states. Each vector in $\mathfrak{H}_S \subset \mathfrak{H}_{P_S}$ is a countable linear combination of product vectors. Each product vector has a form $\otimes \Psi_\mu$ where all Ψ_μ except a countable number is Ω_μ. Therefore there exists a countable set A of μ for each normal state φ such that $\varphi(\mathrm{W}_S(f)) = (\Omega_{S'} \mathrm{W}_S((1-E)f)\Omega_S)\varphi(W_S(Ef))$ for $f \in K_S$ where $E = \sum_{\mu \in A} E_\mu$.

If π_S and $\pi_{S'}$ is quasi-equivalent, then $(\Omega_{S'}, W_{S'}(f)\Omega_{S'})$ has an extension to a normal state of R_S and hence

$$(\Omega_{S'}, \mathrm{W}_{S'}(f)\Omega_{S'}) = (\Omega_S, \mathrm{W}_S((1-E)f)\Omega_S)(\Omega_{S'}, \mathrm{W}_{S'}(Ef)\Omega_{S'}).$$

This implies that

$$S'(f, g) = S([1-E]f, [1-E]g) + S'(Ef, Eg) \tag{6.22}$$

and hence E has the required properties. Q. E. D.

Theorem. *Two primary quasifree representations π_S and $\pi_{S'}$ are quasi-equivalent if and only if the following* 3 *conditions hold:*

(1) *Coincidence of the kernel*: $N_S = N_{S'}$. *(N_S is the set of $f \in K$ such that $S(f, f) + S(\Gamma f, \Gamma f) = 0$.)*

(2) *Coincidence of the induced topology*: $\tau_S = \tau_{S'}$. *(τ_S is the topology induced on K/N_S by $\|f\|_S = [S(f, f) + S(\Gamma f, \Gamma f)]^{1/2}$.)*

(3) $1 - e^{-\chi(S)} e^{\chi(S')}$ *is in the Hilbert Schmidt class, where* $\chi(S) = \tanh^{-1} 2 S^{1/2}(1-S)^{1/2}$, *$S$ is defined by* $S(f, g) = S(f, Sg) + S(\Gamma f, \Gamma Sg)$ *and the positive square root is relative to* $(f, g)_S = S(f, g) + S(\Gamma f, \Gamma g)$.

The condition (3) is equivalent to the condition that $P_S - P_{S'}$ is in the H.S. class.

Proof. By Lemma 2.4, a quasifree state φ_S is primary (i.e. R_S is a factor) if and only if S does not have an eigenvalue $1/2$ (i.e. $S \in \mathfrak{S}_0$). Since $W_S(f) = 1$ if and only if $f \in N_S$, the condition (1) is obviously necessary. By Lemma 6.4 of [1] and due to the equivalence of topologies induced by quasi-equivalent representations, the condition (2) is necessary. The equivalence of (3) and an alternative condition is in Lemma 6.5. By Lemmas 6.9 and 6.8, the condition (3) is necessary. By Lemma 6.2, the three conditions are sufficient. Q. E. D.

Corollary. *A Bogoliubov transformation U is unitarily implementable on a Fock representation π_P if and only if $P - UPU^\dagger$ is in the H.S. class.*

Remark. In general, the condition (3) is not equivalent to the condition that $S^{1/2} - (S')^{1/2}$ is in the Hilbert Schmidt class. They become equivalent if $\chi(S)$ is bounded.

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