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On Global Solutions of the Generalized Korteweg-de Vries Equation

By

Masayoshi TSUTSUMI*

1. Introduction

Various physical phenomena in which nonlinearity and dispersion are important (e.g., shallow water waves [1], hydromagnetic waves [2] and acoustic waves in an anharmonic crystal [3]) lead to the Korteweg-de Vries equation

$$u_t + uu_x + u_{xxx} = 0$$

where subscripts denote partial differentiations. Existence and uniqueness of the global solutions of the Korteweg-de Vries equation for appropriate initial and boundary conditions has been proved by A. Sjöberg [4], R. Teman [5] and Y. Kametaka [6].

The purpose of this note is to investigate the initial-boundary value problem for the generalized Korteweg-de Vries equation:

$$(1.1)\qquad u_t + \gamma u^p u_x + u_{xxx} = 0, \qquad \gamma = \pm 1, p = 1, 2, \cdots\cdots,$$

$$(1.2)\qquad u(x, 0) = f(x), \qquad 0 < x < 1,$$

$$(1.3)\qquad u(0, 1) = u(1, t) \qquad \text{for all } t \geqq 0.$$

Equation (1.1) also arises in the study of an anharmonic crystal [3]. The cases $p=2$ and $p=3$ have been studied by T. Mukasa and R. Iino [7] and K. Masuda [8], respectively. M. Tsutsumi, T. Mukasa and R. Iino

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* Department of Applied Physics, Waseda University, Tokyo, Japan.

[9] have treated more general nonlinear dispersive equation

$$u_t-(f(u))_x+\delta u_{xxx}=0, \qquad \delta>0.$$

But their assumption on the nonlinear term does not include the cases $\gamma>0$, $p\geqq 4$ and $\gamma<0$, $p=$odd number $\geqq 5$.

Below, §2 is devoted to preliminaries. In §3 "potential well" associated with the problem (1.1)–(1.3) is considered and Main Theorem is presented. The proof of this theorem is carried out in the final three sections: Uniqueness is proved in §4, local existence is established in §5, and a priori estimates are stated in §6.

2. Preliminaries

We denote by $L^p(a, b; E)$ the space of E-valued weakly measurable functions $u(t)$ on (a, b) for which

$$\left(\int_a^b \|u(t)\|_E^p dt\right)^{1/p}<\infty, \qquad \text{if } 1\leqq p<\infty,$$

$$\sup_{a\leqq t\leqq b} \|u(t)\|_E<\infty \quad , \qquad \text{if } p=\infty,$$

where E is a Banach space with norm $\|\cdot\|_E$. By $C^m[a, b; E]$[1)] we denote the space of functions which are m times continuously differentiable over $[a, b]$ with values in E. We denote by H_s (s integer) the Hilbert space whose elements are real valued 1-periodic functions with finite norm

$$\|u\|_s^2=\sum(1+(2\pi k)^2)^s|\alpha_k|^2$$

where $u(x)=\sum\alpha_k e^{2\pi ik}$, $\alpha_{-k}=\bar{\alpha}_k$, $i=\sqrt{-1}$. We denote by (f, g) the scalar product and by $\|f\|$ the norm in the space $L^2(0, 1)$ $(=H_0)$, that is,

$$(f, g)=\int_0^1 f(x)g(x)dx \qquad \text{and} \qquad \|f\|^2=(f, f).$$

Then we have, for $u(x)\in H_s$ $(s\geqq 0)$

$$c_1\sum_{0\leqq\rho\leqq s}\|D^\rho u\|\leqq\|u\|_s\leqq c_2\sum_{0\leqq\rho\leqq s}\|D^\rho u\|,$$

1) In the sequel of this note, by m we always denote a nonnegative integer.

where $D^\rho = \partial^\rho/\partial x^\rho$ and c_1, c_2 are positive constants depending on s.

The following lemma is well known.

Lemma 1. *If $u(x) \in H_s$, $s \geqq k+1$ (k nonnegative integer), then u is of class C^k and*

$$\max_{0 \leqq x \leqq 1} |D^\rho u(x)| \leqq c\|u\|_s \qquad for \quad 0 \leqq \rho \leqq k, \tag{2.1}$$

where c denote various positive constants.

Lemma 2. *For any function $u(x) \in H_1$, the inequality*

$$\max_{0 \leqq x \leqq 1} |u(x)| \leqq c\|u\|^{1/2}(\|u\| + \|u_x\|)^{1/2} \tag{2.2}$$

is valid.

In the next lemma we are concerned with invariant integrals associated with the solution of the problem (1.1)–(1.3).

Lemma 3. *Let $u(x, t)$ be a solution of the problem* (1.1)–(1.3). *Then there are constants δ_1, δ_2 such that*

$$I_1(u) = \frac{1}{2}\int_0^1 u^2(x, t)\,dx = \delta_1, \tag{2.3}$$

$$I_2(u) = \int_0^1 (u_x^2/2 - \gamma u^{p+2}/(p+1)(p+2))\,dx = \delta_2. \tag{2.4}$$

Proof. Differentiation with respect to t of the left members of (2.2) and (2.3), use of the differential equation and integration by parts will lead us to prove the desired properties.

3. Potential Well and Main Theorem

Put

$$J(u) = I_1(u) + I_2(u). \tag{3.1}$$

Following the idea of D.H. Sattinger [10] (see also J.-L. Lions [11]) we define "depth of a potential well" by

(3.2) $$d=\inf_{\substack{u\in H_1\\ u\neq 0}} \sup_{\lambda\geqq 0} J(\lambda u).$$

Lemma 4. *We have $d>0$.*

Proof. Evidently we get

$$J(\lambda u)=\frac{\lambda^2}{2}\left(\int_0^1 u_x^2 dx+\int_0^1 u^2 dx\right)-\frac{\lambda^{p+2}}{(p+1)(p+2)}\gamma\int_0^1 u^{p+2}\,dx$$

$$=\frac{\lambda^2}{2}a(u)-\frac{\lambda^{p+2}}{(p+1)(p+2)}b(u),$$

where

$$a(u)=\int_0^1 u_x^2 dx+\int_0^1 u^2 dx,$$

and

$$b(u)=\gamma\int_0^1 u^{p+2} dx.$$

Lemma 1 yields

$$|b(u)|\leqq c\|u\|_1^{p+2}\leqq c\, a(u)^{(p+2)/2}.$$

Hence,

if $b(u)\leqq 0$, $\quad \sup_{\lambda\geqq 0} J(\lambda u)=+\infty$

and if $b(u)>0$, $\quad \sup_{\lambda\geqq 0} J(\lambda u)=J(\{(p+1)a(u)/b(u)\}^{1/p}u)$

$$=\frac{p(p+1)^{2/p}}{2(p+2)}\cdot\frac{a(u)^{(p+2)/p}}{b(u)^{2/p}}$$

$$\geqq\frac{p(p+1)^{2/p}}{2(p+2)}\cdot\frac{1}{c^{2/p}}>0.$$

Q.E.D.

We introduce the potential well W:

$$W=\{u\,|\,u\in H_1,\ 0\leqq J(\lambda u)<d,\ \forall\lambda\in[0,1]\}.$$

Then we can easily obtain the following lemmas (see Lions [11], p. 31):

Lemma 5. *If* $u \in W$, *then we have* $\theta u \in W$, $\forall \theta \in [0, 1]$.

Lemma 6. *We have*

$$W = W_* \cup \{0\}$$

where

$$W_* = \left\{ u \mid \in H_1,\ a(u) - \frac{1}{p+1} b(u) > 0,\ 0 \leqq J(u) < d \right\}.$$

Lemma 7. *The set* W *is bounded in* H_1 *if* d *is finite.*

Now we state the main theorem.

Main Theorem. *For every initial function* $f(x) \in W \cap H_{3(m+1)}$, *the initial-boundary value problem* (1.1)–(1.3) *has a unique solution* $u(x, t)$ *such that for any* $T > 0$

$$u(x, t) \in L^\infty(0, T; H_{3(m+1)}) \cap C[0, T; H_{3m}] \cap \cdots\cdots \cap C^m[0, T; L^2(0, 1)].$$

Corollary. *If* $f(x) \in W \cap H_\infty$, *then* $u(x, t) \in C^\infty[0, T; H_\infty]$,

where $H_\infty = \bigcap_m H_m$.

Remark. In the cases of $p = 1, 2, 3, (\gamma = \pm 1)$ and $p = 2q$ ($q \geqq 2$, integer, $\gamma = -1$), the above theorem (corollary) holds for $f(x) \in H_{3(m+1)}$ ($\in H_\infty$) (see [9]).

4. Uniqueness

Assume that $u(x, t)$ and $v(x, t)$ are two solutions of the problem (1.1)–(1.3) satisfying the same initial condition. Then the difference $w = u - v$ satisfies

$$w_t + w_{xxx} = -\gamma u^p u_x + \gamma v^p v_x,$$

$$w(x, 0) = 0,$$

$$w(0, t) = w(1, t) \qquad \text{for all } t \geqq 0.$$

Hence we get

$$\frac{1}{2}d\|w\|^2/dt=(w,\ w_t)=-\gamma(w,\ u^p u_x)+\gamma(w,\ v^p v_x)$$

$$\leqq \text{const.}\|w\|^2$$

from which it follows that in $0\leqq t\leqq T$

$$w\equiv 0.$$

5. Local Existence

We attempt to construct the local solution of the problem (1.1)–(1.3) by iteration:

$$u_t^{(n)}+\gamma(u^{(n-1)})^p u_x^{(n)}+u_{xxx}^{(n)}=0, \tag{5.1}$$

$$u^{(n)}(x,\ 0)=f(x), \qquad 0<x<1, \qquad (n=1,\ 2,\ \cdots\cdots) \tag{5.2}$$

$$u^{(n)}(0,\ t)=u^{(n)}(1,\ t), \qquad \text{for all } t\geqq 0, \tag{5.3}$$

with initial element

$$u^{(0)}(x,\ t)\equiv 0. \tag{5.4}$$

Lemma 8. *Suppose that* $f(x)\in H_{3(m+1)}$, $\varphi(x,\ t)\in L^\infty(0,\ T;\ H_{3(m+1)})\cap C[0,\ T;\ H_{3m}]\cap\cdots\cdots\cap C^m[0,\ T;\ L^2(0,\ 1)]$. *Then there exists a unique function* $u(x,\ t)$ *satisfying*

$$u\in L^\infty(0,\ T;\ H_{3(m+1)})\cap C[0,\ T;\ H_{3m}]\cap\cdots\cdots\cap C^m[0,\ T;\ L^2(0,\ 1)], \tag{5.5}$$

$$u_t+\varphi(x,\ t)u_x+u_{xxx}=0, \tag{5.6}$$

$$u(x,\ 0)=f(x), \qquad 0<x<1, \tag{5.7}$$

$$u(0,\ t)=u(1,\ t), \qquad \forall t\geqq 0. \tag{5.8}$$

This lemma is easily proved by the method of semi-discrete approximation.

In virtue of Lemma 8, we see that for each $n>0$

$$u^{(n)}(x,\ t)\in L^\infty(0,\ T;\ H_{3(m+1)})\cap C[0,\ T;\ H_{3m}]\cap\cdots\cdots\cap C^m[0,\ T;\ L^2(0,\ 1)].$$

Next we shall show that the family $\{u^{(n)}\}$ is uniformly bounded with respect to n in a certain small time interval.

Lemma 9. *There exists a positive constant t_k such that in $0 \leqq t \leqq t_k < T$*

$$\|u^{(n)}(t)\|_k \leqq L_k, \quad k=0, 1, \cdots\cdots, 3(m+1), \tag{5.9}$$

where L_k are constants independent of n.

Proof. At first we prove the assertion in the case $k=2$.

Multiplication of the equation (5.1) by $u^{(n)}$, integration with respect to x and Lemma 1 give

$$\begin{aligned} d\|u^{(n)}\|^2/dt &= -2\gamma(u^{(n)}, (u^{(n-1)})^p u_x^{(n)}) \\ &= \gamma p((u^{(n)})^2, \ (u^{(n-1)})^{p-1} u_x^{(n-1)}) \\ &\leqq p(\sup|u^{(n-1)}|)^{p-1} \sup|u_x^{(n-1)}| \, \|u^{(n)}\|^2 \\ &\leqq c\|u^{(n-1)}\|_2^p \ \|u^{(n)}\|^2 \end{aligned}$$

from which it follows that

$$\|u^{(n)}\|^2 \leqq \|f\|^2 \exp(c \sup_{0 \leqq \tau \leqq t} \|u^{(n-1)}(\tau)\|_2^p t). \tag{5.10}$$

Multiplication of the equation (5.1) by $u_{xx}^{(n)}$, integration with respect to x and Lemma 1 give

$$\begin{aligned} d\|u_x^{(n)}\|^2/dt &= -\gamma p(u_x^{(n)}, (u^{(n-1)})^{p-1} u_x^{(n-1)} u_x^{(n)}) \\ &\leqq p(\sup|u^{(n-1)}|)^{p-1} \sup|u_x^{(n-1)}| \, \|u_x^{(n)}\|^2 \\ &\leqq c\|u^{(n-1)}\|_2^p \ \|u_x^{(n)}\|^2 \end{aligned}$$

from which it follows that

$$\|u_x^{(n)}\|^2 \leqq \|f_x\|^2 \exp(c \sup_{0 \leqq \tau \leqq t} \|u^{(n-1)}(\tau)\|_2^p t). \tag{5.11}$$

Multiplying the equation (5.1) by $D^4 u^{(n)}$ and integrating with respect to x, we have

$$d\|u_{xx}^{(n)}\|^2/dt = -2\gamma p(p-1)(u_{xx}^{(n)}, (u^{(n-1)})^{p-2}(u_x^{(n-1)})^2 u_x^{(n)})$$
$$-2\gamma p(u_{xx}^{(n)}, (u^{(n-1)})^{p-1} u_{xx}^{(n-1)} u_x^{(n)})$$
$$-3\gamma p(u_{xx}^{(n)}, (u^{(n-1)})^{p-1} u_x^{(n-1)} u_{xx}^{(n)}).$$

Using Lemma 1, we get

$$|(u_{xx}^{(n)}, (u^{(n-1)})^{p-2} (u_x^{(n-1)})^2 u_x^{(n)})|$$
$$\leqq (\sup|u^{(n-1)}|)^{p-2} (\sup|u_x^{(n-1)}|)^2 \|u_{xx}^{(n)}\| \, \|u_x^{(n)}\|$$
$$\leqq c\|u^{(n-1)}\|_2^p (\|u_{xx}^{(n)}\|^2 + \|u_x^{(n)}\|^2),$$

$$|(u_{xx}^{(n)}, (u^{(n-1)})^{p-1} u_{xx}^{(n-1)} u_x^{(n)})|$$
$$\leqq (\sup|u^{(n-1)}|)^{p-1} \sup|u_x^{(n)}| \, \|u_{xx}^{(n)}\| \, \|u_{xx}^{(n-1)}\|$$
$$\leqq c\|u^{(n-1)}\|_2^p (\|u_{xx}^{(n)}\|^2 + \|u_x^{(n)}\|^2),$$

and

$$|(u_{xx}^{(n)}, (u^{(n-1)})^{p-1} u_x^{(n-1)} u_{xx}^{(n)})|$$
$$\leqq (\sup|u^{(n-1)}|)^{p-1} \sup|u_x^{(n-1)}| \, \|u_{xx}^{(n)}\|^2$$
$$\leqq c\|u^{(n-1)}\|_2^p \, \|u_{xx}^{(n)}\|^2.$$

Hence, we have

$$d\|u_{xx}^{(n)}\|^2/dt \leqq c\|u^{(n-1)}\|_2^p (\|u_{xx}^{(n)}\|^2 + \|u_x^{(n)}\|^2)$$

which implies

$$\|u_{xx}^{(n)}\|^2 \leqq (\|f_{xx}\|^2 + c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_2^p \sup_{0\leqq\tau\leqq t} \|u_x^{(n)}(\tau)\|^2 t) \times \exp(c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_2^p t). \tag{5.12}$$

Combining the inequalities (5.10), (5.11) and (5.12), we obtain

$$\|u^{(n)}\|_2^2 \leqq c^2\|f\|_2^2 (1 + \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_2^p t \exp(c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_2^p t)) \times \exp(c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_2^p t). \tag{5.13}$$

Now, let b be an arbitrary fixed positive number and put

$$t_1=\min\left\{\frac{\log(1+b)}{c^{p+1}(1+b)^p\|f\|_2^p+1},\ \frac{b}{(1+b)(c^p(1+b)^p\|f\|_k^p+1)},\ T\right\}. \tag{5.14}$$

Then in the interval $0\leqq t\leqq t_1$, the estimate

$$\|u^{(n)}(t)\|_2^2\leqq L_1^2 \tag{5.15}$$

holds if we take $L_1^2=c^2\|f\|_2^2\,(1+b)^2$.

Indeed we can prove the above assertion by induction on n. When $n=1$, we get

$$\|u^{(1)}(t)\|_2^2\leqq c^2\|f\|_2^2\leqq c^2\|f\|_2^2(1+b)^2.$$

Suppose that the assertion holds when $n=r$, then we have

$$\begin{aligned}\|u^{(r+1)}(t)\|_2^2 &\leqq c^2\|f\|_2^2\,(1+c^p(1+b)^p\|f\|_2^p\,t_1\exp\,(c^{p+1}(1+b)^p\|f\|_2^p\,t_1))\\ &\qquad\times\exp\,(c^{p+1}\,(1+b)^p\|f\|_2^p\,t_1)\\ &\leqq c^2\|f\|_2^2\,(1+b)^2 \qquad\text{for } 0\leqq t\leqq t_1.\end{aligned}$$

Secondly we prove the estimates (5.9) for $k>2$ by induction on k.

Multiplication of the equation (5.1) by $D^{2k}u^{(n)}$ and integration with respect to x give

$$d\|D^k u^{(n)}\|^2/dt=-2\gamma(D^k u^{(n)},\ D^k\{(u^{(n-1)})^p u_x^{(n)}\}).$$

Since

$$\begin{aligned}&|(D^k u^{(n)},\ D^k\{(u^{(n-1)})^p u_x^{(n)}\})|\\ &\quad=C_1(D^k u^{(n)},\ (u^{(n-1)})^{p-1}u_x^{(n-1)}D^k u^{(n)})\\ &\qquad+C_2(D^k u^{(n)},\ (u^{(n-1)})^{p-1}D^k u^{(n-1)}u_x^{(n)})\\ &\qquad+\sum C_{\alpha_1\dots\alpha_{p+1}}(D^k u^{(n)},\ (D^{\alpha_1}u^{(n-1)})\cdots(D^{\alpha_p}u^{(n-1)})(D^{\alpha_{p+1}}u^{(n)})),\end{aligned}$$

(where $\sum$ is taken over all $(\alpha_1,\ \cdots\cdots,\ \alpha_{p+1})$ with $\alpha_1+\cdots\cdots+\alpha_{p+1}=k+1$ and with $\alpha_i\leqq k-1$ for all i and C_1, C_2, $C_{\alpha_1\dots\alpha_{p+1}}$ are constants), we have

$$|(D^k u^{(n)}, D^k \{(u^{(n-1)})^p u_x^{(n)}\})|$$

$$\leqq c\|u^{(n-1)}\|_k^p \{\|D^k u^{(n)}\|^2 + \|u^{(n)}\|_{k-1}^2\} \qquad \text{for } k \geqq 2.$$

Hence

$$d\|D^k u^{(n)}\|^2/dt \leqq c\|u^{(n-1)}\|_k^p \{\|D^k u^{(n)}\|^2 + \|u^{(n)}\|_{k-1}^2\}$$

from which it follows that

$$\|D^k u^{(n)}\|^2 \leqq (\|D^k f\|^2 + c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_k^p \sup_{0\leqq\tau\leqq t} \|u^{(n)}(\tau)\|_{k-1}^2 t)$$

$$\times \exp (c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_k^p t).$$

Summation gives

$$\|u^{(n)}\|_k^2 \leqq (c^2\|f\|_k^2 + c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_k^p \sup_{0\leqq\tau\leqq t} \|u^{(n)}(\tau)\|_{k-1}^2 t) \tag{5.16}$$

$$\times \exp (c \sup_{0\leqq\tau\leqq t} \|u^{(n-1)}(\tau)\|_k^p t) \qquad (k>2).$$

Repeating the same argument as for $k=2$, it is easy to see that the inequality (5.16) implies the validity of the estimates (5.9) for $k>2$ if we take

$$L_k = c\|f\|_k (1+b), \tag{5.17}$$

and

$$t_k = \min\left\{\frac{\log(1+b)}{c^{p+1}(1+b)^p\|f\|_k^p+1}, \frac{b}{(1+b)^2(c^{p+1}(1+b)^p\|f\|_k^p+1)}, t_{k-1}\right\}. \tag{5.18}$$

Q.E.D.

Lemma 10. *There exists a positive constant* T_m *such that in the interval* $0 \leqq t \leqq T_m$

$$\sup_{0\leqq t\leqq T_m} \|u^{(n-1)}(t) - u^{(n)}(t)\|_{3m+2}^2 \tag{5.19}$$

$$< \rho \sup_{0\leqq t\leqq T_m} \|u^{(n)}(t) - u^{(n-1)}(t)\|_{3m+2}^2$$

holds, where $0<\rho<1$.

Proof. The difference $Z^{(n)}=u^{(n+1)}-u^{(n)}(n=1, 2, \cdots\cdots)$ satisfies the equation

$$Z_t^{(n)}+Z_{xxx}^{(n)}=-\gamma v\ u_x^{(n+1)}Z^{(n-1)}+\gamma wZ_x^{(n)}, \tag{5.20}$$

where $v=(u^{(n)})^{p-1}+(u^{(n)})^{p-2}u^{(n-1)}+\cdots\cdots+u^{(n)}(u^{(n-1)})^{p-2}+(u^{(n-1)})^{p-1}$ and $w=(u^{(n-1)})^p$.

Multiplication of the equation (5.20) by $D^{2k}Z^{(n)}$ $(k=0, 1, \cdots\cdots, 3m+2)$ and integration with respect to x give

$$d\|D^kZ^{(n)}\|^2/dt=-2\gamma(D^kZ^{(n)},\ D^k\{vZ^{(n-1)}u_x^{(n+1)}\})+2\gamma(D^kZ^{(n)}, D^k\{wZ_x^{(n)}\}).$$

Using Leibnitz formula, Lemma 1 and Lemma 9, we have, in $0\leqq t\leqq t_k$,

$$|(D^kZ^{(n)}, D^k\{vZ^{(n-1)}u_x^{(n-1)}\})|\leqq c\|D^kZ^{(n)}\|^2+\|Z^{(n-1)}\|_k^2$$

and

$$|(D^kZ^{(n)}, D^k\{wZ_x^{(n)}\})|\leqq c\|D^kZ^{(n)}\|^2+\|Z^{(n)}\|_{k-1}^2.$$

Hence, we get

$$d\|D^kZ^{(n)}\|^2/dt\leqq c\|D^kZ^{(n)}\|^2+c(\|Z^{(n-1)}\|_k^2+\|Z^{(n)}\|_{k-1}^2)$$

from which it follows that

$$\sup_{0\leqq\tau\leqq t}\|D^kZ^{(n)}(\tau)\|^2\leqq c\,t(\sup_{0\leqq\tau\leqq t}\|Z^{(n-1)}(\tau)\|_k^2+\sup_{0\leqq\tau\leqq t}\|Z^{(n)}(\tau)\|_k^2).$$

Summation with respect to k from 0 to $3m+2$ gives

$$\sup_{0\leqq\tau\leqq t}\|Z^{(n)}(\tau)\|_{3m+2}^2\leqq\frac{ct}{1-ct}\sup_{0\leqq\tau\leqq t}\|Z^{(n-1)}(\tau)\|_{3m+2}^2 \tag{5.21}$$

for $0<ct<1$.

If we choose $T_m<\min\{1/2c, t_{3(m+1)}\}$, we establish (5.19).

Q.E.D.

Remark. Proofs of Lemma 9 and Lemma 10 are rather formal. But if initial function $f(x)$ is regularized so that $f\in H_\infty$, then we have $u^{(n)}\in C^\infty[0, T; H_\infty]$ for each $n>0$ and, moreover, the estimates (5.9) are independent of such regularizations, which assures the above calculations in proofs.

From Lemma 9 and Lemma 10 we see that there exist a function $u(x, t)$ and a subsequence of $\{u^{(n)}\}$ (also denoted by $\{u^{(n)}\}$) such that as $n \to \infty$

$$u^{(n)}(x, t) \longrightarrow u(x, t) \text{ weakly star in } L^\infty(0, T_m; H_{3(m+1)})$$

and

$$u^{(n)}(x, t) \longrightarrow u(x, t) \text{ strongly in } L^\infty(0, T_m; H_{3m+2}).$$

In view of the equation (5.1), we easily see that $u(x, t)$ is a solution of (1.1). Thus we obtain the following local existence theorem:

Theorem 1. *For every initial function $f(x) \in H_{3(m+1)}$, there exists a positive constant T_m such that in the interval $0 \leqq t \leqq T_m$ the initial-boundary value problem* (1.1), (1.2), (1.3) *has a unique solution $u(x, t)$ such that*

$$u(x, t) \in L^\infty(0, T_m; H_{3(m+1)}) \cap C[0, T_m; H_{3m}] \cap \cdots \cap C^m[0, T_m; L^2(0, 1)].$$

6. A priori Estimates

Let $u(x, t)$ be a smooth solution of the initial-boundary value problem (1.1), (1.2), (1.3).

Lemma 11. *For any $t > 0$, we have*

$$u(x, t) \in W \tag{6.1}$$

if the initial function $f(x) \in W$.

Proof. Without loss of generality we assume $u(x, t) \neq 0$. Suppose that (6.1) does not hold and let t^* be the smallest time t for which $u(t^*) \notin W$. Then $u(t^*) \in \partial W$ and from Lemma 6, we get

$$J(u(t^*)) = d \tag{6.2}$$

or

$$a(u(t^*)) - \frac{1}{p+1} b(u(t^*)) = 0. \tag{6.3}$$

We see that (6.2) contradicts the fact that $J(u(t))$ is an invariant integral, i.e.,

$$J(u(t^*))=J(f(x))<d.$$

From (6.3) we have

$$J(u(t^*))=J(\{(p+1)a(u(t^*))/b(u(t^*))\}^{1/p}u(t^*))\geqq d$$

which also leads to the contradiction. Q.E.D.

Theorem 2. *If* $f(x)\in W\cap H_{3(m+1)}$, *then we have for any* $T>0$

$$\sup_{0\leqq t\leqq T}\|u(t)\|_k<c,\quad (k=0,1,2,\cdots\cdots,3(m+1)). \tag{6.4}$$

Proof. If d is finite, in virtue of Lemma 7 and Lemma 11, we see that $u(x,t)$ is bounded in H_1 for $\forall t>0$, i.e.,

$$\sup_{0\leqq t\leqq T}\|u(t)\|_1<c. \tag{6.5}$$

If $d=\infty$, in virtue of Lemma 6 and Lemma 11, we can show that if $b(u)\leqq 0$,

$$J(f(x))=J(u)\geqq\frac{1}{2}a(u),$$

and if $b(u)>0$,

$$J(f(x))=J(u)\geqq\frac{p}{2(p+2)}a(u),$$

which imply (6.5).

We now prove (6.1) for $k=2$. Put

$$I_3(t)=\|u_{xx}\|^2+\frac{5\gamma}{3(p+1)}\int_0^1 u^{p+1}u_{xx}dx.$$

Differentiation of $I_3(t)$ with respect to t, use of the differential equation (1.1) and integration by parts give

$$dI_3(t)/dt = -\frac{\gamma p(p-1)(p-2)}{12}\int_0^1 u^{p-3}u_x^5\,dx - \frac{5\gamma}{3}\int_0^1 u^{2p}u_x u_{xx}\,dx.$$

Since

$$\left|\int_0^1 u^{p-3}u_x^5\,dx\right| \leqq (\sup|u|)^{p-3}(\sup|u_x|)^3\|u_x\|^2$$
$$\leqq c(\|u\|_1)^{p-3}\|u\|_1^{3/2}(\|u\|_1+\|u_{xx}\|)^{3/2}\|u\|_1^2$$
$$\leqq c\|u_{xx}\|^2+c,$$

and since

$$\left|\int_0^1 u^{2p}u_x u_{xx}\,dx\right| \leqq (\sup|u|)^{2p}\|u_x\|\,\|u_{xx}\|$$
$$\leqq c(\|u\|_1)^{2p}\|u\|_1\|u_{xx}\|$$
$$\leqq c\|u_{xx}\|^2+c,$$

we have

$$dI_3(t)/dt \leqq c\|u_{xx}\|^2+c. \tag{6.6}$$

Here we have used Lemma 1 and Lemma 2. Integrating (6.6), we get

$$\|u_{xx}\|^2 \leqq c\int_0^t \|u_{xx}\|^2\,dt + ct + \|f_{xx}\|^2 + \frac{5\gamma}{3(p+1)}\int_0^1 f^{p+1}f_{xx}\,dx \tag{6.7}$$
$$- \frac{5\gamma}{3(p+1)}\int_0^1 u^{p+1}u_{xx}\,dx$$
$$\leqq c\int_0^t \|u_{xx}\|^2\,dt + ct + c$$

since

$$\left|\int_0^1 u^{p+1}u_{xx}\,dx\right| = \left|(p+1)\int_0^1 u^p u_x^2\,dx\right| \leqq (p+1)(\sup|u|)^p\|u_x\|^2 \leqq c.$$

The inequality (6.7) yields

$$\sup_{0\leqq t\leqq T}\|u_{xx}(t)\| < c \qquad \text{for any } T>0. \tag{6.8}$$

Combining (6.8) with (6.2), we have the estimate (6.4) for $k=2$.

Next we prove (6.4) for $k\geqq 3$ by induction on k.

Multiplication of the equation (1.1) by $D^{2k}u$ and integration with respect to x give

$$d||D^k u||^2/dt = -2\gamma(D^k u,\ D^k(u^p u_x)).$$

Since

$$|(D^k u,\ D^k(u^p u_x))| \leqq c(1+||u||_{k-1})^{p-1}||D^k u||^2,$$

we get

$$d||D^k u||^2/dt \leqq c(1+||u||_{k-1})^{p-1}||D^k u||^2$$

from which it follows that for any $T>0$

$$||D^k u||^2 \leqq ||D^k f||^2 \exp\,(c(1+\sup_{0\leqq t\leqq T}||u(t)||_{k-1})^{p-1}T).$$

Summation gives

$$||u||_k^2 \leqq c||f||_k^2 \exp\,(c(1+\sup_{0\leqq t\leqq T}||u(t)||_{k-1})^{p-1}T). \tag{6.9}$$

Suppose that the assertion holds for $k-1$ ($k\geqq 3$), then from (6.9), we see that it also holds for k. Q.E.D.

Proofs of Main Theorem and its corollary are easily established, as usual, if we combine Theorem 1 with Theorem 2.

Remark. The Cauchy problem for the equation (1.1) can be treated with suitable modifications of the method in this note.

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