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Mixed Problem for a Hyperbolic System of the First Order

By

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1. Introduction

The present paper is concerned with a mixed problem for a hyperbolic system of the first order which is assumed symmetric only at the boundary.

Let S be a sufficiently smooth compact hypersurface in R^n and Ω be the interior or exterior domain of S. Consider a hyperbolic operator of the first order

$$L=\frac{\partial}{\partial t}-\sum_{j=1}^{n} A_j(t, x)\frac{\partial}{\partial x_j}-C(t, x) \tag{1.1}$$

$$=\frac{\partial}{\partial t}-\mathscr{M}$$

where $A_j(t, x)$ $(j=1, 2, \dots, n)$ and $C(t, x)$ are $m\times m$ matrices. We will assume that $A_j(t, x)$ and $C(t, x)$ are in $\mathscr{B}((0, T)\times R^n)$.[1] We set a boundary condition

$$u(t, x)\in B(t, x) \qquad \text{on } (0, T)\times S \tag{1.2}$$

where the boundary space $B(t, x)$ is a prescribed subspace of C^m depending smoothly on the point $(t, x)\in(0, T)\times S$.

We consider the following mixed (initial-boundary value) problem

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1) $\mathscr{B}(\omega)$, ω being an open set, is the set of all C^∞ functions defined in ω such that their all partial derivatives of any order are bounded.

$$(1.3)\qquad \begin{cases} L[u]=f(t, x) & \text{in } (0, T)\times\Omega \\ u(t, x)\in B(t, x) & \text{on } (0, T)\times S \\ u(0, x)=g(x) \end{cases}$$

where $u(t, x)$, $f(t, x)$ and $g(x)$ are column-vector of length m.

Let us set

$$\mathscr{A}(t, x, \xi)=\sum_{j=1}^{n} A_j(t, x)\,\xi_j$$

We assume the following

CONDITION I. *There exists a symmetric positive $m\times m$ matrix-valued $\mathscr{B}$-function $\mathscr{R}(t, x, \xi)$ defined in $(0, T)\times R^n\times\{\xi;\ |\xi|=1\}$ with the following properties*:

(i) $\mathscr{R}(t, x, \xi)\mathscr{A}(t, x, \xi)$ *is symmetric for* $(t, x, \xi)\in(0, T)\times R^n\times\{\xi;\ |\xi|=1\}$.

(ii) $\mathscr{R}(t, x, \xi)=I$ *when* $x\in S$.

(iii) $\displaystyle\sum_{i,j=1}^{n}\frac{\partial^2\mathscr{R}}{\partial\xi_i\,\partial x_j}(t, x, \xi)\nu_i(x)\nu_j(x)=0$ *on* $(0, T)\times S$

where $\nu(x)=(\nu_1(x), \nu_2(x),\dots, \nu_n(x))$ *is the unit outer normal of* S *at* $x\in S$.

CONDITION II.

$$\mathscr{A}_\nu(t, x)=\mathscr{A}(t, x, \nu(x)) \quad \textit{is not singular on } (0, T)\times S.$$

For two vectors $u=\{u_1, u_2,\dots, u_m\}$, $v=\{v_1, v_2,\dots, v_m\}$ in C^m we set

$$u\cdot\bar{v}=\sum_{i=1}^{m} u_i\cdot\bar{v}_i.$$

CONDITION III. (*Non-negativity of the boundary condition*).

$$u\cdot\overline{\mathscr{A}_\nu(t, x)\,u}\geqslant 0$$

for any $u\in B(t, x)$.

Remark. (ii) of CONDITION I requires that $A_j(t, x)$ $(j=1, 2, \dots, n)$ are symmetric on the boundary.

We will prove

Theorem 1. *Let* CONDITIONS I, II *and* III *be fulfilled. Then for a solution* $u(t, x)\in\mathscr{E}_t^0(H^k(\Omega))\cap\mathscr{E}_t^1(H^{k-1}(\Omega))\cap\cdots\cap\mathscr{E}_t^k(L^2(\Omega))$[2] ($k$: *positive integer*) *of* (1.3), *if* $f(t, x)\in\mathscr{E}_t^0(H^k(\Omega))\cap\mathscr{E}_t^1(H^{k-1}(\Omega))\cap\cdots\cap\mathscr{E}_t^k(L^2(\Omega))$, *the energy inequality*

$$(1.4)\qquad \|u(t, x)\|_{k,L^2(\Omega)}^2+\|\frac{\partial u}{\partial t}(t, x)\|_{k-1,L^2(\Omega)}^2+\cdots+\|\frac{\partial^k u}{\partial t^k}(t, x)\|_{L^2(\Omega)}^2$$

$$\leqslant C_k\Big\{\|g(x)\|_{k,L^2(\Omega)}^2+\|f(0, x)\|_{k-1,L^2(\Omega)}^2+\cdots+\|\frac{\partial^{k-1}f}{\partial t^{k-1}}(0, x)\|_{L^2(\Omega)}^2$$

$$+\int_0^t(\|f(s, x)\|_{k,L^2(\Omega)}^2+\|\frac{\partial f}{\partial t}(s, x)\|_{k-1,L^2(\Omega)}^2+\cdots+\|\frac{\partial^k f}{\partial t^k}(s, x)\|_{L^2(\Omega)}^2)ds\Big\}$$

holds for $t\in[0, T]$, *where* C_k *does not depend on* u *or* t.

In [1], [2], [3], Agranovič treated the mixed problem (1.3) with a boundary condition

$$(1.5)\qquad u\cdot\overline{\mathscr{A}_\nu(t, x)u}\geqslant p_0u\cdot\bar{u}\qquad\text{for all } u\in B(t, x)\qquad (p_0\text{: positive})$$

instead of CONDITION III without assuming (iii) of CONDITION I. He used essentially the strict positivity of the boundary condition. As Agranovič noted in [1], when $A_j(t, x)$ ($j=1,\dots, n$) are symmetric not only at the boundary but also near the boundary, the strict positive boundary condition can be replaced by a non-negative boundary condition, i.e., a boundary condition satisfying CONDITION III. This fact follows from the results of Lax-Phillips [7] on the dissipative boundary problem of symmetric operators and the considerations of Mizohata [8] and Yamaguti [10] on the energy inequality for hyperbolic equations.

Theorem 1 shows that the energy inequality also holds under a non-negative boundary condition without assuming the symmetricity of $A_j(t, x)$ near the boundary. But we assume one additional condition on $A_j(t, x)$, i.e., (iii) of CONDITION I, which is evidently a condition posed on $A_j(t, x)$ only on the boundary.

Concerning CONDITION I we should like to remark that (ii) is necessary

2) $u(t, x)\in\mathscr{E}_t^k(E)$ means that $u(t,x)$ is k-times continuously differentiable as E-valued function of t.

if one want to treat the problem with a non-negative boundary condition. M. Yamaguti pointed out that for a strictly hyperbolic operator with a parameter $\varepsilon>0$

$$L_\varepsilon=\frac{\partial}{\partial t}-\begin{bmatrix}0 & 1 & \varepsilon\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\frac{\partial}{\partial x_1}-\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1/2\end{bmatrix}\frac{\partial}{\partial x_2}$$

$$=\frac{\partial}{\partial t}-A_{1\varepsilon}\frac{\partial}{\partial x_1}-A_2\frac{\partial}{\partial x_2},$$

there exists a non-negative boundary space B for A_2 such that the mixed problem

$$(1.6)\qquad \begin{cases} L_\varepsilon[u]=f & \text{in } (0,\ T)\times\{(x_1,\ x_2);\ x_2>0\}\\ u\,|_{x_2=0}\in B & \\ u(0,\ x)=g(x) & \end{cases}$$

is not well posed for any $\varepsilon>0$.[3] This fact shows a typical difference of the problems with a non-negative boundary condition from the problems with a strictly positive boundary condition, namely, if B is strictly positive (1.6) is well posed in L^2-sense for sufficiently small ε (see Kreiss [4], Rauch [9]). Then we state a sufficient condition for the existence of $\mathscr{R}(t,\ x,\ \xi)$ satisfying CONDITION I.

Proposition 1. *When L is strictly hyperbolic, i.e., all the eigenvalues $\lambda_j(t,\ x,\ \xi)$ of $\mathscr{A}(t,\ x,\ \xi)$ are real and*

$$\inf_{\substack{(t,x)\in[0,T]\times R^n\\ |\xi|=1,\ i\neq j}} |\lambda_j(t,\ x,\ \xi)-\lambda_i(t,\ x,\ \xi)|\geqq c_0>0.$$

Then when $A_j(t,\ x)$ are symmetric on $[0,\ T]\times S$ and

$$(1.7)\qquad \operatorname{Im}\frac{\partial A_j}{\partial \nu}=c_j(t,\ x)\,I \quad \text{for } j=1,\dots,\ n$$

or

$$(1.8)\qquad \operatorname{Im}\frac{\partial A_j}{\partial \nu}=c(t,\ x)A_j(t,\ x) \quad \text{for } j=1,\dots,\ n$$

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3) This was communicated by M. Yamaguti in the seminary on partial defferential equations of Kyoto Univ. several years ago, but this is not published.

hold on $[0, T]\times S$ *where* $c_j(t, x)$ *and* $c(t, x)$ *are smooth scalar functions, we can construct* $\mathscr{R}(t, x, \xi)$ *satisfying* CONDITION I.

To prove Theorem 1 we consider at first the case where the domain is a half-space and the boundary space $B(t,x)$ is independent of t and x. We introduce a suitable norm attached to the given hyperbolic operator which is equivalent to that of $L^2(R^n_+)$. The construction of this norm is the essential part of the present paper.

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2. The Case Where the Domain Is a Half-Space

In this section we show the energy inequality (1.4) under the assumptions that the domain is a half-space and the boundary space $B(t, x)$ is independent of t and x, namely

$$(2.1)\qquad \begin{cases} L[u]=f(t, x) & \text{in } (0, T)\times R^n_+ \\ u(t, x', 0)\in B & \\ u(0, x)=g(x) & \end{cases}$$

where $R^n_+=\{(x', x_n);\ x'\in R^{n-1}, x_n>0\}$ and B is a constant subspace of C^m.

Notations and preliminary lemmas.

Let Σ be Ω or R^n_+. Denote by $\mathscr{E}(k, \Sigma)$ $(k=0, 1, 2,\dots)$ the space $\mathscr{E}^0_t(H^k(\Sigma))\cap\mathscr{E}^1_t(H^{k-1}(\Sigma))\cap\cdots\cap\mathscr{E}^k_t(L^2(\Sigma))$ and for $u(t, x)\in\mathscr{E}(k, \Sigma)$ define $|||u(t, x)|||_{k,\Sigma}$ by

$$|||u(t, x)|||^2_{k,\Sigma}=\sum_{j=0}^{k}\left|\left|\left(\frac{\partial}{\partial t}\right)^j u(t, x)\right|\right|^2_{k-j,L^2(\Sigma)}.$$

We state a simple lemma without proof.

Lemma 2.1. *Let* $\rho(t)$ *and* $\gamma(t)$ *be two non-negative functions defined on* $[0, T]$. *Suppose that* $\gamma(t)$ *is summable on* $(0, T)$ *and* $\rho(t)$ *is non-decreasing. Then the inequality*

$$\gamma(t) \leqslant c\int_0^t \gamma(s)\,ds + \rho(t) \qquad \textit{for all } t \in [0, T]$$

implies

$$\gamma(t) \leqslant e^{ct}\rho(t) \qquad \textit{for all } t \in [0, T].$$

Next we note some results on pseudo-differential operators. We denote ∂/∂_x by ∂_x and $-i\partial_x$ by D_x. Let $\mathscr{P}(x, \xi)$ be $m \times m$ matrix-valued $C^\infty(R^n \times R^n)$ function. Put

$$(2.2) \qquad |\mathscr{P}|_{p,l} = \sup_{\substack{(x,\xi)\in R^n\times R^n \\ |\alpha|+|\beta|\leqslant l}} |(\partial_x^\alpha \partial_\xi^\beta \mathscr{P})(x, \xi)(1+|\xi|^2)^{\frac{-p+|\beta|}{2}}| \quad (l=0, 1, \dots)$$

and $\mathscr{P} \in S^p$ means that $|\mathscr{P}|_{p,l} < +\infty$ for any integer $l \geqslant 0$. Then $|\cdot|_{p,l}$ $(l=0, 1, 2, \dots)$ defines a topology of S^p. For $\mathscr{P}(x, \xi) \in \bigcup_{p=-\infty}^{\infty} S^p$ we define a pseudo-differential operator $\mathscr{P}(x, D_x)$ by

$$\mathscr{P}(x, D_x)u(x) = (2\pi)^{-n}\int e^{ix\xi}\mathscr{P}(x, \xi)\hat{u}(\xi)\,d\xi$$

for $u(x) \in \mathscr{S}(R^n)$[4)] where

$$\hat{u}(\xi) = \int e^{-ix\xi}u(x)\,dx.$$

The following facts are well known (see, for example, Kumano-go [4]).

Lemma 2.2. *Let $\{\mathscr{P}_\iota(x, \xi)\}$ be a bounded set of S^p. Then the point-wise convergence of $\mathscr{P}_\iota(x, \xi)$ implies the convergence in $S^{p+\varepsilon}$ for any $\varepsilon > 0$.*

Lemma 2.3. (i) *Let $\mathscr{P}(x, \xi) \in S^p$ and $\mathscr{Q}(x, \xi) \in S^q$, then*

$$\mathscr{P}(x, D_x)\cdot\mathscr{Q}(x, D_x) = \sum_{|\alpha|<N} \frac{1}{\alpha!}(\partial_\xi^\alpha \mathscr{P} \circ D_x^\alpha \mathscr{Q})(x, D_x) + \mathscr{R}_N(x, D_x)$$

where $\mathscr{R}_N(x, \xi) \in S^{p+q-N}$.

(ii) *For $\mathscr{P}(x, \xi) \in S^p$ there exists $\mathscr{P}^\#(x, \xi) \in S^p$ such that*

$$(\mathscr{P}(x, D_x)u(x), v(x)) = (u(x), \mathscr{P}^\#(x, D_x)v(x))$$

holds for all $u, v \in \mathscr{S}(R^n)$, and the following expansion

4) $\mathscr{S}(R^n)$ is the set of all rapidly decreasing functions defined in R^n.

$$\mathscr{P}^{\sharp}(x, \xi)=\sum_{|\alpha|<N}\frac{1}{\alpha!}\partial_{\xi}^{\alpha}D_{x}^{\alpha}\,{}^{t}\mathscr{P}(x, \xi)+\mathscr{R}_{N}(x, \xi)$$

holds where ${}^{t}\mathscr{P}(x, \xi)$ *denotes the adjoint matrix of* $\mathscr{P}(x, \xi)$ *and* $\mathscr{R}_{N}(x, \xi) \in S^{p-N}$.

(iii) *For* $\mathscr{P}(x, \xi) \in S^{p}$, *there exists* $C_{s}>0$ *such that*

$$\|\mathscr{P}(x, D_{x})u(x)\|_{s}\leqslant C_{s}\|u\|_{p+s}$$

holds for all $u \in \mathscr{S}(R^{n})$ *and* $s \in R$, *and* $C_{s}\leqslant$ const$|\mathscr{P}|_{p,l_{0}}$ *where the constant and* l_{0} *do not depend on* $\mathscr{P}$.

(iv) *Let* $\mathscr{P}(x, \xi) \in S^{0}$ *and* $|\mathscr{P}(x, \xi)|\geqslant c_{0}$ *for all* x, ξ. *Then*

$$\|\mathscr{P}(x, D_{x})u(x)\|\geqslant c_{0}\|u\|-C\|u\|_{-1}$$

holds for any $u \in \mathscr{S}(R^{n})$.

For a function $u(x)$ defined in R_{+}^{n}, denote by $u_{0}(x)$ the function in R^{n} defined as $u_{0}(x)=u(x)$ for $x_{n}>0$ and $u_{0}(x)=0$ for $x_{n}\leqslant 0$.

Lemma 2.4. *Let* $\mathscr{P}(x, \xi) \in S^{0}$ *and* $\mathscr{P}(x', 0, \xi)=0$ *for all* $x' \in R^{n-1}$ *and* $\xi \in R^{n}$. *Then for any* $u(x) \in H^{1}(R_{+}^{n})$, $\mathscr{P}(x, D_{x})u_{0}(x) \in H^{1}(R^{n})$ *and* $\gamma_{\pm}\mathscr{P}(x, D_{x})u_{0}=0$, *where* $\gamma_{+}(\gamma_{-})$ *denotes the trace operator to the boundary* $x_{n}=0$ *for an element in* $H^{\delta}(R_{+}^{n})(H^{\delta}(R_{-}^{n}))$ $\left(\delta>\frac{1}{2}\right)$.

Proof.[5] Let $\chi(l)$ be $C^{\infty}(R^{1})$ function such that

$$\chi(l)=\begin{cases} 1 & l\leqslant 2 \\ 0 & l\geqslant 3. \end{cases}$$

$$\begin{aligned}\mathscr{P}(x, \xi)&=\chi(x_{n})\mathscr{P}(x, \xi)+(1-\chi(x_{n}))\mathscr{P}(x, \xi)\\ &=\mathscr{P}_{1}(x, \xi)+\mathscr{P}_{2}(x, \xi).\end{aligned}$$

$$\begin{aligned}(1-\chi(x_{n}))\mathscr{P}(x, D_{x})u_{0}&=(1-\chi(x_{n}+1))(1-\chi(x_{n}))\mathscr{P}(x, D_{x})u_{0}\\ &=(1-\chi(x_{n}+1))\{\mathscr{P}(x, D_{x})(1-\chi(x_{n}))u_{0}+[\mathscr{P}(x, D), \chi(x_{n})]u_{0}\}.\end{aligned}$$

Evidently $(1-\chi(x_{n}))u_{0} \in H^{1}(R^{n})$ and

$$\|(1-\chi(x_{n}))u_{0}\|_{1,L^{2}(R^{n})}\leqslant C\|u\|_{1,L^{2}(R_{+}^{n})},$$

5) The proof given here is due to H. Kumano-go.

then

$$\mathscr{P}(x, D_x)(1-\chi(x_n))u_0+[\mathscr{P}(x, D_x), \chi(x_n)]u_0\in H^1(R^n),$$

therefore it follows immediately

$$\gamma_{\pm}\mathscr{P}_2(x, D_x)u_0=0. \tag{2.3}$$

By using the assumption $\mathscr{P}(x', 0, \xi)=0$, we have

$$\mathscr{P}(x, \xi)=x_n\int_0^1\frac{\partial\mathscr{P}}{\partial x_n}(x', tx_n, \xi)dt=x_n\mathscr{P}_3(x, \xi),$$

where $\mathscr{P}_3(x, \xi)=\int_0^1\frac{\partial\mathscr{P}}{\partial x_n}(x', tx_n, \xi)\,dt\in S^0$.

Then

$$\begin{aligned}&\mathscr{P}_1(x, D_x)u_0\\ &\quad=\chi(x_n)x_n\mathscr{P}_3(x, D_x)u_0\\ &\quad=\mathscr{P}_3(x, D_x)\chi(x_n)x_nu_0+[\mathscr{P}_3(x, D_x), \chi(x_n)x_n]u_0.\end{aligned}$$

Since $\chi(x_n)x_nu_0\in H^1(R^n)$ and $[\mathscr{P}_3(x, D_x), \chi(x_n)x_n]\in S^{-1}$, $\mathscr{P}_1(x, D_x)u_0\in H^1(R^n)$.

Let $\chi_1(l)$ be a function in $C_0^\infty(R^1)$ such that

$$\chi_1(l)=\begin{cases}1 & |l|\leqslant 1\\ 0 & |l|\geqslant 2.\end{cases}$$

For any $N>0$

$$\begin{aligned}&\mathscr{P}_1(x, D)u_0\\ &=\mathscr{P}_1(x, D_x)\chi_1((1+|D_x|^2)/N)u_0+\mathscr{P}_1(x, D_x)(1-\chi_1((1+|D_x|^2)/N))u_0.\end{aligned}$$

Then

$$\mathscr{P}_1(x, \xi)\chi_1((1+|\xi|^2)/N)\in S^{-\infty}\quad\text{and}\quad\mathscr{P}_1(x', 0, \xi)=0,$$

it follows that

$$\gamma_{\pm}\mathscr{P}_1(x, D_x)\chi_1((1+|D_x|^2)/N)u_0=0. \tag{2.4}$$

$$\mathscr{P}_1(x, D_x)(1-\chi_1((1+|D_x|^2)/N))u_0$$
$$=\mathscr{P}_3(x, D_x)(1-\chi_1((1+|D_x|^2)/N))x_n\chi(x_n)u_0$$
$$+[\mathscr{P}_3(x, D_x)(1-\chi_1((1+|D_x|^2)/N)), x_n\chi(x_n)]u_0.$$

Since $\{\mathscr{P}_3(x, \xi)(1-\chi_1((1+|\xi|^2)/N))\}$ $(N\geqslant 1)$ is a bounded set in S^0 and converges to zero pointwisely when N tends to ∞, it follows that $\mathscr{P}_3(x, \xi)(1-\chi_1((1+|\xi|^2)/N))$ tends to zero in S^{ε_0} $(\varepsilon_0>0)$ when N increases infinitely by applying Lemma 2.2. Then (iii) of Lemma 2.3 shows that

$$||\mathscr{P}_3(x, D_x)(1-\chi_1((1+|D_x|^2)/N))\chi(x_n)x_nu_0||_{1-\varepsilon_0, L^2(R^n)}$$
$$\leqslant 0(1/N)||u||_{1, L^2(R^n_+)}.$$

By the same reasoning we can see

$$||[\mathscr{P}_3(x, D_x)(1-\chi_1((1+|D_x|^2)/N)), x_n\chi(x_n)]u_0||_{1-\varepsilon_0, L^2(R^n)}$$
$$\leqslant 0(1/N)||u_0||_{L^2(R^n)}.$$

Thus we have, if $1-\varepsilon_0>1/2$,

$$||\gamma_\pm\mathscr{P}_1(x, D_x)(1-\chi_1((1+|D_x|^2)/N))u_0||_{L^2(R^{n-1})}$$
$$\leqslant C||\mathscr{P}_1(x, D_x)(1-\chi_1((1+|D_x|^2)/N))u_0||_{1-\varepsilon_0, L^2(R^n)}$$
$$\leqslant 0(1/N)||u||_{1, L^2(R^n_+)},$$

this shows that

$$\gamma_\pm\mathscr{P}_1(x, D_x)(1-\chi_1((1+|D_x|^2)/N))u_0\longrightarrow 0. \tag{2.5}$$

Thus (2.3), (2.4) and (2.5) imply that $\gamma_\pm\mathscr{P}(x, D_x)u_0=0$.

Q.E.D.

Construction of an operator $\mathscr{H}_0$.

Let $\mathscr{R}(x, \xi)$ be a function in $\mathscr{B}(R^n\times(R^n-\{0\}))$ with the following properties:

$$\mathscr{R}(x, \lambda\xi)=\mathscr{R}(x, \xi) \qquad \text{for} \quad \lambda>0$$

(2.6) $\mathscr{R}(x, \xi)$ is symmetric and $\mathscr{R}(x, \xi) \geqslant c_0 > 0$

(2.7) $\mathscr{R}(x', 0, \xi) = I$ for all $(x', \xi) \in R^{n-1} \times \{\xi : |\xi| = 1\}$

(2.8) $$\frac{\partial^2 \mathscr{R}}{\partial x_n \partial \xi_n}(x', 0, \xi) = 0.$$

Let us put

$$\mathscr{N}_0(x, \xi) = \mathscr{R}(x, \xi)^{1/2} \chi_2(\xi)$$

$$\mathscr{N}_1(x, \xi) = \frac{1}{2} \partial_{\xi_n}^2 \partial_{x_n}^2 \mathscr{N}_0(x', 0, \xi)$$

$$\mathscr{N}(x, \xi) = \mathscr{N}_0(x, \xi) + \mathscr{N}_1(x, \xi)$$

where $\chi_2(\xi) = 0$ for $|\xi| \leqslant 1$ and $= 1$ for $|\xi| \geqslant 2$. Evidently $\mathscr{N}_i(x, \xi)$ $(i = 0, 1)$ are symmetric and $\mathscr{N}_0 \in S^0$, $\mathscr{N}_1 \in S^{-2}$. By using (i) and (ii) of Lemma 2.3 we have

$$\begin{aligned}
&\mathscr{N}(x, D_x)^* \cdot \mathscr{N}(x, D_x) \\
&= (\mathscr{N}_0 \circ \mathscr{N}_0)(x, D_x) \\
&\quad + \sum_{i=1}^{n} \{(\partial_{\xi_i} \mathscr{N}_0 \circ D_{x_i} \mathscr{N}_0)(x, D_x) + (\partial_{\xi_i} D_{x_i} \mathscr{N}_0 \circ \mathscr{N}_0)(x, D_x)\} \\
&\quad + [\sum_{|\alpha|=2} \{(\partial_\xi^\alpha \mathscr{N}_0 \circ D_x^\alpha \mathscr{N}_0)(x, D_x) + (\partial_\xi^\alpha D_x^\alpha \mathscr{N}_0 \circ \mathscr{N}_0)(x, D_x)\} \\
&\quad + \sum_{i,j=1}^{n} (\partial_{\xi_i} \partial_{\xi_j} D_{x_i} \mathscr{N}_0 \circ D_{x_j} \mathscr{N}_0)(x, D) + (\mathscr{N}_1 \circ \mathscr{N}_0)(x, D_x) \\
&\quad + (\mathscr{N}_0 \circ \mathscr{N}_1)(x, D_x)] + \mathscr{R}_3'(x, D_x) \\
&= \mathscr{R}_0'(x, D_x) + \mathscr{R}_1'(x, D_x) + \mathscr{R}_2'(x, D_x) + \mathscr{R}_3'(x, D_x)
\end{aligned}$$

where $\mathscr{R}_i'(x, \xi) \in S^{-i}$ $(i = 0, 1, 2, 3)$. Remark that

$$\mathscr{R}_0'(x', 0, \xi) = I \qquad \text{for} \qquad |\xi| \geqslant 2$$

$$\mathscr{R}_i'(x', 0, \xi) = 0 \qquad \text{for} \qquad |\xi| \geqslant 2 \quad i = 1, 2.$$

These follow directly from the assumption on $\mathscr{R}(x, \xi)$ and the definition of $\mathscr{N}(x, \xi)$. Put

$$\mathscr{R}_i(x, \xi)=\mathscr{R}'_i(x, \xi)-\mathscr{R}'_i(x', 0, \xi) \qquad i=0, 1, 2$$

$$\mathscr{R}_3(x, \xi)=\mathscr{R}'_3(x, \xi)+(\mathscr{R}'_0(x', 0, \xi)-I)+\sum_{i=1}^{2}\mathscr{R}'_i(x', 0, \xi)$$

and we have

$$\mathscr{R}_i(x, \xi)\in S^{-i} \qquad \text{for } i=0, 1, 2, 3$$

$$\mathscr{R}_i(x', 0, \xi)=0 \qquad \text{for } i=0, 1, 2, \tag{2.9}$$

and

$$\mathscr{N}(x, D_x)^*\cdot\mathscr{N}(x, D_x)=I+\sum_{i=0}^{3}\mathscr{R}_i(x, D_x). \tag{2.10}$$

Then it follows from (2.9) and (2.10) by using Lemma 2.4

Lemma 2.5. *For any* $u\in H^1(R^n_+)$, *we have*

$$\mathscr{N}(x, D_x)^*\cdot\mathscr{N}(x, D_x)u_0\in H^1(R^n_+),$$

$$\mathscr{N}(x, D_x)^*\cdot\mathscr{N}(x, D_x)u_0\in H^1(R^n_-)$$

and

$$\gamma_\pm\mathscr{N}(x, D_x)^*\cdot\mathscr{N}(x, D_x)u_0=\gamma_\pm u_0+\gamma_\pm\mathscr{R}_3(x, D_x)u_0. \tag{2.11}$$

Denote by $k(x; y)$ the distribution kernel of the operator $\mathscr{N}(x, D_x)^*\cdot\mathscr{N}(x, D_x)-I$. It is well knwon that $k(x; y)\in C^\infty(R^n\times R^n-\{x=y\})$ and for $x\neq y$

$$k(x; y)=\mathscr{F}_\xi^{-1}\Big(\sum_{i=0}^{3}\mathscr{R}_i(x, \xi)\Big)(x-y). \tag{2.12}$$

By taking account that $(\mathscr{N}^*\cdot\mathscr{N}-I)$ is a self-adjoint operator it follows that for any $x\neq y$

$$k(x; y)={}^tk(y; x), \tag{2.13}$$

where ${}^tk(y; x)$ denotes the adjoint matrix of $k(y; x)$. (2.9) implies that

$$k(x', 0; y)=\mathscr{F}_\xi^{-1}(\mathscr{R}_3(x', 0, \xi))(x'-y),$$

where $(x'-y)$ means the point $(x'-y', y_n)$. Then we can easily see that when $n\geqslant 3$

$$|k(x', 0; y)|\leqslant C|x'-y|^{-n+2} \tag{2.14}$$

$$|\partial_{x_i}k(x', 0; y)|\leqslant C|x'-y|^{-n+2} \quad (i=1, 2,\dots, n-1) \tag{2.15}$$

$$|\partial_{y_i}k(x', 0; y)|\leqslant C|x'-y|^{-n+2} \quad (i=1, 2,\dots, n) \tag{2.16}$$

and when $n=2$

$$|k(x', 0; y)|\leqslant C \tag{2.14$'$}$$

$$|\partial_{x_1}k(x', 0; y)|\leqslant C\log(|x'-y|^{-1}) \tag{2.15$'$}$$

$$|\partial_{y_i}k(x', 0; y)|\leqslant C\log(|x'-y|^{-1}) \quad (i=1, 2). \tag{2.16$'$}$$

Let $\theta(s, t)$ be a real-valued function in $C^\infty(R^2-\{0\})$ with the following properties:

(i) $\theta(s, t)=\theta(\lambda s, \lambda t) \qquad \lambda>0, \quad s^2+t^2\neq 0.$

(ii) $\theta(s, t)>0 \qquad$ when $|t(t^2+s^2)^{-1/2}|>1/2$

$\theta(s, t)=0 \qquad$ when $|t(t^2+s^2)^{-1/2}|\leqslant 1/4$

(iii) $\theta(s, t)+\theta(t, s)=1 \qquad$ when $s^2+t^2\neq 0.$

We define $K(x, y)$ by

$$K(x, y)=\begin{cases} \theta(x_n, y_n)k(x', 0; y', y_n)+\theta(y_n, x_n)^t k(y', 0; x', x_n) & \\ \qquad\qquad\qquad\qquad \text{when } x_n^2+y_n^2\neq 0 \\ k(x', 0; y', 0) \qquad \text{for } x_n=y_n=0. \end{cases}$$

Evidently it holds that

$$K(x, y)={}^tK(y, x)$$

and

$$|K(x, y)|\leqslant C|x'-y'|^{-n+2}.$$

Lemma 2.6. *It holds that when* $n\geqslant 3$

(2.17) $\qquad |\partial_{x_i}K(x, y)| \leqslant C|x'-y'|^{-n+2} \qquad$ *for* $i=1, 2, 3, \cdots, n$

and when $n=2$

(2.17)′ $\qquad |\partial_{x_i}K(x, y)| \leqslant C\log(|x'-y'|^{-1}) \qquad$ *for* $i=1, 2.$

Proof. For $i=1, 2, \ldots, n-1$, (2.17)((2.17)′) follows immediately from (2.15) and (2.16) ((2.15)′ and (2.16)′). We show (2.17) ((2.17′)) for $i=n$. At first assume that $x_n^2+y_n^2 \neq 0$, then

$$\begin{aligned}\partial_{x_n}K(x, y) &= \partial_{x_n}\theta(x_n, y_n)k(x', 0; y', y_n)\\ &\quad +\partial_{x_n}\theta(y_n, x_n)^t k(y', 0; x', x_n)\\ &\quad +\theta(y_n, x_n)\partial_{x_n}{}^t k(y', 0; x'. x_n)\\ &= I+II.\end{aligned}$$

$|II| \leqslant C|x'-y'|^{-n-2}$ $(\log|x'-y'|^{-1})$ follows from (2.16) ((2.16)′).

Since $\partial_{x_n}(\theta(x_n, y_n)+\theta(y_n, x_n))=0$

$$\begin{aligned}I &= \partial_{x_n}\theta(x_n, y_n)(k(x', 0; y', y_n)-k(x', 0; y', 0))\\ &\quad +\partial_{x_n}\theta(y_n, x_n)({}^t k(y', 0; x_n)-{}^t k(y', 0; x', 0)),\end{aligned}$$

here we used ${}^t k(x', 0; y', 0)=k(y', 0; x', 0)$. From the homogeneity of $\theta(s, t)$ we have

$$|\partial_{x_n}\theta(x_n, y_n)| \leqslant C(x_n^2+y_n^2)^{-1/2}.$$

And

$$\begin{aligned}&|k(x', 0; y', y_n)-k(x', 0; y', 0)|\\ &\leqslant C\cdot|y_n|\left|\int_0^1(\partial_{y_n}k)(x', 0; y', ty_n)dt\right|\end{aligned}$$

by using (2.16)((2.16)′)

$$\leqslant C\cdot|y_n|\cdot|x'-y'|^{-n+2} \qquad (C|y_n|\cdot\log(|x'-y'|^{-1})).$$

The second term of I can be estimated in the same way. Thus we get

$$|I|\leqslant C|x'-y'|^{-n+2} \qquad (C\log|x'-y'|^{-1}).$$

Next let us consider the case $x_n=y_n=0$. We have from the definition

$$K(x;\ y',0)={}^tk(y',0;\ x',x_n),$$

then (2.17) ((2.17)') follows immediately from (2.16) ((2.16)').

Q.E.D.

Let $\eta(x)$ be $C^\infty(R^n)$ such that $\eta(x)$ is equal to 1 for $|x|\leqslant 1$ and to 0 for $|x|\geqslant 2$. Define an operator $\mathscr{K}$ from $L^2(R^n)$ into $L^2(R^n)$ by

$$(\mathscr{K}u)(x)=\int \eta(x)K(x;\ y)\,\eta(y)u(y)dy \qquad u(x)\in L^2(R^n).$$

Lemma 2.7. *$\mathscr{K}$ is a self-adjoint operator and*

(2.18) $$\|\mathscr{K}u\|_{1,L^2(R^n)}\leqslant C\|u\|_{L^2(R^n)}$$

(2.19) $$\gamma_\pm\mathscr{K}u=\eta(x',0)\gamma_\pm\mathscr{R}_3(x,D_x)(\eta u).$$

Proof. Let us put $\mathscr{K}(x;\ y)=\eta(x)K(x;\ y)\eta(y)$, then we can see that

$$\sup_{x\in R^n}\int|\mathscr{K}(x;\ y)|dy, \qquad \sup_{x\in R^n}\int|\partial_{x_i}\mathscr{K}(x;\ y)|dy$$
$$\sup_{y\in R^n}\int|\partial_{x_i}\mathscr{K}(x;\ y)|dx$$

are bounded with the aid of Lemma 2.6. The above estimates assure (2.18). And

$$\begin{aligned}\gamma_\pm\mathscr{K}u&=\int \eta(x',0)K(x',0;\ y)\eta(y)u(y)dy\\ &=\eta(x',0)\int k(x',0;\ y',y_n)\eta(y)u(y)dy\\ &=\eta(x',0)\int \mathscr{F}_\xi^{-1}(\mathscr{R}_3(x',0,\xi))(x'-y)\eta(y)u(y)dy\\ &=\eta(x',0)\,(\mathscr{R}_3(x,D_x)\eta u)(x',0)\\ &=\eta(x',0)\gamma_\pm\mathscr{R}_3(x,D_x)\eta u,\end{aligned}$$

this shows (2.19). Q.E.D.

Proposition 2.8. *Define an operator* $\mathscr{H}_0$ *by*

$$\mathscr{H}_0=\eta(x)\cdot\mathscr{N}(x, D_x)^*\cdot\mathscr{N}(x, D_x)\eta(x)-\mathscr{K}.$$

Then for any $u\in H^1(R^n_+)$, $\mathscr{H}_0u_0\in H^1(R^n_+), \in H^1(R^n_-)$ *and*

$$\gamma_\pm\mathscr{H}_0u_0=\eta(x', 0)^2\gamma_\pm u_0. \tag{2.20}$$

Proof. $\mathscr{H}_0u_0\in H^1(R^n_\pm)$ follows immediately from lemma 2.5 and (2.18) since $\eta(x)u\in H^1(R^n_+)$ and $\eta(x)u_0=(\eta(x)u)_0$. From (2.11) and (2.19) we have

$$\begin{aligned}\gamma_\pm\mathscr{H}_0u_0&=\eta(x', 0)\{\gamma_\pm(\eta u_0)+\gamma_\pm\mathscr{R}_3(x, D_x)(\eta u)_0\}\\&\qquad-\eta(x', 0)\gamma_\pm\mathscr{R}_3(x, D_x)(\eta u)_0\\&=\eta(x', 0)^2\gamma_\pm u_0.\end{aligned}$$

This proves (2.20). Q.E.D.

Energy inequality.

Let us suppose that there exists $\mathscr{R}(t, x, \xi)$ satisfying CONDITION I taking S as $\{x; x_n=0\}$. Since $\mathscr{R}(t, x, \xi)$ satisfies (2.6), (2.7) and (2.8) for each t, we can construct for each $t\in[0, T]$ the operator $\mathscr{H}_0(t)$ by the method prescribed in the previous paragraph.

Define $(\ ,\ )_{\mathscr{H}(t)}$ by

$$(u, v)_{\mathscr{H}(t)}=(\mathscr{H}_0(t)u_0, v_0)_{L^2(R^n)}+C((1+|D_x|^2)^{-1}u_0, v_0)_{L^2(R^n)} \tag{2.21}$$

for $u, v\in L^2(R^n_+)$ and $\|u\|_{\mathscr{H}(t)}=((u, u)_{\mathscr{H}(t)})^{1/2}$. Suppose that $\operatorname{supp} u$ is contained in $\{x; x_n\geqslant0, |x|\leqslant1\}$, then

$$\|u\|^2_{\mathscr{H}(t)}=\|\mathscr{N}u_0\|^2_{L^2(R^n)}-(\mathscr{K}u_0, u_0)+C\|u_0\|^2_{-1, L^2(R^n)},$$

and from (iv) of Lemma 2.3 and (2.18)

$$\geqslant c_0(\|u_0\|^2-C'\|u_0\|^2_{-1})-\|\mathscr{K}u_0\|_1\|u_0\|_{-1}+C\|u_0\|^2_{-1}.$$

Therefore if we fix C sufficiently large it follows

Lemma 2.9. *For any $u(x)\in L^2(R^n_+)$ whose support is contained in $\{x;\ x_n\geqslant 0,\ |x|\leqslant 1\}$ it holds that*

$$\|u\|_{L^2(R^n_+)}\leqslant C_0\|u\|_{\mathscr{H}(t)}. \tag{2.22}$$

Lemma 2.10. *For any $u(x)\in H^1(R^n_+)$ with the support contained in $\{x;\ x_n\geqslant 0,\ |x|\leqslant 1\}$, the estimate*

$$\begin{aligned}(2.23)\qquad & 2\,\mathrm{Re}((\mathscr{M}[u])_0,\ \mathscr{H}_0(t)u_0) \\ & \leqslant C\|u\|^2_{L^2(R^n_+)}-\int A_n(x',\ 0,\ t)u(x',\ 0)\cdot\overline{u(x',\ 0)}\ dx'\end{aligned}$$

holds where C is independent of u.

Proof. For the simplicity we omit the parameter t. Let us denote the principal part of $\mathscr{M}$ by $\mathscr{M}_0$.

$$\begin{aligned}&\mathscr{F}_x((\mathscr{M}_0[u])_0)(\xi)\\ &=\mathscr{F}_x(i\mathscr{A}(x,\ D_x)u_0(x))(\xi)-\mathscr{F}_{x'}(A_n(x',\ 0)u(x',\ +0))(\xi').^{6)}\end{aligned}$$

We should like to remark that $(\mathscr{M}_0[u])_0$ is in $L^2(R^n)$ but $i\mathscr{A}(x,\ D_x)u_0(x)$ is not in $L^2(R^n)$, it is an element of $H^{-1}(R^n)$. From the Parseval's identity it follows that

$$\begin{aligned}&((\mathscr{M}_0[u])_0,\ \mathscr{H}_0u_0)\\ &=\int\mathscr{F}_x((\mathscr{M}_0[u])_0)(\xi)\cdot\overline{\mathscr{F}_x(\mathscr{H}_0u_0)(\xi)}\ d\xi\\ &=\lim_{\varepsilon\to+0}\int\chi(\xi_n/\varepsilon)^2\,\mathscr{F}_x((\mathscr{M}_0[u])_0)(\xi)\cdot\overline{\mathscr{F}_x(\mathscr{H}_0u_0)(\xi)}\ d\xi\\ &=\lim_{\varepsilon\to 0}\Big\{\int\chi(\xi_n/\varepsilon)^2\,\mathscr{F}_x(i\mathscr{A}(x,\ D_x)u_0)(\xi)\cdot\overline{\mathscr{F}_x(\mathscr{H}_0u_0)(\xi)}\ d\xi\\ &\qquad-\int\chi(\xi_n/\varepsilon)^2\,\mathscr{F}_{x'}(A_n(x',\ 0)u(x',\ +0)\cdot\overline{\mathscr{F}_x(\mathscr{H}_0u_0)(\xi)}\ d\xi\Big\}\\ &=\lim_{\varepsilon\to+0}(I_\varepsilon-II_\varepsilon),\end{aligned}$$

where $\chi(l)\in C^\infty(R)$ such that

6) $\mathscr{F}_x(u(x))(\xi)$ denotes the Fourier image of $u(x)$.

$$\chi(l) = \begin{cases} 1 & |l| \leqslant 1 \\ 0 & |l| \geqslant 2. \end{cases}$$

$$\begin{aligned} I_\varepsilon &= (\chi(D_n/\varepsilon)^2 i\mathscr{A}(x, D_x)u_0, \mathscr{H}_0 u_0) \\ &= (\mathscr{N}^*\mathscr{N}\cdot\chi(D_n/\varepsilon)^2 i\mathscr{A}(x, D_x)u_0, u_0) \\ &\quad -(\chi(D_n/\varepsilon)^2 i\mathscr{A}(x, D_x)u_0, \mathscr{K} u_0). \end{aligned}$$

$$\begin{aligned} &(\mathscr{N}^*\mathscr{N}\chi(D_n/\varepsilon)^2 i\mathscr{A}(x, D_x)u_0, u_0) \\ &= i(\mathscr{N}^*\mathscr{N}\cdot\mathscr{A}(x, D_x)\chi(D_n/\varepsilon)u_0, \chi(D_n/\varepsilon)u_0) \\ &\quad + i(\mathscr{N}^*\mathscr{N}[\chi(D_n/\varepsilon), \mathscr{A}(x, D_x)]u_0, \chi(D_n/\varepsilon)u_0) \\ &\quad + i([\mathscr{N}^*\mathscr{N}, \chi(D_n/\varepsilon)]\mathscr{A}(x, D_x)u_0, \chi(D_n/\varepsilon)u_0). \end{aligned}$$

Since $\mathscr{N}^*\mathscr{N}\mathscr{A} \in S^1$, Re $i\mathscr{N}(x, \xi)^2\mathscr{A}(x, \xi) \in S^0$ and $\chi(D_n/\varepsilon)u_0 \in H^1(R^n)$, we have

$$\begin{aligned} &\text{Re}\,(i\mathscr{N}^*\mathscr{N}\mathscr{A}\chi(D_n/\varepsilon)u_0, \chi(D_n/\varepsilon)u_0) \\ &\qquad \leqslant \text{const}\,\|\chi(D_n/\varepsilon)u_0\|^2_{L^2(R^n)} \\ &\qquad \leqslant \text{const}\,\|u\|^2_{L^2(R^n_+)} \end{aligned}$$

where the constant is independent of ε. And we see easily

$$\|\mathscr{N}^*\mathscr{N}[\mathscr{A}(x, D_x), \chi(D_n/\varepsilon)]u_0\| \leqslant \text{const}\,\|u_0\|$$

$$\|[\mathscr{N}^*\mathscr{N}, \chi(D_n/\varepsilon)]\mathscr{A}(x, D_x)u_0\| \leqslant \text{const}\,\|u_0\|.$$

Therefore we get

$$\text{Re}\,(i\mathscr{N}^*\mathscr{N}\chi(D_n/\varepsilon)^2\mathscr{A}(x, D_x)u_0, u_0) \leqslant \text{const}\,\|u_0\|^2.$$

By taking account of $\{\chi(D_n/\varepsilon)^2\mathscr{A}(x, D_x)\}_{0\leqslant\varepsilon\leqslant1}$ is bounded in S^1 we have from (iii) of Lemma 2.3

$$\|\chi(D_n/\varepsilon)^2\mathscr{A}(x, D_x)u_0\|_{-1} \leqslant C\|u_0\|$$

then by using (2.18)

$$|(\chi(D_n/\varepsilon)^2 i\mathscr{A}(x, D_x)u_0, \mathscr{K}u_0)|$$

$$\leqslant \|\chi(D_n/\varepsilon)^2 i\mathscr{A}(x, D_x)u_0\|_{-1}\|\mathscr{K}u\|_1$$

$$\leqslant \text{const}\,\|u_0\|^2.$$

Thus the estimate

$$\text{Re } I_\varepsilon \leqslant C\|u_0\|^2 \tag{2.24}$$

holds with a constant independent of ε.

Recall Proposition 2.8 and we see

$$\int \chi((\xi_n/\varepsilon)^2 \mathscr{F}_x(\mathscr{H}_0 u_0)(\xi)d\xi_n \text{ tends to}$$

$$\mathscr{F}_{x'}(1/2(\gamma_+\mathscr{H}_0 u_0 + \gamma_-\mathscr{H}_0 u_0))(\xi') = 1/2\mathscr{F}_{x'}(\gamma_+ u)(\xi')$$

in $L^2(R^{n-1})$ when ε tends to zero. Therefore we have

$$II_\varepsilon \longrightarrow \frac{1}{2}\int \mathscr{F}_{x'}(A_n(x', 0)\gamma_+ u_0)(\xi')\cdot\overline{\mathscr{F}_{x'}(\gamma_+ u)(\xi')d\xi'} \tag{2.25}$$

$$= \frac{1}{2}\int A_n(x', 0)\,\gamma_+ u\cdot\overline{\gamma_+ u}\,dx'.$$

Then (2.24) and (2.25) prove (2.23) since it is evident that

$|((C(t, x)u)_0, \mathscr{H}_0 u_0)| \leqslant \text{const}\,\|u_0\|^2$ holds. Q.E.D.

Proposition 2.11. *Suppose that there exists $\mathscr{R}(t, x, \xi)$ satisfying* CONDITION I *taking S as $\{x; x_n=0\}$. Then for any solution $u(t, x)\in \mathscr{E}(k, R_+^n)$ whose support is contained in $\{x; x_n\geqslant 0, |x|\leqslant 1\}$, if $f(t, x)\in H^k((0, T)\times R_+^n)$, the energy inequality*

$$|||u(t, x)|||_{k,R_+^n}^2 \leqslant C_k\left\{|||u(0, x)|||_{k,R_+^n}^2 + \int_0^t |||f(t, x)|||_{k,R_+^n}^2 dt\right\} \tag{2.26}$$

holds for $t\in[0, T]$, where C_k is independent of u.

Proof. At first let us consider the case $k=1$. Assume that $\partial u/\partial t$ and $\partial u/\partial x_j$ $(j=1, 2,\dots, n-1)$ are also in $\mathscr{E}(1, R_+^n)$. The differentiation in t of (2.1) gives

$$(2.27)\qquad \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial t}\right)=\mathscr{M}\left[\frac{\partial u}{\partial t}\right]+\mathscr{M}_t[u]+\frac{\partial f}{\partial t}(t, x)^{7)}$$

$$=\mathscr{M}\left[\frac{\partial u}{\partial t}\right]+f_1(t, x).$$

Then we have

$$(2.28)\qquad \frac{d}{dt}\|u'(t, x)\|^2_{\mathscr{H}(t)}=2\operatorname{Re}\left(\frac{\partial u'}{\partial t}(t, x), u'(t, x)\right)_{\mathscr{H}(t)}$$

$$+(u'(t, x), u'(t, x))_{\dot{\mathscr{H}}(t)}$$

$$=2\operatorname{Re}((\mathscr{M}[u'])_0, \mathscr{H}_0(t)u'_0)$$

$$+2\operatorname{Re}\left((1+|D_x|^2)^{-1}\frac{\partial u'}{\partial t}, u'\right)+\left((u'(t, x))_0, \frac{d\mathscr{H}_0(t)}{dt}u'_0\right)$$

$$+2\operatorname{Re}(f_1(t, x), \mathscr{H}_0(t)u'(t, x)).$$

Since the boundary space is independent of t, $\partial u/\partial t$ also satisfies the boundary condition, therefore we have by using Lemma 2.10 and the non-negativity of the boundary condition

$$2\operatorname{Re}((\mathscr{M}[u'])_0, \mathscr{H}_0(t)u'_0)\leqslant \text{const}\|u'\|^2_{L^2(R^n_+)}.$$

Since $d\mathscr{H}_0(t)/dt$ is a bounded operator in $L^2(R^n)$ it holds

$$|(u'_0(t, x), d\mathscr{H}_0(t)/dt\; u'_0(t, x))|\leqslant \text{const}\|u'\|^2_{L^2(R^n_+)}.$$

It is evident that

$$|(f_1(t, x), u'(t, x))_{\mathscr{H}(t)}|\leqslant \text{const}\{\|u'(t, x)\|^2_{L^2R(^n_+)}+\|f_1\|^2_{L^2(R^n_+)}\},$$

$$2\operatorname{Re}\int_0^t((1+|D_x|^2)^{-1}u''_0(t, x), u'_0(t, x))dt$$

$$=\|u'_0(t, x)\|^2_{-1}-\|u'_0(0, x)\|^2_{-1}$$

and

7) $\mathscr{M}_t$ ($\mathscr{M}_i$) is the differential operator obtained by differentiating the corresponding coefficients of $\mathscr{M}$ in t (in x_i).

$$\|u_0'(t, x)\|_{-1} \leqslant \|(\mathscr{M}[u])_0\|_{-1} + \|f_0\|_{-1}.$$

Remark that

$$\|(\mathscr{M}[u])_0\|_{-1} \leqslant \text{const}\,(\|u_0\|_{L^2(R)} + \langle A_n(t, x', 0)\gamma_+ u(t, x)\rangle)$$

where $\langle\cdot\rangle$ the norm of $L^2(R^{n-1})$.

Thus we get by integration of (2.27) from 0 to t

$$\begin{aligned} &\|u'(t, x)\|^2_{\mathscr{H}(t)} - \|u'(0, x)\|^2_{\mathscr{H}(0)} \\ &\leqslant C\Big\{\int_0^t \|u'(s, x)\|^2_{\mathscr{H}(s)}\, ds + \int_0^t \|f_1(s, x)\|^2_{L^2(R^n_+)}\, ds \\ &\qquad + \|u(t, x)\|^2_{L^2(R^n_+)} + \langle\gamma_+ u(t, x)\rangle^2\Big\}. \end{aligned}$$

Inserting the estimate

$$\|u(t, x)\|^2_{L^2(R^n_+)} \leqslant 2T\Big(\int_0^t \|u'(s, x)\|^2_{L^2(R^n_+)} ds + \|u(0, x)\|^2_{L^2(R^n_+)}\Big)$$

$$\|f_1(s, x)\|^2_{L^2(R^n_+)} \leqslant \text{const}\,(\|u(s, x)\|^2_{1, L^2(R^n_+)} + \|f'(s, x)\|^2_{L^2(R^n_+)})$$

and by taking account of (2.22) it follows

$$\begin{aligned} (2.29)\quad &\|u'(t, x)\|^2_{L^2(R^n_+)} \\ &\leqslant C\Big\{\int_0^t |||u(s, x)|||^2_{1, R^n_+} ds + |||u(0, x)|||^2_{1, R^n_+} \\ &\qquad + \int_0^t |||f(s, x)|||^2_{1, R^n_+} ds + \langle\gamma_+ u(t, x)\rangle^2\Big\} \qquad \text{for } t \in [0, T]. \end{aligned}$$

Differentiate (2.1) in $x_i\ (1 \leq i \leq n-1)$ and we have

$$(2.30)\qquad \frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x_i}\Big) = \mathscr{M}\Big[\frac{\partial u}{\partial x_i}\Big] + \mathscr{M}_i[u] + \frac{\partial f}{\partial x_i}(t, x).$$

Remark $\partial u/\partial x_i$ also satisfies the boundary condition. Then

$$(2.31)\qquad \frac{d}{dt}\Big\|\frac{\partial u}{\partial x_i}(t, x)\Big\|^2_{\mathscr{H}(t)}$$

$$= 2\,\mathrm{Re}\Big(\Big(\mathscr{M}\Big[\frac{\partial u}{\partial x_i}\Big]\Big)_0, \mathscr{H}_0\Big(\frac{\partial u}{\partial x_i}\Big)_0\Big) + 2\,\mathrm{Re}\Big((1+|D_x|^2)^{-1}\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial x_i}\Big)_0, \Big(\frac{\partial u}{\partial x_i}\Big)_0\Big)$$

$$+\left(\left(\frac{\partial u}{\partial x_i}\right)_0, d\mathscr{H}_0(t)/dt\cdot\left(\frac{\partial u}{\partial x_i}(t, x)\right)_0\right)$$

$$+2\,\mathrm{Re}\left(\mathscr{M}_i[u]+\frac{\partial f}{\partial x_i}, \mathscr{H}_0\left(\frac{\partial u}{\partial x_i}\right)_0\right).$$

If we use the estimate

$$\left|\left((1+|D_x|^2)^{-1}\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x_i}\right)_0, \left(\frac{\partial u}{\partial x_i}\right)_0\right)\right|$$

$$=\left|\left((1+|D_x|^2)^{-1}D_{x_i}^2\frac{\partial u_0}{\partial t}, u_0\right)\right|\leqslant\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(R^n_+)}\|u\|^2_{L^2(R^n_+)}$$

it follows by the same reasoning as that of (2.29)

$$\left\|\frac{\partial u}{\partial x_i}(t, x)\right\|^2_{L^2(R^n_+)}\leqslant C\left\{\int_0^t |||u(s, x)|||^2_{1,R^n_+}ds+|||u(0, x)|||^2_{1,R^n_+}\right.$$
$$\left.+\int_0^t |||f(s, x)|||^2_{1,R^n_+}\,ds\right\}\qquad\text{for } t\in[0, T]. \tag{2.32}$$

Since A_n is not singular

$$\frac{\partial u}{\partial x_n}=A_n^{-1}\left(\frac{\partial u}{\partial t}-\sum_{j=1}^{n-1}A_j\frac{\partial u}{\partial x_j}-Cu-f(t, x)\right), \tag{2.33}$$

and

$$\left\|\frac{\partial u}{\partial x_n}(t, x)\right\|^2_{L^2(R^n_+)}\leqslant\text{const}\left(\left\|\frac{\partial u}{\partial t}(t, x)\right\|^2_{L^2(R^n_+)}+\sum_{j=1}^{n-1}\left\|\frac{\partial u}{\partial x_j}(t, x)\right\|^2_{L^2(R^n_+)}\right.$$
$$\left.+\|u(t, x)\|^2_{L^2(R^n_+)}+\|f(t, x)\|^2_{L^2(R^n_+)}\right) \tag{2.34}$$

inserting (2.29) and (2.32)

$$\leqslant\text{const}\left\{\int_0^t |||u(s, x)|||^2_{1,R^n_+}\,ds+\int_0^t |||f(s, x)|||^2_{1,R^n_+}\,ds\right.$$
$$\left.+|||u(0, x)|||^2_{1,R^n_+}+\langle\gamma_+u(t, x)\rangle^2\right\}.$$

Therefore we have

$$|||u(t, x)|||^2_{1,R^n_+}\leqslant C'\left\{|||u(0, x)|||^2_{1,R^n_+}+\int_0^t |||u(s, x)|||^2_{1,R^n_+}\,ds\right.$$

$$+\int_0^t |||f(s, x)|||^2_{1,R^n_+}\, ds + <\gamma_+ u(t, x)>^2\Big\} .$$

Insert the well known estimate

$$<\gamma_+ u(t, x)>^2 \leqslant \varepsilon ||u(t, x)||^2_{1,L^2(R^n_+)} + C(\varepsilon)||u(t, x)||^2_{L^2(R^n_+)}$$

and take $\varepsilon > 0$ as $C'\varepsilon < 1/2$, then we have

$$|||u(t, x)|||^2_{1,R^n_+} \leqslant C''\Big\{|||u(0, x)|||^2_{1,R^n_+} + \int_0^t |||u(s, x)|||^2_{1,R^n_+}\, ds$$
$$+\int_0^t |||f(s, x)|||^2_{1,R^n_+}\, ds\Big\} .$$

The application of Lemma 2.1 by taking

$$\rho(t) = C''\Big(\int_0^t |||f(s, x)|||^2_{1,R^n_+} ds + |||u(0, x)|||^2_{1,R^n_+}\Big)$$

$$\gamma(t) = |||u(t, x)|||^2_{1,R^n_+}$$

leads (2.26).

The energy inequalities of higher order (the case $k \geqslant 2$) are derived step by step. Let $k=2$. Assume that $\partial u/\partial t$ and $\partial u/\partial x_i$ ($i=1, 2,\dots, n-1$) are also in $\mathscr{E}(2, R^n_+)$. Apply the just obtained result to (2.27), then

$$\Big|\Big|\Big|\frac{\partial u}{\partial t}(t, x)\Big|\Big|\Big|^2_{1,R^n_+} \leqslant C\Big\{\Big|\Big|\Big|\frac{\partial u}{\partial t}(0, x)\Big|\Big|\Big|^2_{1,R^n_+} + \int_0^t |||\mathscr{M}_t[u] + f'_t(t, s)|||^2_{1,R^n_+} dt\Big\}$$
$$\leqslant C\Big\{|||u(0, x)|||^2_{2,R^n_+} + \int_0^t |||u(t, x)|||^2_{2,R^n_+} ds$$
$$+\int_0^t |||f(s, x)|||^2_{2,R^n_+}\, ds\Big\} .$$

Similarly it holds that for $i=1, 2,\dots, n-1$

$$\Big|\Big|\Big|\frac{\partial u}{\partial x_i}(t, x)\Big|\Big|\Big|^2_{1,R_+} \leqslant C\Big\{|||u(0, x)|||^2_{2,R^n_+} + \int_0^t |||u(s, x)|||^2_{2,R^n_+} ds$$
$$+\int_0^t |||f(s, x)|||^2_{2,R^n_+}\, ds\Big\} .$$

By using (2.33) we have

$$|||u(t, x)|||^2_{2,R^n_+} \leqslant \text{const}\ \Big\{\Big|\Big|\Big|\frac{\partial u}{\partial t}(t, x)\Big|\Big|\Big|^2_{1,R^n_+} + \sum_{i=1}^{n-1}\Big|\Big|\Big|\frac{\partial u}{\partial x_i}(t, x)\Big|\Big|\Big|^2_{1,R^n_+}$$

$$+ |||f(t, x)|||_{1,R_+^n}^2 + |||u(t, x)|||_{1,R_+^n}^2 \Big\}$$

$$\leqslant C\Big\{|||u(0, x)|||_{2,R_+^n}^2 + \int_0^t |||u(s, x)|||_{2,R_+^n}^2 ds + \int_0^t |||f(s, x)|||_{2,R_+^n}^2 ds\Big\}.$$

The application of Lemma 2.2 derives (2.26) for $k=2$. Repeating this reasoning step by step, (2.26) is proved for any k.

In order to complete the proof we have to remove the additional assumption that $\partial u/\partial t$ and $\partial u/\partial x_i$ ($i=1, 2,\cdots, n-1$) are also in $\mathscr{E}(k, R_+^n)$. So we make use of a mollifier with respect to x' and t. Since the boundary condition is independent of x' and t we can achieve the reasoning by the method used in [5]. Then we omit the proof.

3. Proof of Theorem 1

Proposition 3.1.[8)] *For any* $(t_0, s_0) \in [0, T] \times S$, *there exist a neighborhood* U *of* (t_0, s_0) *in* $(-\delta, T+\delta)\times S$ $(\delta>0)$ *and a smooth unitary matrix-valued function* $T(t, x)$ *defined in* U *such that* $u \in B(t, x)$ *is equivalent to* $T(t, x)u \in \tilde{B}$ *for* $(t, x) \in U$ *where* $\tilde{B}$ *is a subspace of* C^m *independent of* (t, x).

Proof. Let the dimension of $B(t, x)$ be p and $e_i(t, x) = \{e_{i1}(t, x), e_{j2}(t, x),\cdots, e_{im}(t, x)\}$ $(i=1, 2,\cdots, p)$ be a smooth orthogonal base of $B(t, x)$ in U. This is possible when U is sufficiently small. Choose $e_i(t, x)$ $(i=p+1,\cdots, m)$ as $e_i(t, x)$ $(i=1, 2,\cdots, m)$ form a smooth orthonormal base of C^m. Define $T(t, x)$ by

$$T(t, x)=[e_{ij}(t, x)]_{i,j=1, 2,\dots, m}.$$

Evidently $T(t, x)$ is unitary and a smooth function. And if we put $v= T(t, x)\, u$, $u \in B(t, x)$ is equivalent to $v_i=0$ $(i=p+1, p+2,\cdots, m)$. This proves Proposition. Q.E.D.

Let $\{\varphi_j(x)\}_{j=1}^N$ be a partition of unity in a neighborhood of S, namely $\varphi_j(x) \in C_0^\infty(R^n)$ such that

8) This fact is already pointed out in Lax-Phillips [7].

$$\sum_{j=1}^{N} \varphi_j(x)^2 = 1 \qquad \text{in a neighborhood of } S.$$

Assume that the support of φ_j is contained in a sufficiently small neighborhood of $s_j=(s_{j1}, s_{j2},\cdots, s_{jn})$ such that S is represented by an equation

$$x_{i_0}=\psi_j(x_1,\cdots, x_{i_0-1}, x_{i_0+1},\cdots, x_n) \qquad \text{in } U.$$

Define a transformation $\Psi_j(x)$ by

$$y_i=x_i-s_{ji} \qquad i<i_0$$

$$y_i=x_{i+1}-s_{ji+1} \qquad i>i_0$$

$$y_n=x_{i_0}-\psi_j(x_1,\cdots, x_{i_0-1}, x_{i_0+1},\cdots, x_n)$$

and assume that $\Psi_j(U_j)=V_j \subset \{y;\ |y|\leqslant 1\}$ and $\Psi_j(U_j\cap\Omega)=V_j\cap R^n_+$. From Proposition 3.1, we can assume that, if U_j is sufficiently small, there exists a smooth unitary matrix-valued function $T_j(t, y')$ defined in $[0, t_0]\times(V_j\cap\{y_n=0\})$ (t_0 is some positive constant) such that $T_j(t, \Psi_j(x))B(t, x)$ is independent of $(t, x)\in[0, t_0]\times(S\cap U_j)$. For a function $w(x)$ defined in $U_j\cap\Omega$ we denote by $\tilde{w}_j(y)$ a function defined in $V_j\cap R^n_+$ by $\tilde{w}_j(y)=\tilde{w}_j(\Psi_j(x))=w(x)$.

Let us put $u_j(t, y)=T_j(t, y)\widetilde{(\varphi_j u)}_j(t, y)$, then we have

$$(3.1) \qquad \begin{cases} L_j[u_j]=f_j & \text{in } [0, t_0]\times R^n_+ \\ u_j(t, y', 0)\in B_j \end{cases}$$

where, if $i_0=n$,

$$L_j=\frac{\partial}{\partial t}-\sum_{k=1}^{n} A_{jk}(t, y)\frac{\partial}{\partial y_k}-C_j(t, y)$$

$$A_{jk}(t, y)=T_j(t, y')\widetilde{(A_k)}_j(t, y)T_j(t, y')^* \qquad (k=1,\cdots, n-1)$$

$$A_{jn}(t, y)=T_j(t, y')\widetilde{\left(-\sum_{l=1}^{n-1} A_l\frac{\partial\psi_j}{\partial x_l}+A_{nj}\right)}(t, y)T_j(t, y')^*$$

$$C_j(t, y)=T_j(t, y')\tilde{C}_j(t, y)T_j(t, y')^*$$

$$-T_j(t,\ y')\sum_{l=1}^{n} A_{jl}(t,\ y')T_j(t,\ y)^*\frac{\partial T_j}{\partial y_l}(t,\ y')$$

$$f_j(t,\ y)=T_j(t,\ y')(\sum_{l=1}^{n} A_l\frac{\partial \varphi_j}{\partial x_l}u+\ \varphi_j f)\tilde{}_j(t,\ y)$$

and B_j is a p-dimensional subspace of C^m independent of t and y'.

For $\mathscr{R}(t, x, \xi)$ satisfying CONDITION I given in Introduction, let us define

$$\mathscr{R}_j(t,\ y,\ \eta)=T_j(t,\ y')\mathscr{R}\Big(t,\ \Psi^{-1}(y), \frac{\partial\Psi}{\partial x}\cdot\eta\Big)T_j(t,\ y')^*.$$

It is evident that $\mathscr{R}_j(t, y, \eta)$ is symmetric positive, $\mathscr{R}_j(t,\ y,\ \eta)\sum_{k=1}^{n} A_{jk}(t,\ y)\eta_k$ is symmetric and $\mathscr{R}_j(t, y', 0, \eta)=I$. It follows from (ii) of CONDITION I

$$\sum_{j=1}^{n}\frac{\partial\mathscr{R}}{\partial x_j}(t,\ x,\ \xi)\mu_j(x)=0 \qquad \text{on } S$$

if $\sum_{j=1}^{n}\mu_j(x)\cdot\nu_j(x)=0$. Therefore

$$\sum_{j=1}^{n}\frac{\partial}{\partial\xi_i}\Big(\sum_{j=1}^{n}\frac{\partial\mathscr{R}}{\partial x_j}(t,\ x,\ \xi)\,\mu_j\,(x)\Big)\nu_i(x)=0 \qquad \text{on } S.$$

(iii) of CONDITION I means that

$$\sum_{i=1}^{n}\frac{\partial}{\partial\xi_i}\Big(\sum_{j=1}^{n}\frac{\partial\mathscr{R}}{\partial x_j}(t,\ x,\ \xi)\nu_j(x)\Big)\nu_i(x)=0 \qquad \text{on } S.$$

Therefore we have

$$\sum_{i=1}^{n}\frac{\partial}{\partial\xi_i}\frac{\partial\mathscr{R}}{\partial x_n}(t,\ x,\ \xi)\nu_i(x)=0 \qquad \text{on } S. \tag{3.2}$$

By taking account of the definition of Ψ_j we can see

$$\frac{\partial^2\mathscr{R}_j}{\partial\eta_n\partial y_n}(t,\ y,\ \eta)=T_j(t,\ y')\Big(\sum_{i=1}^{n}\frac{\partial^2\mathscr{R}}{\partial x_n\partial\xi_i}\nu_i\Big)T_j(t,\ y')^*,$$

thus from (3.2) we have

$$\frac{\partial^2\mathscr{R}_j}{\partial\eta_n\partial y_n}(t,\ y',\ 0,\ \eta)=0.$$

Therefore Proposition 2.11 is applicable to (3.1) for each j. Suppose $u(t, x) \in \mathscr{E}(k, \Omega)$, then $u_j(t, y) \in \mathscr{E}(k, R_+^n)$, and we have

$$\|\!|u_j(t, y)|\!\|_{k,R_+^n}^2 \leqslant C\left\{\|\!|u_j(0, y)|\!\|_{k,R_+^n}^2 + \int_0^t \|\!|f_j(s, x)|\!\|_{k,R_+^n}^2\, ds\right\} \tag{3.3}$$

for $t \in [0, t_0]$. And it holds that

$$\begin{aligned}\|\!|(1-\sum_{j=1}^N \varphi_j^2)^{1/2} u(t, x)|\!\|_{k,\Omega}^2 \\ \leqslant C\left\{\|\!|u(0, x)|\!\|_{k,\Omega}^2 + \int_0^t \|\!|f(s, x)|\!\|_{k,\Omega}^2\, ds + \int_0^t \|\!|u(s, x)|\!\|_{k,\Omega}^2\, ds\right\}\end{aligned} \tag{3.4}$$

since $\operatorname{supp}((1-\sum \varphi_j^2)^{1/2} u) \cap S = \phi$. And we have

$$\begin{aligned}\|\!|u(t, x)|\!\|_{k,\Omega}^2 \leqslant \text{const}\ \{&\sum_{j=1}^N \|\!|u_j(t, y)|\!\|_{k,R_+^n}^2 \\ &+ \|\!|(1-\sum_{j=1}^N \varphi_j^2)^{1/2} u(t, x)|\!\|_{k,\Omega}^2 + \|\!|u(t, x)|\!\|_{k-1,\Omega}^2\}.\end{aligned} \tag{3.5}$$

Then it follows from (3.3), (3.4) and (3.5) that

$$\begin{aligned}\|\!|u(t, x)|\!\|_{k,\Omega}^2 \leqslant \text{const}\Big\{&\|\!|u(0, x)|\!\|_{k,\Omega}^2 + \int_0^t \|\!|u(s, x)|\!\|_{k,\Omega}^2\, ds \\ &+ \int_0^t \|\!|f(s, x)|\!\|_{k,\Omega}^2\, ds\Big\}\end{aligned} \tag{3.6}$$

with the aid of the estimates

$$\|\!|u(t, x)|\!\|_{k-1,\Omega}^2 \leqslant \text{const}\left\{\int_0^t \|\!|u(s, x)|\!\|_{k,\Omega}^2 ds + \|\!|u(0, x)|\!\|_{k-1,\Omega}^2\right\}$$

$$\|\!|f_j(t, x)|\!\|_{k,R_+^n}^2 \leqslant \text{const}\ \{\|\!|f(t, x)|\!\|_{k,\Omega}^2 + \|\!|u(t, x)|\!\|_{k,\Omega}^2\}.$$

The application of Lemma 2.2 to (3.6) proves Theorem 1. Q.E.D.

4. Proof of Proposition 1

In the previous section we saw that CONDITION I is invariant with a change of variables of the type Ψ_j. Therefore we may restrict ourselves to the case where the domain is a half-space. Remark that it suffices to construct $\mathscr{R}(t, x, \xi)$ locally in x and ξ. Let $e_i(t, x, \xi) = \{e_{i1}, \cdots, e_{im}\}$ be an eigenvector of $\mathscr{A}(t, x, \xi)$ for an eigenvalue $\lambda_i(t, x, \xi)$ such that $|e_i(t, x, \xi)| = 1$. Define $\mathscr{R}(t, x, \xi)$ by

$$\mathscr{R}(t, x, \xi)=[e_i(t, x, \xi)\cdot\overline{e_j(t, x, \xi)}]_{i,j=1,\dots,m}. \tag{4.1}$$

Evidently $\mathscr{R}(t, x, \xi)$ satisfies (i) and (ii) of CONDITION I. Then it suffices to show

$$\frac{\partial^2}{\partial x_n \partial \xi_n} e_i(t, x, \xi)\cdot\overline{e_j(t, x, \xi)}\,|_{x_n=0}=0. \tag{4.2}$$

For $i=j$, (4.2) is trivial. Suppose that $i\neq j$.

$$\frac{\partial}{\partial x_n}\{(\lambda_i-\lambda_j)e_i\cdot\overline{e_j}\}=\frac{\partial}{\partial x_n}\{(\mathscr{A}(t, x, \xi)-{}^t\mathscr{A}(t, x, \xi))e_i\cdot\overline{e_j}\}$$

$$=\frac{\partial(\mathscr{A}-{}^t\mathscr{A})}{\partial x_n}e_i\cdot\overline{e_j}+(\mathscr{A}-{}^t\mathscr{A})\frac{\partial e_i}{\partial x_n}\cdot\overline{e_j}+$$

$$+(\mathscr{A}-{}^t\mathscr{A})e_i\cdot\frac{\partial\overline{e_j}}{\partial x_n}.$$

Put $x_n=0$ and take account of the symmetricity of $\mathscr{A}(x', 0, \xi)$ and we have

$$\frac{\partial}{\partial x_n}\{(\lambda_i-\lambda_j)e_i\cdot\overline{e_j}\}\Big|_{x_n=0}=i\sum_{l=1}^{n}\mathrm{Im}\frac{\partial A_l}{\partial x_n}(t, x', 0)\xi_l e_i\cdot\overline{e_j}. \tag{4.3}$$

Therefore when (1.7) holds

$$\sum_{l=1}^{n}\mathrm{Im}\frac{\partial A_l(t, x', 0)}{\partial x_n}\xi_l e_i\cdot\overline{e_j}$$

$$=(\sum_{l=1}^{n}c_l\xi_l)(e_i\cdot\overline{e_j})=0.$$

When (1.8) holds we see also the right-hand side of (4.3) vanishes. Thus

$$\frac{\partial}{\partial x_n}e_i\cdot\overline{e_j}\Big|_{x_n=0}=0 \qquad \text{for } i\neq j.$$

This proves Proposition. Q.E.D.

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