Construction of Local Elementary Solutions for Linear Partial Differential Operators with Real Analytic Coefficients (II)

——The case with complex principal symbols——

By

Takahiro Kawai

§0. Introduction

In this paper the coefficients of a linear differential operator $P(x, D_x)$ under consideration are assumed to be real analytic functions defined on an open set M in $\mathbf{R}^n$ containing the origin. We denote by S^*M the cotangential sphere bundle of M. The purpose of this paper is to construct locally on S^*M an elementary solution for $P(x, D_x)$ under the assumption that $P(x, D_x)$ is of simple characteristics. When the principal symbol $P_m(x, \xi)$ is real, we have constructed an elementary solution locally for linear differential operator $P(x, D_x)$ and investigated its regularity properties in detail in Kawai [2]. (In this paper we use the same notations as in Kawai [2] for the principal symbol of $P(x, D_x)$ etc., and will not repeat their definitions if there is no fear of confusions). However the celebrated counterexample due to H. Lewy [2] (cf. Schapira [1] concerning the hyperfunction solutions) shows that the construction is possible only under some additional conditions when the principal symbol $P_m(x, \xi)$ is not assumed to be real. Therefore in §1 we construct an elementary solution for $P(x, D_x)$ near (x_0, ξ^0) in S^*M, assuming that $P(x, D_x)$ satisfies condition $(P)_{(x_0, \xi^0)}$, which is given in Definition 1.3. In the sequel we assume that $x_0 = 0$ for the sake of simplicity. The method of constructing a local elementary solution employed in §1 is called the complex method in Kawai

[2] in contrast with the real method employed in §2 of this paper. On the other hand, the condition for the local existence of (distribution) solutions has been recently investigated by Nirenberg and Treves [2], [3] using a priori estimate for pseudo-differential operators (in the sense of Hörmander). Also the same condition is announced by Egorov [1]. Following their way of stating the condition of the solvability of linear differential equations we define condition $(NT)_{f,(0,\xi^0)}$, which is more restrictive than that given by Nirenberg and Treves [2], [3] and Egorov [1], and prove that condition $(NT)_{f,(0,\xi^0)}$ implies condition $(P)_{(0,\xi^0)}$. In fact condition $(NT)_{f,(0,\xi^0)}$ is much more restrictive than condition $(P)_{(0,\xi^0)}$ and we can prove that any hyperfunction solution $u(x)$ of $P(x, D_x)u(x)=0$ is real analytic near $x=0$ if $(NT)_{f,(0,\xi^0)}$ holds for any non-zero cotangent vector ξ^0 at 0. This regularity theorem immediately follows from the regularity properties of the elementary solutions constructed under condition $(NT)_{f,(0,\xi^0)}$.

Though condition $(NT)_{f,(0,\xi^0)}$ is easier to verify than condition $(P)_{(0,\xi^0)}$, it clearly does not cover all the possibilities of the solvable cases in hyperfunction category (Sato [1]) or rather in the framework of Sato's sheaf $\mathscr{C}$ defined on S^*M (Sato [2]~[5]). Hence in §2 we investigate other two extreme cases where we can construct a local elementary solution of $P(x, D_x)$, though condition $(NT)_{f,(0,\xi^0)}$ cannot be applied. In the course of the construction of local elementary solutions we give under some moderate conditions on $P(x, \boldsymbol{D_x})$ an affirmative answer to Sato's conjecture that there should exist an exact sequence:

$$\mathscr{C} \xrightarrow{P} \mathscr{C} \xrightarrow{Q} \mathscr{C}$$

for some pseudo-differential operator Q (in the sense of Sato). Sato [3] proved this fact when $P(x, D_x)$ is the example of H. Lewy [2], i.e.,

$$P(x, D_x) = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2} - 2i(x_1 + ix_2)\frac{\partial}{\partial x_3}.$$ We refer the reader to Sato [5] about the notion of pseudo-differential operators in the sense of Sato. See also Kashiwara and Kawai [1].

When $P(x, D_x)$ is a first order linear partial differential operator, Schapira [2] has obtained the complete result for the local existence of

hyperfunction solutions using the theory of locally convex spaces combined with the results of Nirenberg and Treves [1]. (Cf. Suzuki [1] in the special case that $P(x, D_x)$ is a first order linear differential operator in two independent variables. He treats there the problem of local existence and regularity of hyperfunction solutions for such an operator).

The results of this paper have been announced in Kawai [1], [3].

About the notion of the sheaf $\mathscr{C}$, which is essential in this paper, we refer the reader to the precise and extensive exposition by Kashiwara based on Sato's lectures (Sato [5]). Notes of Sato [2]∼[4] and Kashiwara and Kawai [1] are also available.

The author expresses his hearty gratitude to Professor M. Sato, Professor H. Komatsu and Mr. M. Kashiwara for their advices and constant encouregement.

§1. Construction of Local Elementary Solutions
—— Complex Method ——

In this section we define condition $(NT)_{f,(0,\xi^0)}$ and construct a local elementary solution for a linear differential operator $P(x, D_x)$ satisfying this condition using an existence theorem for Cauchy problems with singular initial data in the complex domain.

As we remarked in the introduction, condition $(NT)_{f,(0,\xi^0)}$ is too restrictive as far as we are concerned with the local existence of solutions for a linear differential operator $P(x, D_x)$. In order to make up for this defect we first give a less restrictive condition, called condition $(P)_{(0,\xi^0)}$, which assures the local existence of elementary solutions for $P(x, D_x)$. Condition $(P)_{(0,\xi^0)}$ is concerned rather with the behaviour of characteristic surfaces of $P(x, D_x)$ than with the operator $P(x, D_x)$ itself.

Before discussing the local existence of solutions for $P(x, D_x)$ in the real domain, we quote from Kawai [2] §1 an existence theorem in the complex domain for Cauchy problems with singular initial data. Since we have assumed that the coefficients of $P(x, D_x)$ are real analytic functions, we may consider that the differential operator $P(x, D_x)$ is defined on a complex domain and we denote the operator by $P(z, D_z)$. Until the end

of this section we assume that the hypersurface $\{z_1=0\}$ (hence $\{z_1=s\}$ with $|s|\ll 1$) is non-characteristic with respect to $P(z, D_z)$ and that $P_m(z. \xi)=0$ implies $(\partial/\partial\xi_1)P_m(z. \xi)\neq 0$ near $(z, \xi)=(0, \xi^0)$.

Theorem 1.1. *Consider the following Cauchy problem (1.1) under the condition (1.2) on the linear differential operator $Q(z, D_z, D_s)$, holomorphic functions $J(z, w, \xi)$ and $\varkappa(z, w, \xi)$.*

$$(1.1) \quad \begin{cases} P(z, D_z)u(z, w, \xi, s)=0 \\ Q(z, D_z, D_s)u(z, w, \xi, s)|_{s=z_1}=\dfrac{J(z, w, \xi)}{\varkappa(z, w, \xi)^l} \end{cases}$$

where l is a positive integer, $Q(z, D_z, D_s)$, $J(z, w, \xi)$ and $\varkappa(z, w, \xi)$ satisfy the following

$$(1.2) \quad \begin{cases} (\mathrm{i}) & \text{\textit{The holomorphic function} } J(z, w, \xi) \text{ \textit{is defined near} } (z, w, \xi)= \\ & =(0, 0, \xi^0) \text{ \textit{and homogeneous of order} } 0 \text{ \textit{with respect to} } \xi. \\ (\mathrm{ii}) & \textit{The holomorphic function } \varkappa(z, w, \xi) \textit{ is defined near } (z, w, \xi)= \\ & =(0, 0, \xi^0), \textit{ homogeneous of order } 1 \textit{ with respect to } \xi \textit{ and} \\ & \textit{has the form } <z-w, \xi>+O(|z-w|^2|\xi|). \\ (\mathrm{iii}) & \textit{The relation } Q_p(z, \mathrm{grad}_{(z,s)}\varphi(z, y, \xi, s))|_{z_1=s}\neq 0 \textit{ holds for the} \\ & \textit{principal part } Q_p \textit{ of } Q, \textit{ where the holomorphic function} \\ & \varphi(z, w, \xi, s) \textit{ denotes the phase function of } P(z, D_z) \textit{ with the} \\ & \textit{initial condition on } \{z_1=s\} \textit{ given by } \varkappa(z, w, \xi), \textit{ i.e.,} \\ & P_m(z, \mathrm{grad}_z\varphi(z, w, \xi, s))=0. \end{cases}$$

Then the Cauchy problem (1.1) admits a solution $u(z, w, \xi, s)$ which has the form

$$(1.3) \quad \begin{cases} u(z, w, \xi, s)=F(z, w, \xi, s)\varphi(z, w, \xi, s)^{-k}+ \\ +G(z, w, \xi, s)\log\varphi(z, w, \xi, s)+H(z, w, \xi, s), \end{cases}$$

where k is a positive integer and F, G and H are holomorphic near $(z, w, \xi, s)=(0, 0, \xi^0, 0)$.

About the proof of this theorem see Kawai [2] §1 Theorem 1.3 and

the corollary and remarks following the theorem. Note that this theorem is a variant of the result of Hamada [1] modified to be suitable for our purpose.

Using the phase function $\varphi(z, w, \xi, s)$, which is used to represent the solution $u(z, w, \xi, s)$ as in (1.3), we give the definition of condition $(P)_{(0,\xi^0)}$.

Definition 1.2. (Sato). A real analytic function $f(x)$ is said to be of positive type if the following condition (1.4) holds.

(1.4)
$$\begin{cases} \text{(i)} \quad \text{Im}f(x) \geqq 0 \text{ whenever } \text{Re}f(x) = 0. \\ \\ \text{(ii)} \quad \text{grad}_x f \neq 0. \end{cases}$$

Definition 1.3. (Condition $(P)_{(0,\xi^0)}$). Choosing a suitable initial datum $\varkappa(x, y, \xi)$ used in Theorem 1.1, which is a real analytic function of positive type in (x, y, ξ), we can find a phase function $\varphi(x, y, \xi, s)$ of P which satisfies the following condition (1.5) for $|x|, |y|, |s| \ll 1$.

(1.5)
$$\begin{cases} \text{Im}\varphi(x, y, \xi, s) \geqq 0 \text{ on } \{(x, y, \xi, s) \text{ real}, \quad x_1 \geqq s, \quad \xi \in I^+ \text{ and} \\ \text{Re}\varphi(x, y, \xi, s) = 0\} \text{ and } \{(x, y, \xi, s) \text{ real}, \quad x_1 \leqq s, \quad \xi \in I^- \text{ and} \\ \text{Re}\varphi(x, y, \xi, s) = 0\}, \text{ where } I^+ \text{ and } I^- \text{ denote some locally closed} \\ \text{sets in an } (n-1)\text{-dimensional unit co-sphere } S^{n-1} \text{ such that} \\ I = I^+ \cup I^- \text{ is a neighbourhood of } \xi^0 \text{ in } S^{n-1}. \end{cases}$$

Remark 1. Afterwards we sometimes omit the subscript $(0, \xi^0)$ and denote this condition by condition (P) for the simplicity of notations.

Remark 2. As in Definition 1.2, condition (P) may be said as follows: the phase function φ can be taken to be of half positive type for a suitable choice of the initial condition $\varkappa$ which is homogeneous of order 1 with respect to ξ and is of positive type.

Remark 3. Since the coefficients of $P_m(x, \xi)$ are complex valued, we cannot integrate the Hamilton-Jacobi equations in the real domain to obtain the phase function φ even when the initial datum $\varkappa(x, y, \xi)$ is real va-

lued. This is the reason why we cannot use the plane waves $<x-y, \xi>$ for the initial data in general but must use the curvilinear waves $\varkappa(x, y, \xi)$, which we use to represent $\delta(x-y)$ by Sato's formula:

$$(1.6) \qquad \delta(x-y) = \frac{(n-1)!}{(-2\pi i)^n} \int_{|\xi|=1} \frac{J(x, y, \xi)}{(\varkappa(x, y, \xi)+i0)^n} \omega(\xi),$$

where $\omega(\xi)$ denotes the volume element on S^{n-1}, i.e.,

$$\omega(\xi) = \sum_{j=1}^{n} (-1)^j \xi_j d\xi_1 \wedge \cdots \wedge d\xi_{j-1} \wedge d\xi_{j+1} \wedge \cdots \wedge d\xi_n$$

and $J(x, y, \xi)$ and $\varkappa(x, y, \xi)$ satisfy the following conditions:

$$(1.7) \left\{ \begin{array}{l} \text{(i)} \quad \text{The real analytic function } \varkappa(x, y, \xi) \text{ is of positive type and} \\ \qquad \text{is positively homogeneous of order 1 with respect to } \xi. \\ \text{(ii)} \quad \text{There are some real analytic functions } \{\varkappa_j(x, y, \xi)\}_{j=1}^{n} \text{ which} \\ \qquad \text{are positively homogeneous of order 1 with respect to } \xi \text{ and} \\ \qquad \text{satisfy } \sum_{j=1}^{n} (x_j - y_j)\varkappa_j(x, y, \xi) = \varkappa(x, y, \xi). \\ \text{(iii)} \quad \text{Using these functions } \varkappa_j \text{ we define } J(x, y, \xi) \text{ by} \\ \qquad \dfrac{\partial(\varkappa_1, \cdots, \varkappa_n)}{\partial(\xi_1, \cdots, \xi_n)} = \det\left(\dfrac{\partial \varkappa_j}{\partial \xi_k}\right)_{1 \le j, k \le n}, \text{ which never vanishes.} \end{array} \right.$$

The expansion of δ-function by the complex valued curvilinear waves given in (1.6) is due to Sato. Note that the ambiguity in the choice of $\varkappa_j(x, y, \xi)$ does not affect the result since the effect of the ambiguity is absorbed into the coboundary part.

Remak 4. We consider that the notion of the half positivity of the phase function would be a key to the local solvability of linear differential equations. Compare the fact that a linear differential operator $P(x, D_x)$ with real prinicipal symbol has a local elementary solution $E(x, y)$ whose "singular support on S^*M" is contained in the union of *half* of the bicharacterstic strips of $P(x, D_x)$. (See Kawai [2] Theorem 3.2 for the precise statement of the above fact). There the notation of half positivity is effectively used to prove the above statement on the singularity of elementary solutions.

Theorem 1.4. *Assume that* $P(x, D_x)$ *satisfies condition* $(P)_{(0, \xi^0)}$. *Then we can construct a local elementary solution* $E(x, y)$ *for which* $P(x, D_x)E(x, y) = \delta(x - y)$ *holds as sections of the sheaf* $\mathscr{C}$ *near* $(x, y; \xi, \eta)$ $=(0, 0; \xi^0, -\xi^0)$. *Here* (ξ, η) *denotes the cotangent vector at* $(x, y) \in M \times M$.

Proof. If $P_m(0, \xi^0) \neq 0$, then the existence of $E(x, y)$ is proved by Sato (Sato [2]~[5]. See also Kashiwara and Kawai [1] Theorem 6.) Hence we assume $P_m(0, \xi^0) = 0$. We first define a linear differential operator $Q(x, D_x, D_s)$ by giving its symbol by $Q(x, \xi, \sigma) =$ $= \{P(x, \xi_1 + \sigma, \xi') - P(x, \xi)\}/\sigma$, where σ stands for D_s. Then we proceed just as in the proof of Theorem 3.1 in Kawai [2] and obtain $E(z, w, \xi, s)$ by Theorem 1.1 for which the following relation (1.8) holds.

(1.8)
$$\begin{cases} P(z, D_z)E(z, w, \xi, s) = 0 \\ Q(z, D_z, D_s)E(z, w, \xi, s) \Big|_{z_1 = s} = \dfrac{(n-1)! J(z, w, \xi)}{(-2\pi i)^n \varkappa(z, w, \xi)^n} \end{cases}$$

where $J(z, w, \xi)$ and $\varkappa(z, w, \xi)$ are chosen so that the condition $(P)_{(0, \xi^0)}$ is satisfied for the operator $P(x, D_x)$. (Cf. Definition 1.3 and Remarks 1 and 2 after the definition).

Then the required elementary solution $E(x, y)$ for $P(x, D_x)$ is given by taking the boundary value of the integral (∗) defined below from the domain $\{\text{Im}\varphi > 0\}$. This integral is real analytic in x_1 and holomorphic in z'.

(∗)
$$\int_\Gamma \omega(\xi) \int_\alpha^{x_1} E(z, y, \xi, s) ds - \int_\Gamma \omega(\xi) \int_{x_1}^\beta E(z, y, \xi, s) ds,$$

where $x = \text{Re} z$ and $y = \text{Re} w$. About the precise notion of taking the boundary values, i.e., its formulation using the representation of relative cohomology group by some Leray coverings, we refer the reader to Komatsu [1]. The half positivity of the phase function $\varphi(x, y, \xi, s)$ required in the definition of condition $(P)_{(0, \xi^0)}$ assures that the hyperfunction $E(x, y)$ is well defined by the above procedure. It immediately follows from the relation (1.8) and the definition of $Q(x, D_x, D_s)$ that

$$P(x, D_x)E(x, \ y) = \int_I \frac{J(x, \ y, \ \xi)}{(\varkappa(x, \ y, \ \xi) + i0)^n}\omega(\xi) = \delta(x - y)$$

holds near $(0, 0 \, ; \, \xi^0, \ -\xi^0)$. The last equality is a corollary of Sato's curvilinear wave expansion of δ-function. (Cf. Remark 2 after Definition 1.3.)

Though Theorem 1.4 is satisfactory from the logical view point, it is not clear a priori whether a given linear differential operator $P(x, D_x)$ satisfies condition $(P)_{(0, \xi^0)}$ or not (unless its principal symbol $P_m(x, \ \xi)$ is real). (Cf. Kawai [2]). Hence we give the definition of condition $(NT)_{f, (0, \xi^0)}$ and prove that condition $(NT)_{f, (0, \xi^0)}$ implies condition $(P)_{(0, \xi^0)}$.

From now on we denote by $A_m(x, \xi)$ and $B_m(x, \xi)$ the real part and the imaginary part of $P_m(x, \ \xi)$, respectively, i.e., we write $P_m(x, \ \xi)$ as $A_m(x, \ \xi) + iB_m(x, \xi)$. Moreover we assume, without loss of generality, that $\mathrm{grad}_\xi A_m(x, \xi) \neq 0$ whenever $P_m(x, \xi) = 0$.

Definition 1.5. (Condition $(NT)_{f, (0, \xi^0)}$, cf. condition $(\mathscr{P})$ given in Nirenberg and Treves [2] p. 460. and the introduction of Treves [1]). Along every bicharactrristic strip of $A_m(x, \xi) = \mathrm{Re}P_m(x, \ \xi)$ which passes through (x, ξ) with $|x| \ll 1$ and $|\xi - \xi^0| \ll 1$, the function $B_m(x, \xi) = \mathrm{Im}P_m(x, \xi)$ has only zeros of finite even order.

Remark. Since the notation of condition $(NT)_{f, (0, \xi^0)}$ seems clumsy, we sometimes omit the subscript $(0, \xi^0)$ in the below.

Theorem 1.6. *Condition* $(NT)_{f, (0, \xi^0)}$ *implies condition* $(P)_{(0, \xi^0)}$.

Proof. If we denote $x_1 - s$ by $\tilde{x}_1$ and $(x_2, \cdots, x_n)$ by x', then the phase function $\varphi(x, \ y, \ \xi, \ s)$ is obtained by solving the first order differential equation $P_m(\tilde{x}_1 + s, \ x', \ \mathrm{grad}_{(\tilde{x}_1, \ x')}\tilde{\varphi}(\tilde{x}_1, \ x', \ y, \ \xi, \ s)) = 0$ with the initial condition on $\{\tilde{x}_1 = 0\}$ is given by $\varkappa(\tilde{x}_1 + s, \ x', \ y, \ \xi)$. Note that $\partial\tilde{\varphi}/\partial\tilde{x}_1 = \partial\varphi/\partial x_1$ holds. Since $|s| \ll 1$, we ca nassume that condition $(NT)_f$ holds for $P_m(\tilde{x}_1, \ x', \ \xi)$. Hence we abbreviate $\tilde{x}_1$ and $\tilde{\varphi}$ to x_1 and φ, respectively, for the simplicity of notations. Now we will choose a suitable initial datum $\varkappa$ so that the half positivity of φ follows. To prove the half positivity of

φ, we shall first decompose $P_m(x, \xi)$ into the from $R_{m-1}(x, \xi)(\xi_1 - a(x, \xi')$ $-ib(x, \xi'))$ near $(0, \xi^0)$. Here $R_{m-1}(x, \xi)$ is a complex valued real analytic function which never vanishes near $(0, \xi^0)$ and is positively homogeneous of order $m-1$ with respect to ξ, and $a(x, \xi')$ and $b(x, \xi')$ are real valued real analytic functions which are positively homogeneous of order 1 with respect to $\xi' = (\xi_2, \cdots, \xi_n)$. Then by the method of the proof of invariance of condition $(\mathscr{P})$ under the contact transformation given in Nirenberg and Treves [2] §2, we can assume that the condition $(NT)_f$ holds for $\xi_1 - a(x, \xi') - ib(x, \xi')$ with A_m replaced by $\xi_1 - a(x, \xi')$ and B_m by $-b(x, \xi')$. (Note that the problem of local existence and regularty of solutions of linear (pseudo)-differential equations can be treated in the most natural way by the contact transformation on S^*M. This is remarked by Egorov [1]. The relation between the contact transformation and the theory of sheaf $\mathscr{C}$ will be explained in a forthcoming paper of Kashiwara and Kawai [2]. See also Kawai [4]). It is obvious from condition $(NT)_f$ that either $b(x, \xi') \geqq 0$ holds in a neighbourhood V of $(0, \xi^{0'})$ or $b(x, \xi') \leqq 0$ holds in V, hence we assume in the sequel that $b(x, \xi') \geqq 0$ holds in V. Now we define the initial datum $\varkappa(x', y, \xi, s)$ by

$$(1.9) \qquad \varkappa(x', y, \xi, s) = (s - y_1)\xi_1 + <x' - y', \xi'> + i|x' - y'|^2.$$

In order to prove that the phase function $\varphi(x, y, \xi, s)$ with the initial datum on $\{x_1 = 0\}$ given by $\varkappa(x', y, \xi, s)$ is of half positive type, we follow the method of estimation given in Nirenberg and Treves [1] Lemma 4.2. It is well known that the phase function $\varphi(x, y, \xi, s)$ *uniquely* exists, since $a(x, \xi')$ and $b(x, \xi')$ are real analytic function in (x, ξ') and can be extended to a complex neighborhood. Therefore it is sufficient to prove $\text{Im}\varphi \geqq 0$ in $\{x_1 \geqq 0\}$ fixing (y, ξ) to (y, ξ_0). Moreover we can assume without loss of generality that $y' = 0$ since condition $(NT)_f$ holds on an open neighbourhood of $(0, \xi^0)$. Thus we are reduced to the estimation of imaginary part of $\varphi(x, y, \xi_0, s)$ which satisfies

$$(1.10) \qquad \frac{\partial \varphi}{\partial x_1} - a(x, \text{grad}_{x'}\varphi) - ib(x, \text{grad}_{x'}\varphi) = 0 \text{ and}$$

$$\varphi(0, x', y_1, 0, \xi_0, s) = (s - y_1)(\xi_0)_1 + <x', \xi_0'> + i|x'|^2.$$

Subtraction of $(s-y_1)(\xi_0)_1$ from φ does not affect the results as far as we are concerned only with the imaginary part of φ, hence we assume that the initial condition of φ is $<x', \xi_0'>+i\,|\,x'\,|^2$ and denote $\varphi=\varphi(x)$ for the simplicity of notations. Thus we need to consider

$$(1.10')\qquad \frac{\partial\varphi}{\partial x_1}-a(x, \mathrm{grad}_{x'}\varphi)-ib(x, \mathrm{grad}_{x'}\varphi)=0 \text{ with}$$

$$\varphi(0, x')=<x', \xi_0'>+i\,|\,x'\,|^2.$$

Following Nirenberg and Treves [2] we straighten out the bicharacteristic strip through $(x, \xi)=(0, a(0, \xi_0'), \xi_0')$ of $\xi_1-a(x, \xi')$. It is achieved by a change of variables from (x) to $(\tilde{x})$ in the neighbourhood of the origin in $\mathbf{R}^n$. The change of variables is defined by solving the following Cauchy problem for first order differential equations (1.11). There we denote $(\tilde{x}_2, \cdots, \tilde{x}_n)$ by $(\tilde{x})'$, by $J\left(\frac{(\tilde{x})'}{x'}\right)$ the Jacobian matrix of the $\tilde{x}_j(2\leq j\leq n)$ with respect to $x_k(2\leq k\leq n)$ and by ${}^tJ\left(\frac{(\tilde{x})'}{x'}\right)$ its transpose.

$$(1.11)\qquad \frac{\partial(\tilde{x})'}{\partial x_1}=J\left(\frac{(\tilde{x})'}{x'}\right)\mathrm{grad}_{\xi'}a\left(x, {}^tJ\left(\frac{(\tilde{x})'}{x'}\right)\xi_0'\right) \text{ with}$$

$$(\tilde{x})'\,|_{x_1=0}=x'.$$

Thus

$$(1.12)\qquad\qquad \tilde{x}_1=x_1 \quad\text{and}\quad (\tilde{x})'=(\tilde{x}(x))'$$

is a real analytic change of variables with non-singular Jacobian matrix $J\left(\frac{\tilde{x}}{x}\right)$ near the origin in $\mathbf{R}^n$. By this coordinate transformation (1.12) $\xi_1-a(x, \xi')$ turns into the form $\tilde{\xi}_1-\tilde{a}(\tilde{x}, (\tilde{\xi})')$, where $\tilde{a}(\tilde{x}, (\tilde{\xi})')=$
$=a(x(\tilde{x}), {}^tJ\left(\frac{(\tilde{x})'}{x'}\right)(\tilde{\xi})')- <\mathrm{grad}_{\xi'}a(x(\tilde{x}), {}^tJ\left(\frac{(\tilde{x})'}{x'}\right)\xi_0'), {}^tJ\left(\frac{(\tilde{x})'}{x'}\right)(\tilde{\xi})'>.$
From this representation of $\tilde{a}(\tilde{x}, (\tilde{\xi})')$ we easily conclude that

$$(1.13)\qquad\qquad \mathrm{grad}_{\xi'}\tilde{a}(\tilde{x}, \xi_0')=0, \text{ hence } \tilde{a}(\tilde{x}, \xi_0')=0$$

and cosequently we have

$$(1.14)\qquad\qquad \mathrm{grad}_{\tilde{x}}\tilde{a}(\tilde{x}, \xi_0')=0.$$

(Cf. Nirenberg and Treves [2] p. 22.) Thus we have straightened out the

bicharacteristic strip through $(0, a(0, \xi'_0), \xi'_0)$ of $\xi_1 - a(x, \xi')$. Note that the hyperplanes $\{\tilde{x}_1 = 0\}$ and $\{x_1 = 0\}$ are identical and that $x' = (\tilde{x})'$ holds on this hyperplane. In the rest of the proof of this theorem we abbreviate $\tilde{x}$, $\tilde{\xi}$, $\tilde{a}$ and $\tilde{b}$ to x, ξ, a and b respectively for the sake of simplicity of notations.

Now we begin the estimation of $\mathrm{Im}\varphi(x)$ in $x_1 \geq 0$. We want to prove

$$(1.15) \qquad |\varphi(x) - \langle x', \xi'_0 \rangle - i|x'|^2 - ib'x_1^{k_0+1}|$$

$$\leq C(|x'| + |x_1|^{1/2})(|x'|^2 + |x_1|^{k_0+1}) \quad \text{for } |x_1| \ll 1,$$

where C is a constant, b' is a positive number and k_0 is the order of zero of $b(x_1, 0, \xi'_0)$ with respect to x_1, which is an even number by the assumption on b. We first expand $\varphi(x)$ in the form

$$(1.16) \qquad \varphi(x) = \langle x', \xi'_0 \rangle + i|x'|^2 + i\sum_{k=0}^{\infty} u_k(x),$$

where $u_k(x)$ is a homogeneous polynomial of order k with respect to x'. By the initial condition on φ imposed in $(1.10')$, we conclude that $u_k(0, x') = 0$ holds for every k. Similarly we expand $T(x, \xi') = a(x, \xi') + ib(x, \xi')$ into the following form

$$(1.17) \qquad T(x, \xi') = \sum_{h=0}^{\infty} T_h(x, \xi'),$$

where $T_h(x, \xi')$ is homogeneous of order h with respect to x' and homogeneous of order 1 with respect to ξ'. Substituting (1.16) and (1.17) into $(1.10')$, we compare both sides according to the order with respect to x' and obtain

$$(1.18) \qquad \frac{d}{dx_1} u_0 = -i T_0(x, \xi'_0 + i\mathrm{grad}_{x'} u_1)$$

and

$$(1.19) \qquad \frac{\partial}{\partial x_1} u_1 = -i T_1(x, \xi'_0 + i\mathrm{grad}_{x'} u_1) +$$

$$+ \langle \mathrm{grad}_{\xi'} T_0(x, \xi'_0 + i\mathrm{grad}_{x'} u_1), \ \mathrm{grad}_{x'} u_2 + 2x' \rangle$$

Now we proceed just as in Nirenberg and Treves [2] pp. 23~25.

From (1.19) we have

$$\frac{\partial}{\partial x_1} u_1 = -i T_1(x, \xi_0') + <\text{grad}_{\xi'} T_0(y, s, \xi_0'), \text{grad}_{x'} u_2 + 2x'> + \Phi_1(x, u_1),$$

where Φ_1 depends only on x and on the coefficients u_{1j} of $u_1 = \sum_{j=2}^{n} u_{1j} x_j$. Note that Φ_1 is real analytic and $\Phi_1(x, 0) = 0$. Therefore we have from (1.13) and (1.14),

$$\frac{\partial}{\partial x_1} u_1 = -i < x', \text{grad}_{x'} b(x_1, 0, \xi_0')> + <\text{grad}_{\xi} b(x_1, 0, \xi_0'), \text{grad}_{x'} u_2 +$$

$$+ 2x'> + \Phi_1(x, u_1).$$

On the other hand we have $|\text{grad}_{(x'\xi')} b|^2 \leq C_0 |b|$ by condition $(NT)_f$ using Lemma 1.1 on real valued C^2-functions of Nirenberg and Treves [2]. Taking this fact into account we see that

$$\left| \frac{\partial}{\partial x_1} u_{1j} \right| \leq C_1 \left(\sum_{l=2}^{n} |u_{1,l}| + |b(x_1, 0, \xi_0')|^{1/2} \right), \quad j = 2, \cdots, n$$

since $\Phi_1(x, 0) = 0$. Therefore we have, for $|s|$ sufficiently small,

$$(1.20) \qquad |u_{1j}| \leq C_2 \int_0^{x_1} |b(x_1', 0, \xi_0')|^{1/2} dx_1', \quad j = 2, \cdots, n.$$

Noting that the bicharacteristic strip through $(0, a(0, \xi_0'), \xi_0')$ of $\xi_1 - a(x, \xi')$ is parallel to x_1-axis by the choice of local coordinate, we use condition $(NT)_f$ and obtain

$$(1.21) \qquad b(x_1, 0, \xi_0') = b_0 x_1^{k_0} + O(|x_1|^{k_0+1}), \text{ where } b_0 > 0$$

and k_0 is an even number. Combining (1.20) and (1.21) we see that

$$(1.22) \qquad |u_{1j}| \leq C_3 |x_1|^{(k_0+2)/2},$$

hence we have

$$(1.23) \qquad |u_1| \leq C_4 |x'| |x_1|^{(k_0+2)/2} \leq C_5 |x_1|^{1/2} (|x'|^2 + |x_1|^{k_0+1}).$$

Now we expand the left side of (1.18) according to the powers of $\text{grad}_{x'} u_1$ and obtain by (1.13) and (1.14)

$$\left| \frac{du_0}{dx_1} - b(x_1, 0, \xi_0') \right| \leq |<\text{grad}_{\xi} b(x_1, 0, \xi_0'), \text{grad}_{x'} u_1>| + C_6 |\text{grad}_{x'} u_1|^2.$$

Using (1.22) and the fact $|\mathrm{grad}_{(x',\xi')}b|^2 \leq C_0|b|$, we have

$$\left|\frac{du_0}{dx_1}-b(x_1, 0, \xi'_0)\right| \leq C_7\{|b(x_1, 0, \xi'_0)^{1/2}|x_1|^{(k_0+2)/2}+|x_1|^{k_0+2}\}.$$

Combining this estimate with (1.21) we conclude that

$$(1.24) \qquad \left|u_0-\frac{b_0}{k_0+1}x_1^{k_0+1}\right| \leq C_8|x_1|^{k_0+2}.$$

On the other hand we have

$$(1.25) \qquad |u_2| \leq C_9|x_1||x'|^2$$

by the definition of $u_2(x)$.

Noting that $\varphi(x)=<x', \xi'_0>+i|x'|^2+i\sum_{k=0}^{2}u_k+O(|x'|^3)$, we obtain the required estimate (1.15) from (1.16), (1.23), (1.24) and (1.25) and conclude that $\mathrm{Im}\varphi(x)\geq 0$ in $x_1\geq 0$ for the solution φ of (1.10'). This ends the proof of the theorem.

Remark. It is obvious from the above proof that $\mathrm{Im}\varphi(x)>0$ if $x_1>s$. This fact implies that condition $(NT)_f$ is much stronger than condition (P). In fact we can prove the following theorem by this fact.

Theorem 1.7. *Let a linear differential operator $P(x, D_x)$ satisfy condition $(NT)_{f,\,(0,\xi^0)}$. Then the elementary solution constructed in Theorem 1.4 defines a kernel function of a pseudo-differential operator (in the sense of Sato) near $(x, y; \xi, \eta)=(0, 0; \xi^0, -\xi^0)$. (We refer the reader to Sato $[3]\sim[5]$ about the definition of pseudo-differential operators in the sense of Sato).*

Proof. It is sufficient to prove $S.S.E(x, y)\subset\{(x, y; \xi, \eta)\in S^*(M\times M)|x=y, \xi=-\eta\}$. By the method of the construction of the phase function given in Theorem 1.6 we conclude that $\mathrm{Im}\varphi(x, y, \xi, s)>0$ if $x_1>s$ or $x_1<s$ under the condition $(NT)_{f,(0\ \xi^0)}$, i.e., that φ is of strictly half positive type, so to speak. We assume without loss of generality that $\mathrm{Im}\varphi>0$ holds if $x_1>s$ in the below. We see also from the definition of φ

that $\varphi(s, x', y, \xi, s) = (s - y_1)\xi_1 + \langle x' - y, \xi' \rangle + i|x' - y'|^2$. Here the norm $|\xi|$ of ξ is assumed to be 1. Now we regard $E(x, y, \xi, s)$ constructed in Theorem 1.4 as a hyperfunction in $(x, y, \xi, s) \in N$ and consider $S.S.E(x, y, \xi, s)$. Let ζ denote the cotangent vector at (x, y, ξ, s). Let $(\zeta_1, \cdots, \zeta_5)$ be the components of a cotangent vector ζ relative to the dual basis of $\left(\dfrac{\partial}{\partial x_1}, \mathrm{grad}_{x'}, \mathrm{grad}_y, \mathrm{grad}_\xi, \dfrac{\partial}{\partial s} \right)$. Then using the strict half positivity of the phase function φ we have

$$S.S.E(x, y, \xi, s) \subset \{(x, y, \xi, s; \zeta) \in S^*N \,|\, x_1 = s,\ x' = y',\ \zeta_1 = \frac{\partial \varphi}{\partial x_1},$$

$$\zeta_2 = \mathrm{grad}_{x'}\varphi, \quad \zeta_3 = \mathrm{grad}_y\varphi, \quad \zeta_4 = \mathrm{grad}_\xi\varphi \ \text{ and } \ \zeta_5 = \frac{\partial \varphi}{\partial s} \}.$$

Note that $\dfrac{\partial \varphi}{\partial x_1} + \dfrac{\partial \varphi}{\partial s}\Big|_{x_1 = s} = \xi_1$ and $\dfrac{\partial \varphi}{\partial \xi_1} = s - y_1$

hold by the definition of the phase function φ. Now we apply to the integration $\int \omega(\xi) \int_\alpha^{x_1} E(x, y, \xi, s)ds$ a lemma due to Sato concerning the regularity property of the integration along fibre (Sato [5] Corollary 6.5.3. Cf. Kawai [2] Lemma 2.2 and Corollary 2.3) and conclude that $S.S.E\ (x, y) \subset \{(x, y; \xi, \eta) \in S^*(M \times M) \,|\, x = y$ and $\xi = -\eta\}$ near $(x, y; \xi, \eta) = (0, 0; \xi_0, -\xi_0)$ by just the same reasoning as in our previous paper Kawai [2]. (Cf. Kawai [2] Theorem 3.2 especially the properties of the phase function φ (3.4)$\sim$(3.8). The proof is easier in this case, since the propagation of the singularities does not occur).

From this theorem we immediately obtain the following theorem.

Theorem 1.8. *Let a linear differential operator $P(x, D_x)$ satisfy condition $(NT)_{f, (0 - \xi^0)}$. Then denoting the solution sheaf of $P(x, D_x)$ by $\mathscr{C}^P$, we have $(0, \xi^0) \notin \mathrm{supp}\,\mathscr{C}^P$.*

Proof. We consider the formal adjoint operator of $P(x, D_x)$ and denote it by $P^*(x, D_x)$. Then it is clear that condition $(NT)_{f, (0, -\xi^0)}$ holds for $P^*(x, D_x)$. In fact condition $(NT)_f$ concerns only with the principal part of a linear differential operator and the principal symbol of $P^*(x, D_x)$ is the same as that of $P(x, D_x)$ except for the signature. Therefore we have $E(x, y)$ for which $P^*(y, D_y)E(x, y) = \delta(x - y)$ holds near $(0, 0, \xi^0, -\xi^0)$. Now consider $u(y)$ for which $P(y, D_y)u(y) = 0$ holds near $(0, \xi^0)$.

Then using the flabbiness of the sheaf $\mathscr{C}$ (Kashiwara [1]) we can assume that $P^*(y, D_y)\tilde{u}(y) = \mu(y)$ with $S.S.\tilde{u}(y)$ compact and $u(y) = \tilde{u}(y)$ near $(0, \xi^0)$. Then we have in a neighbourhood of $(0, \xi^0)$

$$0 = \int \mu(y)E(x, y)\,dy = \int P(y, D_y)\tilde{u}(y)E(x, y)\,dy = \int \tilde{u}(y)P^*(y, D_y)E(x, y)\,dy =$$

$= \tilde{u}(x) = u(x)$. This ends the proof of the theorem.

Corollary 1.9. *Let* $P(x, D_x)$ *satisfy condition* $(NT)_{f, (0, \xi^0)}$ *for any* $\xi^0(\xi^0 \neq 0)$. *Then any hyperfunction solution* $u(x)$ *of* $P(x, D_x)u(x) = 0$ *is real analytic near the origin.*

The proof is immediate from Sato's fundamental exact sequence: $0 \longrightarrow \mathscr{A} \longrightarrow \mathscr{B} \longrightarrow \pi_*\mathscr{C} \longrightarrow 0$, where $\mathscr{A}$ and $\mathscr{B}$ denote the sheaf of germs of real analytic functions and hyperfunctions on M, respectively, and π denotes the projection from S^*M to M. (Cf. Sato [2]~[5]).

Remark 1. The result of Treves [1] shows that Corrolary 1.9 is the best possible one as far as we consider the problem of regularity in the framework of hyperfunctions (or rather distributions in Treves [1]). But in the framework of the sheaf $\mathscr{C}$ Corollary 1.9 (or even Theorem 1.8) is not the best possible one. See for example Theorem 2.3 in the next section. These subjects will be discussed in our forthcoming paper with the aid of the theory of integral operator of finite type developed in Kashiwara and Kawai [2].

Remark 2. If the space dimension n is equal to 2, we can prove Theorem 1.8 by a more elementary method. We refer the reader to Kawai [3] about it.

Remark 3. All the results of this section hold for a linear pseudo-differential operator of finite type $P(x, D_x)$ with its princcipal part a usual partial differential operator, i.e., $P(x, D_x) = P_m(x, D_x) + Q(x, D_x)$, where $P_m(x, D_x)$ is a linear differential operator of order m and $Q(x, D_x)$ is a pseudo-differential operator of finite type of order at most $m-1$. This is a trivial corollary of Kashiwara and Kawai [2] Theorem 1. (Cf. Kashiwara and Kawai [1] Theorem 7). We refer the reader to Kashiwara and Kawai [1] for the notion of pseudo-differential operator of finite type.

This remark holds also for the results of the next section.

Remark 4.   Since a penetrating study on the local existence and regularity of solutions is given by Suzuki [1] for first order linear differential operators in two independent variables, we can give a complete result on these problems for linear differential operators in two independent variables using the above quoted result of Kashiwara and Kawai [1], [2]. This will be discussed somewhere else, because the method of the proof is rather different from the one given in this paper, i.e., via the construction of elementary solutions.

§2.   Construction of Local Elementary Solutions
—— Real Method ——

In this section we consider two cases where condition $(NT)_f$ cannot be applied but the construction of local elementary solutions is possible. The method is close to that employed by Hörmander [2] in the C^∞-category under the assumtion that $P_m(x, \xi)$ is real. Until the end of this section we always assume that

(2.1) $\mathrm{grad}_\xi A_m(x, \xi)$ and $\mathrm{grad}_\xi B_m(x, \xi)$ are linearly

independent whenever $P_m(x, \xi) = A_m(x, \xi) + i B_m(x, \xi) = 0$.

The attempt to weaken this condition will be given in our forthcoming paper using the theory of integral operator of finite type developed in Kashiwara and Kawai [2].

We first give an existence theorem under the conditions expressed in terms of phase functions (Theorem 2.1).   Afterwards we investigate the conditions concerning the differential operator itself which imply the conditions used in Theorem 2.1. (Theorems 2.3 and 2.5).   We hope that the two cases covered by Theorem 1.4 and Theorem 2.1 are complementary unless $P_m(x, \xi)$ is real and that Theorem 1.6, Theorem 2.3 and Theorem 2.5 deal with three typical cases of solvable linear differential equations in the framework of the sheaf $\mathscr{C}$. Remark that the behaviour of the phase functions used to construct local elementary solutions are different from

each other: the phase function used in Theorem 1.6 is of strictly half positive type, that in Theorem 2.3 is of strictly positive type, and that in Theorem 2.5 is real.

Theorem 2.1. *Assume that there exists a phase function* $\varphi(x, y, \xi)$ *of* $P(x, D_x)$ *satisfying the following conditions* (2.2)$\sim$(2.5) *near* (x, y, ξ) $=(0, 0, \xi^0)$. *Then we can construct* $E(x, y)$ *which satisfies* $P(x, D_x)E(x, y)=\delta(x-y)$ *as sections of the sheaf* $\mathscr{C}$.

(2.2) $P_m(x, \mathrm{grad}_x\varphi(x, y, \xi))=P_m(y, \xi)$.

(2.3) $\varphi(x, y, \xi)=\ <x-y, \xi>\ +O(|x-y|^2|\xi|)$

(2.4) $\varphi(x, y, \xi)$ *is real analytic near* $(0, 0, \xi^0)$ *and positively homogeneous of order* 1 *with respect to* ξ.

(2.5) $\varphi(x, y, \xi)$ *is of positive type.* (*Cf. Definition* 1.2).

Proof. We proceed just as in the proof of Theorem 3.2′ in our previous paper Kawai [2].

We first define $\varPhi_j(\tau)$ as follows.

(2.6) $\varPhi_j(\tau)= \begin{cases} \dfrac{(-1)^j(n-j-1)!}{(-2\pi i)^n}\dfrac{1}{\tau^{n-j}} & (j<n) \\[3mm] \dfrac{-1}{(2\pi i)^n(j-n)!}\left\{\tau^{j-n}\log\tau-\left(1+\dfrac{1}{2}+\cdots+\dfrac{1}{j-n}\right)\tau^{j-n}\right\} & (j\geqq n). \end{cases}$

On the other hand $\varphi(x, y, \xi)$ can be analytically extended to a complex domain by its real analyticity and we denote the holomorphic function by $\varphi(z, w, \zeta)$. Using these functions $\varPhi_j(\tau)$ and $\varphi(z, w, \zeta)$, we want to construct a holomorphic function $G(z, w, \zeta)$ in (z, w, ζ) defined in the complex domain $\{(z, w, \zeta)|\sqrt{-1}\ \varphi(z, w, \zeta)$ is not a real non-positive number}, which satisfies

(2.7) $P(z, D_z)G(z, w, \zeta)=P_m(w, \zeta)\{\sum_{j\geqq 0} r_j(z, w, \zeta)\varPhi_j(\varphi(z, w, \zeta)\}$

with $r_0(z, w, \zeta)\neq 0$ near $(0, 0, \xi^0)$ and $r_j(z, w, \zeta)$ being homogeneous of order $(-j)$ with respect to ζ. For that purpose we assume that $G(z, w, \zeta)$

has the form

$$(2.8) \qquad\qquad \sum_{j \geqq 0} f_j(z, w, \zeta) \varPhi_j(\varphi(z, w, \zeta)),$$

where $f_j(z, w, \zeta)$ is a holomorphic function in (z, w, ζ) near $(0, 0, \xi^0)$ and homogeneous of order $(-j)$ with respect to ζ. Then using the condition (2.2) on the phase function $\varphi(x, y, \xi)$, we can determine $f_j(z, w, \zeta)$ formally so that $f_0(w, w, \zeta) = r_0(w, w, \zeta) = 1$ and (2.7) holds by solving transport equations suitably as in the proof of Theorem 3.2′ in Kawai [2]. (Cf. Kawai [2] (1.13) and (1.14)). Then it is easy to see that the following estimate (2.9) holds just as in the proof of the above quoted theorem in Kawai [2] (or rather in the proof of Theorem 1.3 in Kawai [2]).

$$(2.9) \qquad \sup_{(z, w, \zeta) \in V} |f_j(z, w, \zeta)| \leqq C^j j! \quad \text{holds for a constant } C \text{ and}$$

a complex neighbourhood V of $(0, 0, \xi^0)$.

It is obvious from (2.9) that the summation (2.8) converges absolutely and uniformly in $\varOmega = \{(z, w, \zeta) | \sqrt{-1}\, \varphi(z, w, \zeta)$ is not a real non-positive number and $|\varphi(z, w, \zeta)| \leqq C'$ for some $C'(>0)\}$. Therefore $G(z, w, \zeta)$ can be represented in the form

$$(2.10) \qquad G(z, w, \zeta) = \frac{g_0(z, w, \zeta)}{\varphi(z, w, \zeta)^l} + g_1(z, w, \zeta)\log\varphi(z, w, \zeta) + g_2(z, w, \zeta)$$

for an integer l and holomorphic functions $g_j(z, w, \zeta)$ $(j = 0, 1, 2)$.

On the other hand $\dfrac{g_0(z, w, \zeta)}{\varphi(z, w, \zeta)^l}$, $g_1(z, w, \zeta)\log\varphi(z, w, \zeta)$ and $g_2(z, w, \zeta)$

define hyperfunctions $\dfrac{g_0(x, y, \xi)}{(\varphi(x, y, \xi) + i0)^l}$, $g_1(x, y, \xi)\log(\varphi(x, y, \xi) + i0)$

and $g_2(x, y, \xi)$ respectively, if we take their boundary values from the complex domain $\varOmega$, since the phase function $\varphi(x, y, \xi)$ is of positive type by (2.5). That is, $G(z, w, \zeta)$ defines a hyperfunction $g(x, y, \xi) = h_0(x, y, \xi) + h_1(x, y, \xi) + g_2(x, y, \xi)$, where $h_0(x, y, \xi) = \dfrac{g_0(x, y, \xi)}{(\varphi(x, y, \xi) + i0)^l}$, and $h_1(x, y, \xi) = g_1(x, y, \xi)\log(\varphi(x, y, \xi) + i0)$. Moreover we see from (2.7) and (2.9) that

(2.11) $P(x, D_x)G(x, y, \xi) =$

$$= P_m(y, \xi)\{ \sum_{j \geqq 0} r_j(x, y, \xi)\mathit{\Phi}_j(\varphi(x, y, \xi) + i0)\}$$

holds. Here the symbol $\sum_{j \geqq 0} r_j(x, y, \xi)\mathit{\Phi}_j(\varphi(x, y, \xi) + i0)$ denotes a hyper-function defined as the boundary value of holomorphic function $\sum_{j \geqq 0} r_j(z, w, \zeta) \ \mathit{\Phi}_j(\varphi(z, w, \zeta))$ from the domain $\{(z, w, \zeta)\,|\,\sqrt{-1}\varphi(z, w, \zeta)$ is not a real non-positive number$\}$.

   Now we want to define a hyperfunction $G(x, y, \xi)/P_m(y, \xi)$, multiply-ing $G(x, y, \xi)$ by $1/P_m(y, \xi)$. Since $\mathrm{grad}_\xi A_m(y, \xi)$ and $\mathrm{grad}_\xi B_m(y, \xi)$ are linearly independent whenever $P_m(y, \xi) = 0$ by assumption (2.1), the hy-perfunction $1/P_m(y, \xi)$ is well-defined. If we regard $G(x, y, \xi)$ and $1/P_m(y, \xi)$ as hyperfunctions in $(x, y, \xi) \in N$, it immediately follows from $(2.3) \sim (2.5)$ and the definition of $G(x, y, \xi)$ and $1/P_m(y, \xi)$ that $S.S.G(x, y, \xi) \cap \{S.S.(1/P_m(y, \xi))\}^a = \phi$, where a denotes the antipodal mapping of S^*N, which maps $(X; \zeta) \in S^*N$ to $(X; -\zeta) \in S^*N$. Therefore the hyperfunction $F(x, y, \xi) = G(x, y, \xi)/P_m(y, \xi)$ is well defined (cf. Sato [5] §6.4) and satisfies by (2.11):

(2.12) $P(x, D_x)F(x, y, \xi) = \sum_{j \geqq 0} r_j(x, y, \xi)\mathit{\Phi}_j(\varphi(x, y, \xi) + i0).$

   On the other hand it is seen from the conditions $(2.3) \sim (2.5)$ on $\varphi(x, y, \xi)$ combined with Sato's formula (1.6) on the curvilinear wave expansions of δ-functions that the right side of (2.12) defines a kernel function of an elliptic pseudo-differential operator of finite type, since $r_0(x, y, \xi) \neq 0$ near $(0, 0, \xi^0)$. (Cf. Kashiwara and Kawai [1]). Therefore applying Theorem 6 on the invertibleness of elliptic pseudo-differential oprators of finite type in Kashiwara and Kawai [1] we con-clude that there exists $E(x, y)$ for which $P(x, D_x)E(x, y) = \delta(x - y)$ holds near $(0, 0, \xi^0, -\xi^0)$. This ends the proof of the theorem.

   Now we consider the case where the phase function $\varphi(x, \mathrm{y}, \xi)$ used in the above theorem can be chosen so that the singularity of the local elementary solution $E(x, y)$ is described explicitly.

   We first consider the case where $E(x, y)$, or more precisely $E(x, y)dy$, defines a kernel function of a pseudo-differential operator (in the sense of

Sato) near $(0, 0, \xi^0, -\xi^0)$.

In the sequel we use the symbol ξ to represent $\frac{1}{i}D_x$ and denote by $\bar{P}(x, D_x)$ the differential operator whose coefficients are the complex conjugate of those of $P(x, D_x)$, i.e., we define $\bar{P}(x, D_x)$ by giving its symbol as $\sum_{|\alpha|\leq m} \overline{a_\alpha(x)}\xi^\alpha$ if $P(x, \xi)= \sum_{|\alpha|\leq m} a_\alpha(x)\xi^\alpha$. We also denote by $C_{2m-1}(x, \xi)$ the imaginary part of the Poisson bracket of P_m and $\bar{P}_m$, i.e.,

$$\mathrm{Im} \sum_{j=1}^{n}\left(\frac{\partial P_m}{\partial \xi_j}\frac{\partial \bar{P}_m}{\partial x_j} - \frac{\partial P_m}{\partial x_j}\frac{\partial \bar{P}_m}{\partial \xi_j}\right).$$

Definition 2.2. (Condition $(H)_{(0, \xi^0)}$) (Cf. Hörmander [1] §6) A linear differential operator $P(x, D_x)$ is said to satisfy condition $(H)_{(0, \xi^0)}$ if $C_{2m-1}(x, \xi)>0$ near $(0, \xi^0)\in S^*M$ whenever $P_m(x, \xi)=0$.

Theorem 2.3. *Assume that $P(x, D_x)$ satisfies condition $(H)_{(0, \xi^0)}$. Then the elementary solution constructed in Theorem 2.1 defines a kernel function of a pseudo-differential operator near $(0, 0, \xi^0, -\xi^0)$*

Proof. We first construct a suitable phase function $\varphi(x, y, \xi)$ satisfying $(2.2)\sim(2.4)$ and the following condition $(2.5')$ near $(x, y, \xi)=(0, 0, \xi^0)$.

$$(2.5') \qquad\qquad \mathrm{Im}\varphi(x, y, \xi)>0 \text{ if } x\neq y.$$

We say in the below that a phase function φ is of strictly positive type if it satisfies $(2.5')$. To construct such a phase function φ we solve the following non-linear first order partial differential equation (2.13) in the complex domain using the real analyticity of the coefficients of $P(x, D_x)$.

$$(2.13) \qquad P_m(x, \mathrm{grad}_x\varphi(x, y, \xi))=P_m(y, \xi) \text{ with } \varphi(x, y, \xi)=$$
$$= <x-y, \xi> + \sum_{j,k=1}^{n} \alpha_{jk}(y, \xi)(x_j-y_j)(x_k-y_k)+$$
$$+O(|x-y|^3|\xi|),$$

where $\alpha_{jk}(y, \xi)$ is a suitable symmetric matrix which depends real analytically on (y, ξ) and has a positive definite imaginary part. Using the assumption $(H)_{(0, \xi^0)}$ combined with the assumption (2.1) of the linear independence of $\mathrm{grad}_\xi A_m$ and $\mathrm{grad}_\xi B_m$, we can choose the matrix $\alpha_{jk}(y, \xi)$ so that (2.13) has a solution. Note that $C_{2m-1}(x, \xi)>0$ holds near $(0, \xi^0)$

by condition $(H)_{(0,\xi^0)}$. In fact it is obvious from the method of the proof of the famous lemmas due to Hörmander on the existence of symmetric matrix with a positive definite imaginary part. (Hörmander [1] Lemmas 6.1.3 and 6.1.4). Therefore using the classical integration theorem of first order differential equations of Hamilton and Jacobi we have a phase function $\varphi(x, y, \xi)$ which is of strictly positive type.

Using this phase function $\varphi(x, y, \xi)$ we construct $E(x, y)$ near $(0, 0, \xi^0, -\xi^0)$ as in Theorem 2.1. We shall now prove that $E(x, y)dy$ defines a kernel function of a pseudo-differential operator near $(0, 0, \xi^0, -\xi^0)$. For that purpose it is sufficient to prove that

$$(2.14) \qquad S.S.E(x, y) \subset \{(x, y; \xi, \eta) \in S^*(M \times M) \,|\, x = y, \, \xi = -\eta\}$$

$$\text{near } (0, 0, \xi^0, -\xi^0)$$

by the definitition of pseudo-differential operators. (Cf. Sato [2]~[5]).

To prove (2.14) we investigate the regularity property of $E(x, y, \xi)$, regarded as a hyperfunction in (x, y, ξ), and apply Sato's fundamental lemma on regularity to the integration $\int E(x, y, \xi)\omega(\xi)$ (Sato [4] and [5]. See also Kawai [2] Lemma 2.2). We denote by ζ the cotangent vector at (x, y, ξ). Moreover we denote its component relative to the dual basis of $(\mathrm{grad}_x, \mathrm{grad}_y \, \mathrm{grad}_\xi)$ by $(\zeta_1, \zeta_2, \zeta_3)$. By the definition of $E(x, y, \xi)$ we have

$$(2.15) \quad S.S.E(x, y, \xi) \subset \{(x, y, \xi; \zeta) \,|\, x = y, \, P_m(y, \xi) = 0,$$
$$\zeta_1 = a \, \mathrm{grad}_x \varphi(x, y, \xi), \, \zeta_2 = a \, \mathrm{grad}_y \varphi(x, y, \xi) + b(\alpha \, \mathrm{grad}_y A_m(y, \xi) +$$
$$+ \beta \, \mathrm{grad}_y B_m(y, \xi)) \text{ and } \zeta_3 = a \, \mathrm{grad}_\xi \varphi(x, y, \xi) + b(\alpha \, \mathrm{grad}_\xi A_m(y, \xi) +$$
$$+ \beta \, \mathrm{grad}_\xi B_m(y, \xi))\} \cup \{(x, y, \xi; \zeta) \,|\, x \neq y, \, P_m(y, \xi) = 0, \, \zeta_1 = 0 \text{ and }$$
$$(\zeta_2, \zeta_3) = \alpha' \, \mathrm{grad}_{(y, \xi)} A_m(y, \xi) + \beta' \, \mathrm{grad}_{(y, \xi)} B_m(y, \xi)\} \cup \{(x, y, \xi; \zeta) \,|$$
$$x = y, \, P_m(y, \xi) \neq 0, \, \zeta_1 = \xi, \, \zeta_2 = -\xi, \, \zeta_3 = 0\}$$

where $a, b \geq 0$, $a + b > 0$ and α, α', β and β' are real numbers with $|\alpha| + |\beta| \neq 0$, $|\alpha'| + |\beta'| \neq 0$.

Noting that $\mathrm{grad}_\xi \varphi(y, y, \xi) = 0$ we have by (2.15) and Lemma 2.2 in Kawai [2] $x = y$ and $b\alpha \, \mathrm{grad}_\xi A_m + b\beta \, \mathrm{grad}_\xi B_m = 0$ on $S.S. \, E(x, y)$. Therefore we have $b = 0$ by the condition (2.1). On the other hand we have $(\mathrm{grad}_x \varphi(x, y, \xi)|_{x=y}, \mathrm{grad}_y \varphi(x, y, \xi)|_{x=y}) = (\xi, -\xi)$ by the definition of the

phase function $\varphi(x, y, \xi)$. Thus we have proved that $SS.E(x, y) \subset$ $\{(x, y, \xi, \eta) \mid x = y, \xi = -\eta\}$ near $(0, 0, \xi^0, -\xi^0)$. This ends the proof of the theorem.

Remark 1. It should be possible to weaken the condition $(H)_{(0, \xi^0)}$ using Nirenberg and Treves [2], but it seems that we must overcome some technical difficulties and we have proved Theorem 2.3 under this restrictive condition.

Remark 2. We can prove that under the condition $(H)_{(0, \xi^0)}$ $(x, \xi) = (0, -\xi^0) \notin \text{Ker} \mathscr{C}^P$ by the same reasoning employed in the proof of Theorem 1.8. Dr. Naruki has kindly remarked to the author that this fact plays an essential role in studying the problem of imbedding a real manifold into a complex manifold. (Cf. Lewy [1]).

Now we give an affirmative answer to Sato's conjecture that the sheaf of the image of $P(x, D_x)$ can be characterized locally on S^*M as a solution sheaf of some other pseudo-differential operator $Q(x, D_x)$ if $P\mathscr{C} \neq \mathscr{C}$.

Theorem 2.4. *Assume that a linear differential operator $P(x, D_x)$ satisfies condition (2.1) and condition $(H)_{(0, -\xi^0)}$. Then we can find some non-trivial pseudo-differential operator $Q(x, D_x)$ defined near $(0, 0, \xi^0, -\xi^0)$ such that the sequence $\mathscr{C} \xrightarrow{P} \mathscr{C} \xrightarrow{Q} \mathscr{C}$ is exact near $(0, \xi^0)$, i.e., $\text{Im}_{\mathscr{C}}P = \text{Ker}_{\mathscr{C}}Q$.*

Proof. Let P^* be the formal adjoint operator of P. It is obvious by the definition that condition $(H)_{(0, -\xi^0)}$ holds for P^*. Therefore applying Theorem 2.3 to $P^*(y, D_y)$ we can find some pseudo-differential operator Ψ defined near $(0, 0, \xi^0, -\xi^0)$ such that $\Psi P = Id$ holds near $(0, 0, \xi^0, -\xi^0)$. (The symbol Id denotes the identity operator, i.e., the pseudo-differential operator with its kernel function given by $\delta(x-y)dy$). (Cf. the proof of Theorem 1.8.) We define the pseudo-differential operator $Q(x, D_x)$ by $P\Psi$ $-Id = Q$. Thus we have $QP = (P\Psi - Id)P = P - P = 0$, since $\Psi P = Id$. It is also obvious that $Pu = f$ is locally solvable on S^*M if $Qf = 0$. In fact it is sufficient to define u by Ψf, since $Pu = P\Psi f = f + Qf = f$ holds by the assumption. Thus $\text{Im}_{\mathscr{C}}P = \text{Ker}_{\mathscr{C}}Q$ holds near $(0, \xi^0)$. Moreover it is easy to see that $Q(x, D_x)$ defines a non-trivial pseudo-differential operator. We

shall give the proof of this fact following Sato's idea. (The author's original proof was more complicated and close to Hörmander [1] §6). The author expresses his hearty thanks to Professor Sato for his suggestions.

In fact using condition $(H)_{(0, -\xi^0)}$ as in Hörmander [1] Lemma 6.1.3 we can choose $\varphi(y; -\xi^0)$ so that

$$(2.16) \quad \begin{cases} P_m(y, \operatorname{grad}_y\varphi(y; -\xi^0))=0, \ \operatorname{grad}_y\varphi(y; -\xi^0)|_{y=0}=-\xi^0 \ \text{and} \\ \operatorname{Im}\varphi(y; -\xi^0)>0 \ \text{if} \ y\neq0 \end{cases}$$

holds. We define $\psi(x, y; -\xi^0)$ to be $\varphi(y; -\xi^0)-\bar{\varphi}(x; -\xi^0)$, where $\bar{\varphi}$ means the complex conjugate of $\varphi(x)$. Using this function $\psi(x, y; -\xi^0)$ we can construct $W(x, y)=\sum_{j\geq0} a_j(x, y)\varPhi_j(\psi+i0)$ so that it satisfies $P^*(y, D_y)W(x, y)=0$. (The function $\varPhi_j(\tau)$ is defined in (2.6)). The summation $\sum_{j\geq0} a_j(x, y)\varPhi_j(\psi+i0)$ can be endowed with the same meaning as in the right side of (2.11), if we prove the estimate $\sup_{(z, w)\in V} |a_j(z, w)|\leq$ $\leq C^j j!$ for a constant C and a complex neighbourhood V of $(x, y)=(0, 0)$. This estimation is possible as in the proof of Theorem 1.3 in Kawai [2]. (Cf. Mizohata [1]). Note that $S.S.W(x, y)\subset\{(x, y, \xi, \eta)\,|\,x=y=0,$ $\xi=\xi^0, \eta=-\xi^0\}$ holds by the definition of $\psi(x, y)$. Now assume that the pseudo-differential operator Q reduces to zero, i.e., $P(y, D_y)u=f$ is locally solvable near $(0, \xi^0)$ for any f. Choosing $f=\delta(y)$ we have $P(y, D_y)\tilde{u}=$ $=\delta+\mu$, where $S.S.\tilde{u}$ and $S.S.\mu$ are compact in S^*M and that $\tilde{u}=u$ near $(0, \xi^0)$. We have used here the fact that the sheaf $\mathscr{C}$ is flabby. (Cf. Kashiwara [1]). Then we have

$$0\neq\int\delta(y)W(x, y)dy=-\int\mu(y)W(x, y)dy+\int\tilde{u}(y)P^*(y, D_y)W(x, y)dy=0$$

near $(0, \xi^0)$ since $\mu(y)=0$ near $(0, \xi^0)$. Therefore the assumption that $Pu=f$ is locally solvable near $(0, \xi^0)$ has given rise to a contradiction. This proves that the pseudo-differential operator Q is non-trivial.

Remark 1. It is obvious from the method of the proof that we can find under conditions (2.1) and $H_{(0,\xi^0)}$ a pseudo-differential operator $R(x, D_x)$ defined near $(0, 0, \xi^0, -\xi^0)$ such that the sequence $\mathscr{C}\xrightarrow{R}\mathscr{C}\xrightarrow{P}\mathscr{C}$ is exact near $(0, \xi^0)$. In fact we define the pseudo-differential operator R to be

$\Theta P - Id$, where Θ is a pseudo-differential operator for which $P\Theta = Id$ holds. The existence of Θ follows from Theorem 2.3 and the non-triviality of $R(x, D_x)$ follows immeditely from the existence of $\varphi(x; \xi^0)$ which satisfies:

$$(2.16') \quad \begin{cases} P_m(x, \text{grad}_x \varphi(x; \xi^0)) = 0, \ \text{grad}_x \varphi(x; \xi^0)|_{x=0} = \xi^0 \ \text{and} \\ \text{Im}\varphi(x; \xi^0) > 0 \ \text{if} \ x \neq 0. \end{cases}$$

(Cf. Mizohata [1]).

Remark 2. The above proof shows that the non-surjectivity of $P(x, D_x)$ follows from the existence of a suitable characteristic function φ. Thus we can treat the problem of non-existence of local (hyperfunction) solutions in general cases treated by Nirenberg and Treves [2]. We have only to use the analysis of Nirenberg and Treves [2] on the characteristic function φ. Hence we leave the details to the reader. We also hope that we will construct the pseudo-differential Q, the compatibility condition for P, under a less restrictive condition that $H_{(0, -\xi^0)}$.

Remark 3. Since the advent of Lewy's example (Lewy [2]) it has been believed that the image of the space of generalized functions (or even C^∞-functions) under a linear differential operator $P(x, D_x)$ has no "analytical structure" if $P(x, D_x)$ is a differential operator without solutions. (See for example Hörmander [1] §6.1, especially Theorem 6.1.2. Note that $Pu = f$ is locally solvable by the Cauchy-Kowalevsky theorem if f is a real analytic function). And this fact has been used as an excuse for restricting the class of linear dfferential operators in the consideration of overdetermined systems and the resolutions of their solution sheaves. However Sato's conjecture proved above seems to throw light on these problems.

Now we study another extreme case where Theorem 2.1 can be applied.

Theorem 2.5. *Suppose that* $C_{2m-1}(x, \xi) = 0$ *holds on* $\{(x, \xi) \in S^*M \mid P_m(x, \xi) = 0\}$. *We also assume that* $P_m(x, \xi) = A_m(x, \xi) + iB(x, \xi)$ *satis-*

fies condition (2.1) *on the linear independence of* grad $_\xi A_m$ *and* grad $_\xi B_m$. *Then we can apply Theorem* 2.1 *and obtain a local elementary solution for* $P(x, D_x)$ *near* $(0, 0, \xi^0, -\xi^0)$.

Proof. By the assumption on $C_{2m-1}(x, \xi)$ we can choose real valued real anarytic functions $\alpha(x, \xi)$ and $\beta(x, \xi)$ defined near $(0, \xi^0)$, which are positively homogeneous of order 0 with respect to ξ and never vanishes near $(0, \xi^0)$, so that the Poisson bracket $(\alpha(x, \xi)A_m(x, \xi), \beta(x, \xi)B_m(x, \xi))$ vanishes identically near $\{P_m(x, \xi)=0\}$. In fact it is sufficient to choose $\alpha(x, \xi)$ and $\beta(x, \xi)$ so that $\alpha\beta(A_m, B_m)+\alpha B_m(A_m, \beta)+A_m\beta(\alpha, B_m)+$ $+A_mB_m(\alpha, \beta)=0$. On the other hand by the condition on $C_{2m-1}(x, \xi)$ we can find some real valued $\gamma(x, \xi)$ and $\delta(x, \xi)$ which are positively homogeneous of order $m-1$ with respect to ξ and satisfies $(A_m, B_m)=\gamma A_m+$ $+\delta B_m$. Then the assumption (2.1) ensures the existence of the required $\alpha(x, \xi)$ and $\beta(x, \xi)$. Using these functions $\alpha(x, \xi)$ and $\beta(x, \xi)$ we can find by the Hamilton-Jacobi theory a real valued phase function $\varphi(x, y, \xi)$ for which (2.3), (2.4) and the following (2.2′) hold:

$$(2.2')\quad \begin{cases} \alpha(x, \operatorname{grad}_x\varphi(x, y, \xi))A_m(x, \operatorname{grad}_x\varphi(x, y, \xi)) \\ =\alpha(y, \xi)A_m(y, \xi) \text{ and} \\ \beta(x, \operatorname{grad}_x\varphi(x, y, \xi))B_m(x, \operatorname{grad}_x\varphi(x, y, \xi)) \\ =\beta(y, \xi)B_m(y, \xi). \end{cases}$$

Since neither $\alpha(x, \operatorname{grad}_x\varphi(x, y, \xi))$ nor $\beta(x, \operatorname{grad}_x\varphi(x, y, \xi))$ vanishes near $(x, y, \xi)=(0, 0, \xi^0)$, we can determine $G(z, w, \zeta)=\sum\limits_{j\geqq0} f_j(z, w, \zeta)\times$ $\mathit{\Phi}_j(\varphi(z, w, \zeta))$ so that

$$(2.7)\qquad P(z, D_z)G(z, w, \zeta)=P_m(w, \zeta)\{\sum\limits_{j\geqq0} r_j(z, w, \zeta)\mathit{\Phi}_j(\varphi(z, w, \zeta))\}$$

holds in a complex neighbourhood of $(0, 0, \xi^0)$. (The conditions on f_j and r_j are the same as in the proof of Theorem 2.1.) Since the phase function $\varphi(x, y, \xi)$ is real valued, we can proceed as in the proof of Theorem 2.1 and obtain the required local elementary solution $E(x, y)$. This ends the proof of the theorem.

Remark. We can also prove that $S.S.E(x, y)$ is contained in the

union of bicharacteristic strips, which is by definition the 2-dimensional manifold in S^*M with the vector fields

$$H_{A_m} = \sum_{j=1}^{n} \left(\frac{\partial A_m}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial A_m}{\partial x_j} \frac{\partial}{\partial \xi_j} \right) \quad \text{and} \quad H_{B_m} = \sum_{j=1}^{n} \left(\frac{\partial B_m}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial B_m}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$$

as tangents. The proof is just the same as that of Theorem 3.2′ of Kawai [2]. A corollary of this remark is proved by Andersson [1] when $P(x, D_x)$ is a differential operator *with constant coefficients*. But in this case we cannot use this remark to study the propagation of regularities of the solutions of $Pu = 0$ as we have done for $P(x, D_x)$ with a real principal symbol in Theorem 3.3 of Kawai [2], since $S.S.E(x, y)$ is not contained in the union of "half" of the bicharacteristic strips, which is the case of Kawai [2]. This is because $S.S.(1/P_m(y, \xi))$ is rather large in this case, i.e., we can only assert that $S.S.(1/P_m(y, \xi)) \subset \{(y, \xi; \zeta) | P_m(y, \xi) = 0 \text{ and } \zeta = \alpha \mathrm{grad}_{(y,\xi)} A_m(y, \xi) + \beta \mathrm{grad}_{(y,\xi)} B_m(y, \xi), \text{ where } \alpha \text{ and } \beta \text{ are real numbers with } |\alpha| + |\beta| \neq 0\}$.

Hence the problem of propagation of regularities of solutions for the linear differential operator $P(x, D_x)$ satisfying the conditions in Theorem 2.5 will be studied in a forthcoming paper of Kashiwara and Kawai [2] by a different method, i.e., by the use of the theory of integral operators of finite type.

Bibliography

Andersson, K.G., [1], Analyticity of fundamental solutions, *Ark. Mat.* **8** (1969), 73–81.

Egorov, Yu. V., [1], Conditions for the solvability of pseudo-differential operators, *Dokl. Akad. Nauk SSSR*, **187** (1969), 1232–1234. (Russian)

Hamada, Y., [1], The singularities of the solutions of the Cauchy problem, *Publ. RIMS, Kyoto Univ.* **5** (1969), 21–40.

Hörmander, L., [1], *Linear Partial Differential Operators*, Springer, 1963.

Hörmander, L., [2], On the singularities of solutions of partial differential equations, *Proc. Int. Conf. on Functional Analysis and Related Topics*, Univ. of Tokyo Press, 1969, pp. 31–40.

Kashiwara, M., [1], On the flabbiness of the sheaf $\mathscr{C}$, *Sûrikaisekikenkyûsho Kôkyûroku* No. 114, R.I.M.S., Kyoto Univ., 1970, pp. 1–4. (Japanese)

Kashiwara, M. and T. Kawai, [1], Pseudo-differential operators in the theory of hyperfunctions, *Proc. Japan Acad.* **46** (1970), 1130–1134.

Kashiwara, M. and T. Kawai, [2], Pseudo-differential operators in the theory of hyperfunctions, (II), to appear.

Kawai, T., [1], Construction of a local elementary solution for linear partial differential operators, II, *Proc. Japan Acad.* **47** (1971), 147–152.

Kawai, T., [2], Construction of local elementary solutions for linear partial differential operators with analytic coefficients (I)—The case with real principal symbols—, *Publ. RIMS, Kyoto Univ.*, this issue.

Kawai, T., [3], On the local theory of (pseudo)-differential operators, *Sûrikaisekikenkyûsho Kôkyûroku* No. 114, R.I.M.S., Kyoto Univ., 1970, pp. 18–68. (Japanese)

Kawai, T., [4], A survey of the theory of linear (pseudo-) differential equations from the view point of phase functions —— existence, regularity, effect of boundary conditions, transformations of operators, etc., Reports of the symposium on the theory of hyperfunctions and differential equations, R.I.M.S. Kyoto Univ., 1970, pp. 84–92. (Japanese)

Komatsu, H., [1], Relative cohomology of sheaves of solution of differential equations, *Sem. Lions and Schwartz*, 1966. Reprinted in *Reports of Katata Symposium on Algebraic Geometry and Hyperfunctions*, 1969, pp. 1–59.

Lewy, H., [1], On the local character of the solution of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables, *Ann. of Math.* **64** (1956), 514–522.

Lewy, H., [2], An example of a smooth linear partial differential equation without solution, *Ann. of Math.* **66** (1957), 155–158.

Mizohata, S., [1], Solutions nulles et solutions non-analytiques, *J. Math. Kyoto Univ.* **1** (1962), 271–302.

Nirenberg, L. and F. Treves, [1], Solvability of a first order linear partial differential equation, *Comm. Pure Appl. Math.* **16** (1963), 331–351.

Nirenberg, L. and F. Treves, [2], On local solvability of linear partial differential equations, Part, I: Necessary conditions, *Ibid.* **23** (1970) 1–38.

Nirenberg, L. and F. Treves, [3], On local solvability of linear partial differential equations, Part II: Sufficient conditions, *Ibid.* **23** (1970), 459–501.

Sato, M., [1], Theory of hyperfunctions II, *J. Fac. Sci. Univ. Tokyo*, **8** (1960), 387–437.

Sato, M., [2], Hyperfunctions and partial differential equations, *Proc. Int. Conf. on Functional Analysis and Related Topics*, Univ. of Tokyo Press, 1969, pp. 91–94.

Sato, M., [3], Regularity of hyperfunction solutions of partial differential equations, to appear in *Proceedings of Nice Congress* (1970).

Sato, M., [4], Structure of hyperfunctions, *Reports of the Katata Symposium on Algebraic Geometry and Hyperfunctions*, 1969, pp. 4. 1–4. 30. (Notes by Kawai, in Japanese).

Sato, M., [5], Structure of hyperfunctions, *Sûgaku no Ayumi*, **15** (1970), 9–72. (Notes by Kashiwara, in Japanese).

Schapira, P., [1], Une équation aux dérivées partielles sans solutions dans l'espace des hyperfonctions, *C. R. Acad. Sci. Paris*, **265** (1967), 665–667.

Schapira, P., [2], Solutions hyperfonctions des équations aux dérivées partielles du premier ordre, *Bull. Soc. Math. France*, **97** (1969), 243–255.

Suzuki, H., [1], Local existence and analyticity of hyperfunction solution of partial differential equations of first order in two independent variables, *J. Math. Soc. Japan*, **23** (1971), 18–26.

Treves, F., [1], Hypoelliptic partial differential equations of principal type with analytic coefficients, *Comm. Pure Appl. Math.* **23** (1970), 637–651.

Notes Added in Proof on November 26, 1971:

1° Recently Professor François Treves has proved Corollary 1.9 for *distribution* solutions. (Cf. his lecture at A.M.S. Summer Institute on partial differential equations held in August, 1971: On the existence and regularity of solutions of linear partial differential equations. See also Treves, F.: Analytic hypoelliptic partial differential equations of principal type, *Comm. Pure Appl. Math.* **24** (1971), 537–570.)

2° Concerning the resolution of solution sheaves of general overdetermined systems which is touched in Remark 3 in p. 422, a rather complete result is given in Sato, M., T. Kawai and M. Kashiwara: On pseudo-differential equations in hyperfunction theory (in preparation), whose summary will appear in *Proc. A.M.S. Summer Institute on Partial Differential Equations* (1971).

3° Recently Professor Karl Gustav Andersson informed the author that he has proved the analogue of the remark announced in p. 424 of this paper. (See Andersson, K.G.: Propagation of analyticity of solutions of differential equations of principal type, to appear in *Bull. Amer. Math. Soc.*)