Publ. RIMS, Kyoto Univ.
7 (1971/72), 363-397

Construction of Local Elementary Solutions for Linear Partial Differential Operators with Real Analytic Coefficients (I)

——The case with real principal symbols——

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§0. Introduction

In this paper we construct local elementary solutions for linear differential operators $P(x, D_x)$ whose principal symbols are real and of simple characteristics and investigate their regularity properties using Sato's theory of the sheaf $\mathscr{C}$ (Sato [2]~[5]). Throughout this paper we assume that the coefficients of differential operators are analytic.

In §1 we prepare some theorems which extend the classical existence theorem of Cauchy-Kowalevsky in complex domain to cases of singular initial data. (Cf. Hamada [1].)

In §2 we employ the results of §1 to construct local elementary solutions for Cauchy problems for (I-) hyperbolic operators. As an application of the method employed there, we construct a singular solution $u(x)$ of $P(x, D_x)u=0$ whose singular support is "very small". (See Theorem 2.8 for the precise meaning of this statement.)

In §3 we construct a local elementary solution for a linear partial differential operator of real principal symbols with simple characteristics and decide where its singularities locate to conclude that the result of §2 on the existence of singular solutions with small singularities is the best possible one.

Throughout this paper we denote by M an n-dimensional real analytic manifold which we identify with a domain in $\mathbf{R}^n$ containing the origin,

Received March 31, 1971.

and by S^*M its cotangential sphere bundle (or co-sphere bundle for short). About the sheaf $\mathscr{C}$ defined on S^*M we refer the reader to the precise and extensive exposition by Kashiwara based on Sato's lectures (Sato [5]).

The results in this paper have been announced in Kawai [1], [2] and [5].

The author expresses his sincere gratitude to Professor M. Sato, Professor H. Komatsu and Mr. M. Kashiwara for their guidances and advices. Without the atmosphere of the seminars with them this paper would not have been written.

§1. Singular Cauchy Problems in a Complex Domain

In this section we consider an m-th order linear differential operator $P(z, D_z)$ with holomorphic coefficients defined near the origin in $\mathbf{C}^n$. We denote $\partial/\partial z_j$ by D_{z_j}, the symbol of the differential operator by $P(z, \xi)$, and its principal symbol by $P_m(z, \xi)$. The latter is the homogeneous part of $P(z, \xi)$ of order m. Throughout this section we abbreviate $(\xi_2, \cdots, \xi_n)$ to ξ' and $(z_2, \cdots, z_n)$ to z'. We assume in the sequel that $P_m(z, \xi)$ has the form $\sum_{j=0}^{m} a_{m-j}(z, \xi')\xi_1^j$ with $a_0(z, \xi')\equiv 1$.

Under the assumption (1.1) below Hamada [1] has proved the following decisive theorem.

Theorem 1.1. (Hamada [1] p. 23.) *Suppose that*

$$\frac{\partial}{\partial \xi_1}P_m(0; \xi_1, 1, 0, \cdots, 0)\neq 0 \textit{ if } P_m(0; \xi_1, 1, 0, \cdots, 0)=0. \tag{1.1}$$

Then the following Cauchy problem (1.2) *has the unique solution* $u(z)$ *which is analytic in a neighbourhood of* $z=0$ *except for* $K_1\cup\cdots\cup K_m$. *Here* $K_1, \cdots, K_m$ *are the characteristic surfaces passing through* $z_1=z_2=0$, *and each* K_j *is a non-singular hypersurface defined by the equation* $\varphi_j(z)=0$, *where* $\varphi_j(z)$ *are obtained by solving the Hamilton-Jacobi equations.*

$$\begin{cases} P(z, D_z)u(z)=0 \\ \left.\dfrac{\partial^k}{\partial z_1^k}u(z_1, z')\right|_{z_1=0}=w_k(z'), \qquad \textit{where } k=0, 1, \cdots, m-1. \end{cases} \tag{1.2}$$

More precisely, provided that $w_k(z')$ $(k=0, 1, \cdots, m-1)$ *have at most poles along* $z_2=0$, *the solution* $u(z)$ *of* (1. 2) *is expressed in the form*

$$(1.3) \qquad u(z)=\sum_{j=1}^{m}\left\{\frac{F_j(z)}{[\varphi_j(z)]^{p_j}}+G_j(z)\log\varphi_j(z)+H_j(z)\right\},$$

where $F_j(z)$, $G_j(z)$ *and* $H_j(z)$ *are holomorphic in a neighbourhood of* $z=0$ *and* p_j *is a positive integer.*

It is obvious from Hamada's method of the proof that we have the following Theorem 1.1′, which differs from Hamada's theorem only in its form of presentation. Before stating Theorem 1.1′ we prepare some notations.

Throughout this section we assume the following condition:

(1.4) In a neighbourhood of $(z', \xi')=(0, \xi_0')$ $(|\xi_0'|=1)$

$$P_m(0, z', \xi_1, \xi')=0 \text{ implies } \frac{\partial}{\partial\xi_1}P_m(0, z', \xi_1, \xi')\neq 0.$$

Definition 1.1. For any y' with sufficiently small $|y'|$ we denote by $\{K_j(y', \xi')\}_{j=1}^{m}$ the characteristic surfaces of $P(z, D_z)$ passing through the intersection of two hypersurfaces $\{z_1=0\}$ and $\{\langle z'-y', \xi\rangle=0\}$.

Definition 1.2. We denote by $\varphi_j(z, y', \xi')$ the characteristic function corresponding to the characteristic surface $K_j(y', \xi')$ satisfying $P_m(z, \mathrm{grad}_z\varphi_j(z, y', \xi'))=0$ with the initial condition $\langle z'-y', \xi'\rangle$ on $\{z_1=0\}$.

Remark 1. As is well known $\varphi_j(z, y', \xi')$ is obtained by solving the Hamilton-Jacobi equations associated with $P_m(z, \xi)$. Note that $\varphi_j(z, y', \xi')$ is homogeneous of order 1 with respect to ξ'. By this reason we sometimes call $\varphi_j(z, y', \xi')$ a phase function. (Cf. Hörmander [3]).

Remark 2. By the condition (1.4) the roots $\{\xi_1^j(y', \xi')\}_{j=1}^{m}$ of the equation $P_m(0, y', \xi_1, \xi')=0$ are mutually distinct and $\frac{\partial}{\partial z_1}\varphi_j(z, y', \xi')\Big|_{z=(0,y')}=\xi_1^{k_j}(y', \xi')$ for some k_j. Hence afterwards we assume $\left.\dfrac{\partial}{\partial z_1}\varphi_j(z, y', \xi)\right|_{z=(0,y')}=\xi_1^j(y', \xi')$.

Theorem 1.1′. *Consider the following Cauchy problem*:

$$(1.5)\qquad \begin{cases} P(z, D_z)u(z, y', \xi')=0 \\ \left.\dfrac{\partial^k}{\partial z_1^k}u(z_1, z', y', \xi')\right|_{z_1=0}=\left.\dfrac{v_k(z, y', \xi')}{<z'-y', \xi'>^l}\right|_{z_1=0}, \\ k=0, 1, \cdots, m-1, \end{cases}$$

where $v_k(z, y', \xi')$ is a holomorphic function in (z, y', ξ') which is homogeneous of order 0 *with respect to ξ' and l is a positive integer.* (*In this paper it is sufficient to take $v_k\equiv$constant.*)

Then the Cauchy problem (1.5) *has the unique solution $u(z, y', \xi')$, which is expressed in the form*

$$(1.6)\qquad \begin{cases} u(z, y', \xi')=\displaystyle\sum_{j=1}^{m}\left\{\frac{F_j(z, y', \xi')}{[\varphi_j(z, y', \xi')]^{l_j}}+\right. \\ \left.+G_j(z, y', \xi')\log\varphi_j(z, y', \xi')+H_j(z, y', \xi')\right\} \end{cases}$$

where $F_j(z, y', \xi')$, $G_j(z, y', \xi')$ and $H_j(z, y', \xi')$ are holomorphic near $(0, 0, \xi_0')$ and l_j is a positive integer.

Remark. By the method of the proof of Hamada [1] it is obvious that we can choose F_j, G_j and H_j so that $P(z, D_z)u^{(j)}(z, y', \xi')=0$ holds, where $u^{(j)}(z, y', \xi')$ is by definition $\dfrac{F_j(z, y', \xi')}{[\varphi_j(z,y',\xi')]^{l_j}}+G_j(z,y',\xi')\log\varphi_j(z, y',\xi')+$ $+H_j(z, y', \xi')$. This remark motivates the modifications of Theorem 1.1′, which are given in Theorem 1.2 and 1.3.

Theorem 1.1′ is basic in constructing elementary solutions for Cauchy problems for (I-)hyperbolic operators. In order to employ the existence theorem for singular Cauchy problems in a complex domain to develop the local theory of general linear differential operators as in §2 and §3, we must modify Theorem 1.1′ in various ways. Hence in the rest of this section we state the variants of Theorem 1.1. Though they differ from

Theorem 1.1 in the form of presentation, their proofs are essentially the same as in Hamada [1]. We use the same notations as in Theorem 1.1′.

Theorem 1.2. *Consider the following Cauchy problem* $(1.7)_p$ *for some positive integer* p *with* $1 \leqq p \leqq m$ *under the assumption* (1.4).

$$(1.7)_p \qquad \begin{cases} P(z, D_z)u(z, y', \xi')=0 \\ \dfrac{\partial^k}{\partial z_1^k} u(z_1, z', y', \xi')\Big|_{z_1=0} = \dfrac{v_k(z, y', \xi')}{<z'-y', \xi'>^l}\Big|_{z_1=0}, \\ k=0, 1, \cdots, p-1, \end{cases}$$

where l *is a positive integer and* $v_k(z, y', \xi')$ *satisfies the same conditions given in Theorem* 1.1′.

If we choose any p *hypersurfaces of* $\{K_j\}_{j=1}^m$, *say* $K_1, \cdots, K_p$, *then the Cauchy problem* $(1.7)_p$ *admits a solution* $u(z, y', \xi')$ *which is represented in the form*

$$(1.8)_p \qquad \begin{cases} u(z, y', \xi')= \displaystyle\sum_{j=1}^{p} \left\{ \frac{F_j(z, y', \xi')}{[\varphi_j(z, y', \xi')]^{l_j}} + \right. \\ \left. + G_j(z, y', \xi')\log\varphi_j(z, y', \xi') + H_j(z, y', \xi') \right\}, \end{cases}$$

where $F_j(z, y', \xi')$, $G_j(z, y', \xi')$ *and* $H_j(z, y', \xi',)$ *are holomorphic near* $(z, y', \xi')=(0, 0, \xi'_0)$ *and* l_j *is a positive integer.*

Remark. When p is equal to m, the Cauchy problem $(1.7)_p$ reduces to the Cauchy problem (1.5). Except for this special case the solution $u(z, y', \xi')$ is not unique.

Proof of Theorem 1.2. We first construct a formal solution $u(z,y',\xi')$ of $(1.7)_p$ suitably and prove its convergence. We can assume that $v_k=0$ $(k\neq h)$ and that $v_h(z, y', \xi')$ contains no power of $<z'-y', \xi'>$ by the principle of superposition. For the formal construction of $u(z, y', \xi')$ we introduce the following functions $\Phi_j(\tau)$ $(j=-m, -m+1, \cdots)$ satisfying the following (1.9) after Hamada [1].

$$
(1.9)\quad \begin{cases} \dfrac{d}{d\tau}\Phi_j(\tau)=\Phi_{j-1}(\tau) \\[2ex] \Phi_{-h}(\tau)=\dfrac{(-1)^{l-1}(l-1)!}{\tau^l} \\[2ex] \Phi_{-h+l+\alpha}(\tau)=\dfrac{\tau^\alpha}{\alpha!}\log\tau-\dfrac{A_\alpha}{\alpha!}\tau^\alpha, \\[2ex] \text{where } A_\alpha=1+\dfrac{1}{2}+\cdots+\dfrac{1}{\alpha} \text{ and } A_0=0. \end{cases}
$$

We assume by the aid of these functions that the solution $u(z, y', \xi')$ has the form

$$
(1.10)\qquad u(z, y', \xi')=\sum_{j=1}^{p}\left\{\sum_{k=0}^{\infty}\Phi_k(\varphi_j(z, y', \xi'))u_{k,j}(z, y', \xi')\right\}.
$$

Assuming the form of $u(z, y', \xi')$ as above we can proceed just as in Hamada [1] pp. 27–30. The only difference from Hamada's proof is to use the inverse matrix of

$$
p\left\{\overbrace{\begin{pmatrix} 1 & \cdots\cdots & 1 \\ \xi_1^1 & \cdots\cdots & \xi_1^p \\ \vdots & & \vdots \\ (\xi_1^1)^{p-1} & \cdots\cdots & (\xi_1^p)^{p-1} \end{pmatrix}}^{p}\right. \quad ^{1)}
$$

instead of

$$
m\left\{\overbrace{\begin{pmatrix} 1 & \cdots\cdots & 1 \\ \xi_1^1 & \cdots\cdots & \xi_1^m \\ \vdots & & \vdots \\ (\xi_1^1)^{m-1} & \cdots\cdots & (\xi_1^m)^{m-1} \end{pmatrix}}^{m}\right.
$$

to determine the successive initial data on $\{z_1=0\}$ for $u_{k,j}$ by the initial data of u imposed in $(1.7)_p$.

The convergence proof of the formal solution (1.10) is just the same as in Hamada [1] pp. 32–36.

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1) The symobl $(\xi_1^j)^k$ means the k-th power of ξ_1^j.

The most important case in Theorem 1.2 is the case $p=1$ since we shall consider the local theory of linear differential operators in the framework of Sato's sheaf $\mathscr{C}$. For this reason we study a little more general case when $p=1$.

Theorem 1.3. *Consider the following Cauchy problem* (1.11).

$$
(1.11)\qquad \begin{cases} P(z, D_z)u(z, y', \xi')=0 \\ Q(z, D_z)u(z, y', \xi')\Big|_{z_1=0}=\dfrac{v(z, y', \xi')}{<z'-y', \xi'>^l}\Big|_{z_1=0}, \end{cases}
$$

where l is a positive integer, $v(z, y', \xi')$ satisfies the same condition on $v_k(z, y', \xi')$ given in Theorem 1.1′, $Q(z, D_z)$ is a linear differential operator of order p with $p\leqq m-1$ and $Q_p(z, \mathrm{grad}_z\varphi_1(z, y', \xi'))|_{z_1=0}\neq 0$.

Then the Cauchy problem (1.11) *admits a solution $u(z, y', \xi')$ which is expressed in the form*

$$
(1.12)\qquad \begin{cases} u(z)=F(z, y'\,\xi')\varphi_1(z, y', \xi')^{-k}+ \\ +G(z, y', \xi')\log\varphi_1(z, y', \xi')+H(z, y', \xi'), \end{cases}
$$

where k is a positive integer and $F(z, y', \xi')$, $G(z, y', \xi')$ and $H(z, y', \xi')$ are holomorphic near $(z, y', \xi')=(0, 0, \xi'_0)$.

Proof. We abbreviate φ_1 to φ in the proof. We first assume that $u(z, y', \xi')$ has the form

$$
\sum_{k=0}^{\infty}\Phi_k(\varphi(z, y', \xi'))u_k(z, y', \xi'),
$$

where we take $h=p$ in the formula (1.9) which defines $\Phi_k(\tau)$. As in the proof of Theorem 1.2 we can assume without loss of generality that $v(z, y', \xi')$ has no powers of $<z'-y', \xi'>$. Then coefficients $u_k(z, y', \xi')$ of the above expansions is formally defined successively by solving first order linear differential equations (the transport equations) as in Hamada [1] p. 27, i.e., u_k is chosen to satisfy

$$
(1.13)\qquad \mathscr{L}[u_0]=0
$$

$$
(1.14)\qquad \mathscr{L}[u_k]=-\sum_{p=2}^{m}L_p[u_{k+1-p}],
$$

where $\mathscr{L}[u]=\langle \operatorname{grad}_\xi P_m(z, \xi)|_{\xi=\operatorname{grad}_z\varphi}, \operatorname{grad}_z u\rangle + c(z, y', \xi')u$ with some holomorphic function $c(z, y', \xi')$ defined by the lower order terms of $P(z, D_z)$ and L_p are some linear differential operators of order at most p. In fact, by assumption (1.4), the plane $\{z_1=0\}$ is non-characteristic with respect to these differential equations and the condition that $Q_p(z, \operatorname{grad}_z\varphi)\Big|_{z_1=0}\neq 0$ assures that we can determine successively the initial datum for $u_k(z, y', \xi')$ on $\{z_1=0\}$ by the condition imposed in (1.11). Thus we can formally determine $u(z, y', \xi')$. What remains to be proved is to estimate the growth rate of $|u_k|$ with respect to k. In course of the estimation we can assume that $y'=0$ and $\xi'=(1, 0, \cdots, 0)$ without loss of generality because all the constants which appear in the estimates are uniform with respect to parameter y' and ξ' as far as they are sufficiently near $(y', \xi')=(0, \xi'_0)$. The estimation is performed by proving the following estimate (1.15) by induction on k:

$$(1.15)\qquad |D_{z_1}^q D_{z'}^\nu u_k(z)| \leqq c(k)\frac{(k+q+|\nu|)!}{\rho^{k+q+|\nu|}}\exp(\gamma|t|)K(|t|)^{k+q+|\nu|}(\gamma n)^q,$$

where γ and ρ are some constants, $K(t)=\exp(\gamma nt)(1+\gamma nt)$ and $c(k)=c_0^k NA$ with $A=\max\limits_{|z'|\leqq\rho}|v|$ and some constants c_0 and N which depend only on $P(z, D_z)$. The induction is performed by the estimation of $\tilde{u}_k$ and $\tilde{\tilde{u}}_k$, where $u_k=\tilde{u}_k+\tilde{\tilde{u}}_k$ and $\tilde{u}_k$ satisfies $\mathscr{L}[\tilde{u}_k]=0$ with the same initial condition on $\{z_1=0\}$ as that of u_k and $\tilde{\tilde{u}}_k$ satisfies (1.14) with the Cauchy datum 0 on $\{z_1=0\}$. Then the induction proceeds for $\tilde{\tilde{u}}_k$ without any changes of the proof of Hamada [1] p. 35 by the definition. Since γn is larger than 1 we can also prove (1.15) for $\tilde{u}_k$ with k replaced by $k+1$. (Remark that operator $\mathfrak{M}_j$, which appears as in p. 28 of Hamada [1], when we define the Cauchy data of u_k on $\{z_1=0\}$ contains not only $(\partial/\partial z_1)^k$ $(0\leqq k\leqq j)$ but also other differential operators in our case. But the order of the operator $\mathfrak{M}_j$ is also at most j, hence the induction proceeds.)

Remark. It is obvious from the method of the proof that Theorem 1.3 holds under the following localized condition (1.4′) instead of (1.4).

$$(1.4') \qquad \frac{\partial}{\partial \xi_1} P_m(0, \xi) \neq 0 \text{ near } \xi_0, \text{ where } \xi_0 = \operatorname{grad}_z \varphi_1(z, 0, \xi_0')\Big|_{z=0}.$$

This remark applies also to the following corollary.

Corollary 1.4. *Consider the following Cauchy problem* (1.16). *Denote by* $\varphi(z, y, \xi, s)$ *one of the phase functions of* $P(z, D_z)$ *for which* $\varphi(z, y, \xi, s)\Big|_{s=z_1} = \langle z-y, \xi \rangle$.

$$(1.16) \qquad \begin{cases} P(z, D_z)u(z, y, \xi, s) = 0 \\ Q(z, D_z, D_s)u(z, y, \xi, s)\Big|_{s=z_1} = \dfrac{1}{\langle z-y, \xi \rangle^l} \end{cases}$$

where l *is a positive integer and* $Q_p(z, \operatorname{grad}_{(z,s)}\varphi(z, y, \xi, s))\Big|_{z_1=s} \neq 0$. (*Here* p *is the order of* $Q(z, D_z, D_s)$.)

Then the Cauchy problem (1.16) *admits a solution* $u(z, y, \xi, s)$ *which has the form*

$$(1.17) \qquad \begin{cases} u(z, y, \xi, s) = F(z, y, \xi, s)\varphi(z, y, \xi, s)^{-k} + \\ + G(z, y, \xi, s)\log\varphi(z, y, \xi, s) + H(z, y, \xi, s), \end{cases}$$

where k *is a positive integer and* F, G *and* H *are holomorphic near* $(z, y, \xi, s) = (0, 0, \xi_0, 0)$.

Remark 1. This corollary is only a restatement of Theorem 1.3 from the logical view point. However the above form is more convenient in §3, hence we have stated the corollary for the convenience of references.

Remark 2. It is obvious from the method of the proof that instead of $\langle z'-y', \xi' \rangle$ we can use any holomorphic function $\chi(z', y', \xi')$ if $\chi(z', y', \xi')$ is holomorphic near $(z', y', \xi') = (0, 0, \xi_0')$, homogeneous of order 1 with respect to ξ' and satisfies $\chi(z', y', \xi') = \langle z'-y', \xi' \rangle + O(|z'-y'|^2|\xi'|)$. This remark is used in our note Kawai [3] and will be used in our forthcoming paper Kawai [6] in order to construct local elementary solutions, which defines a kernel function of a pseudo-differential operator (in the sense of Sato [4], [5]), under the condition motivated by that given in Nirenberg and Treves [1]. (The condition is called condition $(NT)_f$ in our previous note Kawai [3].)

§2. Construction of Elementary Solutions for Cauchy Problems

Until the end of this paper we consider an m-th order linear differential operator $P(x, D_x)$ with real analytic coefficients defined near the origin of $\mathbf{R}^n$. We assume further that its principal symbol $P_m(x, \xi)$ is real and that $P_m(x, \xi)$ is of simple characteristics, i.e., $\mathrm{grad}_\xi P_m(x, \xi) \neq 0$ whenever $P_m(x, \xi)=0$ except in Theorem 2.6. Moreover we assume that $P_m(x, \xi)$ has the form $\sum_{j=0}^{m} a_{m-j}(x, \xi')\xi_1^j$ with $a_0(x, \xi')\equiv 1$ in this section except in Theorem 2.8. We first construct the elementary solution for Cauchy problem when $P(x, D_x)$ is strictly hyperbolic with respect to $\xi=(1, 0, \cdots, 0)$ near the origin. We mean by the elementary solution for Cauchy problem the hyperfunctions $\{E_k(x_1, x', y')\}_{k=1}^{m}$ which depend real analytically on x_1 and y' respectively and satisfy the following Cauchy problem:

$$(2.1) \qquad \begin{cases} P(x, D_x)E_k(x, y')=0 \\ \left.\dfrac{\partial^j}{\partial x_1^j}E_k(x, y')\right|_{x_1=0}=\delta_{jk}\delta(x'-y'), \quad \text{where } 0\leqq j, k\leqq m-1. \end{cases}$$

(See Sato [1] p. 424 about the notion of real analytic dependence on a parameter of a hyperfunction. See also Sato [5].)

Since a hyperfunction $u(x)$ depending real analytically on x_1 can be specialized (or restricted) to a lower dimensional manifold $\{x_1=s\}$, the second condition in (2.1) is well defined. We also remark that the condition that the hypersurface $\{x_1=0\}$ is non-characteristic with respect to $P(x, D_x)$ implies that $E_k(x, y')$ depends real analytically on x_1 by Sato's fundamental theorem on regularity of hyperfunction solutions of linear differential equations. (See Sato [2]~[5]. See also Kashiwara and Kawai [1].)

If such hyperfunctions $E_k(x, y')$ are obtained, then we can solve the Cauchy problem

$$(2.2)\qquad \begin{cases} P(x, D_x)u(x)=0 \\ \left.\dfrac{\partial^j}{\partial x_1^j}u(x_1, x')\right|_{x_1=0}=\mu_j(x'), \end{cases}$$

where $0\leqq j\leqq m-1$ and $\mu_j(x')$ is a hyperfunction of $(n-1)$ variables with compact support by the formula

$$u(x)=\sum_{k=0}^{m-1}\int E_k(x, y')\mu_k(y')dy'.$$

The assumption of real analytic dependence on y' of $E_k(x, y')$ makes the above integration well-defined. Moreover we can prove that $E_k(x, y')$ has a finite speed of propagation if $P(x, D_x)$ is strictly hyperbolic with respect to $(1, 0, \cdots, 0)$ in the below (Corollary 2.5) using the precise version of Holmgren's uniqueness theorem (Kawai [4], [5], see also Komatsu and Kawai [1]) combined with the regularity properties of $E_k(x, y')$, therefore $\mu_k(x')$ need not have a compact support in this case.

Theorem 2.1. *If $P(x, D_x)$ is a strictly hyperbolic linear differential operator with respect to* $(1, 0, \cdots, 0)$, *then there exist elementary solutions for Cauchy problem $\{E_k\}_{k=1}^{m}$ near the origin.*

Proof. Consider in a complex domain near $(z, y', \xi')=(0, 0, \xi_0')$ the following Cauchy problem:

$$(2.3)\qquad \begin{cases} P(z, D_z)u(z, y', \xi')=0 \\ \left.\dfrac{\partial^l}{\partial z_1^l}u_k(z, y', \xi')\right|_{z_1=0}=\delta_{lk}\dfrac{(n-2)!}{(-2\pi i)^{n-1}\langle z'-y', \xi'\rangle^{n-1}}, \\ \text{where } 0\leqq l, k\leqq m-1. \end{cases}$$

If the domain is sufficiently small, $\dfrac{\partial}{\partial\xi_1}P_m(z, \xi_1, \xi')\neq 0$ holds for ξ with $P_m(z, \xi)=0$, which follows by the assumption of strict hyperbolicity of $P(x, D_x)$. Then Theorem 1.1′ assures the existence of $u_k(z, y', \xi')$, which is expressed in the form (1.5). Moreover, as is remarked after Theorem 1.1′, u_k is decomposed into $\sum_{j=1}^{m} u_k^{(j)}$, so that $P(z, D_z)u_k^{(j)}(z, y', \xi')=0$ holds and $u_k^{(j)}$ are represented in the form $\dfrac{F_j(z, y', \xi')}{[\varphi_j(z, y', \xi')]_j^l}+$

$+G_j(z, y', \xi') \log \varphi_j(z, y', \xi') + H_j(z, y', \xi')$ with holomorphic functions F_j, G_j and H_j. It is obvious that $u_k^{(j)}(z, y', \xi')$ is uniform and analytic in a simply connected complex domain $\{(z, y', \xi') \in \Omega \mid \mathrm{Im}\varphi_j(z, y', \xi') > 0\}$, where Ω is a sufficiently small neighbourhood of $(z, y', \xi') = (0, 0, \xi'_0)$. On the other hand we conclude that the phase function $\varphi_j(x, y', \xi')$ defined in Definition 1.2 is real valued if (x, y', ξ') is real, since we can integrate the Hamilton-Jacobi equations in a real domain to obtain m phase functions $\varphi_1, \cdots, \varphi_m$ by the assumption of strict hyperbolicity of $P(x, D_x)$. Therefore the boundary value of $u_k^{(j)}(z, y', \xi')$ from the complex domain $\{\mathrm{Im}\varphi_j(z, y', \xi') > 0\}$ defines a hyperfunction $u_k^{(j)}(x, y', \xi')$ which depends real analytically on (y', ξ'). The word "boundary value of $u_k^{(j)}$" means logically that the cohomology class defined by $u_k^{(j)}(z, y', \xi')$ using some representation of relative cohomology group by a cohomology group of coverings. (See for example Komatsu [1].)

Generally for a hyperfunction $\mu(x)$ defined on M we denote by $S.S.\mu(x) (\subset S^*M)$ the support of $\beta(\mu(x))$, where β means the cannonical homomorphism: $\mathscr{B} \xrightarrow{\beta} \pi_* \mathscr{C}$. Here $\pi_*\mathscr{C}$ means the direct image of the sheaf $\mathscr{C}$ by the projection $S^*M \xrightarrow{\pi} M$, and Sato's fundamental theorem on the sheaf $\mathscr{C}$ asserts that the mapping β defines a surjection with its kernel the sheaf of germs of real analytic functions on M. Using this notation we conclude from the definition of $u_k^{(j)}(x, y', \xi')$ that

$$(2.4) \qquad S.S.u_k^{(j)}(x, y', \xi') \subset \{(x, y', \xi'; \mathrm{grad}_{(x,y',\xi')}\varphi_j(x, y', \xi')) \mid \varphi_j(x, y', \xi') = 0\}.$$

Here we regard $u_k^{(j)}(x, y', \xi')$ as a hyperfunction of $(x, y', \xi') \in N$ and consider $S.S.u_k^{(j)}$ on S^*N.

The required $E_k(x, y')$ is given by $\sum_{j=1}^{m} \int u_k^{(j)}(x, y', \xi') \omega(\xi')$, where $\omega(\xi')$ denotes the volume element on the $(n-2)$-dimensional unit sphere, i.e., $\omega(\xi') = \sum_{j=2}^{n} (-1)^j \xi_j d\xi_2 \wedge \cdots \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \cdots \wedge d\xi_n$.

In fact $P(x, D_x) u_k^{(j)}(x, y', \xi') = 0$ by the definition of $u_k^{(j)}(x, y', \xi')$ and we have

$$\sum_{j=1}^{m}\int_{|\xi'|=1}\frac{\partial^l}{\partial x_1^l}u_k^{(j)}(x, y', \xi')\bigg|_{x_1=0}\omega(\xi')=$$

$$=\delta_{lk}\int_{|\xi'|=1}\frac{(n-2)!\ \omega(\xi')}{(-2\pi i)^{n-1}(<x'-y', \xi'>+i0)^{n-1}}$$

by the initial condition imposed on $u_k(z, y', \xi')$ in (2.3). On the other hand we have the following well-known formula:

$$\delta(x'-y')=-\frac{(n-2)!}{(-2\pi i)^{n-1}}\int_{|\xi'|=1}\frac{\omega(\xi')}{(<x'-y', \xi'>+i0)^{n-1}}$$

(Cf. Gel'fand and Shilov [1] p. 79.) Thus $E_k(x, y')$ has all the properties required.

We next investigate the singularity of $E_k(x, y')$ constructed in Theorem 2.1. For that purpose the following fundamental lemma due to Sato (Lemma 2.2) plays an essential role. As for the location of singularities on M, not on S^*M, after the integration along fiber, Lemma 2.2 seems to be essentially well known to physicists and Lax [1] applies it to the study of the propagation of singularities in the C^∞-category. Hörmander [3] has contributed to these problems through the geometrical study of phase functions. See also Kashiwara and Kawai [2], where it is shown that the real analytic version of Hörmander's theory can be developed in an analogous way.

Remark that Sato's theory treats the problems in real analytic category.

Lemma 2.2 (*Sato* [4] §6 *and Sato* [5] *Corollary* 6.5.3). *Let $f:N\to M$ be a real analytic mapping from an $(n+r)$-dimensional real analytic manifold N to an n-dimensional real analytic manifold M with maximal rank. Denote by dy the fundamental r-form along the fiber. Suppose that a hyerfunction $\mu(x)$ defined on N satisfies the following condition*:

(2.5) *f is a proper mapping over $S.S.\mu(x)$.*

Then the integration along fiber $\int_{f^{-1}}\mu(x)dy$ is well-defined. (*See Sato* [5] §6.5 *about the integral along fiber on the sheaf $\mathscr{C}$, which is compatible* *with that of hyperfunctions (Sato* [1] §10). *See also Kashiwara and Kawai* [1].) *Moreover*

$$S.S.\int_{f^{-1}}\mu(x)dy \subset \sigma_f(S.S.\mu(x)\cap S^*M\underset{M}{\times}N),$$

where $S^*M\underset{M}{\times}N$ *denotes the fiber product of* S^*M *and* N *over* M *and* σ_f *denotes the natural homomorphism from* $S^*M\underset{M}{\times}N$ *to* S^*M *induced by the mapping* f.

We mention here the following Corollary 2.3, which is an immediate consequence of Lemma 2.2 and useful in applications.

Corollary 2.3. *Assume that a hyperfunction* $\mu(x, y)$ *satisfies the following conditions* (2.6) *and* (2.7).

(2.6) *The projection* $f: N \to M$ *defined by* $(x, y) \mapsto x$ *is a proper mapping over* $S.S.\mu(x, y)$.

(2.7) $S.S.\mu(x, y) \subset \{(x, y; \mathrm{grad}_{(x,y)}\varphi(x, y))\}$ *for some real analytic function* $\varphi(x, y)$.

Denoting by $\nu(x)$ *the integral along fiber* $\int\mu(x, y)dy$, $S.S.\nu(x)$ *is contained in the set of* $(x, \xi)(\in S^*M)$ *for which* $\xi=\mathrm{grad}_x\varphi(x, y)$ *and* $\mathrm{grad}_y\varphi(x, y)=0$ *hold with* $(x, y, \xi, 0)\in S.S.\mu(x, y)$ $(\subset S^*N)$ *for some* y.

As for $E_k(x, y')$ constructed in Theorem 2.1 we have the following

Theorem 2.4. $S.S.E_k(x, y')$ *is contained in the union of the bicharacteristic strips passing through* $x=(0, y')$.

Proof. Let N be a neighbourhood of $(x, y')=(0, 0)\in \mathbf{R}^n\times\mathbf{R}^{n-1}$. Combining Corollary 2.3 and the relation (2.4) about the location of singularities of $u_k^{(j)}(x, y', \xi')$, we conclude that

$$S.S.\int_{|\xi'|=1}u_k^{(j)}(x, y', \xi')\omega(\xi')\subset\{(x, y', \xi'\ \eta')\in S^*N \,\big|\, \mathrm{grad}_{\xi'}\varphi_j(x, y', \xi')=0,$$
$$\xi=\mathrm{grad}_x\varphi_j(x, y', \xi') \text{ and } \eta'=\mathrm{grad}_{y'}\varphi_j(x, y', \xi')\}.$$

By the definition of $\varphi_j(x, y', \xi')$, it is obvious that $\mathrm{grad}_{y'}\varphi_j(x, y', \xi')=-\xi'$

and that $\operatorname{grad}_{\xi'}\varphi_j(x, y', \xi')\big|_{x_1=0}=x'-y'$. This completes the proof by the definition of bicharacteristic strips itself, since $\operatorname{grad}_{\xi'}\varphi$ is invariant on the bicharacteristic strip of $P(x, D_x)$.

Corollary 2.5. *The support of $E_k(x, y')$ regarded as a hyperfunction of x is contained in a conoid with its vertex at y', whose boundary consists of the bicharacteristic curve of $P(x, D_x)$ through y'.*

Proof. By the above theorem $E_k(x, y')$ is real analytic outside the characteristic conoid, which is by definition a conoid formed by all the bicharacteristic curves issuing from $(0, y')$. On the other hand all the Cauchy data of $E_k(x, y')$ on $x_1=0$ vanish except for $x'=y'$. Thus using the precise version of Holmgren's uniqueness theorem for hyperfunction solutions (Kawai [4], [5], a little weaker form of Holmgren's uniqueness theorem is also proved in Schapira [1] by a different method), we conclude that $E_k(x, y')$ vanishes in an open set in $\mathbf{R}^n$ which contains $\{x_1=0\}-\{x=(0, y')\}$. Therefore $E_k(x, y')$ vanishes identically outside a conoid by the unique continuation theorem for analytic functions.

Remark. Up to now we have assumed that $P(x, D_x)$ is strictly hyperbolic with respect to $(1,0, \cdots,0)$ near the origin. But using a result of Hamada [2] it is easy to extend our results to a linear hyperbolic differential operator $P(x, D_x)$ with constant multiplicity satisfying the Levi condition. (See for example Mizohata and Ohya [1] about the Levi condition). The proof is just the same as what has been stated, especially the real analyticity of $E_k(x, y')$ outside the characteristic conoid can be proved. We hope that the Levi condition is redundant in the theory of hyperfunctions at least $P(x, D_x)$ has constant multiplicity, though we have not yet proved this fact. Hence we shall not discuss these any more in this paper. Remark that a complete result is obtained for hyperbolic convolution operators in Kawai [4]. We also remark that an important result of Atiyah, Bott and Gårding [1] that the elementary solution $E(x)$ of hyperbolic operator with constant coefficients $P(D_x)$ is real analytic outside the wave front follows almost automatically from Lemma 2.2

assuming *no conditions on lower order terms* of $P(D_x)$. We hope that we will treat these problems in the future.

The Cauchy problem in a real domain has been considered solely from the view point of well-posedness since the famous book of Hadamard [1]. But using the theory of the sheaf $\mathscr{C}$ we can approach the Cauchy problem in a different way, i.e., from the view point of regularity. Of course such an approach is suggested in Hadamard [1] (see for example his discussions in p. 25 and p. 245), it seems to be difficult to have the precise statement without the theory of the sheaf $\mathscr{C}$ and this may be the reason why such an approach has not been fully developed.

We give some examples of such an approach in the sequel. They are also important in studying the propagation of singularities of the solutions for a linear differential operator, not necessarily hyperbolic. (See for example Theorem 2.8 below). A much more general approach to the Cauchy problem is indicated in Kashiwara and Kawai [2] using the structure of the sheaf $\mathscr{C}$ and the cannonical transformation on S^*M essentially.

We first prove the following Theorem 2.6. Remark that we need not assume that $P(x, D_x)$ is of simple characteristics in this theorem.

Theorem 2.6. *Let M_0 be a hypersurface $\{x_1=0\}$ and I be an open set in S^*M_0. Assume that the following Cauchy problem* (2.8) *is always locally solvable near the origin for any hyperfunctions $\{\mu_j(x')\}_{j=0}^{m-1}$ with $S.S.\mu_j(x')\subset I$:*

$$\left\{\begin{array}{l} P(x, D_x)u(x)=0 \\ \left.\dfrac{\partial^j}{\partial x_1^j}u(x)\right|_{x_1=0}=\mu_j(x'), \text{ where } 0\leqq j\leqq m-1. \end{array}\right. \tag{2.8}$$

Then $P_m(0, x', \xi_1, \xi')=0$ has, as an equation for ξ_1, at least one real solution for any $(x', \xi')\in I$.

Proof. By Sato's theorem on regularity of solutions of linear differential equations (Sato [2]~[5], see also Kashiwara and Kawai [1]), we have $S.S.u(x)\subset\{(x, \xi)\in S^*M \mid P_m(x, \xi)=0\}$. On the other hand we have $S.S.u(0, x')\subset\rho(S.S.u(x))$, where ρ is the cannonical projection

$S^*M\underset{M}{\times}M_0 - S^*_{M_0}M \to S^*M_0$. Here $S^*_{M_0}M$ means the conormal sphere bundle. (Sato [5] Theorem 6.1.1.) Therefore the proof is immediate.

Remark. In the above proof we have used only the condition $u(x)|_{x_1=0}=\mu_0(x')$, hence the above condition on ξ_1 is obviously far from sufficient. We will give a sufficient condition in Theorem 2.7 below. Kashiwara and Kawai [2] give a necessary condition much closer to the sufficient one for a linear differential operator with simple characteristics employing the pseudo-differential operators of finite type (Kashiwara and Kawai [1]).

We next state a generalization of Theorem 2.1. We hope this theorem reveals the connection of the existence theorem of Cauchy-Kowalevsky and the existence theorem for hyperbolic differential operators.

Theorem 2.7. *Let the initial hypersurface M_0 be a domain in $\{x_1=0\}\subset\mathbf{R}^n$, and let I be an open set in S^*M_0. Assume that $P_m(x, \xi)$ has real coefficients and the solutions of $\xi_1^j(0, x', \xi')$ $(j=1, \cdots, m)$ of $P_m(0, x', \xi_1, \xi')=0$ are all real and distinct whenever $(x', \xi')\in I$. Then the following Cauchy problem* (2.9) *is locally solvable.*

$$
(2.9)\qquad \begin{cases} P(x, D_x)u(x)=0 \\ \left.\dfrac{\partial^j}{\partial x_1^j}u(x)\right|_{x_1=0}=\mu_j(x'), \quad \text{where } 0\leqq j\leqq m-1, \end{cases}
$$

provided that $\mu_j(x')$ is a hyperfunction on M_0 satisfying $S.S.\mu_j(x')\Subset I$.

Remark 1. Those differential operators $P(x, D_x)$ which satisfies the conditions posed in Theorem 2.7 are named I-hyperbolic operators in Kawai [1].

Remark 2. This theorem is extended also to the linear differential operators whose principal symbols are not necessarily real under suitable conditions on phase functions in Kawai [5] (Theorem 3.3 and remarks following it). The arguments employed there are closely related to Theorem 1.4 and Theorem 1.6 in Kawai [6].

Proof of Theorem 2.7. We construct hyperfunctions $u_k^{(j)}(x, y', \xi')$ which satisfy the following condition (2.10) as in the proof of Theorem 2.1, as far as $(y', \xi') \in I$ by the assumption on $P_m(x, \xi)$.

$$(2.10)\quad \begin{cases} P(x, D_x)u_k^{(j)}(x, y', \xi')=0 \\ \dfrac{\partial^l}{\partial x_1^l}\left(\sum\limits_{j=1}^{m} u_k^{(j)}(x, y', \xi')\right)\Big|_{x_1=0} = \\ = \dfrac{\delta_{lk}(n-2)!}{(-2\pi i)^{n-1}(<x'-y', \xi'>+i0)^{n-1}}, \quad \text{where } 0\leqq l,\ k\leqq m-1. \end{cases}$$

Then we have

$$(2.11)\quad S.S.u_k^{(j)}(x, y', \xi') \subset \{(x, y', \xi'; \operatorname{grad}_{(x,y',\xi')}\varphi_j(x, y', \xi')) \mid \varphi_j(x, y', \xi')=0\}.$$

Using $u_k^{(j)}(x, y', \xi')$ which satisfies (2.10), we define $v(x)$ by the following formula:

$$v(x)=\sum_{k=0}^{m-1}\sum_{j=1}^{m}\int_{(y',\xi')\in I,|\xi'|=1} u_k^{(j)}(x,y', \xi')\mu_k(y')dy'\omega(\xi').$$

Since $\operatorname{grad}_{y'}\varphi_j(x, y', \xi')=-\xi'$ by the definition, the property of $S.S.u_k^{(j)}(x, y', \xi')$ given in (2.11) secures the above multiplication $u_k^{(j)}(x, y', \xi')\mu_k(y')$. (See Sato [5] §6.4.) Therefore we can perform the above integration as an integration on the sheaf $\mathscr{C}$ by the condition $S.S.\mu_k \Subset I$. It is obvious by the condition (2.10) that $v(x)$ satisfies the following Cauchy problem (2.12) with some real analytic functions $f(x)$ and $g_j(x')$. (Remark that the above integration, which defines $v(x)$, is performed on the sheaf $\mathscr{C}$. Hence real analytic functions must appear as remainder terms on the sheaf $\mathscr{B}$. Remember the following fundamental exact sequence due to Sato $0\to\mathscr{A}\to\mathscr{B}\to\pi_*\mathscr{C}\to 0$, where $\mathscr{A}$ denotes the sheaf of germs of real analytic functions on M.)

$$(2.12)\quad \begin{cases} P(x, D_x)v(x) =f(x) \\ \dfrac{\partial^j}{\partial x_1^j}v(x)\Big|_{x_1=0}=\mu_j(x')+g_j(x'), \quad \text{where } 0\leqq j\leqq m-1. \end{cases}$$

By Cauchy-Kowalevsky theorem we can find some real analytic function $w(x)$ for which $P(x, D_x)w(x)=f(x)$ and $\left.\frac{\partial^j}{\partial x_1^j} w(x)\right|_{x_1=0}=g_j(x')$ hold. Therefore subtracting $w(x)$ from $v(x)$ we obtain the required $u(x)$.

Example of I-hyperbolic operators. Let $P=\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}$ and $I=\{(x_2, x_3, \xi_2, \xi_3) \mid \xi_2>|\xi_3|\}$. Then the linear differential operator P becomes I-hyperbolic. It seems that Volterra treated this example. (See the footnote of Hadamard [1] p. 254.)

Remark. Dr. Morimoto has recently applied the existence theorem for I-hyperbolic operators for the investigation of the relation between the support and the singular support of hyperfunctions motivated by some physical considerations. (Morimoto [1].)

We close this section by proving Theorem 2.8, which extends considerably a result of Zerner [1] and Hörmander [2] using the sheaf $\mathscr{C}$. It is proved in §3 (Theorem 3.4) that the following result is the best possible of that sort.

Theorem 2.8. *Consider one of real bicharacteristic strips of a linear differential operator $P(x, D_x)$ with the real principal symbol and simple characteristics. Donote the bicharacteristic strip by ℓ, which defines a non-singular curve in S^*M by the definition. Then there exists locally a hyperfunction $u(x)$ which satisfies $P(x, D_x)u(x)=0$ and has non-void $S.S.u(x)$ contained in ℓ.*

Proof. Choose a point $p_0=(x, \xi)=(0, \xi^0)$ in ℓ. Since the operator $P(x, D_x)$ is of simple characteristics, we can assume without loss of generality, after a suitable choice of the local coordinate system on M, that $P(x, D_x)$ is non-characteristic with respect to $\{x_1=0\}$ and that the solution $\xi_1(x, \xi')$ of the equation $P_m(x, \xi_1, \xi')=0$ is non-singular near $p_0=(0, \xi^0)$. Then we can apply Theorem 1.2 with $p=1$ and $v_0\equiv 1$. (Cf. Remark after Theorem 1.3.) Hence we obtain some $u(z, y', \xi')$ which satisfies $P(z, D_z)u(z, y', \xi')=0$ and has the form (1.8) with some $\varphi_j(z, y', \xi')$, say $\varphi_1(z, y', \xi')$, for which the characteristic surface $\{\varphi_1(z, 0, \xi')=0\}$

contains the bicharacteristic curve corresponding to the bicharacteristic strip $\mathscr{b}$. The solution $u(z, y', \xi')$ defines a hyperfunction $u(x, y', \xi')$ as in the proof of Theorem 2.1. We also have $S.S.u(x, y', \xi') \subset \{(x, y', \xi'; \operatorname{grad}_{(x,y',\xi')}\varphi_1(x, y', \xi')) \mid \varphi_1(x, y', \xi') = 0\}$ regarding $u(x, y', \xi')$ as a hyperfunction in (x, y', ξ'). On the other hand for any co-vector ξ'_0, we can find a hyperfunction $\mu(y'; \xi'_0)$ in y' for which $S.S.\mu(y'; \xi'_0) = \{(0, \xi'_0) \in S^*M_0\}$. For example we can take $\mu(y'; \xi'_0) = 1/\{<y', \xi'_0> + i(|y'|^2 - <y', \xi'_0>^2) + i0\}$ with $|\xi'_0|$ normalized to 1. Taking $\xi'_0 = (\xi^0)'$ we define $v(x)$ by

$$\int_{(y',\xi')\in V, |\xi'|=1} u(x, y', \xi')\mu(y', \xi')dy'\omega(\xi'),$$

where V is a sufficiently small neighbourfood of $(y', \xi') = (0, \xi'_0)$. Applying Corollary 2.3 as in the proof of Theorem 2.4, we see that $S.S.v(x) \subset \mathscr{b}$. It is obvious from the definition of $u(x, y', \xi')$ and that of $\mu(y'; \xi'_0)$ that $S.S.v(x) \neq \emptyset$. (Moreover it is known from Theorem 3.4 in the next section that $S.S.v(x)$ coincides with $\mathscr{b}$.) It is also trivial by the definition of $v(x)$ that $P(x, D_x)v(x) = f(x)$ is a real analytic function. Of course we can find some real analytic function $w(x)$ which satisfies $P(x, D_x)w(x) = f(x)$ near the origin by the Cauchy-Kowalevsky theorem. Hence, subtracting $w(x)$ from $v(x)$, we obtain the required solution $u(x)$ near the origin.

Remark. Modifying $\mu(y'; \xi'_0)$ given in the proof suitably, Kashiwara [1] has given another short proof of his theorem that the sheaf $\mathscr{C}$ is flabby.

§3. Construction of Local Elementary Solutions for Linear Differential Operators of Real Principal Symbols with Simple Characteristics

In this section we construct local elementary solutions and investigate their properties for linear differential operator $P(x, D_x)$ with real analytic coefficients defined near the origin of $\mathbf{R}^n$ satisfying the following condition:

(3.1) The principal symbol $P_m(x, \xi)$ is real and satisfies $\mathrm{grad}_\xi P_m(x, \xi) \neq 0$ whenever $P_m(x, \xi)=0$.

Theorem 3.1. *For a linear differential operator* $P(x, D_x)$ *satisfying* (3.1), *we can construct in a neighbourhood of* $(x, y, \xi)=(0, 0, \xi^0)$ *a hyperfunction* $E(x, y, \xi)$ *in* (x, y, ξ) *for which the following relation* (3.2) *holds.*

$$P(x, D_x)E(x, y, \xi)=\frac{(n-1)!}{(-2\pi i)^n(<x-y, \xi>+i0)^n}. \tag{3.2}$$

Proof. If $P_m(0, \xi^0)\neq 0$, then the existence of $E(x, y, \xi)$ is proved by Sato. (Sato [2]$\sim$[5]. See also Kashiwara and Kawai [1] Theorem 6.) Hence it is sufficient to construct $E(x, y, \xi)$ near $(0, 0, \xi^0)$ for which $P_m(0, \xi^0)=0$ holds.

By the assumption (3.1) we can choose a local coordinate system on M so that the hypersurface $\{x_1=0\}$, (hence $\{x_1=s\}$ with $|s|\ll 1$) is non-characteristic with respect to $P(x, D_x)$ and $\left.\frac{\partial P_m}{\partial \xi_1}\right|_{(x,\xi)=(0,\xi^0)}\neq 0$. Next we apply Corollary 1.4 defining $Q(x, D_x, D_s)$ by giving its symbol by $Q(x, \xi, \sigma) =\{P(x, \xi_1+\sigma, \xi')-P(x, \xi)\}/\sigma$, where σ stands for D_s. As in Corollary 1.4 we consider the phase function $\varphi(z, y, \xi, s)$ which satisfies

$$\begin{cases} P_m(z, \mathrm{grad}_z\varphi(z, y, \xi, s))=0 \\ \varphi(z, y, \xi, s)|_{z_1=s}=<z-y, \xi>. \end{cases} \tag{3.3}$$

Note that the differential operator $P(x, D_x)$, *a posteriori* $Q(x, D_x, D_s)$ also, has real analytic coefficients, hence can be analytically extended to a complex domain.

It is obvious from the definition of $\varphi(z, y, \xi, s)$ that the following relations (3.4)$\sim$(3.8) hold.

$$\left.\left(\frac{\partial\varphi}{\partial z_1}+\frac{\partial\varphi}{\partial s}\right)\right|_{z_1=s}=\xi_1 \tag{3.4}$$

$$\left.\frac{\partial\varphi}{\partial z_j}\right|_{z_1=s}=\xi_j \qquad (j=2, \dots, n) \tag{3.5}$$

$$\frac{\partial \varphi}{\partial y_j}\Big|_{z_1=s} = -\xi_j \qquad (j=1, 2, \cdots, n) \tag{3.6}$$

$$\frac{\partial \varphi}{\partial \xi_1} = s - y_1 \tag{3.7}$$

$$\frac{\partial \varphi}{\partial \xi_j}\Big|_{z_1=s} = z_j - y_j \qquad (j=2, \cdots, n). \tag{3.8}$$

Since we consider near $(z, y, \xi)=(0, 0, \xi^0)$, we have $|\partial\varphi/\partial s| \ll 1$ by (3.4). Therefore we have $Q_{m-1}(z, \mathrm{grad}_{(z,s)}\varphi)|_{z_1=s} \neq 0$ by the definition of $Q(z, D_z, D_s)$. Hence we can apply Corollary 1.4 and obtain a holomorphic function $E(z, y, \xi, s)$ defined in $\{\mathrm{Im}\varphi(z, y, \xi, s)>0\}$, which satisfies the following singular Cauchy problem (3.9).

$$\begin{cases} P(z, D_z)E(z, y, \xi, s)=0 \\ Q(z, D_z, D_s)u(z, y, \xi, s)|_{s=z_1} = \dfrac{(n-1)!}{(-2\pi i)^n < z-y, \xi >^n} . \end{cases} \tag{3.9}$$

By the assumption (3.1) we can integrate the Hamilton-Jacobi equations in a real domain to obtain φ satisfying (3.3) for real (y, ξ, s). Therefore $E(z, y, \xi, s)$ defines a hyperfunction $E(x, y, \xi, s)$ in $3n$ variables (x, y, ξ, s) satisfying

$$S.S.E(x, y, \xi, s) \subset \{(x, y, \xi, s;\ \mathrm{grad}_{(x,y,\xi,s)}\varphi(x, y, \xi, s))\,|\, \varphi(x, y, \xi, s)=0\}, \tag{3.10}$$

if we take the boundary values of $E(z, y, \xi, s)$ from the complex domain $\{\mathrm{Im}\varphi(z, y, \xi, s)>0\}$ as in Theorem 2.1.

By this regularity property of $E(x, y, \xi, s)$ we can consider the multiplication of $E(x, y, \xi, s)$ by $\theta_{[\alpha, x_1]}(x)$, which is by definition equal to 1 on $[\alpha, x_1]$ and vanishes outside that interval. Therefore we can define the integral $E(x, y, \xi)=\int_\alpha^{x_1} E(x, y, \xi, s)ds$ for some fixed constant $\alpha(|\alpha| \ll 1)$. It is obvious from the definition of the differential operator $Q(x, D_x, D_s)$ that we have

$$P(x, D_x)\, E(x, y, \xi) = \frac{(n-1)!}{(-2\pi i)^n (< x-y, \xi > + i0)^n} .$$

Thus the construction of $E(x, y, \xi)$ is completed.

Errata in Kawai [2]. The operator $Q(x, D_x)$ defined in Theorem 2 in Kawai [2] should be replaced $Q(x, D_x, D_s)$ defined above.

Now we investigate the regularity properties of

$$E(x, y)=\int_U E(x, y, \xi)\omega(\xi),$$

where $\omega(\xi)$ denotes the volume element $\sum_{j=1}^{n}(-1)^j\xi_j d\xi_1\wedge\cdots\cdots\wedge d\xi_{j-1}\wedge d\xi_{j+1}\wedge\cdots\wedge d\xi_n$ and U is a closed neighbourhood of ξ^0 in the unit sphere S^{n-1}.

Thoerem 3.2. *About the location of singularities of* $E(x, y)$ *defined above, the following relation* (3.11) *holds.*

(3.11) *In a neighbourhood of* $(0, 0, \xi^0, -\xi^0)$ *in* $S^*(M\times M)$, *we have* $S.S.E(x, y)\subset\{(x, y, \xi, \eta)\in S^*(M\times M)\,|\,x=y, \xi=-\eta\}\cup\{(x, y, \xi, \eta)\in S^*(M\times M)\,|\,(x, \xi)$ *and* $(y, -\eta)$ *lie in the same bicharacteristic strip with* $x_1\geqq y_1\}$.

Proof. We combine the properties (3.4)$\sim$(3.8) of the phase function $\varphi(z, y, \xi, s)$ with Lemma 2.2. Remark that the effect of the boundary of U and the lower bound α used in the definition of $E(x, y)$ does not appear near $(0, 0, \xi^0, -\xi^0)$, which is also clear from the proof below. Since we perform the integration in s and in ξ to obtain $E(x, y)$, it is important to know the singularities of $E(x, y, \xi, s)\theta(x_1-s)$. ($\theta$ denotes the step function of Heaviside.) Using (3.10) we have the following relation (3.12). There we denote by N a neighbourhood of $(x, y, \xi, s)=(0, 0, \xi^0, 0)$ and by ζ the cotangent vector at (x, y, ξ, s). Moreover we denote by $(\zeta_1, \zeta_2, \zeta_3)$ its components relative to the dual basis of $(\partial/\partial x_1, \mathrm{grad}_{(x', y, \xi)}, \partial/\partial s)$.

(3.12) $S.S.E(x, y, \xi, s)\theta(x_1-s)\subset A\cup B$, where $A=\{(x, y, \xi, s; \zeta)\,|\,x_1=s$ and $\zeta_1=a\dfrac{\partial\varphi}{\partial x_1}\pm b$, $\zeta_2=a\ \mathrm{grad}_{(x', y, \xi)}\varphi$ and $\zeta_3=a\dfrac{\partial\varphi}{\partial s}\mp b$ for some $a, b\geqq 0$ with $a+b>0\}$ and $B=\{(x, y, \xi, s; \zeta)\,|\,x_1\neq s$ and $\zeta==\mathrm{grad}_{(x, y, \xi, s)}\varphi\}$.

We first consider the set A. By (3.4)$\sim$(3.8) we have on A

$(3.12)_A \quad \dfrac{\partial\varphi}{\partial x_1}+\dfrac{\partial\varphi}{\partial s}=\xi_1$, $\mathrm{grad}_{x'}\varphi=\xi'$, $\mathrm{grad}_y\varphi=-\xi$ and $\mathrm{grad}_\xi\varphi=x-y$.

Therefore the contribution of the points in A to $S.S.E(x, y)$ is the following $\tilde{A}$.

$\tilde{A}=\{(x, y, \xi, \eta)\,|\,x=y$ and $\xi=-\eta\}$, i.e., the antidiagonal set of $S^*(M\times M)$. In fact it is sufficient to consider those points in A which satisfy $\dfrac{\partial\varphi}{\partial s}=0$ and $\mathrm{grad}_\xi\varphi=0$, hence we obtain the set $\tilde{A}$ by $(3.12)_A$. In the same way we consider the contribution from the set B. By (3.7) we have $\dfrac{\partial\varphi}{\partial\xi_1}=s-y_1$, and we have only to consider those points satisfying $\dfrac{\partial\varphi}{\partial\xi_1}=0$, thus $s=y_1$. Therefore we can conclude that $x_1\geqq s=y_1$ holds in $S.S.E(x, y)$ by the definition of $E(x, y)$. To decide the contribution to $S.S.E(x, y)$ of the points belonging to B, we consider the bicharacteristic strip ℓ of $P(x, D_x)$ issuing from $(x, y, \xi, s, \mathrm{grad}_{(x,y,\xi,s)}\varphi(x, y, \xi, s))$. Let ℓ meet the surface $\{x_1=s\}$ at $p_0=(\tilde{x}, \tilde{\xi})$. Combining Corollary 2.3 with (3.4)$\sim$(3.8) we conclude that $\tilde{x}=y$ and $\tilde{\xi}=\xi=-\eta$. Thus we have proved (3.11).

Remark. An important fact contained in (3.11) is that $S.S.E(x, y)$ is contained in the union of *half* of the bicharacteristic strips. From this fact it is obvious that for the solution $u(x)$ of $P(x, D_x)u(x)=0$, $S.S.u(x)$ is contained in the set of bicharacteristic strips. More precisely we have

Theorem 3.3. *Assume that the hypersurface $\{x_1=0\}$ is non-characteristic with respect to $P(x, D_x)$ and that $\partial P_m/\partial\xi_1\neq 0$ near $(0, \xi^0)$. Denote by $\xi_1=\lambda(x, \xi')$ a real analytic solution of the characteristic equation $P_m(x, \xi_1, \xi')=0$. Then for each point on the hypersurface $N=\{(x, \xi)\,|\,\xi_1=$ $=\lambda(x, \xi')\}$ we can find an open neighbourhood Ω which satisfies the following: if a hyperfunction solution $u(x)$ of $P(x, D_x)u(x)=0$ vanishes as a section of the sheaf $\mathscr{C}$ outside N and on a set V in N, then $u(x)$ vanishes as a section of the sheaf $\mathscr{C}$ on $\Omega\cap\tilde{V}$, where $\tilde{V}$ is the set of all points connected to a point in V by a bicharacteristic strip of $P(x, D_x)$.*

Proof. Let $P^*(y, D_y)$ be the formal adjoint operator of $P(y, D_y)$. By the preceding theorems we can construct a local elementary solution

$E(x, y)$ for $P^*(y, D_y)$ which satisfies (3.11). We chose an open set ω so small that $E(x, y)$ is defined in $\omega\times\omega^a$. (ω^a denotes the antidiagonal set of ω.) Using the flabbiness of the sheaf $\mathscr{C}$ (Kashiwara [1]) we can assume $P(y, D_y)\tilde{u}(y)=\mu(y)$ holds with $S.S.u(y)$ compact and $\tilde{u}(y)=u(y)$ on ω. By the assumption on $u(y)$, $\mu(y)$ vanishes on an open neighbourhood of V. Therefore using (3.11) and Lemma 2.2 we have on $\omega\cap\tilde{V}$

$$0=\int\mu(y)E(x, y)dy=\int\tilde{u}(y)P^*(y, D_y)E(x, y)dy=\tilde{u}(x)=u(x),$$

where the integration is performed as a section of the sheaf $\mathscr{C}$. Note that we can take as Ω the union of all ω used in the above so that $\Omega\cap\tilde{V}\supset V$ holds.

Since Sato's fundamental theorem on regularity of hyperfunction solutions of linear differential equations (Sato [2]~[5]) implies that $S.S.u(x)\subset\{(x, \xi)\in S^*M\,|\,P_m(x, \xi)=0\}$ if $Pu=0$, we can restate Theorem 3.3 in the following way.

Theorem 3.3′. *Let $u(x)$ satisfy $P(x, D_x)u(x)=0$ as a section of the sheaf $\mathscr{C}$ in an open set Ω in S^*M. Suppose that two points (x, ξ) and (y, η) are connected by a bicharacteristic strip of $P(x, D_x)$ entirely lying in Ω and that $u(x)$ vanishes at (x, ξ) as a section of sheaf $\mathscr{C}$. Then $u(x)$ vanishes at (y, η) as a section of sheaf $\mathscr{C}$.*

Corollary 3.4. *Assume that $P(x, D_x)$ is defined in a neighbourhood M of x_0 and that $\psi(x)$ be a real valued function in $C^2(M)$ such that $\mathrm{grad}_x\psi(x)|_{x=x_0}\neq0$ and the level surface $\{\psi(x)=\psi(x_0)\}$ is pseudo-convex at x_0 in the sense of Hörmander (Hörmander [1] Definition 8.6.1). Then there exists a neighbourhood V of x_0 such that every $u\in\mathscr{B}(M)$ satisfying the conditions that $P(x, D)u=f\in\mathscr{A}(M)$ and that $u\in\mathscr{A}(M^+)$, where $M^+=\{x\in M\,|\,\psi(x)>\psi(x_0)\}$, is in $\mathscr{A}(V)$.*

The proof follows immediately from Theorem 3.3′ if we calculate the second derivative of $\psi(x)$ along the bicharacteristic curve.

Remark. Corollary 3.4 improves Theorem 8.8.1 of Hörmander [1] under assumption (3.1) in two points. The first is that we allow $u(x)$

to be a hyperfunction, not merely a distribution, and the second is that we treat the propagation of real analyticity, not merely infinite differentiability. Observe that Corollary 3.4 trivially follows from Theorem 3.3′ and that we can also consider higher order derivatives of $\psi(x)$ along the bicharacteristics in the same way if we want. These observations will show the effectiveness of Theorem 3.3′ in investigating the propagation of analyticity of the solutions of linear differential equations.

Moreover Mr. Kashiwara suggested to the author that we may improve Theorem 3.3 to the form given below (Theorem 3.5) in an analogous way to the precise version of Holmgren's uniqueness theorem (Kawai [4] and [5]). In fact the proof of Theorem 3.5 is immediate from Theorem 3.3 combined with the theory of pseudo-differential operators of finite type developed in Kashiwara and Kawai [1]. The author expresses his sincere gratitude to Mr. Kashiwara for this suggestion.

Theorem 3.5. *Take the local coordinate system as in the preceding theorem and denote by M_0 the hypersurface $\{x_1=0\}\subset M$. Then the assumption in Theorem* 3.3 *that u vanishes on V can be weakened to the assumption that the restriction of u to $S^*M_0\cap V$ vanishes.*

Proof. We apply a proposition on factorization of polynomial due to Hörmander [4] to $P(x, \xi)$. (Hörmander [4] Proposition 6.1). The proposition asserts that $P(x, D_x)$ can be written near $(0, \xi^0)$ as a composite of two pseudo-differential operators of finite type $Q(x, D_x)$ and $P_1(x, D_x)$ such that $Q(x, D_x)$ is elliptic near $(0, \xi^0)$ and $P_1(x, D_x)$ has the form $\partial/\partial x_1 - r(x, D_{x'})$ with some pseudo-differential operator $r(x, D_{x'})$ of finite type of order 1. Since an elliptic pseudo-differential operator of finite type is invertible (Kashiwara and Kawai [1] Theorem 6), we are allowed to consider $P_1(x, D_x)u(x)=0$ holds near $(0, \xi^0)$. On the other hand $P_1(x, D_x)(u(x)\theta(-x_1))=0$ holds by the assumption on the initial condition of $u(x)$, hence Theorem 3.5 follows from Theorem 3.3.

Remark 1. It may be possible to prove Theorem 3.5 by modifying the elementary solution $E(x, y)$ constructed above for the adjoint operator

$P^*(y, D_y)$ using the result of §2, but the proof employed here is more straightforward.

Remark 2. It is clear that we can restate Theorem 3.5 in the analogous form to Theorem 3.3′.

Remark 3. Theorems 3.2 and 3.3 hold for a linear pseudo-differential operator of finite type, if its principal symbol $P_m(x, \xi)$ is a polynomial in ξ satisfying the assumption (3.1). It is obvious from Theorem 1 of Kashiwara and Kawai [2]. Moreover we can develop an analogous theory for a general linear pseudo-differential operator satisfying the following condition (3.1′).

(3.1′) The principal symbol $P_{-m}(x, \xi)$ is real and $\mathrm{grad}_{(x,\xi)} P_{-m}(x, \xi)$ is not parallel to $(\xi, 0)$ whenever $P_{-m}(x, \xi)=0$.

This case is treated using the real analytic version of the deep theory of Hörmander [3] concerning the equivalence of phase functions. It will be treated in a forthcoming paper of Kashiwara and Kawai. See also Kashiwara and Kawai [2], where the summary is given.

As a corollary of the above uniqueness theorem we have the local representation of the solution sheaf $\mathscr{C}^P$ as an integral of solutions of Cauchy problems for $P(x, D_x)$ in the below if $P(x, D_x)$ satisfies the assumption (3.1).

Taking a local coordinate system as in Theorem 3.3, we consider the problem near $(0, \xi^0)$. Using the same notations as in Theorem 3.3, we denote by $K(y', \xi')$ the characteristic surface of $P(x, D_x)$ which passes through the intersection of two hypersurfaces $\{x_1=0\}$ and $\{<x'-y', \xi'>=0\}$ with its normal direction at $(0, y')$ being $(\lambda(0, y', \xi'), \xi')$. Employing Theorem 1.2 with $p=1$ and $K_1=K$, we obtain $E(x, y', \xi')$ for which

$$\begin{cases} P(x, D_x)E(x, y', \xi')=0 \\ E(x, y', \xi')|_{x_1=0}=\dfrac{(n-2)!}{(-2\pi i)^{n-1}(<x'-y', \xi'>+i0)^{n-1}} \end{cases}$$

hold. (Cf. the proof of Theorem 2.8.) Choosing $\mu_0(x')$ so that $S.S.\mu_0(x')$ is compact in $S^*(\{x_1=0\})$ and $u(0, x')=\mu_0(x')$ holds near $(0, (\xi^0)')$, we define $\mu(x, \xi')$ by

$$\int E(x, y', \xi')\mu_0(y')dy'.$$

Then we have near $(0, \xi^0)$

$$u(x)=\int \mu(x, \xi')\omega(\xi')$$

by Theorem 3.5. By the definition of $\mu(x, \xi')$ we have

$$P(x, D_x)\mu(x, \xi') \;=0 \quad \text{and} \quad \int \mu(x, \xi')\Big|_{x_1=0}\omega(\xi')=u(0, x')$$

near $(0, (\xi^0)')$.

Moreover if we can take the local coordinate system so that

$$\frac{\partial}{\partial\xi_1}P_m(0, \xi_1, \xi')\neq 0 \qquad \text{for any } \xi \text{ with } P_m(0, \xi)=0,$$

then we can represent $u(x)$ as

$$\int_{|\xi'|=1}\mu(x, \xi')\omega(\xi')$$

near the origin by giving p Cauchy data $\{\mu_0(x'), \cdots, \mu_{p-1}(x')\}$ on $\{x_1=0\}$, where p is the number of real roots of the characteristic equation $P_m(0, \xi_1, \xi')=0$ with respect to ξ_1. In fact Theorem 1.2 proves it just in the same way as above.

These remarks may be regarded as a very partial extension of the so called fundamental principle of Ehrenpreis [1] for linear differential operators with constant coefficients. We will also discuss the representation of the solution sheaf $\mathscr{C}^P$ by the integral of boundary values of holomorphic solutions of $P(z, D_z)$ in a forthcoming paper.

Up to now we have developed the local theory of linear differential equations mainly by a complex method, so to speak. We can also develop it using the theory of pseudo-differential operators of finite type (Kashiwara and Kawai [1]). We call such a method the real method in contrast to the name of complex method. Though the complex method is more convenient than the real method in treating the linear differential operators with complex coefficients (Kawai [3]), the real method seems a little simpler as far as the phase function can be taken real valued. Hence we sketch the real method in the below. This method is due to Hörmander [2] and the author expresses his hearty gratitude to Professor Gårding and Professor Kotaké for their kind suggestions that the author should also try to employ Hörmander's method.

In the sequel we take the local coordinate system so that the hypersurface $\{x_1=0\}$ is non-characteristic with respect to $P(x, D_x)$ and $\frac{\partial}{\partial \xi_1} P_m \neq 0$ holds near $(x, \xi)=(0, \xi^0)$.

Theorem 3.2′. *We can construct* $E(x, y)$ *for which* $P(x, D_x)E(x, y) = \delta(x-y)$ *holds near* $(0, 0, \xi^0, -\xi^0) \in S^*(M \times M)$ *and* $S.S.E(x, y) \subset \{(x, y, \xi, \eta) \in S^*(M \times M) \mid x=y, \xi=-\eta\} \cup \{(x, y, \xi, \eta) \in S^*(M \times M) \mid (x, \xi)$ *and* $(y, -\eta)$ *belong to the same bicharacteristic strip of* $P(x, D_x)$ *with* $x_1 \leqq y_1\}$.

Proof. We first consider a phase function $\varphi(x, y, \xi)$ which satisfies the following condition (3.15).

$$
(3.15)\quad \begin{cases} \text{(i)} & P_m(x, \operatorname{grad}_x \varphi(x, y, \xi)) = P_m(y, \xi). \\ \text{(ii)} & \varphi(x, y, \xi) \text{ is real analytic near } (0, 0, \xi^0) \text{ and is positively} \\ & \text{homogeneous of order 1 with respect to } \xi. \\ \text{(iii)} & \varphi(x, y, \xi) \text{ is real valued for real } (x, y, \xi). \\ \text{(iv)} & \varphi(x, y, \xi) = \langle x-y, \xi \rangle + O(|x-y|^2 |\xi|). \end{cases}
$$

The existence of such a function $\varphi(x, y, \xi)$ is well-known by the classical theory of differential equations of first order. The analyticity of the coefficients of $P(x, D_x)$ and that of $\varphi(x, y, \xi)$ required in (3.15) (ii) allow us to consider them in a complex neighbourhood of $(0, 0, \xi^0)$. In an analogous way to the proof of Theorem 1.2 we define $\Phi_j(\tau)$ as follows.

$$(3.16)\quad \Phi_j(\tau)=\begin{cases} \dfrac{(-1)^j(n-j-1)!}{(-2\pi i)^n}\dfrac{1}{\tau^{n-j}} & (j<n) \\ \dfrac{-1}{(2\pi i)^n(j-n)!}\left\{\tau^{j-n}\log\tau-\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{j-n}\right)\tau^{j-n}\right\} & (j\geqq n). \end{cases}$$

We want to construct a holomorphic function $F(z, w, \zeta)$ in (z, w, ζ) defined in $\{\mathrm{Im}\varphi(z, w, \zeta)>0\}$, which satisfies

$$(3.17)\qquad P(z, D_z)F(z, w, \zeta)=\sum_{j\geqq 0} r_j(z, w, \zeta)\Phi_j(\varphi(z, w, \zeta))$$

with $r_0(z, w, \zeta)\neq 0$ near $(0, 0, \xi^0)$ and r_j being homogeneous of order $(-j)$ with respect to ζ. For that purpose we assume $F(z, w, \zeta)$ has the form

$$(3.18)\qquad \sum_{j\geqq 0}\frac{f_j(z, w, \zeta)\Phi_j(\varphi(z, w, \zeta))}{P_m(w, \zeta)},$$

where $f_j(z, w, \zeta)$ is some holomophic function in (z, w, ζ) near $(0, 0, \xi^0)$ and homogeneous of order $(-j)$ with respect to ζ. Then noting that $P_m(z, \mathrm{grad}_z\varphi(z, w, \zeta))=P_m(w, \zeta)$ we can formally determine $f_j(z, w, \zeta)$ so that $f_0(w, w, \zeta)=1$ and

$$(3.19)\qquad P(z, D_z)F(z, w, \zeta)=\sum_{j\geqq 0} r_j(z, w, \zeta)\Phi_j(\varphi(z, w, \zeta))$$

hold. In fact we solve successively the transport equations, i.e., the first order differential equations given in §1,

$$(1.13)\qquad \mathscr{L}[f_0]=0$$

$$(1.14)\qquad \mathscr{L}[f_j]=-\sum_{p=2}^{m} L_p[f_{j+1-p}]\qquad (j\geqq 1)$$

with its Cauchy data equal to 1 for f_0 and to 0 for $f_j (j\geqq 1)$ on a hypersurface $S=\{x_1=y_1\}$, which is non-characteristic with respect to $\mathscr{L}$. Thus the relation (3.19) holds formally. We also prove the following estimate (3.20) just as in the proof of Theorem 1.3. (In fact the estimation is much easier in this case.)

(3.20) $\sup_{(z,w,\zeta)\in V} |f_j(z, w, \zeta)| \leqq c^j j!$ for a complex neighbourhood V of $(0, 0, \xi^0)$ and for a constant c.

It is obvious that this estimate (3.20) implies that the summation (3.18) absolutely converges in $\Omega = \{(z, w, \zeta) \in V \mid \mathrm{Im}\varphi(z, w, \zeta) > 0$, $\mathrm{Im}P_m(w, \zeta) > 0$ and $|\varphi| < c'$ for some $c' > 0\}$. Therefore the relation (3.17) holds in Ω as an equality for holomorphic functions in (z, w, ζ). Taking the boundary values of both sides of (3.17) as in the proof of Theorem 2.1, we have

$$(3.21) \qquad P(x, D_x)F(x, y, \xi) = \sum_{j\geqq 0} r_j(x, y, \xi)\Phi_j(\varphi(x, y, \xi)+i0).$$

On the other hand the right side of (3.21) defines an elliptic pseudo-differential operator of finite type, hence we can find its invese near $(0, 0, \xi^0, -\xi^0)$ as a pseudo-differential operator of finite type. (See Kashiwara and Kawai [1] Theorem 6). Thus we have $E(x, y)$ for which $P(x, D_x)E(x, y) = \delta(x-y)$ holds near $(0, 0, \xi^0, -\xi^0)$, by integrating both sides of (3.21) locally with respect to the volume element on unit sphere $\omega(\xi)$.

Now we investigate the regularity property of $E(x, y)$. It is sufficient to consider $\int F(x, y, \xi)\omega(\xi)$ instead of $E(x, y)$ for that purpose. By the definition of $F(x, y, \xi)$, we have the following relation (3.22). There we use ζ to denote the cotangent vector at (x, y, ξ). Moreover we denote by $(\zeta_1, \zeta_2, \zeta_3)$ its components relative to the dual basis of $(\mathrm{grad}_x, \mathrm{grad}_y, \mathrm{grad}_\xi)$.

(3.22) $S.S.E(x, y, \xi) \subset \{(x, y, \xi; \zeta) \mid \varphi(x, y, \xi) = P_m(y, \xi) = 0$ and $\zeta_1 = a\,\mathrm{grad}_x\varphi(x, y, \xi)$, $\zeta_2 = a\,\mathrm{grad}_y\varphi(x, y, \xi) + b\,\mathrm{grad}_y P_m(y, \xi)$ and $\zeta_3 = a\,\mathrm{grad}_\xi\varphi(x, y, \xi) + b\,\mathrm{grad}_\xi P_m(y, \xi)$ with some $a, b \geqq 0, a+b \neq 0\} \cup \{(x, y, \xi; \zeta) \mid P_m(y, \xi) = 0,\ \varphi(x, y, \xi) \neq 0,\ \zeta_1 = 0,\ (\zeta_2, \zeta_3) = \mathrm{grad}_{(y,\xi)} = P_m(y, \xi)\} \cup \{(x, y, \xi; \zeta) \mid \varphi(x, y, \xi) = 0,\ P_m(y, \xi) \neq 0,\ \zeta = \mathrm{grad}_{(x,y,\xi)}\varphi\}$.

Applying Lemma 2.2 to the integral $F(x, y) = \int F(x, y, \xi)\omega(\xi)$ we conclude that $\zeta_3 = 0$ holds on $S.S.F(x, y)$ near $(0, 0, \xi^0, -\xi^0)$. It obviously follows from the definition of the phase function $\varphi(x, y, \xi)$ given in

(3.15) and the relation $\zeta_3=0$ that (x, ζ_1) and $(y, -\zeta_2)$ belongs to the same bicharacteristic strip with $x_1 \leqq y_1$. Thus we have proved Theorem 3.2′ by a real method.

Remark 1. The above proof of Theorem 3.2′ is very close to Hörmander [2], [3]. (Cf. the method of energy inequatities employed in Hörmander [1] and Shirota [1].) There he treats an operator with C^∞-coefficients and investigate the regularity of distribution solutions in the C^∞-category. Since we have assumed that $P(x, D_x)$ has real analytic coefficients in this paper, many technical difficulties are by-passed. We hope that the employment of the theory of the sheaf $\mathscr{C}$ has made the situation clear.

Remark 2. Recently Hörmander [4] has succeeded in defining an analogue of $S.S.u(x)$ for a distribution $u(x)$ by the aid of his theory of pseudo-differential operators, and has obtained results concerning distribution solutions, which correspond to Theorem 3.3′.

Remark 3. The extension of our theory to a linear differential operator with complex coefficients will be given in our forthcoming paper Kawai [6]. See also Kawai [3], [5].

Bibliography

Atiyah, M.F., R. Bott and L. Gårding, [1], Lacunas for hyperbolic differential operators with constant coefficients I, *Acta Math.* **124** (1970), 109-189.

Ehrenpreis, L., [1], *Fourier Analysis in Several Complex Variables*, Wiley-Interscience, 1970.

Gel'fand, I.M. and G.E. Shilov, [1], *Generalized Functions*, Vol. 1, Kyôritsu, 1963. (Translated from Russian into Japanese)

Hadamard, J., [1], *Lectures on Cauchy Problems*, Reprinted by Dover, 1922.

Hamada, Y., [1], The singularities of the solutions of the Cauchy problem, *Publ. RIMS, Kyoto Univ.* **5** (1969), 21-40.

Hamada, Y., [2], On the propagation of singularities of the solution of the Cauchy problem, *Publ. RIMS, Kyoto Univ.* **6** (1970), 357-384.

Hörmander, L., [1], *Linear Partitial Differential Operators*, Springer, 1963.

Hörmander, L., [2], On the singularities of solutions of partial differential equations, *Proc. Int. Conf. on Functional Analysis and Related Topics*, Univ. of Tokyo Press, 1969, pp. 31-40.

Hörmander, L., [3], Fourier integral operators I, *Acta Math.* **127** (1971), 79-183.

Hörmander, L., [4], Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, to appear in *Comm Pure Appl. Math.*

Kashiwara, M., [1], On the flabbiness of the sheaf $\mathscr{C}$, *Sûrikaiseki-kenkyûsho Kôkyûroku* No. 114, R.I.M.S., Kyoto Univ., 1970, 1-4. (Japanese)

Kashiwara, M. and T. Kawai, [1], Pseudo-differential operators in the theory of hyperfunctions. *Proc. Japan Acad.* **46** (1970), 1130-1134.

Kashiwara, M. and T. Kawai, [2], Pseudo-differential operators in the theory of hyperfunctions (II), to appear.

Kawai, T., [1], Construction of elementary solutions for *I*-hyperbolic operators and solutions with small singularities, *Proc. Japan Acad.* **46** (1970), 912-916.

Kawai, T., [2], Construction of a local elementary solution for linear partial differential operators. I, *Proc. Japan Acad.* **47** (1971), 19-23.

Kawai, T., [3], Construction of a local elementary solution for linear partial differential operators. II, *ibid.* (1971), 147-152.

Kawai, T., [4], On the theory of Fourier transform in the theory of hyperfunctions and its applications to partial differential equations with constant coefficients, Master thesis presented to Univ. of Tokyo, 1970 (Japanese). Reprinted in part in *Sûrikkaiseki-Kenkyûsho Kokyûroku*, No. 108, RIMS, Kyoto Univ., 1969. pp. 84-288. (Japanese)

Kawai, T., [5], On the local theory of (pseudo)-differential operators, *Sûrikaiseki-kenkyûsho Kokyûroku* No. 114, R.I.M.S., Kyoto Univ., 1970, pp. 18-68. (Japanese)

Kawai, T., [6], Contruction of local elementary solutions for linear partial differential operators with real analytic coefficients II —— The case with complex principal symbols——, *Publ. RIMS, Kyoto Univ.*, this issue.

Komatsu, H., [1], Relative cohomology of sheaves of solution of differential equations, *Sem. Lions and Schwartz*, 1966. Reprinted in *Reports of Katata Symposium on Algebraic Geometry and Hyperfunctions*, 1969, pp. 1-59.

Komatsu, H. and T. Kawai, [1], Boundary values of hyperfunction solutions of linear partial differential equations, *Publ. RIMS, Kyoto Univ.* **7** (1971/72), 95-104.

Lax, P.D., [1], Asymptotic solutions of oscillatory initial value problems, *Duke Math. J.* **24** (1957), 627-646.

Mizohata, S. et Y. Ohya, [1], Sur la condition de E.E. Levi concernant des équations hyperboliques, *Proc. Int. Conf. on Functional Analysis and Related Topics*, Univ. of Tokyo Press, 1969, pp. 177-185.

Morimoto, M., [1], On the relation of support and singular support of hyperfunctions, *Reports of the Symposium on the Theory of Hyperfunctions and Partial Differential Equations Held at RIMS, Kyoto Univ.*, 1971, pp. 1-8. (Japanese). See also M. Morimoto, Support et support singulier de l'hyperfonction, *Proc. Japan Acad.* **47** (1971). 648-652.

Nirenberg, L. and F. Treves, [1], On local solvability of linear partial differential equations, Part II, *Comm. Pure Appl. Math.* **23** (1970), 459-510.

Sato, M., [1], Theory of hyperfunctions II, *J. Fac. Sci. Univ. Tokyo* **8** (1960), 387-437.

Sato, M., [2], Hyperfunctions and partial differential equations, *Proc. Int. Conf. on Functional Analysis and Related Topics*, Univ. of Tokyo Press, 1969, pp. 91-94.

Sato, M., [3], Regularity of hyperfunction solutions of partial differential equations, to apper in *Proceedings of Nice Congress* (1970).

Sato, M., [4], Structure of hyperfunctions, *Reports of the Katata Symposium on Algebraic Geometry and Hyperfunctions*, 1969, pp. 4. 1-4. 30. (Notes by Kawai, in Japanese).
Sato, M., [5], Structure of hyperfunctions, *Sûgaku no Ayumi*, **15** (1970), 9-72. (Notes by Kashiwara, in Japanese).

Schapira, P., [1], Théorème d'unicité de Holmgren et opérateurs hyperboliques dans l'espace des hyperfonctions, to appear in *Anais da Academia Brasileira de Ciencias.*

Shirota, T., [1], On the propagation of regularity of solutions of partial differential equations, *Proc. Japan Acad.* **39** (1963), 120-124.

Zerner, M., [1], Solutions singulières d'équations aux dérivées partielles, *Bull. Soc. Math. France*, **91** (1963), 203-226.

Notes Added in Proof on November 25, 1971:

1° Theorem 3.2 of this paper, hence Theorem 3.3, Corollary 3.4 and Theorem 3.5, have been extended recently to the case where the prinicpal symbol $P_m(x, \xi)$ has the form $[Q_q(x, \xi)]^p$, where $Q_q(x, \xi)$ is real and of simple characteristics and p is a positive integer. The proof of this statement uses an asymptotic expansion which is a modification of that used in the proof of Theorem 3.2′. The details will be published somewhere else. We also note that Professor Jean-Michel Bony and Professor Pierre Schapira have recently obtained a rather complete result on the existence of hyperfunction solutions for weakly hyperbolic operators in their joint report at the A.M.S. Summer Institute on partial differential equations held in August 1971. (Prolongement et existence des solutions des systèmes hyperboliques non-stricts à coefficient analytiques). Their results combined with the above remarked improvement of Theorem 3.3 complete the remark in p. 377. See also Sato, M., T. Kawai and M. Kashiwara: On pseudo-differential equations in hyperfunction theory (in preparation).

2° Professor Takeshi Kotaké reported the construction of elementary solutions in the framework of distributions analogous to that given in Theorem 3.2′ at the occasion of the general meeting of the Mathematical Society of Japan held in October 1971, though he does not investigate the cotangential component of the singularities of elementary solutions as we have done in this paper. His result seems to be a rather complete one in order to investigate the propagation of analyticity of *distribution* solutions and is an extension of the recent result of Professor Karl Gustav Andersson to the case of linear differential operators with variable coefficients. (Cf. Andersson, K.G.: Propagation of analiticity of solutions of partial differential equations with constant coefficients, *Ark. Mat.* **8** (1971), 277–302.)