# On Imbedding Theorems for Besov Spaces of Functions Defined in General Regions

By

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## §1. Introduction and Main Results

This paper is a continuation of the author's work [7] concerning Besov spaces of functions defined in general regions; here we treat imbedding theorems. Our method is the same as that of the author's previous papers [5] and [7]. That is, we employ an integral representation which gives us Sobolev type inequalities with the aid of the theory of mean interpolation spaces due to Lions-Peetre [3]. To prove the basic inequalities we shall make use of the idea due to O'Neil [10] and Peetre [11].

We denote by $\Omega$ an open set in $n$-space $\mathbf{R}^n$. Let $1 \leq p$, $\xi \leq \infty$, and let $s$ be a positive integer such that $s \leq n$. For measurable functions defined in $\Omega$ we introduce the norm

$$\|f\|_{L^{p;n-s}(\Omega)} = \text{ess.} \sup_{x'' \in \Omega''} \left[ \int_{\Omega'(x'')} |f(x', x'')|^p dx' \right]^{1/p},$$

where $x = (x', x'')$, $x' \in \mathbf{R}^s$, $x'' \in \mathbf{R}^{n-s}$, $\Omega'(x'') = \{x'; (x', x'') \in \Omega\}$ and $\Omega''$ is the set of all points $x''$ such that $\Omega'(x'')$ is not empty. Also we make use of the norm

$$\|f\|_{L^{p, n-s}_*(\Omega)} = \text{ess.} \sup_{x'' \in \Omega''} \left[ \int_{\Omega'(x'')} |f(x', x'')|^p \frac{dx'}{|x|^s} \right]^{1/p},$$

where $|x|$ denotes the Euclidean norm. For any non-negative integer $m$ and $f \in C^m(\Omega)$ we define the semi-norm

(1.1) $$|f|_{W^m_{p;n-s}(\Omega)} = \sum_{|\alpha|=m} \|D^\alpha f\|_{L^{p, n-s}(\Omega)},$$

and the norm

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(1.2)                    $\|f\|_{W^m_{p;n-s}(\Omega)} = |f|_{W^m_{p,n-s}(\Omega)} + \|f\|_{L^{p;n-s}(\Omega)},$

where

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \quad D_k = \frac{\partial}{\partial x_k}, \qquad |\alpha| = \alpha_1 + \cdots + \alpha_n,$$

and $\alpha$ stands for a multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$. For any non-negative integer $m$ any positive integer $j$, any real $\tau$ with $0 < \tau < j$ and $f \in C^m(\Omega)$ we define the semi-norm

(1.3)      $|f|_{B^{m+\tau;j}_{p,\xi;n-s}(\Omega)}$

$$= \operatorname*{ess.\,sup}_{\substack{x'' \in \Omega'' \\ y'' \in \mathbf{R}^{n-s}}} \sum_{|\alpha|=m} \left\| \left[ \left\| \sum_{\nu=0}^{j} (-1)^\nu \binom{j}{\nu} D^\alpha f(x + \nu y) \right\|_{L^p(\Omega'_{j,y}(x''))} \right] |y|^{-\tau} \right\|_{L^p\left(\mathbf{R}^s, \frac{dy'}{|y|_s}\right)},$$

and norm

(1.4)                    $\|f\|_{B^{m+\tau;j}_{p,\xi;n-s}(\Omega)} = |f|_{B^{m+\tau;j}_{p,\xi;n-s}(\Omega)} + \|f\|_{L^{p,n-s}(\Omega)},$

where  $\Omega_{j,y} = \bigcap_{\nu=0}^{j} (\Omega - \nu y).$

**Definition 1.** The space $W^m_{p;n-s}(\Omega)$ is defined as the completion of the subset of $C^\infty(\Omega)$ consisting of functions whose norm given by (1.2) is finite with respect to this norm. Analogously the space $B^{m+\tau;j}_{p,\xi;n-s}(\Omega)$ is defined as the completion with respect to the norm (1.4).

If $\Omega$ satisfies the cone condition $C(T_0)$, $W^m_{p;0}(\Omega) = W^m_p(\Omega)$ coincides with the usual Sobolev space, and $B^{m+\tau;j}_{p,\xi;0}(\Omega)$ coincides with the usual Besov space $B^{m+\tau}_{p,\xi}(\Omega)$ (see [7]); while $W^m_\infty(\Omega)$ coincides with the space $\mathscr{B}^m(\Omega)$ of all bounded continuous functions defined in $\Omega$ whose all derivatives of orders $\leq m$ exist and are continuous and bounded, and $B^{m+\tau}_{\infty,\infty}(\Omega)$, $0 < \tau < 1$, is the space $\mathscr{B}^{m+\tau}(\Omega)$ of functions $f \in \mathscr{B}^m(\Omega)$ such that for all $|\alpha| = m$ $D^\alpha f$ are uniformly Hölder continuous with exponent $\tau$. Here the condition $C(T_0)$ means that is stated in the beginning of §3. From the results in [7] it is enough to consider the cases where $0 < \tau < 1 = j$ and the case where $\tau = 1$, $j = 2$, so that we set for any positive number $\mu$

$$B_{p,q;n-s}^{\mu}(\Omega) = \begin{cases} B_{p,q;n-s}^{m+\tau,1}(\Omega), & \text{if } \mu = m+\tau, \ 0 < \tau < 1, \ m \text{ is an integer,} \\ B_{p,q;n-s}^{(\mu-1)+1,2}(\Omega), & \text{if } \mu \text{ is an integer.} \end{cases}$$

We state now our main results which are proved in §3:

**Theorem 1.** *Assume that* $1 \leq p \leq q \leq \infty$, $1 \leq s \leq n$.
(i) *The imbedding operator*

$$W_p^m(\Omega) \to W_{q;n-s}^k(\Omega)$$

*exists if* $\Omega$ *satisfies the condition* $C(T_0)$, $m-k-\lambda \geq 0$, *and if either one of the conditions*; (a) $m-k-\lambda > 0$, *or* (b) $1 < p < q < \infty$, *is satisfied, where* $\lambda = n/p - s/q$.

(ii) *The imbedding operator*

$$W_p^m(\Omega) \to B_{q,\eta;n-s}^{\sigma}(\Omega)$$

*exists if* $\Omega$ *satisfies the condition* $C(T_0)$, $m-\sigma-\lambda \geq 0$, *and if either one of the following conditions is satisfied*; (a) $m-\sigma-\lambda > 0$, (b) $1 < p < q < \infty$, $p \leq \eta$, (c) $1 < p \leq \eta \leq \infty$, $s < n$, (d) $\eta = \infty$.
*More precisely, under the above conditions the inequalities*

$$(1.5) \qquad |f|_{W_{q;n-s}^k(\Omega)} \leq C\{T^{m-k-\lambda}|f|_{W_p^m(\Omega)} + T^{-k-\lambda}\|f\|_{L^p(\Omega)}\},$$

$$(1.6) \qquad |f|_{B_{q,\eta,n-s}^{\sigma}(\Omega)} \leq C(1+T^{\sigma-r})\{T^{m-\sigma-\lambda}|f|_{W_p^m(\Omega)} + T^{-\sigma-\lambda}\|f\|_{L^p(\Omega)}\}$$

*hold for every* $f \in W_p^m(\Omega)$ *and* $0 < 3T < T_0$, *where* $r$ *denotes the greatest integer less than* $\sigma$ *and* $C$ *is a constant independent of* $f$ *and* $T$.

**Theorem 2.** *Assume that* $1 \leq p \leq q \leq \infty$, $1 \leq s \leq n$.
(i) *The imbedding operator*

$$B_{p,\xi}^{\tau}(\Omega) \to W_{q;n-s}^k(\Omega)$$

*exists if* $\Omega$ *satisfies the condition* $C(T_0)$, $\tau-k-\lambda \geq 0$ *and if either one*

*of the following conditions holds:* (a) $\tau > k + \lambda$, (b) $\xi \leq q < \infty$, $p < q$, *or* (c) $\xi = 1$; *where* $\lambda = n/p - s/q$.

(ii) *The imbedding operator*

$$B^\tau_{p,\xi}(\Omega) \to B^\sigma_{q,\eta;n-s}(\Omega)$$

*exists if* $\Omega$ *satisfies the condition* $C(T_0)$, $1 \leq \xi$, $\eta \leq \infty$, $\tau \geq \sigma + \lambda$, *and if either* (a) $\tau > \sigma + \lambda$, *or* (b) $\xi \leq \eta$ *holds.*

*That is: under the assumptions stated above the inequalities*

(1.7)     $$|f|_{W^k_{q;n-s}(\Omega)} \leq C\{T^{\tau-k-\lambda}|f|_{B^\tau_{p,\xi}(\Omega)} + T^{-k-\lambda}\|f\|_{L^p(\Omega)}\},$$

(1.8)     $$|f|_{B^\sigma_{q,\eta;n-s}(\Omega)}$$

$$\leq C(1 + T^{\sigma-r})\{T^{\tau-\sigma-\lambda}|f|_{B^\tau_{p,\xi}(\Omega)} + T^{-\sigma-\lambda}\|f\|_{L^p(\Omega)}\},$$

*hold for every* $f \in B^\tau_{p,\xi}(\Omega)$ *and* $0 < 3T < T_0$, *where* $r$ *denotes the greatest integer less than* $\sigma$.


In particular, with the aid of Lemma 2.6 which will be proved in §2, these theorems yield

**Theorem 1′.** *Let* $1 \leq p \leq q \leq \infty$, $1 \leq s < n$, *and let* $x''$ *be any fixed point in* $\Omega''$. *Set* $\lambda = n/p - s/q$.

(i) *Let* $\Omega$ *be an open set satisfying the condition* $C(T_0)$ *and assume that* (a) $m - k - \lambda > 0$, *or* (b) $m - k - \lambda \geq 0$, $1 < p < q < \infty$. *Then the imbedding operator*

$$W^m_p(\Omega) \to W^k_p(\Omega'(x''))$$

*exists and its norm is estimated by a constant independent of* $x''$.

(ii) *Let* $\Omega$ *be an open set satisfying the condition* $C(T_0)$, *and assume that* (a) $m > \sigma + \lambda$, *or* (b) $m \geq \sigma + \lambda$, $1 < p \leq \eta \leq \infty$. *Then the imbedding operator*

$$W^m_p(\Omega) \to B^\sigma_{q,\eta}(\Omega'(x''))$$

*exists and its norm is estimated by a constand independent of* $x''$.

**Theorem 2′.** *Let* $1 \leq s < n$, *and let* $x''$ *be any fixed point in* $\Omega''$. *Under the assumption stated in Theorem 2, there exist the imbedding operators*

$$B_{p,\xi}^{\tau}(\Omega) \to W_q^k(\Omega'(x'')),$$

*and*

$$B_{p,\xi}^{\tau}(\Omega) \to B_{q,\eta}^{\sigma}(\Omega'(x''))$$

*with norms not greater than a constant independent of* $x''$.

The main parts of Theorem 1 and Theorem 1′ have been essentially proved already by Il'in [2]. However his conditions imposed on the region are somewhat complicated, and his proof is too long, in the author's opinion.

Here we shall remark on the norm of Besov spaces. Besov [1] introduced the space $B_{p,\xi}^{(\sigma_1, \cdots, \sigma_n)}(\mathbf{R}^n)$ whose norm is given by

$$(1.9) \quad \|f\|_{L^p(\mathbf{R}^n)} + \sum_{i=1}^{n} \left[ \int_{-\infty}^{\infty} \left\| \sum_{\nu=0}^{j_i} (-1)^\nu \binom{j_i}{\nu} D_i^{m_i} f(x + t\nu e_i) \right\|_{L^p(\mathbf{R}^n)} t^{-\tau_i \xi - 1} dt \right]^{1/\xi},$$

where $\sigma_i = m_i + \tau_i$, $m_i \geq 0$, $j_i > \tau_i > 0$; $m_i$, $j_i$ are integers, and $e_i = (0, \cdots, 1, \cdots, 0)$. He showed that this space depends only on $\sigma_1, \cdots, \sigma_n$; $p, \xi$, and independent of the choice of $m_1, \cdots, m_n$; $j_1, \cdots, j_n$. Il'in [2] employed the analogous norm for functions defined in a region $\Omega$ and proved the imbedding theorems. The norm of this kind is dependent on the choice of the coordinate system, furthermore the imbedding theorems are proved for a region more restricted than that discussed in this paper. On the other hand, Taibleson [12] showed that

$$B_{p,\xi}^{(\sigma, \cdots, \sigma)}(\mathbf{R}^n) = B_{p,\xi}^{\sigma}(\mathbf{R}^n),$$

where the right hand side is the space defined at the beginning of this section (an operator theoretical proof of this identity is given in [6]). From our investigation it seems that for a general region $\Omega$ the norm given by (1.4) is more natural than that given by (1.9).

Recently, by means of an operator theoretical method Yoshikawa [14], [15] discussed the imbedding theorems for the case where $\Omega = \mathbf{R}^n$.

## §2.  Interpolation Spaces

We shall reproduce here some theorems concerning mean interpolation spaces which will be applied later on.

First we shall introduce some notations. Let $(M, \mu)$ be a measure space and let $X$ be a Banach space. By $L^p(M, \mu; X)$ we denote the Banach space of all strongly $\mu$-measurable $X$-valued functions $u(\xi)$ such that

$$(2.1) \qquad \|u\|_{L^p(M,\mu;X)} = \left[ \int_M \|u(\xi)\|_X^p \mu(d\xi) \right]^{1/p} \qquad (1 \leq p < \infty),$$

$$= \operatorname*{ess.\,sup}_{\xi} \|u(\xi)\|_X \qquad (p = \infty)$$

is finite.  In particular, the space $L^p(\mathbf{R}^+, t^{-1}dt; X)$ is denoted by $L_*^p(\mathbf{R}^+; X)$, where $\mathbf{R}^+$ is the set of all positive numbers, and the space $L^p(M, \mu; \mathbf{C})$ is denoted by $L^p(M, \mu)$, or $L^p(M)$.

Let $X$, $Y$ be two Banach spaces continuously imbedded in a Hausdorff linear topological space $\mathscr{E}$.  Assume that $1 \leq p, q \leq \infty$, $-\infty < \xi, \eta < \infty$ and that $\xi \cdot \eta < 0$.  Then the mean interpolation spaces $S(p, \xi, X; q, \eta, Y)$ is the space of elements such that

$$(2.2) \qquad\qquad f = \int_0^\infty u(t) \frac{dt}{t}$$

for some function $u(t)$ of $t > 0$ with $t^\xi u(t) \in L_*^p(\mathbf{R}^+; X)$ and $t^\eta u(t) \in L_*^q(\mathbf{R}^+; Y)$.  The norm of $f$ is given by

$$(2.3) \qquad \inf_u \{ \max(\|t^\xi u(t)\|_{L_*^p(\mathbf{R}^+;X)}, \|t^\eta u(t)\|_{L_*^q(\mathbf{R}^+;Y)}) \},$$

where $u$ ranges over all functions satisfying the identity (2.2).  It is known that

$$S(p, \lambda\xi, X; q, \lambda\eta, Y) = S(p, \xi, X; q, \eta, Y) \quad (\lambda \neq 0)$$

(cf. Lions-Peetre [3]) with equivalent norms.  In the following we shall

write $S(p, \theta, X; p, \theta-1, Y)$ by $(X, Y)_{\theta,p}$, $0 < \theta < 1$.

**Lemma 2.1.** (Peetre [11])  *Let $(M, \mu)$ be a measure space. Assume that $1 \leqq p_1, p_2, q \leqq \infty$, $0 < \theta < 1$.   Then*

$$(L^{p_1}(M), L^{p_2}(M))_{\theta,q} = L^{(p,q)}(M)$$

*with equivalent norms, where*

(2.4)                          $$\frac{1}{p} = \frac{1-\theta}{p_1} + \frac{\theta}{p_2}$$

*and $L^{(p,q)}$ denotes the Lorentz space* (cf. [4], [9], [10]).

**Lemma 2.2.** (Peetre [11])  *Let $(M, \mu)$ be a measure space. Assume that $1 \leqq p, q \leqq \infty$.   Then*

$$L^{(p,1)}(M) \subset L^{(p,q)}(M) \subset L^{(p,\infty)}(M)$$

*with continuous injections.   And $L^{(p,p)} = L^p$ with equivalent norms.*

Combining these lemmas with the reiteration theorem (Lions-Peetre [3]), we obtain

**Lemma 2.3.** *Let $(M, \mu)$ be a measure space and assume that $1 \leqq p_1, p_2, q_1, q_2, q \leqq \infty$, $0 < \theta < 1$.   Then*

$$(L^{(p_1,q_1)}(M), L^{(p_2,q_2)}(M))_{\theta,q} = L^{(p,q)}(M)$$

*with equivalent norms,   where $p$ is given by (2.4).*

**Lemma 2.4.** *Let $\Omega$ be an open set in $\mathbf{R}^n$, and assume that $1 \leqq p < \infty$, $1 \leqq \xi \leqq \infty$, $1 \leqq s \leqq n$, $0 < \theta < 1$.   Then*

$$(L^{p;n-s}(\Omega), L^{q;n-s}(\Omega))_{\theta,\xi} \subset L^{(r,\xi);n-s}(\Omega),$$

*with continuous injection, where $1/r = (1-\theta)/p + \theta/q$.*

*Proof.*   Let

$$f(x) = \int_0^\infty u(t, x) t^{-1} dt,$$

where $t^\theta u(t,x) \in L_*^\xi(\mathbf{R}^+; L^{p;n-s}(\Omega))$, $t^{\theta-1}u(t, x) \in L_*^\xi(\mathbf{R}^+; L^{q;n-s}(\Omega))$. Then, for almost every $x''$ in $\Omega''$, $t^\theta u(t, x', x'') \in L_*^\xi(\mathbf{R}^+; L^p(\Omega'(x'')))$ and $t^{\theta-1}u(t, x', x'') \in L_*^\xi(\mathbf{R}^+; L^q(\Omega'(x'')))$. Hence, we have

$$f(x', x'') = \int_0^\infty u(t, x', x'') t^{-1} dt \in L^{(r,\xi)}(\Omega'(x''))$$

and

$$\|f(x', x'')\|_{L^{(r,\xi)}(\Omega'(x''))} \leq \|f\|_{(L^{p;n-s}(\Omega), L^{q;n-s}(\Omega))_{\theta,\xi}}.$$

This completes the proof of the lemma.

In the next section we shall frequently make use of the following well known fact:

**Lemma 2.5.** *Let $(M_1, \mu_1)$, $(M_2, \mu_2)$ be two $\sigma$-finite measure spaces, and let $K(x, y)$ be a $(\mu_1 \times \mu_2)$-measurable function such that*

$$\int_{M_1} |K(x, y)|^r \mu_1(dx) \leq C_1^r \qquad \text{for almost all } y \text{ in } M_2,$$

$$\int_{M_2} |K(x, y)|^r \mu_2(dy) \leq C_2^r \qquad \text{for almost all } x \text{ in } M_1,$$

$$(1 \leq r \leq \infty).$$

*Then the integral operator with kernel $K(x, y)$ is a bounded linear operator from $L^p(M_2, \mu_2)$ into $L^q(M_1, \mu_1)$ with norm not greater than $C_1^{1-r/q} C_2^{r/q}$, where $1/r + 1/p - 1/q = 1$. In particular, if*

$$\operatorname*{ess.\,sup}_{x,y} |K(x,y)|, \ \operatorname*{ess.\,sup}_y \int_{M_1} |K(x, y)| \mu_1(dx), \ \operatorname*{ess.\,sup}_x \int_{M_2} |K(x, y)| \mu_2(dy)$$

*are finite, then the integral operator with kernel $K(x, y)$ is bounded from $L^p(M_2)$ into $L^q(M_1)$, where $1 \leq p \leq q \leq \infty$.*

Finally we shall prove the following

**Lemma 2.6.** *For every function $f \in B_{p;\xi;n-s}^{\tau;j}(\Omega)$ and for any fixed point $x''$ in $\Omega''$*

(2.5) $$|f|_{B_{p;\xi}^{\tau,j}(\Omega'(x''))} \leq |f|_{B_{p;\xi;n-s}^{\tau,j}(\Omega)}$$

*Proof.* Set

$$F(y', y'') = \int_{\Omega'_{j,y}(x'')} \left| \sum_{\nu=0}^{j} (-1)^{\nu} \binom{j}{\nu} f(x+\nu y) \right|^{p} dx'.$$

Then by Fatou's lemma we have

$$F(y', 0) \leqq \varliminf_{y'' \to 0} F(y', y''),$$

so that, applying Fatou's lemma again, we obtain

$$\int \frac{F(y', 0)^{\xi/p}}{|y'|^{\xi\tau+s}} dy' \leqq \varliminf_{y'' \to 0} \int \frac{F(y', y'')^{\xi/p}}{|y|^{\xi\tau+s}} dy' \leqq |f|^{\xi}_{B^{\tau, j}_{p, \xi; n-s}(\Omega)}.$$

Thus the lemma is proved.

## §3. Basic Inequalities

We say that an open set $\Omega$ satisfies the condition $C(T_0)$ if there exists an $\mathbf{R}^n$-valued function $\Psi(x)$ such that (a) $x+tz+t\Psi(x) \in \Omega$ if $x \in \Omega$, $0 < t < T_0$, $|z| \leqq 1$, and that (b) $\Psi(x)$ is bounded and uniformly continuous in $\mathbf{R}^n$. If $\Omega$ satisfies the above condition, then, replacing $T_0$ by $T_0/\eta$, $\Psi(x)$ by $\eta\Psi * \varphi(x)$, where $\eta > 1$ and $\int \varphi(x)dx = 1$, $\varphi \in C_0^{\infty}(\mathbf{R}^n)$, the support of $\varphi$ is sufficiently small, we may assume that $\Psi \in \mathscr{B}^{\infty}(\mathbf{R}^n) = \bigcap_{m=1}^{\infty} \mathscr{B}^m(\mathbf{R}^n)$. In the following we always assume that $\Omega$ satisfies the condition $C(T_0)$ with $\Psi(x) \in \mathscr{B}^{\infty}(\mathbf{R}^n)$.

In the following of this section we shall discuss some inequalities, which, combining with the integral representations given in [5] and [7], prove the imbedding theorems.

We shall first prove

**Lemma 3.1.** *For $f \in L^p(\Omega)$ we define*

$$(3.1) \qquad U_0(t, x) = \int_B |f(x+tz+t\Psi(x))| \, dz.$$

*Let $1 \leqq p \leqq q \leqq \infty$ and let $1 \leqq \eta \leqq \infty$. Set $\lambda = n/p - s/q$. Then:*

(i) *For every $f \in L^p(\Omega)$ and $0 < t < T_0$*

$$(3.2) \qquad \|U_0(t, x)\|_{L^{q; n-s}(\Omega)} \leqq Ct^{-\lambda}\|f\|_{L^p(\Omega)},$$

where $g(u'') = \|f(u', u'')\|_{L^p(\Omega'(u''))}$, we have

$$\iint \left[ \int_{bB'} |f(x'+tz', x''+tz'')|^p dz' \right]^{q/p} dx'$$

$$\leq [g(x''+tz'')^p t^{-s}]^{(q-p)/p} \iint_{bB'} |f(x'+tz', x''+tz'')|^p dx' dz'$$

$$= a'b^s t^{-s(q-p)/p} g(x''+tz'')^q,$$

which gives

$$(3.6) \qquad \|U_0(t, x)\|_{L^q(\Omega'(x''))} \leq (a'b^s)^{1-1/p+1/q} t^{s/q-s/p} \int_{bB''} g(x''+tz'') dz''.$$

The desired inequality (3.2) now follows from this and Hölder's inequality.

The cases (a) and (d) in (ii) and in (iii) are immediate consequence of (i). Therefore it is sufficient to prove the other cases. We may assume that $l = \lambda$ or $l = \lambda + \sigma$.

We shall prove first (3.4). Consider the case (b). Let $f \in L^{(p,\eta)}(\Omega)$. Then there exist functions $v(t, x)$ and $w(t, x)$ such that

$$f(x) = v(t, x) + w(t, x) \qquad\qquad \text{a.e. } t,$$

$$t^{\kappa\theta} v(t, x) \in L^\eta_*(\mathbf{R}^+; L^1(\Omega)), \ t^{\kappa(\theta-1)} w(t, x) \in L^\eta_*(\mathbf{R}^+; L^q(\Omega)),$$

where

$$\theta = \frac{(p-1)q}{(q-1)p} \quad, \qquad \kappa = -n + \frac{n}{q} \quad.$$

Clearly by (3.3) we have

$$\|[\|t^\lambda U_0(t, x)\|_{L^{q, n-s}(\Omega)}]\|_{L^\eta_*([0,T])}$$

$$\leq C_1 \{ \|t^{\kappa\theta} v(t, x)\|_{L^\eta_*(\mathbf{R}^+; L^1(\Omega))} + \|t^{\kappa(\theta-1)} w(t, x)\|_{L^\eta_*(\mathbf{R}^+; L^q(\Omega))} \},$$

which gives

$$(3.4') \qquad \left\{ \int_0^T \|U_0(t, x) t^\lambda\|^\eta_{L^{q, n-s}(\Omega)} \frac{dt}{t} \right\}^{1/\eta} \leq C \|f\|_{L^{(p,\eta)}(\Omega)},$$

and (3.4) follows from this, since $L^p \subset L^{(p,\eta)}$ if $p \leq \eta$.

Next consider the case (c). It follows from (3.6) that for every $x'' \in \Omega''$,

$$\|[\| U_0(t, x) t' \|_{L^q(\Omega'(x''))}]\|_{L^\eta_*(\mathbf{R}^+)}$$

$$\leq C_1 \| t^{(n-s)/p} \int_{bB''} g(x'' + tz'') dz'' \|_{L^\eta_*(\mathbf{R}^+)}$$

$$= C_1 \left\| \int_{btB''} K(|u''|/t)\{|u''|^{(n-s)/p} g(x'' + u'')\}|u''|^{-n+s} du'' \right\|_{L^\eta_*(\mathbf{R}^+)},$$

where $g(u'') = \|f(u', u'')\|_{L^p(\Omega'(u''))}$ and $K(t) = t^{(n-s)(1-1/p)}$. Noting that

$$\int_{btB''} K(|u''|/t)|u''|^{-n+s} du'' = \frac{a'' b^{(n-s)(1-1/p)}}{1-1/p},$$

$$\int_{|u''|/b}^{\infty} K(|u''|/t) t^{-1} dt = \frac{b^{(n-s)(1-1/p)}}{(n-s)(1-1/p)},$$

$$K(|u''|/t) \leq b^{(n-s)(1-1/p)} \quad \text{when} \quad u'' \in btB'',$$

we have (3.4) with the help of Lemma 2.5.

*Proof of* (3.3). Let $1 < p < q < \infty$. Setting $u(t, x) = \{t^\lambda U_0(t, x)\}^\xi$ for $0 < t \leq T$, $u(t, x) = 0$ otherwise, the left hand side of (3.3) is equal to

$$\left\| \int_0^\infty u(t, x) \frac{dt}{t} \right\|^\xi_{L^{q/\xi; n-s}(\Omega)}.$$

Take, $\theta$, $q_1$, $q_2$ and $\kappa$ so that

$$1 < p < q_1 < q < q_2 < \infty, \quad \xi \leq q_1, \quad 0 < \theta < 1,$$

$$\frac{1}{q} = \frac{1-\theta}{q_1} + \frac{\theta}{q_2}, \qquad \kappa = -\frac{s}{q_1} + \frac{s}{q_2}.$$

Then we have from (3.4)

$$\| t^{\kappa\theta\xi} u(t, x) \|_{L^{q/\xi}_*(\mathbf{R}^+; L^{q_1/\xi; n-s}(\Omega))},$$

$$\| t^{\kappa(\theta-1)\xi} u(t, x) \|_{L^{q/\xi}_*(\mathbf{R}^+; L^{q_2/\xi; n-s}(\Omega))}$$

$$\leq \{C\|f\|_{L^p(\Omega)}\}^\xi.$$

Since $(L^{q_1/\xi;n-s}(\Omega), L^{q_2/\xi;n-s}(\Omega))_{\theta,q/\xi} \subset L^{q/\xi;n-s}(\Omega)$, we obtain

$$\left\| \int_0^\infty u(t,\ x)\frac{dt}{t} \right\|_{L^{q/\xi;n-s}(\Omega)} \leq (C\|f\|_{L^p(\Omega)})^\xi,$$

which is the desired inequalities (3.3).

*Proof of part* (iii). By Lemma 2.5 we can easily see that the integral operator with the kernel

$$\left\{ h\Big(\frac{|y|}{t}\Big)^k \Big(\frac{t}{|y|}\Big)^\sigma \right\}^\xi$$

is bounded and linear from $L_*^{\eta/\xi}(\mathbf{R}^+)$ into $L_*^{\eta/\xi;n-s}(\mathbf{R}^n)$, where $0<\sigma<k$, $\xi \leq \eta$. Thus (3.5) is a consequence of (3.4) and Jessen's inequality (generalized Minkowski inequality). This completes the lemma.

Next consider the analogous inequalities for functions belonging to a Besov space:

**Lemma 3.2.** *Let $\Omega$ be an open set in $\mathbf{R}^n$, and assume that $\Psi(x)$ is bounded and continuous. For every function $f \in B_{p;\xi}^{\tau;j}(\Omega)$, $0<\tau<j$, $0<(2j-1)t<T_0$, we define*

$$(3.7) \qquad U_j(t,\ x) = \int_B \int_{(2j-1)B} \left| \sum_{\nu=0}^j (-1)^\nu \binom{j}{\nu} f\Big(x + t\frac{(j-\nu)z+\nu w}{j} + t\Psi(x)\Big) \right| dz\,dw.$$

*Let $1 \leq p \leq q \leq \infty$, $1 \leq \xi$, $\eta \leq \infty$, $0 < \sigma < k$. Set $\lambda = n/p - s/q$. Then:*
   (i) *For any $f \in B_{p;\xi}^{\tau;j}(\Omega)$ and $0<(2j-1)t<T_0$*

$$(3.8) \qquad \|U_j(t,\ x)\|_{L^{q,n-s}(\Omega)} \leq C_1 t^{-\lambda} \int_{2B} F_j(tz)dz,$$

$$\leq Ct^{\tau-\lambda}|f|_{B_{p;\xi}^{\tau;j}(\Omega)},$$

*where $C$ is a constant independent of $f$ and $t$, and*

$$F_j(y) = \left\| \sum_{\nu=0}^j (-1)^\nu \binom{j}{\nu} f(x+\nu y) \right\|_{L^p(\Omega_{j,y})}, \quad \Omega_{j,y} = \bigwedge_{\nu=0}^j (\Omega - \nu y).$$

(ii)   *Assume  that  $l+\tau-\lambda\geqq0$.   For  every  $f\in B_{p,\xi}^{\tau,j}(\Omega)$  and  $0<$
$(2j-1)T<T_0$*

(3.9)        $\left\{\int_0^T\left[\|U_j(t,x)t^l\|_{L^{q;n-s}(\Omega)}\right]^\eta\dfrac{dt}{t}\right\}^{1/\eta}\leqq CT^{l+\tau-\lambda}|f|_{B_{p,\xi}^{\tau,j}(\Omega)},$

*where  $C$  is  a  constant  independent  of  $f$  and  $T$,  if  $l-\lambda+\tau>0$  or  if  $\xi\leqq\eta$.
And*

(3.10)      $\left\|\left\{\int_0^T(U_j(t,x)t^l)^r\dfrac{dt}{t}\right\}^{1/r}\right\|_{L^{q,n-s}(\Omega)}\leqq CT^{l+\tau-\lambda}|f|_{B_{p,\xi}^{\tau,j}(\Omega)},$

*where  $C$  is  a  constant  independent  of  $f$  and  $T$,  if  $r\leqq q$  and  if  either  one
of  the  conditions  (a)  $l+\tau>\lambda$,  (b)  $\xi\leqq q<\infty$,  $p<q$,  $r<q$,  or  (c)  $\xi\leqq r$  is
satisfied.*

(iii)   *For  any  $f\in B_{p,\xi}^{\tau,j}(\Omega)$  and  $0<(2j-1)T<T_0$*

(3.11)   $\left\|\left[\int_0^T\left\{\|U_j(t,x)\|_{L^{q,n-s}(\Omega)}h\Big(\dfrac{|y|}{t}\Big)^k|y|^{-\sigma}t^l\right\}^r\dfrac{dt}{t}\right]^{1/r}\right\|_{L_k^{\eta,n-s}(\mathbf{R}^n)}$

$\leqq CT^{l+\tau-\sigma-\lambda}|f|_{B_{p,\xi}^{\tau,j}(\Omega)},$

*where  $C$  is  a  constant  independent  of  $f$  and  $T$,  if  $r\leqq\eta$,  $l+\tau\geqq\sigma+\lambda$,  $\xi\leqq\eta$
or  if  $l+\tau>\sigma+\lambda$.*

*Proof.*   Denoting  the  characteristic  function  of  $\Omega$  by  $e(x)$,  we  find
that  the  function  $U_j(t,x)$  is  not  greater  than

$$j^n\int_{bB}\int_{2B}\Big\{\prod_{\nu=0}^j e(x+tz+t\nu w)\Big\}\Big|\sum_{\nu=0}^j\binom{j}{\nu}(-1)^\nu f(x+tz+t\nu w)\Big|\,dz\,dw,$$

$$=j^n\int_{2B}dw\int_{bB}G_{j,tw}(x+tz)\,dz,$$

where

$$G_{j,w}(u)=\Big[\prod_{\nu=0}^j e(u+\nu w)\Big]\Big|\sum_{\nu=0}^j(-1)^\nu\binom{j}{\nu}f(u+\nu w)\Big|,$$

and  hence  we  get

$$\|U_j(t,x)\|_{L^{q;n-s}(\Omega)}\leqq j^n\int_{2B}dw\left\|\int_{bB}G_{j,tw}(x+tz)\,dz\right\|_{L^{q;n-s}(\Omega)}.$$

From (3.2) it follows that

$$\left\|\int_{bB} G_{j,tw}(x+tz)\,dz\right\|_{L^{q;n-s}(\Omega)} \leq C_1 t^{-\lambda}\|G_{j,tw}(u)\|_{L^p(\Omega)}$$

$$= C_1 t^{-\lambda} F_j(tw).$$

Therefore we obtain the first inequality in (3.8), which implies the second inequality in virtue of Hölder's inequality.

(ii) *Proof of* (3.9). We may consider only the case where $\xi \leq \eta$ and $l + \tau = \lambda$. From (3.8) it follows that

$$\|U_j(t,x)t^l\|_{L^{q,n-s}(\Omega)} \leq C t^{-\tau}\int_{2B} F_j(tz)\,dz,$$

$$= C\int_{2tB} (|u|/t)^{n+\tau}|u|^{-\tau}F_j(u)|u|^{-n}\,du.$$

Applying Lemma 2.5 to the last integral transformation, we obtain (3.9).

Assume that $p < q < \infty$ and $l + \tau = \lambda$. Note the

$$g(x) = \int_0^T \{U_j(t,x)t^l\}^r \frac{dt}{t} = \int_0^\infty u(t,x)t^{-1}\,dt,$$

where $u(t,x)$ is the function which is equal to $\{t^l U_j(t,x)\}^r$ for $0 < t \leq T$ and vanishes otherwise. Set $\theta = 1 - r\rho/q$ and $\kappa = -s/\rho$, where $r\rho = \max(p,r)$. From (3.9) we obtain

$$\|[\|t^{\kappa(\theta-1)}u(t,x)\|_{L^\infty(\Omega)}]\|_{L^\zeta(\mathbf{R}^+)} \leq \{C|f|_{B^{\tau,\eta}_{p,\xi}(\Omega)}\}^r,$$

where $r\zeta = \max(\xi,r)$, and

$$\|[\|t^{\kappa\theta}u(t,x)\|_{L^{\rho,n-s}}]\|_{L^\zeta(\mathbf{R}^+)} \leq \{C|f|_{B^{\tau,\eta}_{p,\xi}(\Omega)}\}^r.$$

Hence $g \in (L^{\rho;n-s}(\Omega), L^\infty(\Omega))_{\theta,\zeta} \subset L^{(q/r,\zeta);n-s}(\Omega)$, in view of Lemma 2.4, and we can conclude that

$$(3.10^*)\qquad \left\|\left[\int_0^T \{U_j(t,x)t^l\}^r \frac{dt}{t}\right]^{1/r}\right\|_{L^{(q,\xi),n-s}(\Omega)} \leq C T^{l+\tau-\lambda}|f|_{B^{\tau,\eta}_{p,\xi}(\Omega)}.$$

If $\xi \leq q$, this implies (3.10), since $L^{(q,\xi)} \subset L^q$.

When $\xi \leq r$, (3.10) is verified by (3.9) and Jessen's inequality.

(iii)   We may assume that $l+\tau=\sigma+\lambda$, $r\leq\eta$.   From the fact stated in the proof of Lemma 3.1 (iii) it follows that

$$\left\|\left[\int_0^T\left\{\|U_j(t,\ x)\|_{L^{q;n-s}(\Omega)}h\left(\frac{|y|}{t}\right)^k|y|^{-\sigma}t^l\right\}^r\frac{dt}{t}\right]^{1/r}\right\|_{L_*^{\eta,n-s}(\mathbf{R}^n)}$$

$$\leq C\left[\int_0^T\|U_j(t,\ x)t^{l-\sigma}\|_{L^{q;n-s}(\Omega)}^\eta\frac{dt}{t}\right]^{1/\eta}.$$

This combined with (3.9) gives (3.11), and the lemma is established.

By $\mathscr{B}^{j-}(\Omega)$ we denote the space of all functions which have bounded and uniformly Lipschitz continuous derivatives up to order $j-1$.

From the preceding lemmas we obtain

**Lemma 3.3.**   *Let $\Omega$ be an open set in $\mathbf{R}^n$, and let $K(x,z)\in\mathscr{B}^{i-}(\mathbf{R}^n\times\mathbf{R}^n)$, supp $K\subset\mathbf{R}^n\times B$.   Assume that $1\leq p\leq q\leq\infty$, $1\leq\xi,\eta\leq\infty$, $0<\tau<j$, $0<\sigma<i$.   Set $\lambda=n/p-s/q$.*

*For any function $f\in C^1(\Omega)$ and $0<t<T_0$ we define*

$$(3.12)\qquad V(t,\ x)=\int K(t,\ x)f(x+tz+t\mathit{\Psi}(x))dz.$$

*Then:*

(A) (i)   *For any $f\in L^p(\Omega)\cap C^1(\Omega)$ and $0<t<T_0$*

$$(3.13)\qquad \|V(t,\ x)\|_{L^{q;n-s}(\Omega)}\leq Ct^{-\lambda}\|f\|_{L^p(\Omega)},$$

$$(3.14)\qquad |V(t,\ x)|_{B^{\sigma;i}_{q;\eta;n-s}(\Omega)}\leq C(t^{-\lambda-\sigma}+t^{-\lambda})\|f\|_{L^p(\Omega)},$$

*where $C$ is a constant independent of $f$ and $t$.*

(ii)   *There is a constant $C$ such that for any $f\in L^p(\Omega)\cap C^1(\Omega)$ and $0<T<T_0$*

$$(3.15)\qquad \left\|\int_0^T V(t,\ x)t^{l-1}dt\right\|_{L^{q,n-s}(\Omega)}\leq CT^{l-\lambda}\|f\|_{L^p(\Omega)},$$

*if either one of the condition* (a) $l>\lambda$, *or* (b) $l\geq\lambda$, $1<p<q<\infty$ *is satisfied.*

(iii)   *There is a constant $C$ such that for any $f\in L^p(\Omega)\cap C^1(\Omega)$ and $0<T<T_0$*

(3.16) $\quad \left| \int_0^T V(t, x) t^{l-1} dt \right|_{B_{q;\eta;n-s}^{\sigma,i}(\Omega)} \leq C(1+T^\sigma) T^{l-\sigma-\lambda} \|f\|_{L^p(\Omega)}$,

*if $\sigma+\lambda \leq l$, and if either one of the conditons (a) $\sigma+\lambda < l$, (b) $1 < p < q \leq \infty$, $p \leq \eta$, (c) $1 < p \leq \eta \leq \infty$, $s < n$, or (d) $\eta = \infty$ is satisfied.*

(B) *Under the additional assumption that*

$$K(x, z) = \sum_{|\alpha| = j-1} D_z^\alpha H_\alpha(x, z),$$

*where $D_z^\beta H_\alpha(x, z) \in \mathscr{B}^{i-}(\mathbf{R}^n \times \mathbf{R}^n)$ for any $\alpha$ with $|\alpha| = j-1$ and any $\beta$ with $|\beta| \leq j-1$, and that*

$$\text{supp } H_\alpha \subset \mathbf{R}^n \times B, \quad \int H_\alpha(x, z) dz = 0$$

*for any $|\alpha| = j-1$, there is a constant $C$ such that for every $f \in B_{p;\xi}^{\tau,j}(\Omega) \cap C^j(\Omega)$ and $0 < t < T$ (or $0 < T < T_0$)*

(i)

(3.17) $\quad \|V(t, x)\|_{L^{q;n-s}(\Omega)} \leq C t^{\tau-\lambda} |f|_{B_{p;\xi}^{\tau,j}(\Omega)}$,

(3.18) $\quad |V(t, x)|_{B_{q;\eta,n-s}^{\sigma,i}(\Omega)} \leq C(t^{\tau-\sigma-\lambda} + t^{\tau-\lambda}) |f|_{B_{p;\xi}^{\tau,j}(\Omega)}$;

(ii)

(3.19) $\quad \left\| \int_0^T V(t, x) t^{l-1} dt \right\|_{L^{q;n-s}(\Omega)} \leq C T^{l+\tau-\lambda} |f|_{B_{p;\xi}^{\tau,j}(\Omega)}$,

*if $l+\tau \geq \lambda$ and if either one of the conditions (a) $l+\tau > \lambda$, (b) $\xi \leq q < \infty$, $p < q$, or (c) $\xi = 1$ is satisfied; and*

(iii)

(3.20) $\quad \left| \int_0^T V(t, x) t^{l-1} dt \right|_{B_{q;\eta;n-s}^{\sigma,i}(\Omega)} \leq C(1+T^\sigma) T^{l+\tau-\sigma-\lambda} |f|_{B_{p;\xi}^{\tau,j}(\Omega)}.$

*if either (a) $l+\tau-\sigma-\lambda > 0$, or (b) $l+\tau-\sigma-\lambda \geq 0$, $\xi \leq \eta$ is satisfied.*

*Proof.* Note that

$$\left| \sum_{\nu=0}^i (-1)^\nu \binom{i}{\nu} V(t, x+\nu y) \right| \leq C_1 \left\{ h\left(\frac{|y|}{t}\right)^i + h(|y|)^i \right\} \sum_{\nu=0}^i U_j(t, x+\nu y)$$

(see proof of [7] Lemma 2.6), then the assertions follow from Lemma 3.1 and Lemma 3.2.

*Proof of Theorem* 1 *and Theorem* 2. To prove Theorem 1 it is enough to show that the inequalities (1.5) and (1.6) hold for every $f \in W_p^m(\Omega) \cap C^\infty(\Omega)$. From Lemma 2.1 in [5] it follows that for any $f \in C^m(\Omega)$ and $|\beta| < m$

$$D^\beta f(x) = \sum_{|\alpha|=m, \alpha \geq \beta} \int_0^T t^{m-|\beta|-1} dt \int \omega_{\alpha-\beta}(x, z) D^\alpha f(x+tz+t\Psi(x)) dz$$

$$+ T^{-|\beta|} \int \omega_{m,\beta}(x, z) f(x+Tz+T\Psi(x)) dz,$$

which gives the desired inequalities with the aid of Lemma 3.3 (A).

In the same way Theorem 2 is verified by Lemma 3.3 (B) and the integral representation (2.1′) in [7].

## Appendix A

Let $\Phi(t)$, $t \in \mathbf{R}^+$, be a convex non-decreasing, positive function such that $\Phi(t)/t \to 0$ as $t \to 0$ and that $\Phi(t)/t \to +\infty$ as $t \to +\infty$. By $L^\Phi(\Omega)$ we write the space of measurable functions $f(x)$ such that there is $\lambda > 0$ with

$$\int \Phi(\lambda |f(x)|) dx \leq 1.$$

With respect to the norm

$$\|f\|_\Phi = \inf\left\{\lambda^{-1}; \int \Phi(\lambda |f(x)|) dx \leq 1\right\}$$

$L^\Phi(\Omega)$ is a Banach space, and is called a *Orlicz space*.

As a supplement to the imbedding theorem, we have the following theorem concerning the limiting case.

**Theorem A.** *The imbedding operator*

$$B_{p,\infty}^\tau(\Omega) \to L^{\Phi_p}(\Omega)$$

*exists if* $\tau = n/p$, *where*

$$\Phi_p(t) = \sum_{j \geq p} \frac{t^j}{j!}.$$

*Proof.* Let $q \geq p$. It follows from our proof of Theorem 2 that

$$\|f\|_{L^q(\Omega)} \leqq C_1 q\, T^{\tau-\lambda}\, |f|_{B^\tau_{p,\infty}(\Omega)} + C_2\, T^{-\lambda}\|f\|_{L^p(\Omega)},$$

where $\lambda = n/p - n/q$, $0 < T < T_0/3$, and where $C_1$, $C_2$ are constant independent of $f$, $T$, and $q$. Take a fixed $T_1 < T_0/3$, and put

$$A = C_1\,|f|_{B^\tau_{p,\infty}(\Omega)}, \qquad B = C_2\|f\|_{L^p(\Omega)},$$

$$k = \max(p,\ BA^{-1}T_1^{-n/p}).$$

Then we have,

$$\|f\|_{L^q(\Omega)} \leqq 2B^{p/q}(qA)^{1-p/q} \qquad \text{if } q \geqq k,$$

$$(\text{take } T = (B/qA)^{p/n})$$

$$\|f\|_{L^q(\Omega)} \leqq 2\, T_1^{n/q} B\, T_1^{-n/p} \qquad \text{if } p \leqq q < k,$$

$$(\text{take } T = T_1).$$

Consequently we obtain

$$\sum_{j \geqq p} \frac{\lambda^j}{j!} \int |f(x)|^j\, dx \leqq T_1^n \sum_{k > j \geqq p} \frac{(2\lambda B)^j}{j!}\, T_1^{-nj/p}$$

$$+ \left(\frac{B}{A}\right)^p \sum_{j \geqq k} \frac{(2\lambda A)^j}{j!} j^{j-p}$$

$$\leqq T_1^n \sum_{k > j \geqq p} (2\lambda B\, T_1^{-n/p})^j + \left(\frac{B}{A}\right)^p \sum_{j \geqq k} (2e\lambda A)^j.$$

Therefore

$$\int \Phi_p(\lambda\,|f(x)|)\,dx \leqq 1,$$

if

$$4eA\lambda \leqq 1 \text{ and } 4(e + T_1^{-n/p})\lambda B$$

This implies that

$$\|f\|_{\Phi_p} \leqq 4eA + 4(e + T_1^{-n/p})B$$

which is the desired inequality.

## Appendix B

In this appendix we prove a theorem, half of which is contained in Theorem 1.4 in [7].

**Theorem B.** *Let* $\mu_0$, $\mu_1 > 0$, $1 \leq p_0$, $p_1$, $q_0$, $q_1 \leq \infty$ *and let* $0 < \theta < 1$. *Set*

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad \mu = (1-\theta)\mu_0 + \theta\mu_1,$$

$$\mathscr{B}_0 = B_{p_0,q_0}^{\mu_0}(\Omega), \quad \mathscr{B}_1 = B_{p_1,q_1}^{\mu_1}(\Omega).$$

*Then*, (i) *if* $r \geq p$ *and* $r \geq q$,

$$(\mathscr{B}_0, \mathscr{B}_1)_{\theta,r} \supset B_{p,q}^{\mu}(\Omega).$$

(ii) *If* $r \leq p$ *and* $r \leq q$,

$$(\mathscr{B}_0, \mathscr{B}_1)_{\theta,r} \subset B_{p,q}^{\mu}(\Omega).$$

*All injections above are continuous.*

The theorem is proved in virtue of the following lemmas.

**Lemma B.1.**

  (a)  $W_p^m(\Omega) \subset (B_{p_0,\infty}^m(\Omega), B_{p_1\infty}^m(\Omega))_{\theta,q}$, *if* $q \geq p$.
  (b)  $W_p^m(\Omega) \subset [B_{p_0,\infty}^m(\Omega), B_{p_1,\infty}^m(\Omega)]_\theta$.

*Proof.* Set $k = n^m + 1$. Define the linear operator $L : \{L^p(\Omega)\}^k \to B_{p,\infty}^m(\Omega)$ by

$$L\{f_\alpha\} = \sum_{|\alpha|=m}(-1)^m m \int_0^T t^{m-1} dt \int K_\alpha(x, z) f_\alpha(x + tz + t\Psi(x)) dz$$

$$+ \int \omega_{m+2}(x, z) f_0(x + Tz + T\Psi(x)) dz,$$

where

$$K_\alpha(x, z) = \frac{2}{m(m+1)} \sum_{|\gamma|=2} \binom{\alpha+\gamma}{\gamma} D_z^\gamma \omega_{\alpha+\gamma}(x, z).$$

It follows from Lemma 3.3 that $L$ is bounded. Also with the aid of Lemma 2.1 corollary in [7] we have

$$L(\{f^{(\alpha)}\}_{|\alpha|=m \text{ or } 0}) = f$$

for every $f \in W_p^m(\Omega) \cap C^\infty(\Omega)$.

Applying the interpolation theorem, we see

$$L: (\{L^{p_0}(\Omega)\}^k, \{L^{p_1}(\Omega)\}^k)_{\theta,q} \to (B_{p_0,\infty}^m(\Omega), B_{p_1,\infty}^m(\Omega))_{\theta,q}$$

is bounded. Combining this with

$$(L^{p_0}, L^{p_1})_{\theta,q} = L^{(p,q)} \supset L^p,$$

and that

$$\{f^{(\alpha)}\}_{|\alpha|=m,0} \in (L^p(\Omega))^k$$

for every $f \in W_p^m(\Omega)$, we have part (a). Part (b) can be proved similarly.

Let $(M, \mu)$ be a measure space, and let $\rho(x)$ be a non-negative measurable function. By $L^{p,\rho}(M; X)$ we denote the Banach space of functions $f$ such that $\rho f \in L^p(M; X)$.

**Lemma B.2.** *Let $X_0$, $X_1$ be two Banach spaces contained in a Hausdorff linear topological space, and let $1 \le q_0, q_1, p \le \infty$, $0 < \theta < 1$. Set*

$$\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \quad X_{\theta,p} = (X_0, X_1)_{\theta,p},$$

$$\rho(x) = \rho_0(x)^{1-\theta} \rho_1(x)^\theta,$$

*where $\rho_0$, $\rho_1$ be non-negative measurable functions,*

   (i) *If $p \le q$, then*

$$(L^{q_0,\rho_0}(M; X_0), L^{q_1,\rho_1}(M; X_1))_{\theta,p} \subset L^{q,\rho}(M; X_{\theta,p})$$

*with continuous injection.*

   (ii) *If $(M, \mu)$ is $\sigma$-finite and if $q \le p$, $q < +\infty$, then*

$$L^{q,\rho}(M; X_{\theta,p}) \subset (L^{q_0,\rho_0}(M; X_0), L^{q_1,\rho_1}(M; X_1))_{\theta,p}$$

*with continuous injection.*

*Proof.*  First we note that

$$(Y_0, Y_1)_{\theta,p} = S(p_0, \theta, Y_0; p_1, \theta-1, Y_1)$$

where $1/p = (1-\theta)/p_0 + \theta/p_1$ (Peetre [12]).

(i)  Since $p \leq q$, we can choose $p_0 \leq q_0$, $p_1 \leq q_1$.  Let

$$f(x) = \int_0^\infty u(t, x) \frac{dt}{t}$$

where

$$u(t, x) \in L_*^{p_0, \theta}(\mathbf{R}^+; L^{q_0, \rho_0}(M; X_0)) \cap L_*^{p_1, \theta-1}(\mathbf{R}^+; L^{q_1, \rho_1}(M; X_1))$$

$$(L_*^{p, \alpha}(\mathbf{R}^+; X) = L^{p, t^\alpha}(\mathbf{R}^+, t^{-1}dt; X)).$$

By a lemma due to Lions-Peetre [3]

$$\|f\|_{X_{\theta,p}} \leq \|t^\theta u(t, x)\|_{L_*^{p_0}(\mathbf{R}^+; X_0)}^{1-\theta} \|t^{\theta-1} u(t, x)\|_{L_*^{p_1}(\mathbf{R}^+; X_1)}^\theta.$$

So that, in view of Hölder's inequality and Jessen's inequality,

$$\|\rho(x)f(x)\|_{L^q(M; X_{\theta,p})}$$

$$\leq \|\rho_0(x)t^\theta u(t, x)\|_{L_*^{q_0}(M; L_*^{p_0}(\mathbf{R}^+; X_0))}^{1-\theta}$$

$$\times \|\rho_1(x)t^{\theta-1} u(t, x)\|_{L^{p_1}(M; L_*^{p_1}(\mathbf{R}^+; X_1))}^\theta$$

$$\leq \|t^\theta u(t,x)\|_{L_*^{p_0}(\mathbf{R}^+; L^{q_0,\rho_0}(M; X_0))}^{1-\theta}$$

$$\times \|t^{\theta-1} u(t, x)\|_{L_*^{p_1}(\mathbf{R}^+; L^{q_1,\rho_1}(M; X_1))}^\theta,$$

which implies part (i).

*Proof of part* (ii).  It is sufficient to show that for any simple $X_{\theta,p}$-valued function $f$ there exists $u(t, x)$ such that

$$\|t^\theta \rho_0(x) u(t, x)\|_{L_*^{p_0}(\mathbf{R}^+; L^{q_0}(M; X_0))},$$

$$\|t^{\theta-1} \rho_1(x) u(t, x)\|_{L_*^{p_1}(\mathbf{R}^+; L^{q_1}(M; X_1))} \leq C \|f\|_{L^{q,\rho}(M; X_{\theta,p})},$$

$$f(x) = \int_0^\infty u(t, x) \frac{dt}{t}.$$

Let

$$f(x) = \sum_\nu a_\nu \psi_\nu(x),$$

where $a_\nu \in X_{\theta,p}$, and $\{\psi_\nu\}$ are characteristic functions of measurable, disjoint sets. By definition for any $c > 1$ we can find $u_\nu(t)$ such that

$$a_\nu = \int_0^\infty u_\nu(t) \frac{dt}{t}$$

with

$$\|t^\theta u_\nu\|_{L_*^{p_0}(\mathbf{R}^+; X_0)}, \quad \|t^{\theta-1} u_\nu\|_{L_*^{p_1}(\mathbf{R}^+; X_1)} \leqq c \|a_\nu\|_{X_{\theta,p}}.$$

Then

$$u(t, x) = \sum_\nu u_\nu(\varphi(x)t)\psi_\nu(x),$$

where $\varphi(x) = \{\|f(x)\|_{X_{\theta,p}} \sigma^{-1}\}^\lambda \rho_0(x)^{q/q_1} \rho_1(x)^{-q/q_0}$,

$$\sigma = \|f\|_{L^{q,\rho}(M; X_{\theta,p})}, \quad \lambda = \frac{q_0 - q}{q_0 \theta},$$

has the required property.

**Lemma B.3.** (a generalization of the commutativity theorem due to Grisvard). *Let* $X_0$, $X_1$, $Y_0$, $Y_1 \subset \mathcal{X}$ *be Banach spaces.*

(i) *Assume that there exists a linear operator* $E$ *from* $\mathcal{X}$ *into* $L^1_{loc}(\mathbf{R}^+; \mathcal{X})$ *such that* $E$ *is continuous from* $S_i$ *into* $W_i$, $(i = 0, 1)$ *and* $PEf = f$ *for every* $f \in S_0 + S_1$, *where*

$$S_i = (X_i, Y_i)_{\sigma_i, q_i},$$

$$W_i = L_*^{q_i, \lambda_i \sigma_i}(\mathbf{R}^+; X_i) \cap L_*^{q_i, \lambda_i(\sigma_i - 1)}(\mathbf{R}^+; Y_i), \qquad (i = 0, 1)$$

$$Pu = \int_0^\infty u(t) \frac{dt}{t}.$$

*Set* $X_{\theta,p} = (X_0, X_1)_{\theta,p}$, $Y_{\theta,p} = (Y_0, Y_1)_{\theta,p}$. *If* $p \leqq q$, *then*

$$(S_0, S_1)_{\theta,p} \subset (X_{\theta,p}, Y_{\theta,p})_{\sigma,q} = S_\theta,$$

*where* $\lambda=(1-\theta)\lambda_0+\theta\lambda_1$, $\lambda\sigma=(1-\theta)\lambda_0\sigma_0+\theta\lambda_1\sigma_1$.

(ii)    *Assume that there exists a linear operator* $J$ *from* $\mathscr{L}_0+\mathscr{L}_1$ *into* $\mathscr{X}$ *such that* $J$ *is continuous from* $\mathscr{L}_i$ *into* $S_i$ $(i=0, 1)$, *and* $Ju=f$ *if* $u(t)$ $=f$ *a.e.t. Here*

$$\mathscr{L}_i=L_*^{q_i,\lambda_i\sigma_i}(\mathbf{R}^+; X_i)+L_*^{q_i,\lambda_i(\sigma_i-1)}(\mathbf{R}^+; Y_i).$$

*If* $p\geq q$, *then* $S_\theta\subset(S_0, S_1)_{\theta,p}$.

*Proof.* (i) From the interpolation theorem it follows that $E$ is continuous from $(S_0, S_1)_{\theta,p}$ into $(W_0, W_1)_{\theta,p}$. In view of Lemma B.2.(i) we see

$$(W_0, W_1)_{\theta,p}\subset L_*^{q,\lambda\sigma}(\mathbf{R}^+; X_{\theta,p})\cap L_*^{q,\lambda(\sigma-1)}(\mathbf{R}^+; Y_{\theta,p}).$$

Therefore, $PE$ is a continuous injection from $(S_0, S_1)_{\theta,p}$ into $S_\theta$.

(ii)    By interpolation we see that $J$ maps continuously $(\mathscr{L}_0, \mathscr{L}_1)_{\theta,p}$ into $(S_0, S_1)_{\theta,p}$. It follows from Lemma B.2. (ii) that $J$ is a bounded linear operator from

$$L_*^{q,\lambda\sigma}(\mathbf{R}^+; X_{\theta,p})+L_*^{q,\lambda(\sigma-1)}(\mathbf{R}^+; Y_{\theta,p})$$

into $(S_0, S_1)_{\theta,p}$, so that we have (ii). This completes the lemma.

*Proof of the theorem. Part* (i). Combining Lemma B.1, Lemma B.3.(ii) and Lemma 4.3 in [7] (this lemma is still valid if we replace $W_p^m$ by $B_{p,\infty}^m$), we have

$$(\mathscr{B}_0, \mathscr{B}_1)_{\theta,r}$$

$$\supset((L^{p_0}, L^{p_1})_{\theta,r}, (B_{p_0,\infty}^m, B_{p_1,\infty}^m)_{\theta,r})_{\sigma,q}$$

$$\supset(L^p, W_p^m)_{\sigma,q}=B_{p,q}^\mu,$$

where $\mu_i=\sigma_i m$ $(i=0, 1)$, $\mu=\sigma m$.

*Part* (ii).

In view of Lemma B.3.(i), Lemma 4.2 in [7] and the fact that

$$(W_{p_0}^m, W_{p_1}^m)_{\theta,r}\subset W_p^m$$

(apply the interpolation theorem to the operator $f \to D^\alpha f$), we see that

$$(\mathscr{B}_0, \mathscr{B}_1)_{\theta,r} \subset ((L^{p_0}, L^{p_1})_{\theta,r}, (W^m_{p_0}, W^m_{p_1})_{\theta,r})_{\sigma,q}$$

$$\subset (L^p, W^m_p)_{\sigma,q} = B^\mu_{p,q}.$$

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